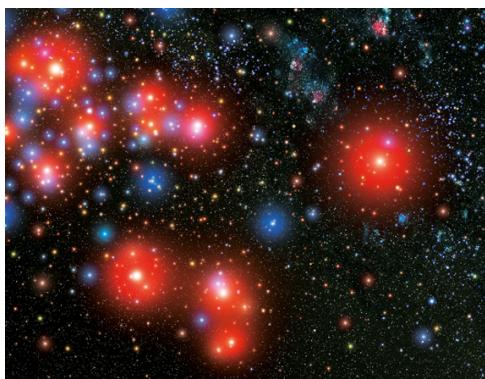


# Constraining violations of the weak equivalence principle in the dark sector

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# Constraining violations of the weak equivalence principle in the dark sector

**Sina Bahrami**

Institute for Gravitation and the Cosmos, Pennsylvania State University,  
State College, PA 16803, U.S.A.

Department of Physics, Cornell University,  
Ithaca, NY 14853, U.S.A.

E-mail: [sina.bahrami@psu.edu](mailto:sina.bahrami@psu.edu)

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**Abstract.** The effective field theory of dark energy is generalized to incorporate dark matter, which is modelled using a complex scalar field with a global U(1) symmetry. The dark matter model used here has similarities to models of ultralight axion. Generic interaction terms in the dark matter sector violate the weak equivalence principle. The degree to which such violations can occur is constrained using a number of recent analyses and forecasts in cosmology.

**Keywords:** dark energy theory, dark matter theory

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## 1 Introduction

### 1.1 Background and motivation

Over the past few decades, evidence for a recent era of cosmic acceleration has accumulated from a diverse set of cosmological observations and is now overwhelming. These observations include type Ia supernovae [1–4], anisotropies of the cosmic microwave background radiation (CMB) [5, 6], and large scale structure (LSS) surveys [7–9]. The most common explanation

for the current phase of cosmic acceleration is dark energy, with a cosmological constant being its simplest realization.<sup>1</sup> Besides dark energy, which accounts for about 70 percent of the Universe's overall energy, observations have long supported the notion that about a quarter of the energy content of the Universe is in the form of non-luminous, non-relativistic, and weakly interacting matter that is commonly referred to as (cold) dark matter. As for dark energy, the evidence for dark matter has various origins, with the oldest pieces of evidence coming from the observations of galaxy rotation curves [11, 12] and the more recent though somewhat indirect ones coming from observational cosmology, e.g. the CMB anisotropies [5]. Non-dynamical dark energy, or cosmological constant, along with cold dark matter form the so-called  $\Lambda$ CDM model of cosmology, which is by far the most successful model describing the history of our Universe from well before the epoch of matter-radiation equality. In conjunction with the standard predictions of inflation, the  $\Lambda$ CDM model provides nearly perfect theoretical fits to the CMB anisotropy data [13].

Despite its overwhelming success, the  $\Lambda$ CDM model faces some theoretical and observational challenges. A long standing theoretical challenge to non-dynamical dark energy is the well known cosmological constant problem [14]. With observations indicating that the cosmological constant has a present value  $\Lambda_0 \sim (10^{-3} \text{ eV})^4$  [5], a large amount of fine tuning is necessary to cancel out loop corrections of about  $(10^2 \text{ GeV})^4$  that it receives from the Standard Model fields alone over the energy ranges at which the Standard Model is known to be valid. On the observational end, a prominent challenge to the  $\Lambda$ CDM model is the discrepancy in the deduced values of the Hubble constant  $H_0$  from observations of Cepheid variables [15] and the CMB data collected and analysed by the Planck collaborations [5].

Dynamical dark energy models, of which there are many [10], are a useful theoretical foil for analysing dark energy observations. They can also potentially address the Hubble discrepancy problem [16]. Generically they do not address the fine tuning problems of the cosmological constant [14], and in fact generically have additional fine tuning problems associated with matter couplings [17].

If dark energy is indeed dynamical and driven by a single degree of freedom, then its dynamics can be described by an effective field theory (EFT) where the dark energy perturbations are the pseudo Nambu-Goldstone bosons that spontaneously break the approximate de Sitter symmetry of the background spacetime [18–20].<sup>2</sup> A question then arises as to how the dark energy field couples to other forms of matter, including dark matter. If the matter sectors are incorporated in the EFT of dark energy, all generic interaction terms that are compatible with the assumed internal symmetries of the matter sectors must be included. Such interaction terms amount to the dark energy field mediating a new kind of force, the so-called “fifth” force, among particles of various kind, in violation of the weak equivalence principle (WEP) or the universality of free fall for test particles in an external gravitational field.<sup>3</sup>

More specifically, one can classify violations of the WEP into three types:

1. *Baryonic-Baryonic (BB)*: this is the type of WEP violation most commonly considered in the past. An additional scalar field (here the dynamical dark energy field) couples in different ways to different sectors of the standard model. Then the total gravitational force between two bodies (tensor plus fifth force) can depend on for example baryon

<sup>1</sup>See [10] for a comprehensive review of the many theories of cosmic acceleration.

<sup>2</sup>This EFT formulation is analogous to the effective field theory of inflation developed in refs. [21, 22].

<sup>3</sup>The weak equivalence principle is expected to be violated in generic theories where dynamical dark energy interacts with dark matter. See [23] and [24] for some of the first studies of WEP violation for dark matter.

number as well as total mass. This scenario is now tightly constrained thanks to a variety of satellite and Earth-based experiments.<sup>4</sup>

2. *Baryonic-Dark Matter (BD)*: this occurs when dark matter particles and baryonic matter experience different accelerations. As an example,<sup>5</sup> suppose that the action for the baryonic fields  $\Psi_b$  and dark matter fields  $\Psi_d$  can be written as

$$S_b[e^{2\alpha_b(\phi)}g_{ab}, \Psi_b] + S_d[e^{2\alpha_d(\phi)}g_{ab}, \Psi_d] \quad (1.1)$$

for two different coupling functions  $\alpha_b$  and  $\alpha_d$  to the spacetime metric  $g_{ab}$ . The effective Newton's constant then becomes a matrix with elements [26]

$$G_{ij} \propto 1 + 2m_p^2 \alpha'_i(\phi) \alpha'_j(\phi), \quad (1.2)$$

where  $i$  and  $j$  run over the dark and baryonic sectors  $d$  and  $b$ . Since the dark matter is observed only through its gravitational effects, the individual constants  $G_{dd}$  and  $G_{db}$  are not directly observable. A rescaling of the form  $\rho_d \rightarrow e^\nu \rho_d$ ,  $G_{dd} \rightarrow e^{-2\nu} G_{dd}$ ,  $G_{db} \rightarrow e^{-\nu} G_{db}$ , where  $\rho_d$  is the dark matter density, has no effect on the dynamics of visible matter and is thus not observable. Hence only the combination  $G_{db}/\sqrt{G_{dd}G_{bb}}$  is observable, and its deviations from unity are a measure of the WEP violation [27].

3. *Dark Matter-Dark Matter (DD)*: finally, if the couplings between the dark energy and dark matter fields have generic forms, then not only is there a violation of the WEP between dark matter and baryons, but also within the dark matter sector alone. This means that the dark matter couples not just to a combination of the metric and dark energy field, as in eq. (1.1) above, but to the metric and dark energy fields individually. In this case, if there is more than one species of dark matter particle, the acceleration experienced by a freely falling particle can vary from one species to another.

In the absence of direct detection of dark matter, the status of WEP violations of the DD type is currently unknown. Violations of the BD type can be constrained by a variety of astrophysical and cosmological observations. One constraint comes from observations of tidal streams of tidally disrupted satellite galaxies [27]. An analysis of dark matter clustering in the Abell Cluster A586 also hinted at the possibility of a dark energy-enhanced gravitational interaction for dark matter particles over baryons [28, 29]. On the other end, in observational cosmology, it has been suggested that weak lensing and redshift space distortions can be used to search for the WEP violation between dark matter and baryons if the dark matter density perturbations obey the  $\Lambda$ CDM continuity equation [30]. Moreover, existing and upcoming CMB and LSS observations provide a large amount of information that can be used to constrain dark energy interactions with dark matter, which in turn can be used to constrain WEP violations between the dark and baryonic matter sectors [31–41].

## 1.2 Outline of the paper

In this paper, we generalize the EFT of dark energy by incorporating a dark matter sector. We model dark matter as a complex scalar field with a global U(1) symmetry,<sup>6</sup> which has

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<sup>4</sup>The so-called Eötvös parameter, which is a dimensionless measure of the amount of the WEP violation for baryons, is currently constrained down to  $[-1 \pm 9(\text{syst}) \pm 9(\text{stat})] \times 10^{-15}$  by the recent MICROSCOPE experiment [25].

<sup>5</sup>More general WEP violating interactions are discussed in the following sections.

<sup>6</sup>The number of the EFT operators are reduced if fermions are used in place of scalar fields.

some similarities to axion dark matter models.<sup>7</sup> Coherent excitations of this field can act like a pressureless fluid, at the level of the background cosmology and of linearized perturbations, as we discuss in more detail below. Since the WEP is not a symmetry, it is generically violated by EFT operators in the dark matter sector in this and similar models [43].

The structure of this paper is as follows. In section 2 we introduce the basics of the dark matter model under study. After a brief review of the systematics of the WEP in a field theoretic Lagrangian, we show in sections 2.1.1 and 2.1.2 that with a suitable choice of parameters, the basic dark matter model gives rise to the same background and linear perturbations phenomenology as cold dark matter. We then move on to formulating the EFT of dark matter interactions with dark energy in section 3. After identifying the EFT operators and expressing the action in sections 3.2 and 3.3, we discuss the regime of validity of the EFT in section 3.4. We work out the EFT dynamical equations for the background cosmology and linear perturbations in section 4. Finally, as an example of how the WEP violation modifies the cosmological and astrophysical observables, we derive the effective Newton’s constant for dark matter in section 5.

### 1.3 Summary of the main results

The following summarizes our main findings in the present analysis:

- In the space of solutions of the background equations, there are fairly generic solutions for which the background stress energy tensor of the dark matter candidate converges exactly to that of a pressureless fluid in the limit of complete cosmological constant domination, without the need to fine tune its mass. This is in contrast to axion models, where the equation of state parameter for dark matter becomes nearly zero after averaging over a period large compared to the period of oscillations [42].
- As for axions, the sound speed of dark matter linear perturbation modes<sup>8</sup> is close to zero when averaged over a period large compared to the period of oscillations. For this to hold today, the dark matter mass should be at least a few orders of magnitude larger than the Hubble constant. If this is to be true at the time when the shortest linear mode became subhorizon, then the dark matter mass is required to be at least ten orders of magnitude larger than the Hubble constant. This is consistent with the lower bound on the mass of ultralight axion dark matter that was found in [44].
- We have not analysed the dark matter perturbations in non-linear regime to see if they continue to behave as perturbations of a pressureless fluid. This is an open question.
- There are thirteen dark energy-dark matter interaction operators in the EFT with dimension  $\leq 4$  that are relevant for analysing the background and linear perturbations. The results of analyses performed in [38] and [39] are used to constrain the coefficients of some of these operators. However, most of these constraints are weak.
- Some fine tuning in the EFT coefficients is necessary if the model is to successfully reproduce the  $\Lambda$ CDM background cosmology. The fine tuning becomes more severe as the dark matter mass becomes lighter.

<sup>7</sup>For a recent review of axion cosmology and axion models of dark matter see [42].

<sup>8</sup>I.e. modes with physical momenta  $k_{\text{phys}} \gtrsim 0.1 \text{ Mpc}^{-1}$ .

- As far as observations are concerned, we can only talk about the WEP violations of the BD type. An astrophysical signature of this effect is a modification of the Newton's constant for dark matter as compared to baryons. However, we show that a similar modification can be obtained from a non-minimal coupling of dark matter to gravity. In order to differentiate the two effects, one can potentially exploit the fact that the latter correction to the Newton's constant is scale dependent.

## 2 A first step: a simplified model of gravity-dark matter-dark energy interactions

Before introducing our more comprehensive effective action for the interactions of dark matter and dark energy in section 3, we find it instructive to study the preliminary aspects of such interactions using a simpler model. We model the dark energy field using a single real scalar field  $\phi$  and the dark matter field using a complex scalar field  $\Pi$ . We require the dark matter sector to respect a global  $U(1)$  symmetry.

The action is given by<sup>9</sup>

$$S[g_{ab}, \phi, \Pi^\dagger, \Pi, \psi] = S_{\text{gravity}}[g_{ab}] + S_{\text{DE}}[g_{ab}, \phi] + S_{\text{DM-DE}}[g_{ab}, \phi, \Pi^\dagger, \Pi] + S_b[g_{ab}, \psi], \quad (2.1)$$

where the gravitational action, dark energy and dark matter actions are

$$S_{\text{gravity}}[g_{ab}] \equiv S_{\text{EH}}[g_{ab}] = \int d^4x \sqrt{-g} \frac{m_p^2}{2} R, \quad (2.2a)$$

$$S_{\text{DE}}[g_{ab}, \phi] = \int d^4x \sqrt{-g} [X - V_1(\phi)], \quad (2.2b)$$

$$S_{\text{DM-DE}}[g_{ab}, \phi, \Pi^\dagger, \Pi] = \int d^4x \sqrt{-g} e^{-2\alpha} [e^\alpha Y - V_2(\phi, X, \Pi^\dagger \Pi)]. \quad (2.2c)$$

Also  $S_b$  is the baryonic action of the Standard Model fields collectively denoted by  $\psi$ . Here we have defined

$$X \equiv -\frac{1}{2} g^{ab} \nabla_a \phi \nabla_b \phi, \quad (2.3a)$$

$$Y \equiv -g^{ab} \nabla_a \Pi^\dagger \nabla_b \Pi. \quad (2.3b)$$

In eq. (2.2) ‘‘EH’’ stands for Einstein-Hilbert,  $R$  is the Ricci scalar, and  $\alpha$  is a smooth but otherwise arbitrary function of  $\phi$ .

If the dark matter potential  $V_2$  does not depend on  $\phi$  and  $X$ ,

$$V_2(\phi, X, \Pi^\dagger \Pi) = V_2(\Pi^\dagger \Pi), \quad (2.4)$$

then the conformal transformation

$$\hat{g}_{ab} = e^{-\alpha} g_{ab} \quad (2.5)$$

removes all direct couplings between the dark matter and the dark energy fields. The action is then cast in the form

$$\hat{S}[\hat{g}_{ab}, \phi, \Pi^\dagger, \Pi, \psi] = \hat{S}_{\text{gravity}}[\hat{g}_{ab}, \phi] + \hat{S}_{\text{DE}}[\hat{g}_{ab}, \phi] + \hat{S}_{\text{DM}}[\hat{g}_{ab}, \Pi^\dagger, \Pi] + S_b[e^\alpha \hat{g}_{ab}, \psi], \quad (2.6)$$

---

<sup>9</sup>Our convention is to set  $c = \hbar = 1$  and define the reduced Planck mass  $m_p^2 = (8\pi G)^{-1}$ . The metric has signature  $(- + + +)$ .

where

$$\hat{S}_{\text{gravity}}[\hat{g}_{ab}, \phi] = \int d^4x \sqrt{-\hat{g}} \frac{e^\alpha m_p^2}{2} \left[ \hat{R} - 3\hat{g}^{ab} \hat{\nabla}_a \hat{\nabla}_b \alpha - \frac{3}{2} \hat{g}^{ab} \hat{\nabla}_a \alpha \hat{\nabla}_b \alpha \right], \quad (2.7a)$$

$$\hat{S}_{\text{DE}}[\hat{g}_{ab}, \phi] = \int d^4x \sqrt{-\hat{g}} e^{2\alpha} \left[ e^{-\alpha} \hat{X} - V_1(\phi) \right], \quad (2.7b)$$

$$\hat{S}_{\text{DM}}[\hat{g}_{ab}, \Pi^\dagger, \Pi] = \int d^4x \sqrt{-\hat{g}} \left[ \hat{Y} - V_2(\Pi^\dagger \Pi) \right], \quad (2.7c)$$

and  $\hat{X}$  and  $\hat{Y}$  are the same as  $X$  and  $Y$  with  $g^{ab}$  replaced with  $\hat{g}^{ab}$ . We will refer to the hatted frame as the Jordan frame and the unhatted frame as the Einstein frame, in a slight generalization of the common terminology. Note that dark matter has no violations of the WEP of the DD type in the model (2.6) satisfying the condition (2.4), since from the form of the action dark matter particles freely fall along the geodesics of the metric  $\hat{g}_{ab}$ . However, it will have violations of the BD type whenever  $\alpha'(\phi) \neq 0$ . Allowing the potential  $V_2$  to depend on  $\phi$  and  $X$  will generically result in additional violations of the DD type. The dark energy field  $\phi$  then mediates an extra force on dark matter particles. In section 3.3 we will express the EFT action in the hatted (generalized Jordan) frame associated with  $\hat{g}_{ab}$ . However, we will conduct most of the following analysis in the unhatted (Einstein) frame.

In what follows we shall neglect the baryonic contributions to the equations of motion. The equations of motion derived from the action (2.1), neglecting baryons, are the coupled Klein-Gordon equations for the dark energy and the dark matter fields,

$$-\nabla_a \nabla^a \phi + \nabla_a \left[ e^{-2\alpha} \nabla^a \phi V_{2,X} \right] + \alpha_{,\phi} e^{-\alpha} \left[ Y - 2e^{-\alpha} V_2 \right] + V_{1,\phi} + e^{-2\alpha} V_{2,\phi} = 0, \quad (2.8a)$$

$$-\nabla^a \left[ e^{-\alpha} \nabla_a \Pi^\dagger \right] + e^{-2\alpha} V_{2,\Pi} = 0, \quad -\nabla^a \left[ e^{-\alpha} \nabla_a \Pi \right] + e^{-2\alpha} V_{2,\Pi^\dagger} = 0, \quad (2.8b)$$

and the Einstein's equations

$$m_p^2 G_{ab} \equiv m_p^2 \left[ R_{ab} - \frac{1}{2} R g_{ab} \right] = T_{ab}^{\text{DE}} + T_{ab}^{\text{DM-DE}}. \quad (2.9)$$

Here the dark energy and dark matter stress-energy tensors are

$$T_{ab}^{\text{DE}} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta S_{\text{DE}}}{\delta g^{ab}} = X_{ab} + g_{ab} \left[ X - V_1 \right], \quad (2.10a)$$

$$T_{ab}^{\text{DM-DE}} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta S_{\text{DM-DE}}}{\delta g^{ab}} = e^{-\alpha} \left[ 2Y_{ab} - e^{-\alpha} V_{2,X} X_{ab} + g_{ab} (Y - e^{-\alpha} V_2) \right]. \quad (2.10b)$$

Also  $R_{ab}$  is the Ricci tensor,  $X_{ab} \equiv \nabla_a \phi \nabla_b \phi$ ,  $Y_{ab} \equiv \nabla_a \Pi^\dagger \nabla_b \Pi$ , and we use the notation  $A_{,B} \equiv \partial_B A$ . Note that all indices are raised and lowered using the metric  $g_{ab}$ .

