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A unified input-output approach for networked control problems with decentralized and selfish optimality

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Abstract— **In this paper, we take an input-output approach to enhance the study of cooperative multiagent optimization problems that admit decentralized and selfish solutions, hence eliminating the need for an interagent communication network. The framework under investigation is a set of n independent agents coupled only through an overall cost that penalizes the divergence of each agent from the average collective behavior.**

In the case of identical agents, or more generally agents with identical essential input-output dynamics, we show that optimal decentralized and selfish solutions are possible in a variety of standard input-output cost criteria. These include the cases of ℓ_1, ℓ_2, ℓ_∞ induced, and $\mathcal{H}_2$ norms for any finite n. Moreover, if the cost includes non-deviation from average variables, the above results hold true as well for ℓ_1, ℓ_2, ℓ_∞ induced norms and any n, while they hold true for the normalized, per-agent square $\mathcal{H}_2$ norm, cost as $n \to \infty$.

We also consider the case of nonidentical agent dynamics and prove that similar results hold asymptotically as $n \to \infty$ in the case of ℓ_2 induced norms (i.e., $\mathcal{H}_\infty$) under a growth assumption on the $\mathcal{H}_\infty$ norm of the essential dynamics of the collective.

I. Introduction

The study of networked systems has become a main area of research in recent years and there is a plethora of open questions pertaining to various aspects such as structural properties like controllability/observability [1], [2] and [3], performance, or noise and uncertainty amplification [4], [5], [6], distributed controller design [9] —[13] among others.

In this paper we are interested in originally disconnected multi-agent systems that share a common social cost. The cost represents an input-output performance measure of the overall system where performance is measured as a function of the distance of each agent's output from the corresponding average output of the collective. This second characteristic is similar to that of cost functions used in Mean Field games e.g., [18], [15], [14], [19],[17], [20].

In general, the optimal controller is expected to be distributed over the network which requires each agent to exchange information with all the others agents. For this reason, such optimal controller is called centralized. However, this solution does not scale well when the number of agents, n, is large. Hence it is of interest to know when decentralized solutions are a good approximation of the optimal centralized ones. Some answers are available in specific, state-space based, problem formulations. For example, the approximation method used in the $\mathcal{H}_2$ norm minimization of spatially invariant systems [7] shows that the optimal controller has exponentially decaying tails in the spatial domain and hence it can be approximated by a local one by truncating the tails. In the mean field approaches (e.g., [15]), a decentralized suboptimal strategy is obtained by replacing the actual average measurements with a deterministic input representing the mass behavior. This input under appropriate assumptions is shown to well approximate, in the limit of large n, the expected value of the average measurement signals, and can be computed independently by each agent.

In this paper, we focus on input-output approaches to investigate how decentralization, and hence lack of communication, between a large number of dynamically decoupled LTI agents affects the overall closed loop performance, captured by input-output norms that encode deviation from some average type of behavior. In particular, we present a unified and streamlined approach that enriches with insights and expands our initial developments in [22], [23] for a variety of standard input-output norms.

For the case of identical agents, we show that optimality is achieved by decentralized and selfish solutions in the cases of ℓ_1, ℓ_∞ induced, $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms for any finite n. Moreover, if the cost includes non-deviation from average variables, the above results also hold true for ℓ_1, ℓ_∞ induced and $\mathcal{H}_\infty$ norms for any n. For the normalized, per-agent square $\mathcal{H}_2$ norm, they also hold true as $n \to \infty$. By a decentralized and selfish solution we mean that each agent can totally disregard its deviation from average, i.e., the social coupling, and perform an optimization of its own local regulated variables based on its own local measurements. This turns out to be optimal for the collective. In fact, this is also the case for nonidentical agents which have the same essential input-output dynamics and hence the underlying model matching problems become identical. For the case of nonidentical essential dynamics, we prove that similar results hold asymptotically as $n \to \infty$ in the case of ℓ_2 induced norms (i.e., $\mathcal{H}_\infty$) under a sublinear growth assumption on the $\mathcal{H}_\infty$ norm of the essential dynamics of the collective.

The paper is organized as follows. In section 2, a basic norm minimization problem is posed in terms of a Youla-Kucera (Y-K) parametrization of all stabilizing, possibly centralized, controllers. In section 3, we proceed to its solution as well as to the solutions of more complicated

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versions for the case of identical agents. In section 4, we analyze the case of nonidentical agents and conclude in section 5.

Some basic notation is as follows. For a real sequence $M = \{M(k)\}_{k=0}^{\infty}$ we use the ℓ_1-norm $||M||_1 := \sum_k |M(k)|$, the ℓ_2 or $\mathcal{H}_2$ norm $||M||_2 := [\sum_k M(k)^2]^{1/2}$, and the ℓ_∞-norm $||M||_\infty := \sup_k |M(k)|$. For a real sequence of matrices $M = [M_{ij}] = \{M(k)\}_{k=0}^{\infty}$ we use the ℓ_1-induced norm $||M||_{1i} := \max_j \sum_i ||M_{ij}||_1$; the ℓ_∞-induced norm $||M||_{\infty i} := \max_i \sum_j ||M_{ij}||_1$ and the ℓ_2 or $\mathcal{H}_2$ norm $||M||_2 := [\sum_{i,j} ||M_{ij}||_2^2]^{1/2}$. These norms will be used for norms of LTI systems when viewed in terms of their pulse response. If M is a transfer function $||M||_{\mathcal{H}_\infty} := \sup_\omega \sigma_{\max}[M(e^{\jmath\omega})]$, where $\sigma_{\max}$ stands for the maximum singular value; we note that $\mathcal{H}_\infty$ is the ℓ_2-induced norm of the map M. Also, in the discussions that follow we often use the $|| \bullet ||$ notation generically up to a point before we clearly specify what are the norms that the results apply.

II. Problem Definition

We consider n dynamically decoupled systems $\{G_i\}_{i=1}^n$. Each G_i has control input u_i, measurement output y_i, disturbance w_i and regulated variable z_i. Let $z = \Phi w$ where $z = [z_i]_{1\le i\le n}$, $w = [w_i]_{1\le i\le n}$ are respectively the vectors of regulated and disturbance signals, and Φ is the closed loop when each G_i is in feedback with its corresponding controller K_i. We allow at this point each K_i to be connected to the other controllers K_j, thus the overall controller K, given by the relation $u = Ky$ where y and u are the concatenated measurements and control signals y_i and u_i respectively, can be a full matrix. For any K that stabilizes the overall system of G_i's the corresponding Φ can be obtained via a Youla-Kucera parametrization as

$$\Phi = w \mapsto z = H - UQV$$

where $H = \mathrm{diag}(H_1, \dots, H_n)$, $U = \mathrm{diag}(U_1, \dots, U_n)$, $V = \mathrm{diag}(V_1, \dots, V_n)$ are diagonal stable systems the elements of which can be obtained from standard factorizations of the individual G_i's. The system Q is also stable but can be a full matrix of stable systems $[Q_{ij}]_{1\le i,j\le n}$. In view of this we have

$$\Phi = \begin{bmatrix} H_1 - U_1Q_{11}V_1 & -U_1Q_{12}V_2 & \cdots \\ -U_2Q_{21}V_1 & H_2 - U_2Q_{22}V_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \quad (1)$$

We are interested in optimizing system's performance with respect to a variable that measures deviations from population average. In this sense, defining

$$e_i := z_i - \bar{z}, \quad \bar{z} := \frac{1}{n}(z_1 + \cdots + z_n), \quad e := [e_i]_{1\le i\le n}$$

we are interested in the map

$$\Psi := w \mapsto e$$

which can be expressed as

$$\Psi = (I - \frac{1}{n}\mathbf{1}\mathbf{1}^T)\Phi = \Phi - \bar{\Phi}$$

with

$$\bar{\Phi} = \frac{1}{n}\mathbf{1}\mathbf{1}^T\Phi = [\mathbf{1}\bar{\Phi}_1 \dots \mathbf{1}\bar{\Phi}_n]$$

where $\mathbf{1}$ is a vector of 1's and

$$\bar{\Phi}_j = \frac{1}{n}H_j - \frac{1}{n}(U_1Q_{1j}V_j + U_2Q_{2j}V_j + \cdots + U_nQ_{nj}V_j)$$