Note that by virtue of the Bianchi identity  $\nabla^a G_{ab} = 0$ , we have

$$\nabla^a (T_{ab}^{\text{DE}} + T_{ab}^{\text{DM-DE}}) = 0, \quad (2.11)$$

despite each individual stress energy tensor not being covariantly conserved. However, if  $V_2$  did not depend on  $\phi$  and  $X$ , we could analyse the model in the Jordan frame using the action given in eqs. (2.6) and (2.7). Then it is not difficult to see that the dark matter stress energy tensor defined with respect to  $\hat{g}_{ab}$ ,

$$\hat{T}_{ab}^{\text{DM}} \equiv \frac{-2}{\sqrt{-\hat{g}}} \frac{\delta \hat{S}_{\text{DM}}}{\delta \hat{g}^{ab}}, \quad (2.12)$$

is covariantly conserved, i.e.  $\hat{\nabla}^a \hat{T}_{ab}^{\text{DM}} = 0$ , by virtue of the dark matter equations of motion derived from the action (2.6).

## 2.1 A study of the minimally coupled dark matter model

In the following subsections, we show that the dark matter model (2.2c) admits a fluid description which comports with two of the main properties of cold dark matter, namely that it has negligible equation of state parameter and sound speed at the level of the background and the linear perturbations respectively. To simplify this analysis, we assume that the dark matter field is minimally coupled to gravity, so that  $\alpha = 0$  in the action (2.2c). We also assume that dark energy is non dynamical and described by a cosmological constant with a value of  $\Lambda_0 \sim 10^{-120} m_p^4$ . We take the dark matter potential to be

$$V_2(\Pi^\dagger \Pi) = \pm m^2 \Pi^\dagger \Pi + \lambda (\Pi^\dagger \Pi)^2, \quad (2.13)$$

where  $m$  and  $\lambda$  are some constants. There are several ways of expressing  $\Pi$  in terms of two real scalar fields. The parametrization

$$\Pi \equiv \mathcal{R} e^{i\zeta}, \quad (2.14)$$

in terms of the radial and angular variables  $\mathcal{R}$  and  $\zeta$  makes the presence of the global U(1) symmetry in the dark matter sector manifest, and simplifies the analysis of the background dark matter equations of motion. The dark matter potential  $V_2$  in this case becomes

$$V_2(\mathcal{R}^2) = \pm m^2 \mathcal{R}^2 + \lambda \mathcal{R}^4. \quad (2.15)$$

Another parametrization is

$$\Pi \equiv \frac{1}{\sqrt{2}} (\varphi_1 + i\varphi_2), \quad (2.16)$$

which is useful for deriving an effective fluid description beyond the background level (particularly when  $\lambda = 0$ ). We will return to this latter point in section 2.1.2.

### 2.1.1 Background dynamics

We assume that the background geometry is described by the spatially flat Friedmann-Robertson-Walker metric

$$ds^2 = a(\eta)^2 [-d\eta^2 + dr^2 + r^2 d\Omega^2], \quad (2.17)$$

where  $a$  is the scale factor,  $\eta$  is the conformal time, and  $d\Omega^2$  is the metric on the unit 2-sphere. At the background level, all fields are functions of  $\eta$  alone. We denote the background values of all quantities using subscript 0.

The background Einstein equations are the two Friedmann equations that are derived from eqs. (2.9) and (2.10a) (set  $\alpha = 0$ ),

$$3m_p^2 \frac{\mathcal{H}^2}{a^2} = \frac{1}{a^2} (T_{\eta\eta,0}^{\text{DE}} + T_{\eta\eta,0}^{\text{DM}}) = \rho_0^{\text{DE}} + \rho_0^{\text{DM}}, \quad (2.18a)$$

$$m_p^2 \left( \frac{\mathcal{H}^2}{a^2} - \frac{2a''}{a^3} \right) = \frac{1}{a^2} (T_{ii,0}^{\text{DE}} + T_{ii,0}^{\text{DM}}) = p_0^{\text{DE}} + p_0^{\text{DM}}, \quad (2.18b)$$

where  $i$  denotes a spatial index,  $\mathcal{H} \equiv a'/a$ , and

$$\rho_0^{\text{DE}} = \Lambda_0, \quad p_0^{\text{DE}} = -\Lambda_0, \quad (2.19\text{a})$$

$$\rho_0^{\text{DM}} = \frac{1}{a^2} \left[ \mathcal{R}_0'^2 + \mathcal{R}_0^2 \zeta_0'^2 \right] \pm m^2 \mathcal{R}_0^2 + \lambda \mathcal{R}_0^4, \quad (2.19\text{b})$$

$$p_0^{\text{DM}} = \frac{1}{a^2} \left[ \mathcal{R}_0'^2 + \mathcal{R}_0^2 \zeta_0'^2 \right] \mp m^2 \mathcal{R}_0^2 - \lambda \mathcal{R}_0^4. \quad (2.19\text{c})$$

In addition, the following background Klein-Gordon equations are derived from eq. (2.8) for the dark matter fields

$$\frac{\mathcal{R}_0''}{\mathcal{R}_0} + 2\mathcal{H} \frac{\mathcal{R}_0'}{\mathcal{R}_0} - \zeta_0'^2 \pm a^2 m^2 + 2a^2 \lambda \mathcal{R}_0^2 = 0, \quad (2.20\text{a})$$

$$2\mathcal{R}_0' \zeta_0' + \mathcal{R}_0 \zeta_0'' + 2\mathcal{H} \mathcal{R}_0 \zeta_0' = 0. \quad (2.20\text{b})$$

It is also helpful to define the following quantities

$$w^{\text{DM}} \equiv \frac{p_0^{\text{DM}}}{\rho_0^{\text{DM}}}, \quad \Omega^{\text{DM}} \equiv \frac{a^2 \rho_0^{\text{DM}}}{3m_p^2 \mathcal{H}^2}, \quad (2.21)$$

where the quantity defined on the left is the equation of state parameter and the one on the right is the density parameter for the background dark matter.

Equation (2.20b) can be integrated to give

$$a^2 \mathcal{R}_0^2 \zeta_0' = c, \quad (2.22)$$

for some constant  $c$ . The integrability of this equation is the result of the U(1) symmetry of the dark matter action.<sup>10</sup> The conserved quantity given in eq. (2.22) can be thought of as angular momentum in the dark matter field space. Thus, the background dark matter equations of motion reduce to a single non-linear ODE given by

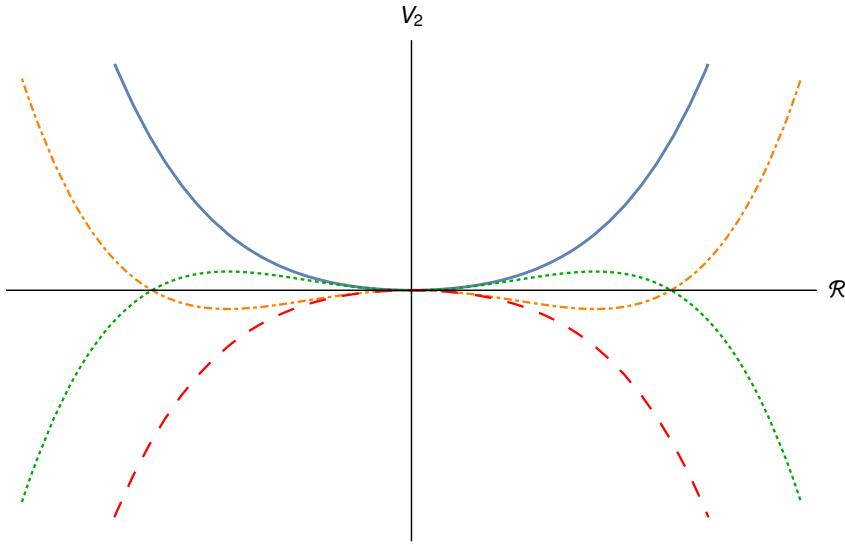
$$\frac{\mathcal{R}_0''}{\mathcal{R}_0} + 2\mathcal{H} \frac{\mathcal{R}_0'}{\mathcal{R}_0} - \frac{c^2}{a^4 \mathcal{R}_0^4} \pm a^2 m^2 + 2a^2 \lambda \mathcal{R}_0^2 = 0. \quad (2.23)$$

Prior to conducting a detailed analysis of the background equations (2.18) and (2.20), we find it illuminating to discuss some overall aspects of the solutions to these equations. This is largely determined by the form of the dark matter potential given in eq. (2.13). See figure 1 for the four different possibilities for the dark matter potential.

Since we expect the dark matter energy density to continue to get diluted into the future, we look for background solutions where the dark matter field  $\mathcal{R}_0$  eventually settles at a minimum of its potential. Thus, we discard the potential  $-m^2 \mathcal{R}^2 - |\lambda| \mathcal{R}^4$  as it has no minima. Also note that one needs to add the constant term  $m^4/(4|\lambda|)$  to the Higgs-like potential  $-m^2 \mathcal{R}^2 + |\lambda| \mathcal{R}^4$  in order to set the minimum of the potential to zero. Otherwise the dark matter's energy density will eventually be dominated by its non-zero potential, which in turn renders its equation of state parameter  $w^{\text{DM}} \rightarrow -1$ . Adding this constant is compatible with the internal U(1) symmetry of the dark matter action, though doing so is a fine-tuning of the potential.

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<sup>10</sup>Cosmological dark matter models with internal U(1) symmetries have previously been previously explored in ref. [45].



**Figure 1.** Different shapes of the dark matter potential  $V_2$  are drawn here. we have suppressed the angular dependence of the potential. The solid blue potential corresponds to  $m^2\mathcal{R}^2 + |\lambda|\mathcal{R}^4$ , the dashed red potential to  $-m^2\mathcal{R}^2 - |\lambda|\mathcal{R}^4$ , the orange dashed-dotted one to  $-m^2\mathcal{R}^2 + |\lambda|\mathcal{R}^4$ , and the dotted green one to  $m^2\mathcal{R}^2 - |\lambda|\mathcal{R}^4$ .

We first analyse the future asymptotics of eq. (2.23) for the potentials  $m^2\mathcal{R}^2 \pm |\lambda|\mathcal{R}^4$ . Assuming that the overall background energy density is dominated by  $\Lambda_0$  in this limit, we approximate  $a$  and  $\mathcal{H}$  using

$$a(\eta) = \frac{-\sqrt{3}m_p}{\sqrt{\Lambda_0}\eta} + \mathcal{O}(\eta), \quad \mathcal{H} = \frac{-1}{\eta} + \mathcal{O}(\eta), \quad (2.24)$$

where  $-\infty < \eta < 0$  and  $\eta \rightarrow 0^-$  is the future limit. Moreover, as the dark matter fields roll towards the minimum of their potential at  $\mathcal{R}_0 = 0$ , we ultimately have  $\mathcal{R}_0^2 \ll m^2/|\lambda|$  (unless  $m = 0$ ). Using this approximation and eq. (2.24), the background dark matter equation (2.23) reduces to

$$\frac{\mathcal{R}_0''}{\mathcal{R}_0} - \frac{2}{\eta} \frac{\mathcal{R}_0'}{\mathcal{R}_0} - \frac{\Lambda_0^2 c^2 \eta^4}{9m_p^4 \mathcal{R}_0^4} + \frac{3m_p^2}{\Lambda_0 \eta^2} m^2 = 0. \quad (2.25)$$

If  $c \neq 0$  we can find a solution  $\mathcal{R}_0 = \alpha\eta^{3/2}$  where  $\alpha$  satisfies

$$-\frac{9}{4} - \frac{\Lambda_0^2 c^2}{9m_p^4 \alpha^4} + \frac{3m_p^2 m^2}{\Lambda_0} = 0. \quad (2.26)$$

It is easy to check that for this solution, we have

$$p_0^{\text{DM}} = \frac{1}{a^2} \left[ \mathcal{R}_0'^2 + \frac{c^2}{a^4 \mathcal{R}_0^2} \right] - m^2 \mathcal{R}_0^2 = 0. \quad (2.27)$$

This set of solutions therefore correspond to pressureless dark matter solutions in the asymptotic future. Although the choice of the value of  $\alpha$  satisfying eq. (2.26) is a fine tuning, one can check that linearized perturbations around this solution are stable.<sup>11</sup>

Turning to the Higgs-like potential  $V_2 = -m^2\mathcal{R}^2 + |\lambda|\mathcal{R}^4 + m^4/4|\lambda|$ , eq. (2.23) in the vicinity of the minima of the potential at  $\mathcal{R} = m/\sqrt{2|\lambda|}$  is approximately

$$\delta_0'' - \frac{2}{\eta}\delta_0' - \frac{2^{3/2}|\lambda|^{3/2}\Lambda_0^2 c^2 \eta^4}{9m^3 m_p^4} + \frac{6m^2 m_p^2}{\Lambda_0 \eta^2} \delta_0 = 0, \quad (2.30)$$

where we used eq. (2.24) and defined  $\delta_0 \equiv \mathcal{R}_0 - m/\sqrt{2|\lambda|}$ . If  $c = 0$ , one arrives at a conclusion similar to the previous case; namely that by fine-tuning the value of  $m$ , one can find asymptotically pressureless background solutions for the dark matter fields. Alternatively, if  $c \neq 0$ , one finds that  $\delta_0 \propto \eta^6$ . However, it is not difficult to check that  $w^{\text{DM}} \rightarrow 1$  in this case. We therefore disregard the Higgs-like potential for the following discussions in this section.

We now use  $\mathcal{R}_0 = \alpha\eta^{3/2}$  as an approximation for  $\mathcal{R}_0$  when  $\mathcal{R}_0^2 \ll m^2/|\lambda|$ . We set  $c^2/\Lambda_0 m_p^2 = 1$  and require  $\alpha$  to satisfy (2.26). We then numerically integrate the background equations given in (2.20) away from  $\mathcal{R}_0 = 0$ . We do this for the potentials  $V_2 = m^2\mathcal{R}^2 \pm |\lambda|\mathcal{R}^4$  for two separate cases:

- First, we set  $\lambda = 0$  while varying  $m^2 m_p^2 / \Lambda_0$  (see figure 2).
- Next, we assume  $\lambda \neq 0$  and vary  $m^2 / (m_p^2 \lambda)$  (see figure 3).

Note that all numerical results are plotted as functions of the numberfold  $N \equiv \log a / \log a_0$ , where  $a_0$  is the value of the scale factor today.

### 2.1.2 Dynamics of linear scalar perturbations

We now turn to analysing the linear scalar perturbation theory of the minimally coupled dark matter. We take the dark matter potential to be  $V_2 = m^2\Pi^\dagger\Pi$  and show that the sound speed for the dark matter perturbation modes in the linear regime can be averaged to near zero values for sufficiently large values of the dark matter mass  $m$ .