We let $\mathbf{T}$ denote the operator $\mathbf{T} \triangleq (I - \frac{1}{n}\mathbf{1}\mathbf{1}^T)$. Note that $\mathbf{T}^2 = \mathbf{T}$ and $\mathbf{T}\mathbf{1} = 0$.

A basic problem of interest is to find the controller to minimize some norm of Ψ i.e.,

$$\psi^o := \inf_Q ||\Psi|| \quad (2)$$

In later sections we consider more elaborate problems which involve norms of other than deviation form average signals. Also, since the case of MIMO problems follow the same path, and to avoid unnecessary complexity, we will assume that both w_i and z_i are scalar signals.

III. Identical Systems

We first consider the case where the systems G_i are identical in which case $H_i = \bar{H}$, $U_i = \bar{U}$, and $V_i = \bar{V}$ for all $i = 1, \dots n$. Looking at a fixed j,

$$\bar{\Phi}_j = \frac{1}{n}\bar{H} - \bar{U}\bar{Q}_j\bar{V}$$

where

$$\bar{Q}_j := \frac{1}{n}(Q_{1j} + \cdots + Q_{nj})$$

and hence

$$\begin{aligned} \Psi_{jj} &= \bar{H} - \bar{U}Q_{jj}\bar{V} - (\tfrac{1}{n}\bar{H} - \bar{U}\bar{Q}_j\bar{V}) \\ &= \tfrac{n-1}{n}(\bar{H} - \bar{U}[\tfrac{n}{n-1}(Q_{jj} - \bar{Q}_j)]\bar{V}) \end{aligned}$$

and for any $i \neq j$

$$\begin{aligned} \Psi_{ij} &= -\bar{U}Q_{ij}\bar{V} - (\tfrac{1}{n}\bar{H} - \bar{U}\bar{Q}_j\bar{V}) \\ &= -\tfrac{1}{n}(\bar{H} - \bar{U}[n(\bar{Q}_j - Q_{ij})]\bar{V}) \end{aligned}$$

Because of symmetry the following can be argued about the optimizing Q.

Lemma 1: For Problem (2) it is enough to search for Q with structure

$$Q = \begin{bmatrix} q & 0 & \dots & 0 \\ 0 & q & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & q \end{bmatrix} - \begin{bmatrix} \theta & \theta & \dots & \theta \\ \theta & \theta & \ddots & \vdots \\ \vdots & \ddots & \ddots & \theta \\ \theta & \dots & \theta & \theta \end{bmatrix}.$$

or we can write Q in a compact form

$$Q = I\,q - \mathbf{1}\theta\mathbf{1}^T.$$

the diagonal part of Q, (which depends on q) represents the decentralized part of the controller. While the second term, function of θ represents the part of the controller that couples all nodes.

Let

$$\varphi^o := \inf_q ||f(q)|| \quad (3)$$

where

$$f(q) := \bar{H} - \bar{U} q \bar{V}$$

then, the following can be claimed.

Proposition 1: Consider Problem (2). Then, decentralized control K^o obtained by solving Problem (3) optimizes performance in Problem (2). Moreover, $\psi^o = \kappa\varphi^o$ where $\kappa = ||I - \frac{1}{n}\mathbf{1}\mathbf{1}^T||$.

Note for the various choices of norms we have that

$$\kappa = (n-1)^{1/2};\ 1;\ 2(n-1)/n;\ 2(n-1)/n$$

respectively for the

$$\mathcal{H}_2;\ \mathcal{H}_\infty;\ \ell_1\text{-induced};\ \ell_\infty\text{-induced}$$

norm performance. The above expressions for κ can be easily obtained by direct computation; the case of the $\mathcal{H}_\infty$ norm follows from the fact that $\sigma_{\max}[aI - \frac{1}{n}\mathbf{1}b\mathbf{1}^T] = \max(|a|, |a-b|)$ (e.g., [21].)

Summarizing, the above result says that Problem (3), although the the cost function couples the nodes, is completely solved by each agent independently where each agent minimizes its own local cost disregarding the mass average $\bar{z}$.

We next study the extension of the above result to constrained and two-block problems.

A. Constrained Problems

Herein we consider the same problem as above with the addition of a norm constraint. In particular, if we let $\xi = \Xi w$ be an additional signal of interest with Ξ the corresponding closed loop map, we are considering

$$\psi^o := \inf_{Q:||\Xi|| \le \gamma} ||\Psi|| \quad (4)$$

where $\gamma > 0$ is a positive constant, and the norms in the cost and constraints are the same, for simplicity. The map Ξ can capture regular, not necessarily deviation from average, signals such as absolute control actions (not relative to the mass average). For instance, in the case of stable systems G_i with $\xi = u$ as the signal of interest, $\Xi = Q$. In general, we will have that

$$\Xi = H_\xi - U_\xi Q V_\xi$$

where $H_\xi = \text{diag}(\bar{H}_\xi, \ldots, \bar{H}_\xi)$, $U_\xi = \text{diag}(\bar{U}_\xi, \ldots, \bar{U}_\xi)$, $V_\xi = \text{diag}(\bar{V}_\xi, \ldots, \bar{V}_\xi)$ are, as in before in expressing Φ, diagonal stable systems the elements of which can be obtained from standard factorizations of the individual G_i's; as before, Q can be a full matrix of stable systems $[Q_{ij}]_{1\le i,j \le n}$.

Define now

$$g(q) := \bar{H}_\xi - \bar{U}_\xi q \bar{V}_\xi$$

and

$$\varphi^o := \inf_{q:||g(q)|| \le \gamma} ||f(q)|| \quad (5)$$

with $f(q) = \bar{H} - \bar{U} q \bar{V}$ as before. Then, the following can be claimed.

Theorem 1: Consider Problem (4).

1) In the cases of ℓ_1-induced and ℓ_∞-induced norms, decentralized control K^o obtained by solving Problem (5) optimizes performance in Problem (4) for each n. Moreover, we have $\psi^o = 2\varphi^o(n-1)/n$.
2) In the case of $\mathcal{H}_\infty$, decentralized control K^o obtained by solving Problem (5) optimizes performance in Problem (4) for each n, with optimal cost $\psi^o = \varphi^o$.
3) In the in the case of $\mathcal{H}_2$, decentralized control K^o obtained by solving Problem (5) optimizes a normalized by the number of agents performance in Problem (4) asymptotically as n increases in the sense $\lim_{n\to\infty} \frac{1}{n}|(\psi^o)^2 - (\varphi^o)^2| = 0$.

Remark 1: We see that the decentralized results of Proposition 1 directly extends to the constrained case when the ℓ_1, ℓ_∞ and $\mathcal{H}_\infty$ norm are considered. In these settings, we say that the controllers are decentralized do not depend on n and are selfish, as they do not care about the social objective. It is interesting that selfishness is socially optimal in these cases. In the case of $\mathcal{H}_2$ norm, the optimal controllers are centralized (or require to measure a signal about the mass behavior) and depend on n. However, in the limit of large n, the decentralized selfish controller is asymptotically optimal in terms energy per agent.

Remark 2: If the norm constraint $||\Xi|| \le \gamma$ is in terms of induced ℓ_1, ℓ_∞ or $\mathcal{H}_\infty$ then solving Problem (5) with $||f(q)||$ in any norm, including $\mathcal{H}_2$, will lead to a decentralized controller for finite n.