For the purpose of analysing perturbations, we find it convenient to express  $\Pi$  in terms of  $\varphi_1$  and  $\varphi_2$  using eq. (2.16). Then the dark matter stress energy tensor can be written as a sum of two separate stress energy tensors, each associated with one of the real scalar fields. Using eq. (2.10a) (with  $\alpha = 0$ ) we have

$$T_{ab}^{\text{DM}}[g_{ab}, \Pi, \Pi^\dagger] = {}^{(1)}T_{ab}^{\text{DM}}[g_{ab}, \varphi_1] + {}^{(2)}T_{ab}^{\text{DM}}[g_{ab}, \varphi_2], \quad (2.31)$$

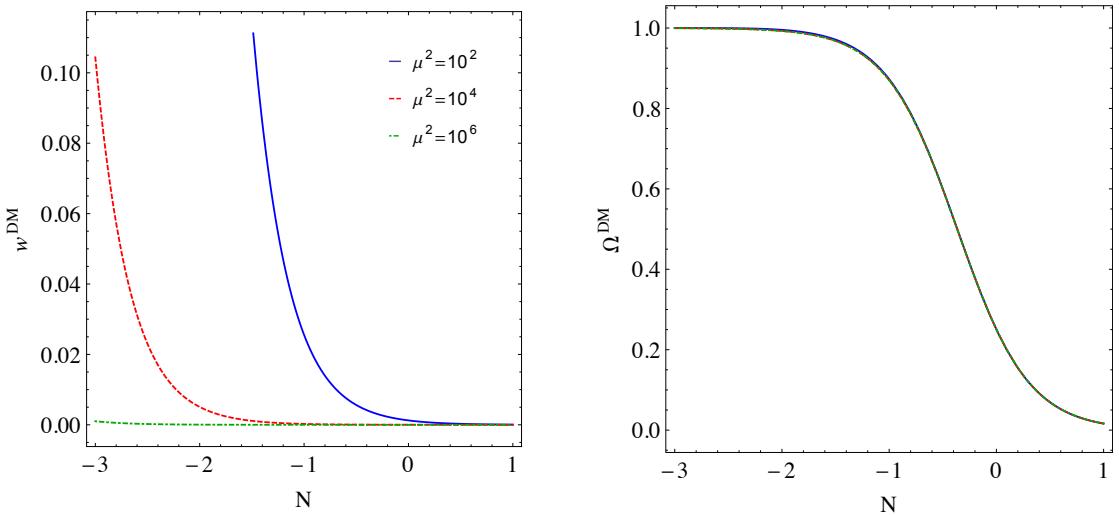
<sup>11</sup>On the other hand if  $c = 0$ , the general solution to eq. (2.25) is

$$\mathcal{R}_0 = \alpha_1 \eta^{\frac{3+\nu}{2}} + \alpha_2 \eta^{\frac{3-\nu}{2}}, \quad (2.28)$$

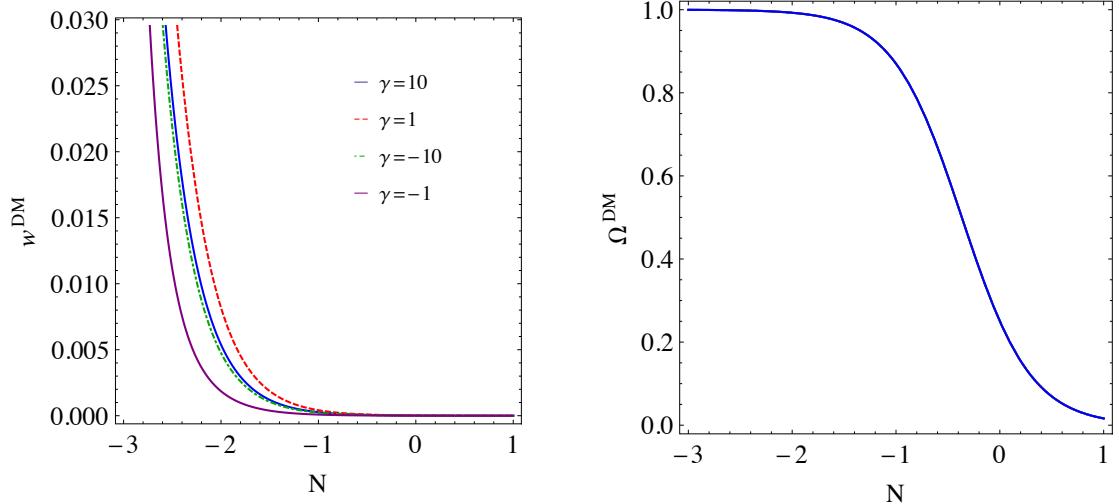
for constants  $\alpha_i$ , where  $\nu = \sqrt{9 - 12 \frac{m_p^2 m^2}{\Lambda_0}}$ . Using this solution, we compute the background equation of state parameter for dark matter in this limit to be

$$w^{\text{DM}} \approx \frac{\frac{\mathcal{R}_0'^2}{a^2} - m^2 \mathcal{R}_0^2}{\frac{\mathcal{R}_0'^2}{a^2} + m^2 \mathcal{R}_0^2} = \frac{\eta^{2\nu} \alpha_1^2 [3\Lambda_0 - 4m^2 m_p^2 + \Lambda_0 \nu] + \alpha_2^2 [3\Lambda_0 - 4m^2 m_p^2 - \Lambda_0 \nu]}{\eta^{2\nu} \alpha_1^2 [3\Lambda_0 + \Lambda_0 \nu] + 8m^2 m_p^2 \alpha_1 \alpha_2 \eta^\nu + \alpha_2^2 [3\Lambda_0 - \Lambda_0 \nu]}. \quad (2.29)$$

It can be seen from above that as  $\eta \rightarrow 0^-$ ,  $w^{\text{DM}} \rightarrow 0$  if and only if  $m^2 \rightarrow 3\Lambda_0/(4m_p^2)$ . In other words, one is required to fine-tune the value of the dark matter mass to be on the order of the Hubble constant today in order to find asymptotically pressureless solutions in this case.



**Figure 2.** We plot the equation of state parameter and the density parameter for the background dark matter when  $\lambda = 0$  and dark energy is approximated using the cosmological constant at all times.  $N = 0$  corresponds to today and  $N < 0$  corresponds to the past. In particular,  $N = -3$  roughly corresponds to the redshift  $z \approx 19$ . We also defined  $\mu^2 \equiv m^2 m_p^2 / \Lambda_0$ . As can be seen from the plots, the larger the value of  $\mu^2$ , the smaller the equation of state parameter will be as we go to the past. Also note that the density parameters almost completely coincide in all cases.



**Figure 3.** Here we plot the equation of state parameter and the density parameter of the background dark matter for different values of  $\gamma \equiv m^2 / (m_p^2 \lambda)$  when  $\lambda \neq 0$  and the dark energy is approximated using the cosmological constant at all times. While it is evident from the above plot that the negative values of  $\gamma$  result in a smaller equation of state parameter for the background dark matter, this observation need not hold asymptotically to the past.

where

$${}^{(1)}T_{ab}^{\text{DM}}[g_{ab}, \varphi_1] = \nabla_a \varphi_1 \nabla_b \varphi_1 + g_{ab} \left[ -\frac{1}{2} \nabla_a \varphi_1 \nabla^a \varphi_1 - \frac{1}{2} m^2 \varphi_1^2 \right], \quad (2.32a)$$

$${}^{(2)}T_{ab}^{\text{DM}}[g_{ab}, \varphi_2] = \nabla_a \varphi_2 \nabla_b \varphi_2 + g_{ab} \left[ -\frac{1}{2} \nabla_a \varphi_2 \nabla^a \varphi_2 - \frac{1}{2} m^2 \varphi_2^2 \right]. \quad (2.32b)$$

The above stress energy tensors can be cast in the perfect fluid form.<sup>12</sup> The energy density, pressure, and four-velocity associated with  $\varphi_1$  are given by<sup>13</sup>

$${}^{(1)}\rho = -\frac{1}{2}\nabla_a\varphi_1\nabla^a\varphi_1 + \frac{1}{2}m^2\varphi_1^2, \quad (2.33a)$$

$${}^{(1)}p = -\frac{1}{2}\nabla_a\varphi_1\nabla^a\varphi_1 - \frac{1}{2}m^2\varphi_1^2, \quad (2.33b)$$

$${}^{(1)}u_a = \frac{\nabla_a\varphi_1}{\sqrt{-\nabla_a\varphi_1\nabla^a\varphi_1}}, \quad (2.33c)$$

with similar expressions holding for  $\varphi_2$ . We can then compute the sound speed associated with each dark matter degree of freedom for any given mode  $k$  by

$${}^{(1)}c_s^2 \Big|_k = \frac{{}^{(1)}\delta p}{{}^{(1)}\delta\rho} \Big|_k = \frac{\delta(-\nabla_c\varphi_1\nabla^c\varphi_1 - m^2\varphi_1^2)}{\delta(-\nabla_c\varphi_1\nabla^c\varphi_1 + m^2\varphi_1^2)} \Big|_k, \quad (2.34a)$$

$${}^{(2)}c_s^2 \Big|_k = \frac{{}^{(2)}\delta p}{{}^{(2)}\delta\rho} \Big|_k = \frac{\delta(-\nabla_c\varphi_2\nabla^c\varphi_2 - m^2\varphi_2^2)}{\delta(-\nabla_c\varphi_2\nabla^c\varphi_2 + m^2\varphi_2^2)} \Big|_k. \quad (2.34b)$$

Another quantity of interest is the relative boost  $\Delta u$  between the two components of dark matter, which can be parametrized in terms of

$$\Delta u \Big|_k \equiv \sqrt{\left| \delta\left({}^{(1)}u_a {}^{(2)}u^a + 1\right) \right|_k}. \quad (2.35)$$

To compute these quantities, we need to track the evolution of the scalar perturbations in the gravity-dark matter system from the moment that they enter the Hubble horizon up until today.<sup>14</sup> Here we are interested in doing this for the perturbation modes that are in the linear regime, i.e. for the modes with wavenumbers  $1 \lesssim k/\mathcal{H}_0 \lesssim 450$ , where  $\mathcal{H}_0$  is the conformal Hubble constant today. To begin, recall that the linearly perturbed Friedmann-Robertson-Walker metric in the Newtonian gauge is given by

$$ds^2 = a(\eta)^2 \left[ - (1 + 2\Phi[\eta, x^i]) d\eta^2 + (1 - 2\Psi[\eta, x^i]) (dx^i)^2 \right], \quad (2.36)$$

where  $i$  is a spatial index and  $\Phi$  and  $\Psi$  are the metric perturbation functions. We also perturb the dark matter fields  $\varphi_1$  and  $\varphi_2$  by writing

$$\varphi_1(\eta, x^i) = \varphi_{1,0}(\eta) + \delta\varphi_1(\eta, x^i), \quad \varphi_2(\eta, x^i) = \varphi_{2,0}(\eta) + \delta\varphi_2(\eta, x^i), \quad (2.37)$$

where  $\varphi_{1,0}$  and  $\varphi_{2,0}$  are their background values.

When there is no anisotropic shear in the stress energy tensor of dark matter, as is the case for the minimally coupled dark matter model under study, it follows from the Einstein equations that  $\Phi = \Psi$ .<sup>15</sup> Therefore, a complete dynamical description of the linear scalar

<sup>12</sup>although generally not an isentropic fluid.

<sup>13</sup>There are ambiguities with defining a comoving frame for the stress energy tensor of a scalar field. For instance, if the background solutions  $\varphi_{1,0}$  are oscillatory, then  ${}^{(1)}u_{a,0}$  is discontinuous. In addition it is required that  $\nabla_a\varphi_1$  and  $\nabla_a\varphi_2$  be timelike.

<sup>14</sup>We ignore radiation and other forms of matter for this analysis.

<sup>15</sup>See any introductory book on the cosmological perturbation theory, e.g. [46].

perturbations can be obtained by solving for  $\Phi$ ,  $\delta\varphi_1$ , and  $\delta\varphi_2$ . To do this, we linearize the dark matter Klein-Gordon equations given in (2.8) as well as

$$m_p^2 \sum_{i=1}^3 G_{ii} = \sum_{i=1}^3 (-\Lambda_0 g_{ii} + T_{ii}^{\text{DM}}) \quad (2.38)$$

in  $\Phi$ ,  $\delta\varphi_1$ , and  $\delta\varphi_2$ . The resultant equations, after Fourier decomposing the perturbation functions<sup>16</sup> are

$$\delta\varphi_1'' + 2\mathcal{H}\delta\varphi_1' + \delta\varphi_1 \left[ k^2 + m^2 a^2 \right] + 2m^2 a^2 \varphi_{1,0} \Phi - 4\varphi_{1,0}' \Phi' = 0, \quad (2.39a)$$

$$\delta\varphi_2'' + 2\mathcal{H}\delta\varphi_2' + \delta\varphi_2 \left[ k^2 + m^2 a^2 \right] + 2m^2 a^2 \varphi_{2,0} \Phi - 4\varphi_{2,0}' \Phi' = 0, \quad (2.39b)$$

$$\begin{aligned} m_p^2 \Phi'' + 3m_p^2 \mathcal{H} \Phi' + \Phi \left[ 2m_p^2 \left( 2 \frac{a''}{a} - \mathcal{H}^2 \right) + \varphi_{1,0}'' + \varphi_{2,0}'' - \frac{1}{2} m^2 a^2 \left( \varphi_{1,0}^2 + \varphi_{2,0}^2 \right) - a^2 \Lambda_0 \right] \\ + \frac{1}{2} m^2 a^2 \left( \varphi_{1,0} \delta\varphi_1 + \varphi_{2,0} \delta\varphi_2 \right) - \frac{1}{2} \left( \varphi_{1,0}' \delta\varphi_1' + \varphi_{2,0}' \delta\varphi_2' \right) = 0, \end{aligned} \quad (2.39c)$$

where  $k$  is the wavenumber associated with a given mode. We integrate the system of equations (2.39) for a given mode  $k$  from the time it enters the Hubble horizon, i.e. at the time  $\eta_i$  given by  $k = \mathcal{H}(\eta_i)$ , up until today. If no entropy perturbations were generated during the inflationary period, the initial conditions for the perturbation functions  $\Phi$ ,  $\delta\varphi_1$ , and  $\delta\varphi_2$  at the time of horizon entry are given by [46]

$$\Phi(k) \Big|_{k=\mathcal{H}(\eta_i)} = \mathcal{C}(k) \left[ -1 + \frac{\mathcal{H}(\eta_i)}{a(\eta_i)^2} \int_{-\infty}^{\eta_i} a^2 d\eta \right], \quad (2.40a)$$

$$\delta\varphi_1(k) \Big|_{k=\mathcal{H}(\eta_i)} = -\mathcal{C}(k) \frac{\varphi_{1,0}'(\eta_i)}{a(\eta_i)^2} \int_{-\infty}^{\eta_i} a^2 d\eta, \quad (2.40b)$$

$$\delta\varphi_2(k) \Big|_{k=\mathcal{H}(\eta_i)} = -\mathcal{C}(k) \frac{\varphi_{2,0}'(\eta_i)}{a(\eta_i)^2} \int_{-\infty}^{\eta_i} a^2 d\eta, \quad (2.40c)$$

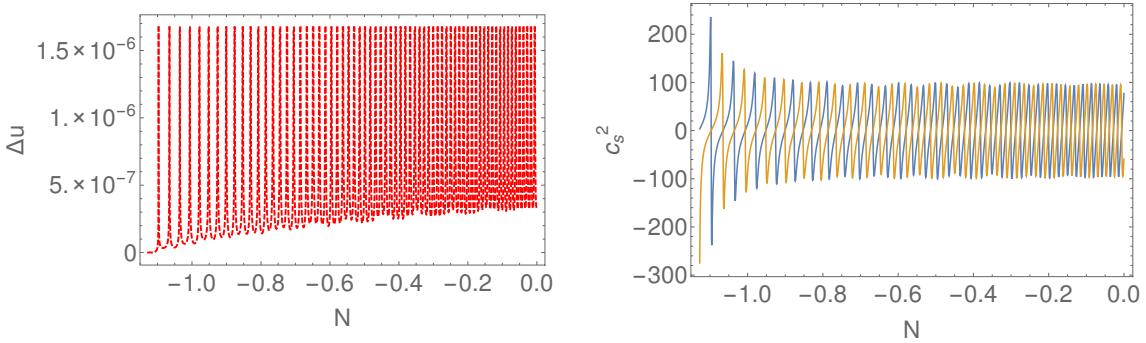
where  $\mathcal{C}(k) \approx i10^{-4}k^{-3/2}$  is the superhorizon value of the so-called comoving curvature perturbation for a single field slow roll model of inflation at the energy scale of  $\sim 10^{-2}m_p$  [47].

Upon solving eqs. (2.39), we compute the sound speeds and the relative four-velocity using eqs. (2.34) and (2.35) for any given mode  $k$ . Evaluated in the Newtonian gauge, the results are

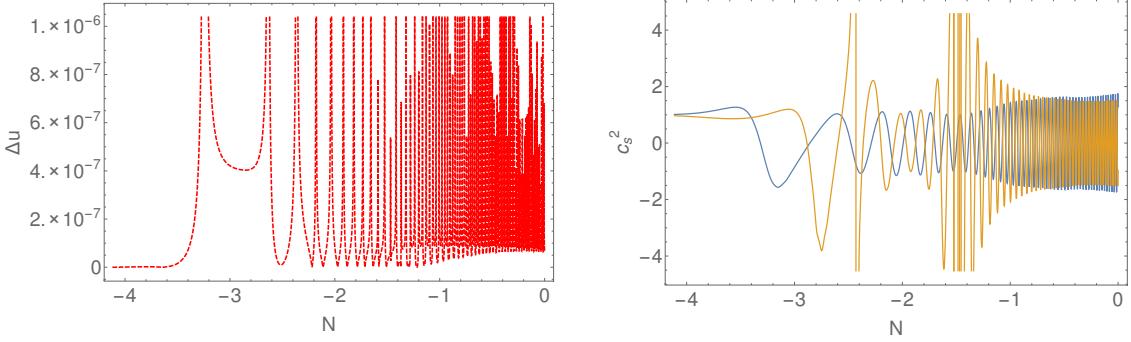
$$\begin{aligned} {}^{(1)}c_s^2 \Big|_k &= \frac{\varphi_{1,0}'(\delta\varphi_1' - \varphi_{1,0}'\Phi) - m^2 a^2 \varphi_{1,0} \delta\varphi_1}{\varphi_{1,0}'(\delta\varphi_1' - \varphi_{1,0}'\Phi) + m^2 a^2 \varphi_{1,0} \delta\varphi_1} \Big|_k, \\ {}^{(2)}c_s^2 \Big|_k &= \frac{\varphi_{2,0}'(\delta\varphi_2' - \varphi_{2,0}'\Phi) - m^2 a^2 \varphi_{2,0} \delta\varphi_2}{\varphi_{2,0}'(\delta\varphi_2' - \varphi_{2,0}'\Phi) + m^2 a^2 \varphi_{2,0} \delta\varphi_2} \Big|_k, \\ \Delta u \Big|_k &= \sqrt{\frac{k^2}{2} \left( \frac{\delta\varphi_1}{\varphi_{1,0}'} \Big|_k - \frac{\delta\varphi_2}{\varphi_{2,0}'} \Big|_k \right)^2}. \end{aligned} \quad (2.41)$$

<sup>16</sup>We Fourier decompose a perturbation function using

$$\mathcal{F}(\eta, k^i) = \int \frac{d^3x}{(2\pi)^{3/2}} e^{-i\delta_{ij}x^i k^j} \mathcal{F}(\eta, x^i).$$



**Figure 4.** This is the plot of  $\Delta u$  (left),  ${}^{(1)}c_s^2$  (right-yellow), and  ${}^{(2)}c_s^2$  (right-blue) for the mode  $k/\mathcal{H}_0 = 1$ . Here we have taken  $m/H_0 = 10^2$ . Note that the sound speeds are highly oscillatory with a nearly vanishing mean value throughout. Also, the relative four-velocity between the dark matter components is negligible.



**Figure 5.** This is the plot of  $\Delta u$  (left),  ${}^{(1)}c_s^2$  (right-yellow), and  ${}^{(2)}c_s^2$  (right-blue) for the mode  $k/\mathcal{H}_0 = 11$ . Here we have taken  $m/H_0 = 10^2$ . Note that the sound speeds begin with a value of about one, and eventually become highly oscillatory with a nearly vanishing mean value. Also, the relative four-velocity between the dark matter components is negligible.

As an example, we numerically compute the two sound speeds and the relative four-velocity for the modes with  $k = \mathcal{H}_0$  and  $k = 11\mathcal{H}_0$ . We plot the results in figures 4 and 5 respectively. Note that in doing so we have made use of the background solutions found in section 2.1.1.

The qualitative features of the plots in figures 4 and 5 can be explained as follows. First, note that the sound speeds for both modes exhibit highly oscillatory behaviour as they evolve towards the present time (and in fact, well into the future). This is the result of the oscillatory behaviour of the background dark matter fields  $\varphi_{1,0}$  and  $\varphi_{2,0}$  and their linear perturbations  $\delta\varphi_1$  and  $\delta\varphi_2$ . In fact, for the background solutions that we specialized to in section 2.1.1, it follows from eqs. (2.22) and (2.26) that  $\varphi_{1,0}$  and  $\varphi_{2,0}$  roughly oscillate with frequencies  $\zeta'_0/a = c/(a^3\mathcal{R}_0^2) \sim m$  in the  $\eta \rightarrow 0^-$  limit. This observation remains valid as long as  $p_0^{\text{DM}} \approx 0$ , as this relation implies  $\zeta'_0/a \sim m$  for  $m \gg H_0$ . On the other hand, we have already seen in the numerical results presented in the previous section that for values of  $m$  that are large compared to the Hubble constant today, the dark matter model under study maintains a near zero background pressure for a longer duration as we evolve the background solutions towards the past.<sup>17</sup> Therefore, for sufficiently large values of  $m$ , we can

<sup>17</sup>See figure 2.

find background solutions for which  $\varphi_{1,0}$  and  $\varphi_{2,0}$  oscillate with frequencies comparable to  $m$  and larger than the Hubble parameter from the moment that the shortest mode in the linear regime enters the Hubble horizon up until today.

Similarly, the oscillatory behaviour of the perturbations  $\delta\varphi_1$  and  $\delta\varphi_2$  in the  $\eta \rightarrow 0^-$  limit can be surmised from eq. (2.39). By taking the future limit of the latter set of equations, we arrive at

$$\delta\varphi_1'' + 2\mathcal{H}\delta\varphi_1' + m^2a^2\delta\varphi_1 + 2m^2a^2\varphi_{1,0}\Phi - 4\varphi_{1,0}'\Phi' = 0, \quad (2.42a)$$

$$\delta\varphi_2'' + 2\mathcal{H}\delta\varphi_2' + m^2a^2\delta\varphi_2 + 2m^2a^2\varphi_{2,0}\Phi - 4\varphi_{2,0}'\Phi' = 0, \quad (2.42b)$$

$$m_p^2\Phi'' + 3m_p^2\mathcal{H}\Phi' + a^2\Lambda_0\Phi + \frac{1}{2}m^2a^2(\varphi_{1,0}\delta\varphi_1 + \varphi_{2,0}\delta\varphi_2) - \frac{1}{2}(\varphi_{1,0}'\delta\varphi_1' + \varphi_{2,0}'\delta\varphi_2') = 0, \quad (2.42c)$$

where we used the background solutions found in section 2.1.1. It is not difficult to see that in the future limit,  $\delta\varphi_1$  and  $\delta\varphi_2$  decouple from  $\Phi$ . Indeed, if we ignore the couplings in eq. (2.42), then

$$\delta\varphi_i, \delta\varphi_2 \rightarrow c\eta^{3/2}\eta^{\pm i\frac{m}{H_0}\sqrt{1-\frac{9H_0^2}{4m^2}}}, \quad (2.43a)$$

$$\Phi \rightarrow c_1\eta + c_2\eta^3, \quad (2.43b)$$

where  $c$ ,  $c_1$ , and  $c_2$  are some constants. Notice the frequency of oscillations for  $\delta\varphi_1$  and  $\delta\varphi_2$  which is identical to their background counterparts  $\varphi_{1,0}$  and  $\varphi_{2,0}$  when  $m \gg H_0$ .<sup>18</sup> To see that this decoupling is consistent with the set of equations (2.42), observe that the corrections due to the  $\Phi$  terms to the first two equations become

$$2m^2a^2\varphi_{1,0}\Phi - 4\varphi_{1,0}'\Phi' \rightarrow \eta^{1/2} \text{ or } \eta^{5/2}, \quad (2.44)$$

which induce corrections in  $\delta\varphi_1$  and  $\delta\varphi_2$  that decay as  $\eta^{5/2}$  or  $\eta^{9/2}$ . Thus, the corrections due to  $\Phi$  are subleading in the  $\eta \rightarrow 0^-$  limit. On the other hand, the corrections due to  $\delta\varphi_1$  and  $\delta\varphi_2$  to the  $\Phi$  equation decay at least as  $\eta$ , which is the same decay form as given by the  $\Phi$  terms if  $\Phi \rightarrow c_2\eta^3$  solution is selected. If  $\Phi \rightarrow c_1\eta$  is selected instead, then the  $\Phi$  terms in the same equation blow up as  $\eta^{-1}$ , which implies that the asymptotic dynamics of  $\Phi$  is unaffected by its couplings to  $\delta\varphi_1$  and  $\delta\varphi_2$ . In fact, the leading corrections to the asymptotic solution for  $\Phi$  in this latter case are of  $\eta^3$  form. Numerical solutions to eq. (2.39) confirm the asymptotic decoupling of the perturbation functions. Also, the asymptotic solutions for  $\Phi$  turn out to be of the  $\Phi \rightarrow c_1\eta$  form.

Next, note that the qualitative form of the sound speeds for  $k = \mathcal{H}_0$  and  $k = 11\mathcal{H}_0$  modes at the onset of horizon entry are quite different, with the former quickly becoming oscillatory and the latter remaining close to one for roughly an e-fold. Numerical investigations suggest that if  $ma \lesssim \mathcal{H}$  at the time of horizon entry for a given mode, then all three sound speeds associated with that mode initially remain close to one, whereas a sharply oscillatory behaviour results for modes with  $ma \gg \mathcal{H}$  at the time of their horizon entry. On the other hand, we observe that the sound speeds for both modes end up oscillating with a nearly constant amplitude as they evolve to the present time. This is consistent with the

<sup>18</sup>Note that  $a \propto \eta^{-1} \propto e^{H_0 t}$  in the  $\eta \rightarrow 0^-$  limit. Thus,

$$\eta^{\pm i\frac{m}{H_0}\sqrt{1-\frac{9H_0^2}{4m^2}}} \propto e^{\pm imt\sqrt{1-\frac{9H_0^2}{4m^2}}}.$$

previously found asymptotic forms for the perturbation functions. In fact, using  $\Phi \propto \eta$  and  $\delta\varphi_1, \delta\varphi_2 \propto \eta^{3/2}$  along with the asymptotic forms of the background solutions, one finds that the numerators and denominators of the sound speeds given in eq. (2.41) decay as  $\eta$  to the leading order. This implies that the amplitude of oscillations approaches a constant value in the  $\eta \rightarrow 0^-$  limit. Note that it is more difficult to analytically infer the period of oscillation in this limit, though numerically we have found it to be much shorter than  $m^{-1}$ .

The other noteworthy feature seen in figures 4 and 5 is the negligible relative velocity between the two dark matter fields. This is expected theoretically, because  $\Delta u$  defined in (2.35) is roughly  $k/(m_p \mathcal{H}_0) \times \delta\varphi \ll k\varphi/(m_p \mathcal{H}_0) \sim k_{\text{phys}}/m$ . Therefore, for a sufficiently large value of dark matter mass  $m$ , this quantity remains small for linear perturbations.

To close this discussion, we remind the reader that an averaged out near zero sound speed for the dark matter candidate is achievable should one select the dark matter mass  $m$  to be sufficiently large. More specifically, in order that the modes with wavenumbers  $1 \lesssim k/\mathcal{H}_0 \lesssim 450$  have negligible sound speeds at the present time, one needs  $m/\mathcal{H}_0 \gtrsim 450$ . Requiring these modes to have negligible sound speeds as early as they become subhorizon necessitates  $m/\mathcal{H}_0 \gtrsim \mathcal{H}(\eta_*)/\mathcal{H}_0 \sim 10^{10}$ , where  $\eta_*$  is the conformal time at which the mode  $k/\mathcal{H}_0 = 450$  becomes subhorizon, assuming that the dark matter candidate generates the CDM background cosmology up until then. This is similar to the conclusion of a detailed study in [44] that uses ultralight axions for dark matter.

### 3 The effective field theory of interacting dark energy and dark matter

#### 3.1 Overview

In this section we expand the analysis of the preceding section to allow for more general couplings between the dark matter and dark energy fields. We use the techniques that were previously employed in constructing the EFT of single clock inflation [21] to construct an EFT of both dark energy and dark matter.<sup>19</sup> Such methods have already been applied in constructing EFTs of dark energy [19, 20, 49, 52, 53].<sup>20</sup> The successful application of EFT methods in these scenarios is based on the fact that the degrees of freedom driving the current and the primordial phases of cosmic acceleration determine a preferred choice of spacetime foliation.

We now review the basic idea of the construction as applied to the present context, in a language that does not refer to coordinate choices. Suppose that we start with a general action  $S = S[g_{ab}, \Pi, \phi]$  for the metric  $g_{ab}$ , dark energy field  $\phi$  and dark matter field  $\Pi$ . This action is diffeomorphism invariant:

$$S[\psi_* g_{ab}, \psi_* \Pi, \psi_* \phi] = S[g_{ab}, \Pi, \phi] \quad (3.1)$$

where  $\psi$  is any smooth diffeomorphism and  $\psi_*$  is the pullback operation. Suppose now that we are given a background solution  $g_{ab,0}, \Pi_0, \phi_0$ . We choose to use the foliation determined by the dark energy background solution  $\phi_0$ , and define a modified action functional

$$S_1[g_{ab}, \Pi, \delta\phi] = S[g_{ab}, \Pi, \phi_0 + \delta\phi]. \quad (3.2)$$

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<sup>19</sup>See [22] for an extension of the single clock formalism to multifield models of inflation. There have also been more recent extensions of this EFT [48, 50]. Also see [51] for a different EFT formalism for inflation.

<sup>20</sup>A different kind EFT valid under certain assumptions about the UV physics can be found in [54].

which is a functional of the scalar field perturbation  $\delta\phi$ . The action  $S_1$  is invariant under the transformations

$$g_{ab} \rightarrow \psi_* g_{ab}, \quad \Pi \rightarrow \psi_* \Pi, \quad \delta\phi \rightarrow \psi_* \delta\phi + \psi_* \phi_0 - \phi_0. \quad (3.3)$$

The action of these symmetries is linearly realized only for foliation preserving diffeomorphisms, those for which

$$\psi_* \phi_0 = \phi_0. \quad (3.4)$$

More general diffeomorphisms are realized nonlinearly, from eq. (3.3).

We next define a new action functional of the metric and dark matter fields by setting  $\delta\phi$  to zero:

$$S_2[g_{ab}, \Pi] = S_1[g_{ab}, \Pi, 0]. \quad (3.5)$$

Because of the gauge redundancy in the description of the theory, the action  $S_2$  still contains complete information about the theory. However it is no longer invariant under the full diffeomorphism group: the relation

$$S_2[\psi_* g_{ab}, \psi_* \Pi] = S_2[g_{ab}, \Pi] \quad (3.6)$$

is valid only for the foliation preserving diffeomorphisms that satisfy (3.4). The action  $S_2$  describes the theory specialized to unitary gauge.

We can reconstruct from the action  $S_2$  the fully covariant version of the theory by performing the so-called Stueckelberg trick.<sup>21</sup> We define a new action functional that depends on  $g_{ab}$ ,  $\Pi$  and an arbitrary smooth diffeomorphism  $\xi : M \rightarrow M$  by

$$S_3[g_{ab}, \Pi, \xi] = S_2[\xi_* g_{ab}, \xi_* \Pi]. \quad (3.7)$$

This action is invariant under general diffeomorphisms:

$$S_3[\psi_* g_{ab}, \psi_* \Pi, \psi^{-1} \circ \xi] = S_3[g_{ab}, \Pi, \xi]. \quad (3.8)$$

From the action  $S_3$  one can obtain the original action  $S$ , coming a full circle, as follows. We have

$$S_3[g_{ab}, \Pi, \xi] = S_2[\xi_* g_{ab}, \xi_* \Pi] = S_1[\xi_* g_{ab}, \xi_* \Pi, 0], \quad (3.9)$$

from eqs. (3.7) and (3.5). Now applying the invariance (3.3) with  $\psi = \xi^{-1}$  gives

$$S_3[g_{ab}, \Pi, \xi] = S_1[g_{ab}, \Pi, (\xi^{-1})_* \phi_0 - \phi_0] = S[g_{ab}, \Pi, \phi]. \quad (3.10)$$

Here we have used (3.2) and defined  $\phi = (\xi^{-1})_* \phi_0$ ; the action  $S_3$  depends on  $\xi$  only through  $\phi$ .

The key idea of the EFT of inflation/dark energy is to apply the usual rules of EFT to the unitary gauge form (3.5) of the action. One proceeds by writing down all possible operators that are invariant under foliation preserving diffeomorphisms, which for perturbations about FRW solutions includes all time dependent spatial diffeomorphisms. This can be done efficiently using the 3+1 ADM splitting of the metric, which splits the spacetime metric  $g_{ab}$  into a spatial metric  $h_{ij}$ , lapse  $N$ , and shift vector  $N^i$  via

$$ds^2 = h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) - N^2 dt^2. \quad (3.11)$$

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<sup>21</sup>See [55] for an excellent review of this technique and its applications in gauge theory as well as gravitational theories.

Here the surfaces of constant  $t$  are chosen to coincide with surfaces of constant  $\phi_0$ . Neglecting for the moment the dependence on the dark matter field  $\Pi$ , the most general unitary gauge action that is invariant under foliation preserving diffeomorphisms is

$$S_2[g_{ab}] = \int d^3x \int dt \sqrt{h} \mathcal{L}[t, h_{ij}, N, K_{ij}, D_i, \partial_t - \mathcal{L}_{\vec{N}}], \quad (3.12)$$

where  $D_i$  is the 3D covariant derivative associated with  $h_{ij}$  and  $K_{ij} = (h_{ij} - D_i N_j - D_j N_i)/(2N)$  is the extrinsic curvature, with  $N_i = h_{ij} N^j$ . This process is more efficient than other approaches since the field  $\phi$  does not appear anywhere in the action (3.5). After all the appropriate operators have been identified, one can always return to the fully covariant form of the theory using the Stueckelberg trick.

The action (3.12) can depend on the three dimensional Riemann tensor  ${}^{(3)}R_{ijkl}$  through the derivative  $D_i$ . It is convenient to re-express this dependence and the dependence on the lapse and the extrinsic curvature in terms of the perturbations

$$\delta N \equiv N - \alpha(t), \quad (3.13)$$

$$\delta K_{ij} \equiv K_{ij} - \beta(t) h_{ij}, \quad (3.14)$$

$$\delta {}^{(3)}R_{ijkl} \equiv {}^{(3)}R_{ijkl} - \gamma(t) [h_{ik}h_{jl} - h_{jk}h_{il} - h_{il}h_{jk} + h_{jl}h_{ik}]. \quad (3.15)$$

Here the functions  $\alpha(t)$ ,  $\beta(t)$  and  $\gamma(t)$  parametrize the background solution, and are regarded as fixed functions of time, which are allowed in the action (3.5). Finally the dependence on three dimensional curvature can be re-expressed in terms of a dependence on four dimensional curvature and on the extrinsic curvature using the Gauss-Codazzi equations [21].

In the following subsections we follow this approach, but generalize previous treatments by including the dark matter field  $\Pi$ . For simplicity, we disregard baryonic matter. In section 3.2 we identify all the relevant (dimension  $\leq 4$ ) and marginally relevant (dimension = 4) operators in the dark matter sector.<sup>22</sup> Just as in the EFT of dark energy, we only include the geometric objects that are either spacetime tensors or tensors intrinsic to the surfaces of constant dark energy field. We find it convenient to implement the Stueckelberg trick from the beginning by expressing all terms in the action in their covariant forms. We present the EFT action in section 3.3, discuss the EFT regime of validity in section 3.4, and derive the relevant equations of motion in section 4. Our present work is intended to complement previous attempts to formulate a generalized interacting theory of dark energy and dark matter [56–60].<sup>23</sup>

### 3.2 The effective field theory operators for the dark matter sector

In this section we identify all the operators of  $\text{dim} \leq 4$  that appear in the dark matter effective action up to the second order in perturbations. The goal is to replace the action (2.1) with a more general action. Following [21] we perform a field redefinition

$$\phi \rightarrow \bar{\phi} = \bar{\phi}(\phi) \quad (3.16)$$

so that in the new coordinates on field space the background solution is just  $\bar{\phi}_0(t) = t$ . We denote the perturbation to the scalar field in these field space coordinates by

$$\tau(t, x^i) = \delta \bar{\phi}(t, x^i) = \bar{\phi}(t, x^i) - \bar{\phi}_0(t) = \bar{\phi}(t, x^i) - t. \quad (3.17)$$

<sup>22</sup>Irrelevant operators (dimension  $> 4$ ) could be important if one is interested in probing the dynamics of energy scales that are comparable to the cutoff scale of the EFT. See section 3.4 for a discussion.

<sup>23</sup>See [61, 62] for studies on some aspects of the effective field theory of dark energy in presence of matter.

	Background	Perturbations	dimension
$\Pi^\dagger \Pi$	✓	✓	2
$\delta f \Pi^\dagger \Pi$	✗	✓	2
$n^a \nabla_a \delta f \Pi^\dagger \Pi$	✗	✓	3
$\delta K \Pi^\dagger \Pi$	✗	✓	3
$(\Pi^\dagger \Pi)^2$	✓	✓	4
$g^{ab} \nabla_a \Pi^\dagger \nabla_b \Pi$	✓	✓	4
$h^{ab} \nabla_a \Pi^\dagger \nabla_b \Pi$	✗	✓	4
$g^{ab} \nabla_a \nabla_b \delta f \Pi^\dagger \Pi$	✗	✓	4
$h^{ab} \nabla_a \nabla_b \delta f \Pi^\dagger \Pi$	✗	✓	4
$n^a \nabla_a \delta K \Pi^\dagger \Pi$	✗	✓	4
$\delta K^2 \Pi^\dagger \Pi$	✗	✓	4
$\delta [K_{ab} K^{ab}] \Pi^\dagger \Pi$	✗	✓	4
$h^{ab} \delta R_{ab} \Pi^\dagger \Pi$	✗	✓	4
$R \Pi^\dagger \Pi$	✓	✓	4

**Table 1.** Operators that are relevant for the effective linear perturbation theory of interacting dark energy-dark matter. Here we have defined  $f \equiv g^{ab} \bar{\phi}_{,a} \bar{\phi}_{,b}$ . Also, for any object  $X$ ,  $\delta X \equiv X - X_0$  with  $X_0$  being its background value.