B. 2-Block Problems

In the previous section we considered constrained problems. As a consequence, single norm minimization problems of the form

$$\inf_Q || \begin{bmatrix} \Psi \\ \Xi \end{bmatrix} || \quad (6)$$

lead to similar results (see Appendix). In particular, the corresponding problem to solve to obtain an optimal decentralized solution for the case of ℓ_1-induced is

$$\inf_q \left(\frac{2(n-1)}{n} ||f(q)||_1 + ||g(q)||_1\right).$$

For the case of ℓ_∞-induced, it is

$$\inf_q || \begin{bmatrix} \frac{2(n-1)}{n} f(q) \\ g(q) \end{bmatrix} ||_{\infty i}$$

or, equivalently,

$$\inf_q \max\left(\frac{2(n-1)}{n} ||f(q)||_1, ||g(q)||_1\right).$$

For the case of $\mathcal{H}_\infty$ and $\mathcal{H}_2$, the corresponding problems are respectively

$$\inf_q || \begin{bmatrix} f(q) \\ g(q) \end{bmatrix} ||_{\mathcal{H}_\infty}, \quad \inf_q || \begin{bmatrix} f(q) \\ g(q) \end{bmatrix} ||_2.$$

In all the above, if q^o represents an optimal solution the corresponding decentralized controller is obtained by using $Q^o = \text{diag}(q^o, \ldots, q^o)$. We note for the $\mathcal{H}_2$ case these controllers are optimal only asymptotically as $n \to \infty$.

IV. Nonidentical Systems

A. Systems with Identical Inner Factors

Herein we consider the possibility of having different dynamical systems G_i, yet the Y-K parametrization leads to identical model matching problems. There are a number of problems of this sort that can be related to different G_i's. For example, in relation to the previous section, suppose that Φ is such that $H_1 = \cdots = H_n = \bar{H}$ and $\{U_i\}$ and $\{V_i\}$, while not the same, have common inner and co-inner factors $\bar{U}$ and $\bar{V}$ respectively. That is, $U_i = \bar{U}X_i$ and $V_i = Y_i\bar{V}$ with $\bar{U}$, $\bar{V}$ inner and co-inner and X_i, Y_i outer and co-outer respectively. Then, by defining a new variable $Z := \text{diag}(X_1, \ldots, X_n)Q\text{diag}(Y_1, \ldots, Y_n)$ the problem reverts to the solution for identical systems.

1) A stable systems example: Suppose we are interested in a weighted sensitivity map $\Phi = W(I - GK)^{-1}$ where $G := \text{diag}(G_1, \ldots, G_n)$ with the G_i's being stable, SISO, and having the same unstable zeros, and $W := \text{diag}(\bar{W}, \ldots, \bar{W})$ is a stable weight. Factoring out the unstable zeros of each G_i we can represent $G_i = BX_i$ with B inner and X_i outer. Then, letting $\bar{H} = \bar{W}$ and $\bar{U} = WB$, minimizing $||\Psi||$ can be done by decentralized control that can be obtained by solving

$$\varphi^o = \inf_{\bar{Z}} ||\bar{H} - \bar{U}\bar{Z}||$$

with $Q^o_{ii} = (X_i)^{-1}\bar{Z}^o$ corresponding to the optimal $\bar{Z}^o$ solution.

2) Constrained and 2-block problems: Similar results hold for the constrained and 2-block problems of the last subsection whenever the corresponding model matching expressions are the same. Consider, for example, the previous case where we impose a constraint on the complementary sensitivity $\Xi = GK(I - GK)^{-1}$ of the form $||\Xi|| \leq \gamma$. Then, following the notation of the previous subsection, $\bar{H}_\xi = 0$, $\bar{U}_\xi = \bar{U}$ and $\bar{V}_\xi = 1$. The problem to solve is therefore

$$\varphi^o = \inf_{\bar{Z}: ||\bar{U}\bar{Z}|| \leq \gamma} ||\bar{H} - \bar{U}\bar{Z}|| \quad (7)$$

with $Q^o_{ii} = (X_i)^{-1}\bar{Z}^o$ corresponding to the optimal $\bar{Z}^o$ solution, which is decentralized.