As discussed above, the allowed operators can be constructed from scalar functions such as the dark matter field  $\Pi$ , spacetime tensors such as  $g_{ab}$  and  $R_{abcd}$ , and foliation dependent tensors such as the unit normal  $n^a$ , induced metric  $h_{ab}$ , and extrinsic curvature tensor  $K_{ab}$ . After covariantizing using the Stueckelberg trick, these quantities can be written as

$$n^a = \frac{g^{ab} \phi_{,b}}{\sqrt{-g^{cd} \bar{\phi}_{,c} \bar{\phi}_{,d}}}, \quad (3.18)$$

$$h_{ab} = g_{ab} + n_a n_b, \quad (3.19)$$

$$K_{ab} = h_a{}^c h_b{}^d \nabla_c n_d. \quad (3.20)$$

The independent operators in the dark matter sector are given in table 1. Some of the operators are relevant for both the background and the linear perturbation theory as discussed in the subsequent sections. [See appendix C for a more detailed discussion of all possible operators.]

### 3.3 The effective field theory action

We are now ready to express the EFT action. This action provides a complete picture for the dynamics of the background and the linear scalar perturbation theory of interacting dark energy and dark matter. The general EFT action in the “hatted frame”<sup>24</sup> with metric  $\hat{g}_{ab}$  is

<sup>24</sup>This is a slight generalization of the commonly used Jordan frame. For a general interacting theory of dark energy and dark matter, there could be no conformal frame in which all couplings between the two sectors are eliminated. The hatted frame here coincides with the standard Jordan frame if we set  $\alpha_i = \beta_i = 0$ .

given by [see eqs. (2.5) and (2.6)]

$$\hat{S}[\hat{g}_{ab}, \bar{\phi}, \Pi^\dagger, \Pi, \psi] = \hat{S}_{\text{gravity}}[\hat{g}_{ab}, \bar{\phi}] + \hat{S}_{\text{DE}}[\hat{g}_{ab}, \bar{\phi}] + \hat{S}_{\text{DM-int}}[\hat{g}_{ab}, \bar{\phi}, \Pi^\dagger, \Pi] + S_b[e^\alpha \hat{g}_{ab}, \psi]. \quad (3.21)$$

Here  $\hat{S}_{\text{gravity/DE}}$  are the effective gravitational / dark energy actions that were worked out in ref. [19, 20], and  $\hat{S}_{\text{DM-int.}}$  is the effective dark matter action which includes its interactions with the dark energy field and gravity. Since our focus in this paper is on the dynamics of dark matter, we choose the following simple forms for the effective gravitational and dark energy actions

$$\begin{aligned} \hat{S}_{\text{gravity}}[\hat{g}_{ab}, \bar{\phi}] &= \int d^4x \sqrt{-\hat{g}} \frac{e^\alpha m_p^2}{2} \left[ \hat{R} - 3\hat{g}^{ab} \hat{\nabla}_a \hat{\nabla}_b \alpha - \frac{3}{2} \hat{g}^{ab} \hat{\nabla}_a \alpha \hat{\nabla}_b \alpha \right], \\ \hat{S}_{\text{DE}}[\hat{g}_{ab}, \bar{\phi}] &= \int d^4x \sqrt{-\hat{g}} e^{2\alpha} \left[ \frac{-e^{-\alpha}}{2} \Lambda_0 \hat{f} - \tilde{\Lambda} \right], \end{aligned} \quad (3.22)$$

where

$$\hat{f} = \hat{g}^{ab} \hat{\nabla}_a \bar{\phi} \hat{\nabla}_b \bar{\phi} \quad (3.23)$$

and  $\tilde{\Lambda} \equiv \Lambda_0(1 + \beta_1)$  for some function  $\beta_1$  of  $\bar{\phi}$ . This is essentially the same action for gravity and dark energy that was given in eq. (2.7). Using the operators listed in table 1, the effective dark matter action takes the following form

$$\begin{aligned} \hat{S}_{\text{DM-int}}[\hat{g}_{ab}, \bar{\phi}, \Pi^\dagger, \Pi] &= \int d^4x \sqrt{-\hat{g}} \left[ -\hat{g}^{ab} \hat{\nabla}_a \Pi^\dagger \hat{\nabla}_b \Pi + \alpha_1 \hat{h}^{ab} \hat{\nabla}_a \Pi^\dagger \hat{\nabla}_b \Pi - \tilde{\lambda} (\Pi^\dagger \Pi)^2 \right. \\ &\quad + \left( \mp \tilde{m}^2 + \tilde{\gamma} \hat{R} + \mu^2 \alpha_2 \delta \hat{f} + \mu \alpha_3 \hat{n}^a \hat{\nabla}_a \delta \hat{f} + [\alpha_4 \hat{g}^{ab} + \alpha_5 \hat{h}^{ab}] \hat{\nabla}_a \hat{\nabla}_b \delta \hat{f} + \mu \alpha_6 \delta \hat{K} \right. \\ &\quad \left. \left. + \alpha_7 \hat{n}^a \hat{\nabla}_a \delta \hat{K} + \alpha_8 \delta [\hat{K}^2] + \alpha_9 \delta [\hat{K}_{ab} \hat{K}^{ab}] + \alpha_{10} \hat{h}^{ab} \delta \hat{R}_{ab} \right) \Pi^\dagger \Pi \right], \end{aligned} \quad (3.24)$$

where  $\alpha_i$  are some functions of  $\bar{\phi}$  and  $\mu$  is some constant of mass dimension one. Also,  $\tilde{m}^2 \equiv m^2(1 + \beta_2)$ ,  $\tilde{\lambda} \equiv \lambda(1 + \beta_3)$ , and  $\tilde{\gamma} \equiv \gamma(1 + \beta_4)$  for some constants  $m^2$ ,  $\lambda$ , and  $\gamma$  and some functions  $\beta_i$  of  $\bar{\phi}$ .

The action (3.24) incorporates different possibilities for dark matter interactions. The term proportional to  $\alpha_1$  is associated with local Lorentz violation for the dark matter perturbations. The term proportional to  $\tilde{\gamma}$  encodes non-minimal coupling to gravity. Note that this term does not result in the violation of the WEP as long as  $\beta_4 = 0$ . All terms proportional to  $\alpha_i$  result in the violation of the WEP by virtue of being foliation dependent.

Finally, note that expressing  $\tilde{\Lambda}$ ,  $\tilde{m}$ ,  $\tilde{\lambda}$ , and  $\tilde{\gamma}$  in terms of constant terms and the  $\beta_i$  functions is a gauge dependent procedure. To compare the predictions of this model with observations, one would need to fix the values of the  $\beta_i$  functions at a fiducial redshift.

### 3.4 The effective field theory regime of validity

We now determine the domain of validity of the EFT constructed in the previous section. We first examine the bounds on the UV cutoff of the EFT by estimating the magnitude of the dark matter irrelevant operators and requiring them to be suppressed compared to the relevant operators. We then complement our estimates by demanding radiative stability for the coefficients of the main operators, namely the masses and the leading coupling constants of the dark matter and dark energy fields. Here we do not speculate on the nature of the UV physics from which this EFT results in the infrared limit.



**Figure 6.** An illustration of the various energy scales  $E$  in the effective field theory, where  $H$  is the Hubble parameter. Here  $\langle \Pi \rangle$  and  $\langle \phi \rangle$  denote the present values of the dark matter and dark energy fields respectively. The UV cutoff  $\mathcal{C}$  of the EFT is generically expected to be larger than  $\langle \Pi \rangle$  and  $\langle \phi \rangle$  and the masses  $m$  and  $m_\phi$  of the dark matter and dark energy fields. Radiative stability for the parameters of the dark matter-dark energy Lagrangian places an upper bound on the cutoff  $\mathcal{C}$ . One such upper bound is  $m\lambda^{-1/2}$  which comes from the one loop contribution of  $\lambda(\Pi^\dagger \Pi)^2$  to the mass  $m$  of the dark matter fields.

Since we have multiple degrees of freedom in the EFT, it is possible that the scales at which different sectors of the EFT become strongly coupled are substantially different. In this work, our estimates for the range of the UV cutoff are based on the dark matter sector, including dark matter-dark energy interactions. Our main conclusion is that in order to minimize the amount of fine tuning for the parameters of the EFT Lagrangian,

- the hierarchy between the dark matter mass  $m$  and the dark energy mass  $m_\phi$  should not be more than a few orders of magnitude,
- the dark matter mass should be many orders of magnitude larger than the Hubble parameter today, and
- the UV cutoff  $\mathcal{C}$  should be several orders of magnitude larger than  $\Lambda_0^{1/4} \sim 10^{-3}$  eV.

See figure 6 for an illustration of these conclusions.

### 3.4.1 Cutoff estimation based on the dark matter potential

Let us begin by ignoring the dark matter interactions with dark energy and gravity. Focusing on the operators with  $\text{dim} \leq 4$ , the effective dark matter Lagrangian that we formulated in section 3.3 has two parameters, namely the mass  $m$  of the dark matter fields and the dark matter coupling constant  $\lambda$ . As a first step, it is reasonable to demand that the UV cutoff  $\mathcal{C}$  be within the range  $m \ll \mathcal{C} \ll m_p$ . Note that  $m \gg H_0$  as the dark matter mass is required to be several orders of magnitude heavier than the Hubble constant.<sup>25</sup>

It is necessary in any EFT that irrelevant operators be suppressed compared to relevant and marginally relevant operators in the regime where EFT is expected to be predictive. To examine this more closely, consider non-derivative self-interactions in the dark matter potential which we parametrize as

$$\sum_{n \geq 1} \sigma_n \frac{(\Pi^\dagger \Pi)^n}{\mathcal{C}^{2n-4}}, \quad (3.25)$$

where  $\sigma_n$  are some dimensionless constants. The dark matter mass  $m$  and coupling constant  $\lambda$  are expressed as  $\sqrt{\sigma_1} \mathcal{C}$  and  $\sigma_2$  respectively. The condition  $m \lesssim \mathcal{C}$  requires  $\sigma_1$  to be small

<sup>25</sup>See the analysis given in section 2.1.2. Also, the lightest dark matter candidates that can account for the entire CDM content of the Universe have  $m \gtrsim 10^{-24}$  eV  $\sim 10^9 H_0$  [44].

compared to unity. If we assume that dark matter is weakly interacting with itself, then

$$\lambda(\Pi^\dagger \Pi)^2 \sim \lambda \frac{\Lambda_0^2}{m^4} \ll \Lambda_0, \quad (3.26)$$

where we used  $m^2 \Pi^\dagger \Pi \sim \Lambda_0$ . This implies that  $\lambda \ll m^4/\Lambda_0$ . If  $m \lesssim \Lambda_0^{1/4}$ , then  $\lambda$  must be suppressed compared to unity.<sup>26</sup> Setting  $\lambda$  to values much smaller than unity is a fine tuning if no assumptions regarding the UV physics have been made.

Turning to terms in eq. (3.25) with  $n \geq 3$  (irrelevant operators), we have

$$\sigma_n \frac{(\Pi^\dagger \Pi)^n}{\mathcal{C}^{2n-4}} \sim \sigma_n \frac{\Lambda_0^n}{m^{2n} \mathcal{C}^{2n-4}} \ll \Lambda_0 \quad (3.27)$$

which using  $\Lambda_0 \sim m_p^2 H_0^2 \sim (10^{-3} \text{ eV})^4$  requires the cutoff to be bounded below by

$$\mathcal{C} \gg \sigma_n^{\frac{1}{2n-4}} \sqrt{m_p H_0} \left( \frac{m}{\sqrt{m_p H_0}} \right)^{-\frac{n}{n-2}}. \quad (3.28)$$

Assuming  $\sigma_n \sim \mathcal{O}(1)$  this gives a lower bound below  $10^{-3} \text{ eV}$  for  $m \gtrsim 10^{-3} \text{ eV}$ , but a more stringent lower bound above  $10^{-3} \text{ eV}$  for larger  $m$ . If the dark matter fields are similar to the QCD axions, then  $m \gtrsim 10^{-6} \text{ eV}$ ,<sup>27</sup> which implies  $\mathcal{C} \gtrsim \mathcal{O}(1) \text{ MeV}$ . Note that  $m \gtrsim 10^{-13} \text{ eV}$  as  $\mathcal{C}$  cannot exceed  $m_p$ .

If one allows  $\sigma_n \ll 1$ , then smaller values for  $m$  are feasible at the expense of fine tuning. This may be necessary if one is interested in significant interactions between dark matter and dark energy fields when the dark energy field has a mass  $m_\phi \sim H_0$ .<sup>28</sup> We can rearrange eq. (3.28) to derive the following upper bound on  $\sigma_n$

$$\sigma_n \ll \frac{H_0^2}{m_p^2} \left( \frac{\mathcal{C}}{m_p} \right)^{2n-4} \left( \frac{m}{H_0} \right)^{2n}. \quad (3.29)$$

We expect  $m/H_0 \geq 10^9$  if the dark matter candidate accounts for all of dark matter in the Universe [44]. For  $n = 3$  and using  $\mathcal{C} \lesssim m_p$  this gives  $\sigma_3 \ll 10^{-68}$ , which is an extreme fine tuning. If we lower the cutoff scale to  $\mathcal{C} \sim \text{eV}$ , the fine tuning of  $\sigma_3$  becomes more severe, where now it becomes bounded above by  $10^{-120}$ . Overall, the smaller the hierarchy between the cutoff scale and the dark matter mass, and the lighter the dark matter fields, more severe fine tunings should be expected. However, this is not particularly surprising because fine tuning is a generic problem of dynamical dark energy models which has now become more severe as a result of dark energy coupling to dark matter. This level of fine tuning may not harm the self consistency of the EFT model as long as the EFT parameters are stable under radiative corrections. Whether extreme fine tuning of parameters can prevent the EFT from admitting a well defined UV completion is a relevant question which we will not address in the present work.

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<sup>26</sup>Recall that our analysis in section 2.1.1 indicated that smaller values of  $\lambda$  result in  $w^{\text{DM}} \ll 1$  for higher redshifts and therefore are better suited for reproducing the background  $\Lambda\text{CDM}$  cosmology.

<sup>27</sup>See [63–66]. The lower bound on the mass of the QCD axions can be significantly lowered if one allows for fine tuning of its parameters [67, 68].

<sup>28</sup>Ultralight dark matter fields can interact significantly with dark energy if the hierarchy between their respective masses is not too large [69].

### 3.4.2 Bounds on the cutoff from radiative corrections

Unless prohibited or strongly constrained by a symmetry, radiative corrections can induce large changes in the parameters of a Lagrangian. Such corrections, which are cutoff dependent, are then absorbed by carefully tuned “bare” Lagrangian parameters in order to produce “physical” Lagrangian parameters, which are cutoff independent.<sup>29</sup> This process requires fine tuning of bare Lagrangian parameters. If this fine tuning is undesirable, then the cutoff needs to be suitably lowered in order to keep radiative corrections under control. By demanding radiative stability at one loop order for a number of primary EFT parameters, we derive an upper bound for the EFT cutoff  $\mathcal{C}$ .

For simplicity, we ignore direct couplings of gravity to dark energy and dark matter.<sup>30</sup> We restrict attention to modes with momenta  $k \gg \mathcal{H}$  for which we can neglect the influence of background geometry on the mode dynamics. The EFT action that we formulated in section 3.3 can then be written as (ignoring irrelevant operators)<sup>31</sup>

$$S_{\text{DE-DM}}^{k \gg \mathcal{H}}[\phi, \Pi^\dagger, \Pi] = \int d^4x \left[ -\frac{1}{2} \partial_a \phi \partial^a \phi - V(\phi) - \partial_a \Pi^\dagger \partial^a \Pi - m^2 \Pi^\dagger \Pi - \lambda (\Pi^\dagger \Pi)^2 - \mu \phi \Pi^\dagger \Pi - \epsilon \phi^2 \Pi^\dagger \Pi \right], \quad (3.30)$$

where  $\mu$  and  $\epsilon$  are constants of dimension one and zero respectively. The dark energy potential  $V(\phi)$  is assumed to be sufficiently flat with a current value of approximately  $\Lambda_0$ . If we take this potential to be  $V(\phi) = m_\phi^2 \phi^2 / 2$ , we have

$$V(\langle \phi \rangle) = \frac{1}{2} m_\phi^2 \langle \phi \rangle^2 \approx \Lambda_0, \quad (3.31)$$

where  $\langle \phi \rangle$  is the present value of the dark energy field. Potential flatness then requires  $m_\phi \lesssim \Lambda_0^{1/4} \lesssim \langle \phi \rangle$ .<sup>32</sup> The interaction terms in action (3.30) result from expanding  $\tilde{m}^2$  in eq. (3.24) in powers of  $\phi$ . If  $\tilde{m}^2$  has an analytic dependence on  $\phi$ , we can write

$$\tilde{m}^2(\phi) = m^2 \left[ 1 + \sum_{i=1}^{\infty} \xi_i \left( \frac{\phi}{\mathcal{C}} \right)^i \right], \quad (3.32)$$

for some constants  $\xi_i$ . In the EFT framework, the strong coupling limit occurs when  $\phi \sim \mathcal{C}/\xi_i^{1/i}$ . This parametrization gives  $\mu \equiv \xi_1 m^2 / \mathcal{C}$  and  $\epsilon \equiv \xi_2 m^2 / \mathcal{C}^2$ . Note that we should require  $\epsilon, \lambda \ll 1$  in order to rely on perturbation methods for computing radiative corrections. This latter requirement implies in particular that  $\xi_2 \ll \mathcal{C}^2 / m^2$ , which is not a stringent constraint given that  $\mathcal{C} \gtrsim m$ .

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<sup>29</sup>We are using brute-force cutoff as a method of regularization.

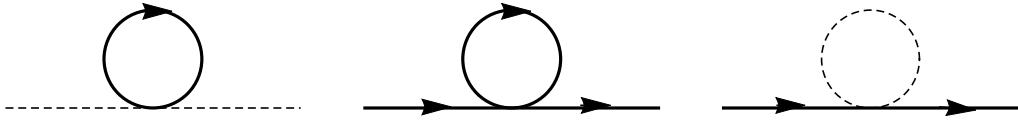
<sup>30</sup>In our case, these direct couplings take the forms  $e^\alpha R$  and  $\tilde{\gamma} R \Pi^\dagger \Pi$  that appear in eqs. (3.22) and (3.24) respectively.

<sup>31</sup>We ignore foliation dependent operators as they result in local Lorentz violation. Note, however, that we have no direct evidence that dark matter or dark energy respect local Lorentz symmetry. Here we do so to simplify our analysis.

<sup>32</sup>For the dark energy potential to be flat, we require  $|V'/V^{3/4}|, |V''/V^{1/2}| \ll 1$ .

	DM	DE
$\delta m^2$	$\sim \lambda \mathcal{C}^2$	$\sim \epsilon \mathcal{C}^2$
$\delta m_\phi^2$	$\sim \epsilon \mathcal{C}^2$	
$\delta \mu$	$\sim \lambda \mu \log(\frac{\mathcal{C}}{m})$	
$\delta \lambda$	$\sim \lambda^2 \log(\frac{\mathcal{C}}{m})$	$\sim \epsilon^2 \log(\frac{\mathcal{C}}{m_\phi})$
$\delta \epsilon$	$\sim \lambda \epsilon \log(\frac{\mathcal{C}}{m})$	

**Table 2.** Leading one loop corrections to  $m_\phi$ ,  $m$ ,  $\mu$ ,  $\epsilon$ , and  $\lambda$  due to the dark matter (DM) and dark energy (DE) loops based on the interactions terms that appear in eq. (3.30). One loop corrections that are inversely proportional to  $\mathcal{C}$  are omitted.



**Figure 7.** One loop corrections to  $m_\phi$  (left) and  $m$  (right) from  $\phi^2 \Pi^\dagger \Pi$  operator, as well as the one loop correction to  $m$  (center) from  $(\Pi^\dagger \Pi)^2$ .

Our objective here is to keep the one loop radiative corrections to  $m_\phi$ ,  $m$ ,  $\mu$ ,  $\lambda$ , and  $\epsilon$  not large compared to their bare values. The estimates for radiative corrections to these parameters are provided in table 2. Among these parameters,  $m$  and  $m_\phi$  are most sensitive to the UV cutoff where  $\delta m^2, \delta m_\phi^2 \propto \mathcal{C}^2$ . The most stringent bounds on  $\mathcal{C}$  result from these corrections (see figure 7 for the Feynman diagrams associated with these loop corrections). Requiring  $\delta m^2 \lesssim m^2$  and  $\delta m_\phi^2 \lesssim m_\phi^2$  gives

$$\mathcal{C} \lesssim \min \left\{ \frac{m}{\sqrt{\lambda}}, \frac{m}{\sqrt{\epsilon}}, \frac{m_\phi}{\sqrt{\epsilon}} \right\}. \quad (3.33)$$

Several remarks should be made regarding the above relation. First, using the definition given in eq. (3.32) for  $\epsilon$  we have

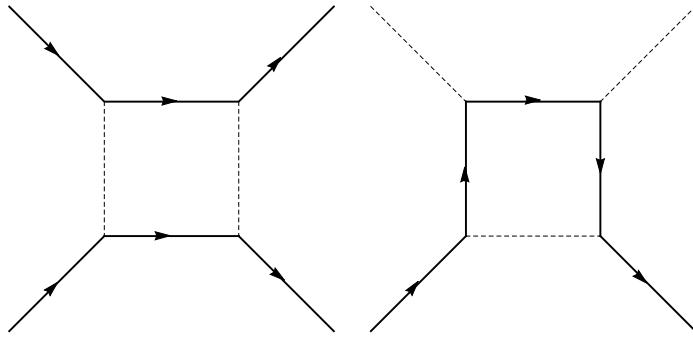
$$\frac{\min\{m, m_\phi\}}{\sqrt{\xi_2} m} \gtrsim 1, \quad (3.34)$$

which necessitates fine tuning of  $\xi_2$  to values smaller than order unity unless  $m_\phi \gtrsim m$ . Additionally, while lowering  $\lambda$  increases  $m\lambda^{-1/2}$  classically, the quantum corrections to  $\lambda$  from dark energy loops require  $\lambda_{\text{phys}} \equiv \lambda_{\text{bare}} + \delta\lambda \gtrsim \epsilon^2 \log(\mathcal{C}/m_\phi)$ . if we assume  $m_\phi \sim m$ , this lower bound on  $\lambda_{\text{phys}}$  gives

$$\mathcal{C} \lesssim m \times \min \left\{ \frac{1}{\sqrt{\epsilon}}, \frac{1}{\epsilon \sqrt{\log(\frac{\epsilon^2}{2})}} \right\} = \frac{m}{\sqrt{\epsilon}} \quad (3.35)$$

for  $\epsilon \ll 1$ . Finally, note that  $\epsilon$  too is not protected from quantum corrections even if we classically set  $\epsilon = 0$ . Indeed, one loop corrections to  $\lambda$  and  $\epsilon$  from  $\phi \Pi^\dagger \Pi$  operator are roughly  $\mu^4/\mathcal{C}^4$ , which using the definition of  $\mu$  become roughly  $\xi_1^4 m^8/\mathcal{C}^8$  (see figure 8 for the relevant Feynman diagrams). This gives

$$\mathcal{C} \lesssim \frac{m}{\sqrt{\epsilon_{\text{phys}}}} = \frac{\mathcal{C}^4}{\xi_1^2 m^3} \Rightarrow \mathcal{C} \gtrsim \xi_1^{2/3} m, \quad (3.36)$$



**Figure 8.** One loop corrections to  $\lambda$  (left) and  $\epsilon$  (right) from  $\phi\Pi^\dagger\Pi$  operator.

which, while consistent with our expectations that  $m \lesssim \mathcal{C}$  for  $\xi_1 \sim \mathcal{O}(1)$ , does not provide any upper bounds on  $\mathcal{C}$ . Therefore, the UV cutoff cannot be constrained using eq. (3.33) if  $\lambda$  and  $\epsilon$  are tuned to zero classically and  $\phi\Pi^\dagger\Pi$  is the only interaction operator. In that case, an upper bound for  $\mathcal{C}$  is obtained by requiring the one loop correction  $\delta m_\phi^2$  to the mass of the dark energy field to be bounded by  $m_\phi^2$ , which gives  $\log(\mathcal{C}/m) \lesssim m_\phi^2/\mu^2$ . In particular, if  $\mathcal{C}$  is larger than  $m$  by at least a few orders of magnitude, then  $\mu^2 \lesssim m_\phi^2$ .

#### 4 The effective field theory dynamical equations

In this section we derive the effective dynamical equations for the gravitational and the dark matter fields by varying the action (3.21) with respect to  $g_{ab}$ ,  $\Pi^\dagger$ , and  $\Pi$ . Given the parameterization (3.17) of the dark energy field as  $\bar{\phi}$ , we find it more convenient to derive the dark energy background and linear perturbation equations separately in sections 4.1 and 4.2 respectively.

The Einstein equations are (we drop the hat from the operators in section 3.3)

$$\mathcal{G}_{ab} = T_{ab}^{\text{DE}} + T_{ab}^{\text{DM}} + T_{ab}^{\text{int}} \quad (4.1)$$

where the gravitational tensor  $\mathcal{G}_{ab}$  is defined as

$$\begin{aligned} \mathcal{G}_{ab} \equiv \frac{2}{\sqrt{-g}} \frac{\delta S_{\text{gravity}}}{\delta g^{ab}} &= e^\alpha m_p^2 \left[ G_{ab} + \frac{1}{2} g_{ab} \left( 2g^{ab} \nabla_a \nabla_b \alpha + \frac{1}{2} g^{ab} \nabla_a \alpha \nabla_b \alpha \right) - \nabla_a \nabla_b \alpha \right. \\ &\quad \left. + \frac{1}{2} \nabla_a \alpha \nabla_b \alpha \right], \end{aligned} \quad (4.2)$$

the effective dark energy stress energy tensor  $T_{ab}^{\text{DE}}$  is given by

$$T_{ab}^{\text{DE}} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta S_{\text{DE}}}{\delta g^{ab}} = e^\alpha \Lambda_0 f_{ab} + e^{2\alpha} g_{ab} \left( -\frac{1}{2} e^{-\alpha} \Lambda_0 f - \tilde{\Lambda} \right), \quad (4.3)$$

the effective dark matter stress energy tensor  $T_{ab}^{\text{DM}}$  is given by

$$\begin{aligned} T_{ab}^{\text{DM}} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta S_{\text{DM-int}}}{\delta g^{ab}} \bigg|_{\alpha_i=\beta_i=\gamma=0} &= 2\nabla_{(a} \Pi^{\dagger} \nabla_{b)} \Pi + g_{ab} \left[ -g^{cd} \nabla_c \Pi^{\dagger} \nabla_d \Pi \mp m^2 \Pi^{\dagger} \Pi \right. \\ &\quad \left. - \lambda (\Pi^{\dagger} \Pi)^2 \right], \end{aligned} \quad (4.4)$$

and the effective stress energy tensor  $T_{ab}^{\text{int}}$  for the dark matter interactions with dark energy and gravity is given by

$$\begin{aligned}
T_{ab}^{\text{int}} \equiv & \frac{-2}{\sqrt{-g}} \frac{\delta S_{\text{DM-int}}}{\delta g^{ab}} - T_{ab}^{\text{DM}} = -2\alpha_1 \nabla_{(a} \Pi^{\dagger} \nabla_{b)} \Pi + g_{ab} \left[ \alpha_1 h^{cd} \nabla_c \Pi^{\dagger} \nabla_d \Pi + (\mp m^2 \beta_2 + \tilde{\gamma} R) \right. \\
& \times \Pi^{\dagger} \Pi - \lambda \beta_3 (\Pi^{\dagger} \Pi)^2 - 2g^{cd} \nabla_c \nabla_d (\tilde{\gamma} \Pi^{\dagger} \Pi) \left. \right] - 2\tilde{\gamma} R_{ab} \Pi^{\dagger} \Pi - 2\alpha_1 n^c \left[ \nabla_c \Pi^{\dagger} \nabla_{(a} \Pi n_{b)} + \nabla_c \Pi \right. \\
& \times \nabla_{(a} \Pi^{\dagger} n_{b)} \left. \right] + 2\alpha_1 J_{ab} n^c n^d \nabla_c \Pi^{\dagger} \nabla_d \Pi + 2\nabla_a \nabla_b (\tilde{\gamma} \Pi^{\dagger} \Pi) + \left[ \mu^2 \alpha_2 \left\{ g_{ab} \delta f - 2\delta f_{ab} \right\} + \mu \alpha_3 \left\{ g_{ab} \right. \right. \\
& \times n^c \nabla_c \delta f - 2n_{(a} \nabla_{b)} \delta f + J_{ab} n^c \nabla_c \delta f \left. \right\} + \alpha_5 \left\{ g_{ab} h^{cd} \nabla_c \nabla_d \delta f - 2\nabla_{(a} \nabla_{b)} \delta f - 4n^c n_{(a} \nabla_{b)} \nabla_c \delta f \right. \\
& + 2J_{ab} n^c n^d \nabla_c \nabla_d \delta f \left. \right\} + \mu \alpha_6 \left\{ g_{ab} \delta K - 2\delta \left( K_{ab} - \frac{1}{2} J_{ab} K \right) \right\} + \alpha_7 \left\{ g_{ab} n^c \nabla_c \delta K + 2\delta \left[ K \left( K_{ab} \right. \right. \right. \\
& \left. \left. \left. - \frac{1}{2} J_{ab} K \right) \right] \right\} + \alpha_8 \left\{ g_{ab} \delta K^2 - 4\delta \left[ K \left( K_{ab} - \frac{1}{2} J_{ab} K \right) \right] \right\} + \alpha_9 \left\{ g_{ab} \delta (K_{cd} K^{cd}) - 4\delta \left( K^c (a K_b)_c \right. \right. \\
& \left. \left. - \frac{1}{2} J_{ab} K^{cd} K_{cd} \right) \right\} + \alpha_{10} \left\{ g_{ab} h^{cd} \delta R_{cd} - 2 \left( \delta R_{ab} + 2n^c \delta R_{c(a} n_{b)} - n^c n^d \delta R_{cd} J_{ab} \right) \right\} \left. \right] \Pi^{\dagger} \Pi - 2\delta f_{ab} \\
& \times \left[ -\mu \nabla_c (\alpha_3 n^c \Pi^{\dagger} \Pi) + g^{cd} \nabla_c \nabla_d (\alpha_4 \Pi^{\dagger} \Pi) + \nabla_c \nabla_d (\alpha_5 h^{cd} \Pi^{\dagger} \Pi) \right] - g_{ab} g^{cd} \nabla_c \delta f \nabla_d [\alpha_4 \Pi^{\dagger} \Pi] \\
& + 2\nabla_{(a} [\alpha_4 \Pi^{\dagger} \Pi] \nabla_{b)} \delta f - g^{cd} \nabla_d [\alpha_5 h_{ab} \nabla_c \delta f \Pi^{\dagger} \Pi] + 2\nabla_c [\alpha_5 \nabla_{(a} \delta f h^c b) \Pi^{\dagger} \Pi] + 2\delta \left\{ \nabla_c \left[ (n_{(a} h_{b)}^c \right. \right. \\
& \left. \left. - \frac{1}{2} h_{ab} n^c \right) \right] \left[ [\mu \alpha_6 - \alpha_7 K] \Pi^{\dagger} \Pi - n^c \nabla_c [\alpha_7 \Pi^{\dagger} \Pi] \right] - 2\nabla_c \left[ \alpha_7 \left( n_{(a} h_{b)}^c - \frac{1}{2} h_{ab} n^c \right) \delta K \Pi^{\dagger} \Pi \right] \right. \\
& + 2\nabla_c [\alpha_7 \Pi^{\dagger} \Pi] \left[ n^c \delta \left( K_{ab} - \frac{1}{2} J_{ab} K \right) + \delta K \left( \delta_{(a}^c n_{b)} - \frac{1}{2} n^c J_{ab} \right) \right] + 4\delta \left\{ \nabla_c \left[ K \left( n_{(a} h_{b)}^c - \frac{1}{2} h_{ab} n^c \right) \right. \right. \\
& \times \alpha_8 \Pi^{\dagger} \Pi \left. \right] + 4\delta \left\{ \nabla_c \left[ \left( n_{(a} K_{b)}^c - \frac{1}{2} n^c K_{ab} \right) \right] \alpha_9 \Pi^{\dagger} \Pi \right\} + \delta \left\{ 2\nabla_c \nabla_{(b} \right\} [\alpha_{10} h_{a)}^c \Pi^{\dagger} \Pi] - \delta \left\{ g^{cd} \nabla_c \nabla_d \right\} \\
& \times [\alpha_{10} h_{ab} \Pi^{\dagger} \Pi] - \delta \left\{ g_{ab} \nabla_c \nabla_d \right\} [\alpha_{10} h^{cd} \Pi^{\dagger} \Pi]. \tag{4.5}
\end{aligned}$$

In eqs. (4.2), (4.3), (4.4), and (4.5) we defined  $f_{ab} \equiv \bar{\phi}_{,a} \bar{\phi}_b$ ,  $\delta f_{ab} \equiv (\bar{\phi}_{,a} \bar{\phi}_{,b} - a^2 \delta_{\eta a} \delta_{\eta b})$ , and  $J_{ab} \equiv \bar{\phi}_{,a} \bar{\phi}_{,b} / (\bar{\phi}_{,c} \bar{\phi}^{,c})$ . We also used the notation  $X_{(a} Y_{b)} \equiv (X_a Y_b + X_b Y_a) / 2$ .

For much of the subsequent analysis we use the dark matter parameterization in terms of  $\mathcal{R}$  and  $\zeta$  defined in eq. (2.14). We therefore find it convenient to express the dark matter and interactions stress energy tensors as a sum of stress energy tensors each associated with  $\mathcal{R}$  and  $\zeta$ .<sup>33</sup> The dark matter stress energy tensor given in eq. (4.4) is written as

$$T_{ab}^{\text{DM}} = \mathcal{R} T_{ab}^{\text{DM}} + \zeta T_{ab}^{\text{DM}} \tag{4.6}$$

where

$$\begin{aligned}
\mathcal{R} T_{ab}^{\text{DM}} &= 2\nabla_a \mathcal{R} \nabla_b \mathcal{R} + g_{ab} \left[ -\nabla_c \mathcal{R} \nabla^c \mathcal{R} \mp m^2 \mathcal{R}^2 - \lambda \mathcal{R}^4 \right], \\
\zeta T_{ab}^{\text{DM}} &= \mathcal{R}^2 \left[ 2\nabla_a \zeta \nabla_b \zeta - g_{ab} \nabla_c \zeta \nabla^c \zeta \right]. \tag{4.7}
\end{aligned}$$

<sup>33</sup>The  $\zeta$  pieces of both stress energy tensors actually include factors of  $\mathcal{R}^2$ . This is expected because  $\zeta$  is dimensionless.