B. Strongly Nonidentical Systems

By strongly nonidentical we refer to the case that the factorizations involved are not providing identical model matching problems. Herein we consider the case of nonidentical systems G_i where *the norm of interest is the ℓ_2-induced, i.e., the $\mathcal{H}_\infty$ norm.*

Furthermore, we impose a uniform bound $\gamma > 0$ on $||Q||$ for all n (or, on $||\Phi||$ for that matter) to guarantee that the closed loop is ℓ_2 stable for the infinite agent case as $n \to \infty$ (string stability). We also continue to consider scalar w_i's to ease the exposition. Recall

$$\Psi = (I - \frac{1}{n}\mathbf{1}\mathbf{1}^T)\Phi = \mathbf{T}\Phi.$$

The problem of interest is

$$\psi^o := \inf_{Q: ||Q|| \leq \gamma} ||\Psi|| \quad (8)$$

We are further assuming that $||H_i||$, $||U_i||$ and $||V_i||$ are uniformly bounded by $\bar{\gamma}$ for all n.

For a given $M \leq n$ let Π_M represent the Mth truncation operator i.e, $\Pi_M\Psi = [\Psi_{ij}]_{1 \leq i,j \leq M}$ for $M \leq n$. Then,

$$||\Psi|| \geq ||\Pi_M\Psi||.$$

Now

$$\Pi_M\Psi = \Pi_M\Phi - \Pi_M\frac{1}{n}\mathbf{1}\mathbf{1}^T\Phi$$

Lemma 2: Assuming

$$||[U_1 \ldots U_n]|| \leq \gamma_u n^\rho \quad (9)$$

for some $\gamma_u \geq 0$ and $0 \leq \rho < 1$ it holds that

$$||\Pi_M\frac{1}{n}\mathbf{1}\mathbf{1}^T\Phi|| \to 0$$

as $n \to \infty$.

We note that assumption (9) provides a condition on the growth of the norm of the collective. It is easy to show that this condition is satisfied for the case of identical systems. Indeed, if $U_i = \bar{U}$ all i then

$$||[U_1 \ldots U_n]|| = \sqrt{n}||\bar{U}||$$

which is satisfies (9) for $\rho = 1/2$ and $\gamma_u = ||\bar{U}||$.
Let

$$\mu_M := \inf_{||Z|| \leq \gamma} ||\Pi_M H - \Pi_M UZ\Pi_M V|| \quad (10)$$

and let $Z^{o,M}$ be optimal [1] for the above problem and let

$$\Phi^{o,M} = \Pi_M H - \Pi_M UZ^{o,M}\Pi_M V$$

We note $Z^{o,M}$ is decentralized (diagonal) and the same holds for $\Phi^{o,M}$. Let

$$\Psi^M = \mathbf{T}_M\Phi^{o,M}$$

where

$$\mathbf{T}_M := (I - \frac{1}{M}\mathbf{1_M}\mathbf{1_M}^T)$$

and $\mathbf{1_M}$ is a vector of 1's of dimension M. Let also

$$\mu^o := \limsup_{M \to \infty} \mu_M$$

[1] we assume existence to avoid standard technicalities that do not change the results and replace optimal with arbitrarily close to optimal in case when existence is not guaranteed

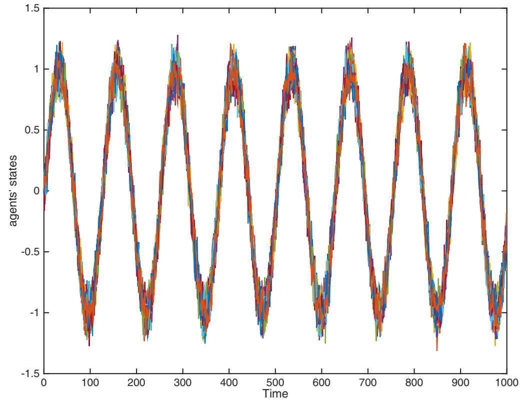

Fig. 1. Agents's states in response to noisy inputs with a common sinusoidal component. Decentralized controller for 30 agents

then the following holds.

Theorem 2: Under assumption (9) we have

$$\lim_{n\to\infty} \psi^o = \mu^o$$

and an arbitrarily close to optimal decentralized controller can obtained by $Z^{o,M}$ for sufficiently large M.

V. Examples

In this section. we present two examples in support of the results. We consider $\mathcal{H}_\infty$ problems.

A. Agents with identical dynamics

We first consider identical agents. Each agent, i, is a single integrator (adder). The model for each open loop agent is

$$\begin{aligned} x_i(k+1) &= x_i(k) + w_i(k) + u_i(k) \\ y_i(k) &= -x_i(k) + v_i(k) \end{aligned}$$

where w_i represents input noise, v_i a disturbance input, and u is the control input. As indicated in Section III-B, we compute the optimal $\mathcal{H}_\infty$ controller for each agent, according to the following generalized single agent plant.

$$\begin{aligned} x_i(k+1) &= x_i(k) + w_i(k) + u_i(k) \\ z_i(k) &= -x_i(k) + v_i(k) \\ \xi_i(k) &= u_i(k) \\ y_i(k) &= -x_i(k) + v_i(k) \end{aligned}$$

where the controller must minimize the $\mathcal{H}_\infty$ norm from $\begin{bmatrix} w_i \\ v_i \end{bmatrix}$ to $\begin{bmatrix} z_i \\ \xi_i \end{bmatrix}$. The cost focuses on rejecting v from y while keeping the control input "small".

The optimal $\mathcal{H}_\infty$ norm is 1.9021. The optimal controller is a static gain $K_i = 0.61803$ mapping y_i to u_i.

An optimal centralized controller for 2 agents that solves Problem (6), is given by

$$K(z) = \begin{bmatrix} 0.34515\frac{z+0.2056}{z+0.1118} & -0.27288\frac{z-0.006807}{z+0.1118} \\ -0.27288\frac{z-0.006807}{z+0.1118} & 0.34515\frac{z+0.2056}{z+0.1118} \end{bmatrix}$$

which has the parallel structure, and the same optimal norm 1.9021. Figure 1 shows the state response of each agent (for 30 agents) using the decentralized optimal controller $K_{dec} = IK_i$, when v has a common sinusoidal component. We see that the common component in v is reasonably tracked by $x_i's$ (rejected from $y_i's$.) The optimal $\mathcal{H}_\infty$ norm for $n = 30$ for both decentralized and centralized controllers is still 1.9021 as predicted by the theory.

B. Agents with non identical dynamics

Here we consider each agent having the following dynamics.

$$\begin{aligned} x_{i1}(k+1) &= x_{i1}(k) + x_{i2}(k) \\ x_{i2}(k+1) &= a_{i2}x_{i2}(k) + w_i(k) + b_i u_i(k) \\ y_i(k) &= -x_{i1}(k) + v_i(k) \end{aligned}$$

where $a_i \in [0, 2]$ and $b_i \in [0, 1]$, uniformly distributed. As before, each agent computes its own selfish $\mathcal{H}_\infty$ controller for the generalized plant

$$\begin{aligned} x_{i1}(k+1) &= x_{i1}(k) + x_{i2}(k) \\ x_{i2}(k+1) &= a_{i2}x_{i2}(k) + w_i(k) + b_i u_i(k) \\ z_i(k) &= -x_{i1}(k) + v_i(k) \\ \xi_i(k) &= u_i(k) \\ y_i(k) &= -x_{i1}(k) + v_i(k) \end{aligned}$$