Similarly, we write the interactions stress energy tensor given in eq. (4.5) as

$$T_{ab}^{\text{int}} = {}^{\mathcal{R}}T_{ab}^{\text{int}} + \zeta T_{ab}^{\text{int}} \quad (4.8)$$

where

$$\begin{aligned} \zeta T_{ab}^{\text{int}} &= \mathcal{R}^2 \alpha_1 \left[ -2\nabla_a \zeta \nabla_b \zeta - 4n^c \nabla_c \zeta \nabla_{(a} \zeta n_{b)} + 2J_{ab} n^c n^d \nabla_c \zeta \nabla_d \zeta + g_{ab} h^{cd} \nabla_c \zeta \nabla_d \zeta \right], \\ {}^{\mathcal{R}}T_{ab}^{\text{int}} &= -2\alpha_1 \nabla_a \mathcal{R} \nabla_b \mathcal{R} - 4\alpha_1 n^c \nabla_c \mathcal{R} \nabla_{(a} \mathcal{R} n_{b)} + 2\alpha_1 J_{ab} n^c n^d \nabla_c \mathcal{R} \nabla_d \mathcal{R} - 2\tilde{\gamma} R_{ab} \mathcal{R}^2 \\ &\quad + 2\nabla_a \nabla_b [\tilde{\gamma} \mathcal{R}^2] + g_{ab} [\alpha_1 h^{cd} \nabla_c \mathcal{R} \nabla_d \mathcal{R} + (\mp m^2 \beta_2 + \tilde{\gamma} R) \mathcal{R}^2 - \lambda \beta_3 \mathcal{R}^4 - 2\nabla^c \nabla_c (\tilde{\gamma} \mathcal{R}^2)] \\ &\quad + \left. {}^{\text{non-cov}}T_{ab}^{\text{int}} \right|_{\Pi^\dagger \Pi = \mathcal{R}^2}. \end{aligned} \quad (4.9)$$

In the above equation,  ${}^{\text{non-cov}}T_{ab}^{\text{int}}$  refers to the operators in the interaction stress energy tensor that are expressed in the non-covariant format.

Lastly, the dark matter equations of motion are

$$\begin{aligned} \nabla^a \nabla_a \Pi^\dagger - \nabla_a \left[ \alpha_1 h^{ab} \nabla_b \Pi^\dagger \right] + \left[ \mp \tilde{m}^2 - 2\tilde{\lambda} [\Pi^\dagger \Pi] + \tilde{\gamma} R + \mu^2 \alpha_2 \delta f + \mu \alpha_3 n^a \nabla_a \delta f \right. \\ \left. + (\alpha_4 g^{ab} + \alpha_5 h^{ab}) \nabla_a \nabla_b \delta f + \mu \alpha_6 \delta K + \alpha_7 n^a \nabla_a \delta K + \alpha_8 \delta K^2 + \alpha_9 \delta (K_{ab} K^{ab}) \right. \\ \left. + \alpha_{10} h^{ab} \delta R_{ab} \right] \Pi^\dagger = 0, \end{aligned} \quad (4.10)$$

along with its complex conjugate. Expressed in terms of the radial and angular fields  $\mathcal{R}$  and  $\zeta$ , they become

$$\begin{aligned} (g^{ab} - \alpha_1 h^{ab}) \nabla_a \nabla_b \mathcal{R} - \nabla_a (\alpha_1 h^{ab}) \nabla_b \mathcal{R} - \mathcal{R} (g^{ab} - \alpha_1 h^{ab}) \nabla_a \zeta \nabla_b \zeta + \left[ \mp \tilde{m}^2 - 2\tilde{\lambda} \mathcal{R}^2 \right. \\ \left. + \tilde{\gamma} R + \mu^2 \alpha_2 \delta f + \mu \alpha_3 n^a \nabla_a \delta f + (\alpha_4 g^{ab} + \alpha_5 h^{ab}) \nabla_a \nabla_b \delta f + \mu \alpha_6 \delta K + \alpha_7 n^a \nabla_a \delta K + \alpha_8 \delta K^2 \right. \\ \left. + \alpha_9 \delta (K_{ab} K^{ab}) + \alpha_{10} h^{ab} \delta R_{ab} \right] \mathcal{R} = 0, \\ \nabla_a \left( \mathcal{R}^2 [g^{ab} - \alpha_1 h^{ab}] \nabla_b \zeta \right) = 0. \end{aligned} \quad (4.11)$$

#### 4.1 Background equations revisited

The EFT action that was worked out in section 3.3 subsumes the original action of section 2. Due to the dark energy parametrization, the effective action's dependence on the dark energy field is partly disguised in our choices for the EFT functions  $\alpha$ ,  $\alpha_i$ , and  $\beta_i$ . This dark energy parametrization significantly simplifies the study of the background dynamics. We now use the results of the previous subsection to derive the effective background equations.

The Einstein equations (4.1) evaluated on the flat FRW background reduce to the following two Friedmann equations

$$\begin{aligned} \frac{1}{a^2} \mathcal{G}_{\eta\eta,0} &= \frac{1}{a^2} T_{\eta\eta,0}^{\text{DE}} + \frac{1}{a^2} T_{\eta\eta,0}^{\text{DM}} + \frac{1}{a^2} T_{\eta\eta,0}^{\text{int}} = \rho_0^{\text{DE}} + \rho_0^{\text{DM}} + \rho_0^{\text{int}} \\ &\Rightarrow e^\alpha m_p^2 \left( 3 \frac{\mathcal{H}^2}{a^2} + \frac{3\alpha'^2}{4a^2} + 3 \frac{\mathcal{H}\alpha'}{a^2} \right) = e^{2\alpha} \left( \frac{e^{-\alpha}}{2} \Lambda_0 + \tilde{\Lambda} \right) + \frac{\mathcal{R}'_0^2}{a^2} + \mathcal{R}_0^2 \frac{\zeta'_0^2}{a^2} - \frac{6\mathcal{H}}{a^2} (\tilde{\gamma} \mathcal{R}_0^2)' \\ &\quad + \left( \pm \tilde{m}^2 + \tilde{\lambda} \mathcal{R}_0^2 - \frac{6\mathcal{H}^2}{a^2} \tilde{\gamma} \right) \mathcal{R}_0^2, \end{aligned}$$

$$\begin{aligned}
\frac{1}{a^2} \mathcal{G}_{ii,0} &= \frac{1}{a^2} T_{ii,0}^{\text{DE}} + \frac{1}{a^2} T_{ii,0}^{\text{DM}} + \frac{1}{a^2} T_{ii,0}^{\text{int}} = p_0^{\text{DE}} + p_0^{\text{DM}} + p_0^{\text{int}} \\
\Rightarrow e^\alpha m_p^2 \left( \frac{\mathcal{H}^2}{a^2} - \frac{2a''}{a^3} - \frac{\alpha''}{a^2} - \frac{\mathcal{H}\alpha'}{a^2} - \frac{\alpha'^2}{4a^2} \right) &= e^{2\alpha} \left( \frac{e^{-\alpha}}{2} \Lambda_0 - \tilde{\Lambda} \right) + \frac{\mathcal{R}_0'^2}{a^2} + \mathcal{R}_0^2 \frac{\zeta_0'^2}{a^2} + \frac{2}{a^2} (\tilde{\gamma} \mathcal{R}_0^2)'' \\
+ \frac{2\mathcal{H}}{a^2} (\tilde{\gamma} \mathcal{R}_0^2)' + \left( \mp \tilde{m}^2 - \tilde{\lambda} \mathcal{R}_0^2 + \left[ 4 \frac{a''}{a^3} - 2 \frac{\mathcal{H}^2}{a^2} \right] \tilde{\gamma} \right) \mathcal{R}_0^2, &
\end{aligned} \tag{4.12}$$

where  $i$  is a spatial index and we used the dark matter parametrization in terms of  $\mathcal{R}$  and  $\zeta$  given in eq. (2.14).

The background dark matter equations of motion evaluated using eq. (4.10) are

$$\frac{\mathcal{R}_0''}{\mathcal{R}_0} + 2\mathcal{H} \frac{\mathcal{R}_0'}{\mathcal{R}_0} - \zeta_0'^2 + \left( \pm a^2 \tilde{m}^2 + 2a^2 \tilde{\lambda} \mathcal{R}_0^2 - 6 \frac{a''}{a} \tilde{\gamma} \right) = 0 \tag{4.13}$$

and the same equation as in (2.20) for  $\zeta_0$ .

Finally, to derive the background equation associated with the dark energy field we expand the effective action (3.21) to linear order in the dark energy perturbation function  $\tau$ , then vary the resultant expression with respect to  $\tau$ . Doing this we get

$$\begin{aligned}
\frac{m_p^2}{2a} e^\alpha \alpha' \left( 6 \frac{a''}{a^3} + 3 \frac{\alpha''}{a^2} + \frac{3\alpha'/2}{2a^2} + 6 \frac{\mathcal{H}\alpha'}{a^2} \right) - e^{2\alpha} \left( \frac{e^{-\alpha}}{2a} \Lambda_0 [\alpha' + 6\mathcal{H}] + \frac{\tilde{\Lambda}}{a} \left[ 2\alpha' + \frac{\tilde{\Lambda}'}{\tilde{\Lambda}} \right] \right) \\
+ \left( \mp m^2 \beta_2' - \lambda \beta_3' \mathcal{R}_0^2 + 6\gamma \beta_4' \frac{a''}{a^3} \right) \frac{\mathcal{R}_0^2}{a} + 2\mu^2 a^3 \alpha_2 \mathcal{R}_0^2 \left( \frac{\alpha_2'}{\alpha_2} + 3\mathcal{H} + 2 \frac{\mathcal{R}_0'}{\mathcal{R}_0} \right) \\
+ 2\mu a^2 \alpha_3 \mathcal{R}_0^2 \left( 3\mathcal{H}^2 + 10\mathcal{H} \frac{\mathcal{R}_0'}{\mathcal{R}_0} + 2 \frac{\mathcal{R}_0'^2}{\mathcal{R}_0^2} + 5\mathcal{H} \frac{\alpha_3'}{\alpha_3} + 4 \frac{\mathcal{R}_0' \alpha_3'}{\mathcal{R}_0 \alpha_3} + 3 \frac{a''}{a} + 2 \frac{\mathcal{R}_0''}{\mathcal{R}_0} + \frac{\alpha_3''}{\alpha_3} \right) \\
- 2a \alpha_4 \mathcal{R}_0^2 \left( 6\mathcal{H} \frac{\mathcal{R}_0'^2}{\mathcal{R}_0^2} + 12\mathcal{H} \frac{\mathcal{R}_0' \alpha_4'}{\mathcal{R}_0 \alpha_4} + 6 \frac{\mathcal{R}_0'^2 \alpha_4'}{\mathcal{R}_0^2 \alpha_4} + 4 \frac{a'' \mathcal{R}_0'}{a \mathcal{R}_0} + 2 \frac{a'' \alpha_4'}{a \alpha_4} + 6\mathcal{H} \frac{\mathcal{R}_0''}{\mathcal{R}_0} + 6 \frac{\mathcal{R}_0' \mathcal{R}_0''}{\mathcal{R}_0^2} \right. \\
\left. + 6 \frac{\alpha_4' \mathcal{R}_0''}{\alpha_4 \mathcal{R}_0} + 3\mathcal{H} \frac{\alpha_4''}{\alpha_4} + 6 \frac{\mathcal{R}_0' \alpha_4''}{\mathcal{R}_0 \alpha_4} + 2 \frac{\mathcal{R}_0'''}{\mathcal{R}_0} + \frac{\alpha_4'''}{\alpha_4} \right) + 24a \alpha_5 \mathcal{R}_0^2 \left( -\mathcal{H}^3 + 2\mathcal{H}^2 \frac{\mathcal{R}_0'}{\mathcal{R}_0} + \mathcal{H}^2 \frac{\alpha_5'}{\alpha_5} \right. \\
\left. + 2\mathcal{H} \frac{a''}{a} \right) = 0.
\end{aligned} \tag{4.14}$$

As expected, the above equation is non-dynamical, rather it enforces a constraint that is to be satisfied by the background and the EFT functions. It can be seen from the above derivations that we have a total of five background equations for a total of three background functions which are the FRW scale factor  $a$  and the dark matter fields  $\mathcal{R}_0$  and  $\zeta_0$ , as well as all the EFT functions in the action (3.21) except  $\alpha_1, \alpha_6, \alpha_7, \alpha_8, \alpha_9$ , and  $\alpha_{10}$ . Therefore, thirteen of fourteen EFT functions can be freely specified.

Lastly, we find it worthwhile to relate the results of our EFT formalism to the phenomenological models for interacting dark matter and dark energy, where such interactions are typically represented via the non-conservation of the dark matter stress energy tensor [30, 70–72]. To derive the background continuity equation for the dark matter, we begin by noting that

$$\begin{aligned}
\nabla_a T^{\text{DM}}{}^a{}_\eta &= \nabla_a \mathcal{G}^a{}_\eta - \nabla_a T^{\text{DE}}{}^a{}_\eta - \nabla_a T^{\text{int}}{}^a{}_\eta \text{ (evaluated on background)} \\
\Rightarrow \rho_0^{\text{DM}}{}' + 3\mathcal{H}(\rho_0^{\text{DM}} + p_0^{\text{DM}}) &= -\rho_0^{\text{int}}{}' - 3\mathcal{H}(p_0^{\text{int}} + p_0^{\text{int}}) - \rho_0^{\text{DE}}{}' - 3\mathcal{H}(\rho_0^{\text{DE}} + p_0^{\text{DE}}) \\
- (\mathcal{G}^\eta{}_{\eta,0})' - 3\mathcal{H}(\mathcal{G}^\eta{}_{\eta,0} - \mathcal{G}^i{}_{i,0}).
\end{aligned} \tag{4.15}$$

Using eqs. (4.12) and (4.14) we find

$$\begin{aligned}
\rho_0^{\text{DM}}' + 3\mathcal{H}(\rho_0^{\text{DM}} + p_0^{\text{DM}}) &= -2\mathcal{R}_0\mathcal{R}_0' \left( \pm m^2\beta_2 + 2\lambda\beta_3\mathcal{R}_0^2 - 6\tilde{\gamma}\frac{a''}{a^3} \right) \\
&- 2\mu^2 a^4 \alpha_2 \mathcal{R}_0^2 \left( \frac{\alpha_2'}{\alpha_2} + 3\mathcal{H} + 2\frac{\mathcal{R}_0'}{\mathcal{R}_0} \right) \\
&- 2\mu a^3 \alpha_3 \mathcal{R}_0^2 \left( 3\mathcal{H}^2 + 10\mathcal{H}\frac{\mathcal{R}_0'}{\mathcal{R}_0} + 2\frac{\mathcal{R}_0'^2}{\mathcal{R}_0^2} + 5\mathcal{H}\frac{\alpha_3'}{\alpha_3} + 4\frac{\mathcal{R}_0'\alpha_3'}{\mathcal{R}_0\alpha_3} + 3\frac{a''}{a} + 2\frac{\mathcal{R}_0''}{\mathcal{R}_0} + \frac{\alpha_3''}{\alpha_3} \right) \\
&+ 2a^2 \alpha_4 \mathcal{R}_0^2 \left( 6\mathcal{H}\frac{\mathcal{R}_0'^2}{\mathcal{R}_0^2} + 12\mathcal{H}\frac{\mathcal{R}_0'\alpha_4'}{\mathcal{R}_0\alpha_4} + 6\frac{\mathcal{R}_0'^2\alpha_4'}{\mathcal{R}_0^2\alpha_4} + 4\frac{a''\mathcal{R}_0'}{a\mathcal{R}_0} + 2\frac{a''\alpha_4'}{a\alpha_4} + 6\mathcal{H}\frac{\mathcal{R}_0''}{\mathcal{R}_0} + 6\frac{\mathcal{R}_0'\mathcal{R}_0''}{\mathcal{R}_0^2} \right. \\
&\left. + 6\frac{\alpha_4'\mathcal{R}_0''}{\alpha_4\mathcal{R}_0} + 3\mathcal{H}\frac{\alpha_4''}{\alpha_4} + 6\frac{\mathcal{R}_0'\alpha_4''}{\mathcal{R}_0\alpha_4} + 2\frac{\mathcal{R}_0'''}{\mathcal{R}_0} + \frac{\alpha_4'''}{\alpha_4} \right) \\
&- 24a^2 \alpha_5 \mathcal{R}_0^2 \left( -\mathcal{H}^3 + 2\mathcal{H}^2\frac{\mathcal{R}_0'}{\mathcal{R}_0} + \mathcal{H}^2\frac{\alpha_5'}{\alpha_5} + 2\mathcal{H}\frac{a''}{a} \right). \tag{4.16}
\end{aligned}$$

As expected from our choice of frame in the effective action (3.21), the background dark matter continuity equation deviates from the standard CDM result<sup>34</sup> by terms that either break the WEP or are associated with non-minimal gravitational interactions.

In ref. [39], the current value of the right hand side of eq. (4.16) is roughly parametrized as  $\xi\mathcal{H}_0\Lambda_0$ , where the CMB anisotropy maps together with the matter power spectrum are used to show that  $|\xi| \ll 0.1$ . We apply this constraint individually to the terms appearing on the right hand side of eq. (4.16), upon which we arrive at the following set of constraints:

$$\begin{aligned}
|\tilde{\gamma}| &\ll 0.1 \frac{m^2}{H_0^2}, \\
|\beta_2| &\ll 0.1, \quad |\beta_3| \ll 0.1 \frac{m^4}{\lambda\Lambda_0}, \\
|\alpha_2| &\ll 0.1 \frac{m^2}{\mu^2}, \quad |\alpha_3| \ll 0.1 \frac{m^2}{\mu H_0}, \quad |\alpha_4|, |\alpha_5| \ll 0.1 \frac{m^2}{H_0^2}, \tag{4.17}
\end{aligned}$$

where we estimated  $m^2\mathcal{R}_0^2 \sim \Lambda_0$ ,  $\partial_\eta^n \mathcal{R}_0 \sim \mathcal{H}^n \mathcal{R}_0$ ,  $\partial_\eta^n \alpha_i \sim \mathcal{H}^n \alpha_i$ , and set  $a_0 = 1$ . Of the above constraints, only the ones on  $\beta_2$  and possibly  $\alpha_2$  are stringent. Indeed, following the arguments provided in section 3.4, we have

$$H_0 \ll m, \quad \frac{m^4}{\lambda\Lambda_0} \gg 1, \tag{4.18}$$

which implies that the above upper bounds on  $\tilde{\gamma}$ ,  $\beta_3$ ,  $\alpha_4$ , and  $\alpha_5$  are large compared to unity. Furthermore, as we argued in section 3.4, we require  $\mu \lesssim m_\phi$ . Given that a small hierarchy between  $m$  and  $m_\phi$  is preferred based on the fine tuning arguments previously outlined, we can have  $\mu \lesssim m$ . This latter implies that while the above upper bound on  $\alpha_3$  is not stringent at all, the upper bound on  $\alpha_2$  could be somewhat stringent if  $m^2/\mu^2 \lesssim 10$ .