The optimal controller has order 2. We considered 50 agents. We applied this decentralized selfish solution $K_{dec} = \text{diag}\{K_i\}$ $i = 1, \ldots, 50$ and computed the $\mathcal{H}_\infty$ norm to the closed loop multi agent plant

$$\begin{aligned} x_1(k+1) &= x_1(k) + x_2(k) \\ x_2(k+1) &= ax_2(k) + w_i(k) + bu(k) \\ z(k) &= -\mathbf{T}x_1(k) + \mathbf{T}v(k) \\ \xi(k) &= u(k) \\ y(k) &= -x_1(k) + v(k) \end{aligned}$$

where $a = \text{diag}\{a_i\}$ and $b = \text{diag}\{b_i\}$ $i = 1, \ldots, 50$, $u = K_{dec}y$ and the vector signals are the stacking of the signals of each agent i.

Note that z now measures the deviation from the average of all agents.

The $\mathcal{H}_\infty$ norm using the decentralized selfish solution turned out to be 13.1455. While the optimal centralized solution leads to a norm equal to 13.1422, which is very close to the cost with decentralized controller as predicted.

Figure 2 and 3 show the response of the two closed loop systems when v has a common sinusoidal component. Note that the decentralized closed loop response is naturally "reasonably" tracking the common sinusoidal "low frequency" input while obtaining a norm close to optimal with respect to variations from average. The optimal centralized solution, does not directly react to averages by construction. Thus, it is not surprising that the tracking performance of a common sinusoidal input is not as good.

VI. Concluding Remarks

By considering a network of dynamically decoupled LTI agents with performance measured in terms of a variety input-output norms that capture deviation from some average type of behavior, we presented a unified input-output approach to analyze issues related to the question of when

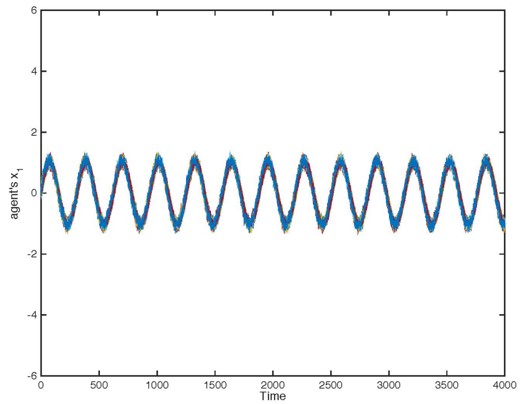

Fig. 2. Agents's responses to noisy inputs with a common sinusoidal component. Decentralized controller for 50 nonidentical agents

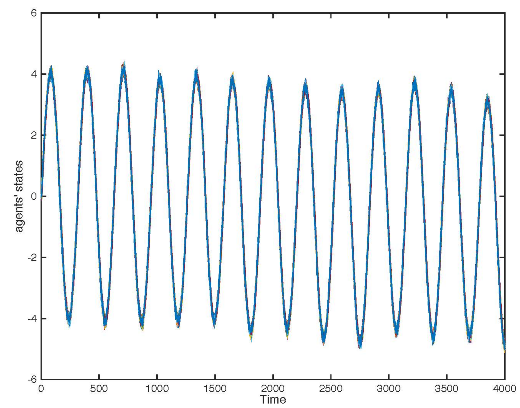

Fig. 3. Agents's responses to noisy inputs with a common sinusoidal component. Optimal centralized controller for 50 nonidentical agents

a selfish behavior is socially optimal. For identical agents it was shown that complete decentralization and selfish behavior does not degrade optimal performance in the ℓ_1, ℓ_∞ induced, $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms for any finite n. Moreover, if the cost includes non-deviation from average variables, the above results hold true as well for ℓ_1, ℓ_∞ induced and $\mathcal{H}_\infty$ norms and any n, while they hold true for the normalized, per-agent square $\mathcal{H}_2$ norm, cost as $n \to \infty$. We also consider the case of nonidentical agent dynamics and prove that similar results hold asymptotically as $n \to \infty$ in the case of ℓ_2 induced norms (i.e., $\mathcal{H}_\infty$) under a sublinear growth assumption on the $\mathcal{H}_\infty$ norm of the essential dynamics of the collective.

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