## 4.2 Linear perturbation equations in the Newtonian gauge

We now derive the linearly perturbed Einstein equations (4.1), the dark matter equations (4.11), and the dark energy equation in the Newtonian gauge. The latter is derived

<sup>34</sup>Note that for CDM one has  $p_0^{\text{DM}} \ll \rho_0^{\text{DM}}$ . Here we did not omit pressure from the continuity equation. However, in section 2.1.1 we showed that pressureless solutions can be found.

by expanding the EFT action (3.21) to second order in the scalar perturbations and then varying the resultant expression with respect to  $\tau$ .

We express the components of the Einstein equations in the following way:

$$\begin{aligned} \delta(\mathcal{G}_{ab}n^a n^b) &= \delta(T_{ab}^{\text{DE}} n^a n^b + T_{ab}^{\text{DM}} n^a n^b + T_{ab}^{\text{int}} n^a n^b) \\ &\Rightarrow e^\alpha m_p^2 \left[ \Phi \left( -6 \frac{\mathcal{H}^2}{a^2} - 6 \frac{\mathcal{H}\alpha'}{a^2} - \frac{3\alpha'^2}{2a^2} \right) + 3\tau \left( \frac{\mathcal{H}\alpha'^2}{2a^3} + \frac{\alpha'^3}{4a^3} + \frac{\mathcal{H}\alpha''}{a^3} + \frac{\alpha'\alpha''}{2a^3} \right) \right. \\ &\quad \left. - \frac{\alpha'}{a^3} \partial^2 \tau + \frac{2}{a^2} \partial^2 \Psi + 3 \left( \frac{\mathcal{H}\alpha'}{a^3} + \frac{\alpha'^2}{2a^3} \right) \tau' - 3 \left( 2 \frac{\mathcal{H}}{a^2} + \frac{\alpha'}{a^2} \right) \Psi' \right] = \delta\rho^{\text{DE}} + \delta\rho^{\text{DM}} + \delta\rho^{\text{int}}, \end{aligned} \quad (4.19)$$

$$\begin{aligned} \delta(\mathcal{G}_{ab}h^{ab}) &= \delta(T_{ab}^{\text{DE}} h^{ab} + T_{ab}^{\text{DM}} h^{ab} + T_{ab}^{\text{int}} h^{ab}) \\ &\Rightarrow e^\alpha m_p^2 \left[ \Phi \left( -6 \frac{\mathcal{H}^2}{a^2} + 6 \frac{\mathcal{H}\alpha'}{a^2} + \frac{3\alpha'^2}{2a^2} + 12 \frac{\alpha''}{a^3} + 6 \frac{\alpha'''}{a^2} \right) + 3\tau \left( - \frac{\mathcal{H}\alpha'^2}{2a^3} - \frac{\alpha'^3}{4a^3} \right. \right. \\ &\quad \left. \left. - \frac{\alpha'a''}{a^4} + \frac{\mathcal{H}\alpha''}{a^3} - \frac{3\alpha'\alpha''}{2a^3} - \frac{\alpha'''}{a^3} \right) + 2 \frac{\alpha'}{a^3} \partial^2 \tau + \frac{2}{a^2} \partial^2 [\Phi - \Psi] + 3 \left( \frac{\mathcal{H}\alpha'}{a^3} - \frac{\alpha'^2}{2a^3} - 2 \frac{\alpha''}{a^3} \right) \tau' \right. \\ &\quad \left. + 3 \left( \frac{\alpha'}{a^2} + 2 \frac{\mathcal{H}}{a^2} \right) [\Phi' + 2\Psi'] - 3 \frac{\alpha'}{a^3} \tau'' + \frac{6}{a^2} \Psi'' \right] = 3(\delta\rho^{\text{DE}} + \delta\rho^{\text{DM}} + \delta\rho^{\text{int}}), \end{aligned} \quad (4.20)$$

$$\begin{aligned} \delta(\mathcal{G}_{ab}h^{(a} n^{b)}) &= \delta(T_{ab}^{\text{DE}} h^{(a} n^{b)} + T_{ab}^{\text{DM}} h^{(a} n^{b)} + T_{ab}^{\text{int}} h^{(a} n^{b)}) \\ &\Rightarrow \frac{e^\alpha m_p^2}{2} \left[ \left( 4 \frac{\mathcal{H}^2}{a^2} - 2 \frac{a''}{a^3} \right) \partial_i \tau - \left( \frac{2\mathcal{H} + \alpha'}{a} \right) \partial_i \Phi + \frac{\alpha'}{a^2} \partial_i \tau' - \frac{2}{a} \partial_i \Psi' \right] = \left( \partial_i \tau - \frac{a \partial_i \delta \mathcal{R}}{\mathcal{R}'_0} \right) [\mathcal{R} \rho_0^{\text{DM}} \right. \\ &\quad \left. + \mathcal{R} p_0^{\text{DM}} + \mathcal{R} \rho_0^{\text{int}} + \mathcal{R} p_0^{\text{int}}] + \left( \partial_i \tau - \frac{a \partial_i \delta \zeta}{\zeta'} \right) [\zeta \rho_0^{\text{DM}} + \zeta p_0^{\text{DM}}] + \frac{1}{4} \mathcal{R} \mathfrak{C}_i^{\text{int}}, \end{aligned} \quad (4.21)$$

$$\begin{aligned} \delta(\mathcal{G}_{ab}h^a_i h^b_i) - \frac{1}{3} \delta(\mathcal{G}_{ab}h^{ab} h_{ii}) &= \delta(T_{ab}^{\text{DE}} h^a_i h^b_i + T_{ab}^{\text{DM}} h^a_i h^b_i + T_{ab}^{\text{int}} h^a_i h^b_i) \\ &\quad - \frac{1}{3} \delta(T_{ab}^{\text{DE}} h^{ab} h_{ii} + T_{ab}^{\text{DM}} h^{ab} h_{ii} + T_{ab}^{\text{int}} h^{ab} h_{ii}) \\ &\Rightarrow \frac{e^\alpha m_p^2}{6} \left[ \partial^2 \left( \Phi - \Psi + \frac{\alpha'}{a} \tau \right) - 3 \partial_i^2 \left( \Phi - \Psi + \frac{\alpha'}{a} \tau \right) \right] = \mathcal{R} \Sigma_{ii}^{\text{int}}, \end{aligned}$$

$$\begin{aligned} \delta(\mathcal{G}_{ab}h^a_i h^b_j) - \frac{1}{3} \delta(\mathcal{G}_{ab}h^{ab} h_{ij}) &= \delta(T_{ab}^{\text{DE}} h^a_i h^b_j + T_{ab}^{\text{DM}} h^a_i h^b_j + T_{ab}^{\text{int}} h^a_i h^b_j) \\ &\quad - \frac{1}{3} \delta(T_{ab}^{\text{DE}} h^{ab} h_{ij} + T_{ab}^{\text{DM}} h^{ab} h_{ij} + T_{ab}^{\text{int}} h^{ab} h_{ij}) \\ &\Rightarrow -\frac{e^\alpha m_p^2}{2} \partial_i \partial_j \left[ \Phi - \Psi + \frac{\alpha'}{a} \tau \right] = \mathcal{R} \Sigma_{ij}^{\text{int}}, \end{aligned} \quad (4.22)$$

where we used eq. (4.2) and the results of the last subsection. The functions  $\delta\rho$ ,  $\delta p$ ,  $\delta\rho$ ,  $\mathfrak{C}_i$ , and  $\Sigma_{ij}$  are the energy density perturbation, the pressure perturbation, the “generalized pressure” perturbation, the “heat flow” covector, and the “anisotropic stress” tensor for the dark matter, dark energy, and interaction sectors. We have borrowed this terminology from imperfect relativistic fluids, which we briefly review in appendix B. The complete definition for these quantities and their expressions in the Newtonian gauge is provided in appendix B.1.

Similarly, we derive the linearized dark matter equations using eq. (4.11). For the radial dark matter perturbation  $\delta\mathcal{R}$  we find

$$\begin{aligned}
& -\frac{1}{a^2}(\partial_\eta^2 - [1 - \alpha_1]\partial^2)\delta\mathcal{R} - \frac{\mathcal{H}}{a^2}\delta\mathcal{R}'(2 + 9\alpha_1) - \delta\mathcal{R}\left(\pm\tilde{m}^2 + 6\tilde{\lambda}\mathcal{R}_0^2 - 6\tilde{\gamma}\frac{a''}{a^3} - \frac{\zeta_0'^2}{a^2}\right) \\
& + 2\frac{\mathcal{R}_0\zeta_0'}{a^2}\delta\zeta' + \Phi\left(2\mu^2\mathcal{R}_0 + 6\frac{\mathcal{H}^2}{a^2}\mathcal{R}_0[\alpha_7 - 3\alpha_8 - \alpha_9 - \alpha_{10}] + 3\frac{\mathcal{H}}{a}\mu\alpha_6\mathcal{R}_0 - 2\mathcal{R}_0\frac{\zeta_0'^2}{a^2}\right. \\
& \left. + 2\frac{\mathcal{R}_0''}{a^2} + 2\frac{\mathcal{H}\mathcal{R}_0'}{a^2}(2 + 9\alpha_1) - 3\mathcal{R}_0\frac{a''}{a^3}[4\tilde{\gamma} + \alpha_7 + 2\alpha_{10}]\right) - 6\alpha_{10}\mathcal{R}_0\left(\frac{\mathcal{H}^2}{a^2} + \frac{a''}{a^3}\right)\Psi \\
& + \frac{\tau}{a}\left(\mp m^2\mathcal{R}_0\beta_2' - 2\lambda\mathcal{R}_0^3\beta_3' - 9\mathcal{R}_0'\alpha_1'\frac{\mathcal{H}}{a^2} + 6\gamma\mathcal{R}_0\beta_4'\frac{a''}{a^3}\right) + \frac{\Phi'}{a}\left(-2\mu\alpha_3\mathcal{R}_0 + \frac{\mathcal{R}_0'}{a}\right. \\
& \left. - \frac{\mathcal{H}}{a}\mathcal{R}_0[6\tilde{\gamma} + 4\alpha_4 + 6\alpha_5 + 3\alpha_7 + 3\alpha_{10}]\right) + \frac{\mathcal{R}_0}{a}\Psi'\left(3\mu\alpha_6 + 3[1 + 3\alpha_1]\frac{\mathcal{R}_0'}{a\mathcal{R}_0}\right. \\
& \left. + \frac{\mathcal{H}}{a}[-18\tilde{\gamma} + 3\alpha_7 - 18\alpha_8 - 6\alpha_9 - 15\alpha_{10}]\right) + \frac{\mathcal{R}_0}{a}\tau'\left(-6\frac{\mathcal{H}^2}{a^2}\alpha_5 - 2\mu^2\alpha_2 - 2\alpha_4\frac{a''}{a^3}\right) \\
& + \frac{\mathcal{R}_0}{a^2}\tau''(2\mu\alpha_3 + 6\alpha_5) - 2\frac{\mathcal{R}_0}{a^2}\alpha_4\Phi'' - 3\frac{\mathcal{R}_0}{a^2}(2\tilde{\gamma} + \alpha_7 + \alpha_{10})\Psi'' + \frac{\mathcal{R}_0}{a^2}\partial^2\tau\left(\alpha_1\frac{\mathcal{R}_0'}{a\mathcal{R}_0}\right. \\
& \left. + \mu\alpha_6 + 2\alpha_7 - 6\alpha_8 - 2\alpha_9\right) - \frac{\mathcal{R}_0}{a^2}\partial^2\Phi(2\tilde{\gamma} - 2\alpha_4 - 2\alpha_5 + \alpha_{10}) + 4\frac{\mathcal{R}_0}{a^2}\partial^2\Psi(\tilde{\gamma} + \alpha_{10}) \\
& + 2\frac{\mathcal{R}_0}{a^3}\alpha_4\tau''' - \frac{\mathcal{R}_0}{a^3}\partial^2\tau'(2\alpha_4 + 2\alpha_5 + \alpha_7) = 0, \tag{4.23}
\end{aligned}$$

and for the angular dark matter perturbation  $\delta\zeta$  we find

$$\begin{aligned}
& -\frac{\mathcal{R}_0^2}{a^2}(\partial_\eta^2 - [1 - \alpha_1]\partial^2)\delta\zeta - 2\frac{\mathcal{R}_0^2}{a^2}\left(\mathcal{H} + \frac{\mathcal{R}_0'}{\mathcal{R}_0}\right)\delta\zeta' - 2\frac{\mathcal{R}_0\zeta_0'}{a^2}\delta\mathcal{R}' + 2\frac{\mathcal{R}_0'\zeta_0'}{a^2}\delta\mathcal{R} + \frac{\mathcal{R}_0^2\zeta_0'}{a^2}(\Phi' + 3\Psi') \\
& + \frac{\mathcal{R}_0^2\alpha_1\zeta_0'}{a^3}\partial^2\tau = 0, \tag{4.24}
\end{aligned}$$

where we used eq. (2.22).

Finally, we derive the linear perturbation equation for the dark energy field by varying the perturbed effective action with respect to  $\tau$ . In doing so, for simplicity and due to our focus being on the dark energy interactions with dark matter, we only keep the second order perturbation terms in  $S_{\text{DM-int}}$  that depend on the dark matter perturbations. Other terms that are either quadratic in  $\tau$  or the metric perturbations, or are products of  $\tau$  and the metric perturbations, are discarded as they would be generated by the EFT of dark energy. Doing this we find

$$\begin{aligned}
& 4\frac{e^\alpha}{a^2}\left(2\Lambda_0 - \frac{3m_p^2\alpha'^2}{a^2}\right)[\tau'' - \partial^2\tau] + 4\frac{e^\alpha}{a^2}\left(2\Lambda_0[2\mathcal{H} + \alpha'] - 3\frac{m_p^2}{a^2}[\alpha'^3 + 2\alpha'\alpha'']\right)\tau' \\
& + 2\frac{e^\alpha}{a^2}\left(\Lambda_0[10\mathcal{H}\alpha' + 2\alpha'^2 + 2\alpha'' - 4e^\alpha\{\mathcal{H}\beta_1' - 4\alpha'\beta_1' - \beta_1''\}] - 8e^\alpha\tilde{\Lambda}[\mathcal{H}\alpha' - 2\alpha'^2 - \alpha'']\right. \\
& \left. + 3\frac{m_p^2}{a^2}\left[4\mathcal{H}^2\alpha'^2 - \mathcal{H}\alpha'^3 - \alpha'^4 + 4\mathcal{H}\alpha'\frac{a''}{a} - 2\alpha'^2\frac{a''}{a} - 2\mathcal{H}\alpha'\alpha'' - 5\alpha'^2\alpha'' - 4\alpha''\frac{a''}{a} - 2\alpha''^2\right.\right. \\
& \left.\left. - 2\alpha'\alpha'''\right]\right)\tau + \frac{e^\alpha}{a}\left(\Lambda_0[-24\mathcal{H} - 4\alpha'] + 16\tilde{\Lambda}e^\alpha\alpha' + 8e^\alpha\Lambda_0\beta_1' + 6\frac{m_p^2}{a^2}\left[2\mathcal{H}\alpha'^2 + \alpha'^3 + 2\alpha'\frac{a''}{a}\right.\right. \\
& \left.\left. + 2\alpha'\alpha'''\right]\right) \tag{4.25}
\end{aligned}$$

$$\begin{aligned}
& + 4\alpha'\alpha'' + 2\alpha'''] \Big) \Phi + 3\frac{e^\alpha}{a} \left( \Lambda_0[-24\mathcal{H} - 4\alpha'] - 16\tilde{\Lambda}e^\alpha\alpha' - 8e^\alpha\Lambda_0\beta'_1 + 6\frac{m_p^2}{a^2} \left[ 2\mathcal{H}\alpha'^2 + \alpha'^3 \right. \right. \\
& \left. \left. + 2\alpha'\frac{a''}{a} + 4\alpha'\alpha'' + 2\alpha'''] \right] \right) \Psi + \frac{32\mathcal{R}_0}{a} \left( \mu^2\alpha_2 \left\{ -3\mathcal{H} - \frac{\mathcal{R}'_0}{\mathcal{R}_0} - \frac{\alpha'_2}{\alpha_2} \right\} + \mu\alpha_3 \left\{ -3\frac{\mathcal{H}^2}{a} - 5\frac{\mathcal{H}\mathcal{R}'_0}{a\mathcal{R}_0} \right. \right. \\
& \left. \left. - 5\frac{\mathcal{H}\alpha'_3}{a\alpha_3} - 2\frac{\mathcal{R}'_0\alpha'_3}{a\mathcal{R}_0\alpha_3} - 3\frac{a''}{a^2} - \frac{\mathcal{R}''_0}{a\mathcal{R}_0} - \frac{\alpha''_3}{a\alpha_3} \right\} + \alpha_4 \left\{ 2\frac{a''\mathcal{R}'_0}{a^3\mathcal{R}_0} + 3\frac{\mathcal{H}\mathcal{R}''_0}{a^2\mathcal{R}_0} + \frac{\mathcal{R}'''_0}{a^2\mathcal{R}_0} + \frac{6\mathcal{H}\mathcal{R}'_0\alpha'_4}{a^2\mathcal{R}_0\alpha_4} \right. \right. \\
& \left. \left. + 2\frac{a''\alpha'_4}{a^3\alpha_4} + 3\frac{\mathcal{R}''_0\alpha'_4}{a^2\mathcal{R}_0\alpha_4} + 3\frac{\mathcal{H}\alpha''_4}{a^2\alpha_4} + 3\frac{\mathcal{R}'_0\alpha''_4}{a^2\mathcal{R}_0\alpha_4} + \frac{\alpha'''_4}{a^2\alpha_4} \right\} + \alpha_5 \left\{ 3\frac{\mathcal{H}^3}{a^2} - 3\frac{\mathcal{H}^2\mathcal{R}'_0}{a^2\mathcal{R}_0} - 6\frac{\mathcal{H}a''}{a^3} - 6\frac{a''\mathcal{R}'_0}{a^3\mathcal{R}_0} \right. \right. \\
& \left. \left. - 3\frac{\mathcal{H}\mathcal{R}''_0}{a^2\mathcal{R}_0} - 3\frac{a'''}{a^3} - 3\frac{\mathcal{H}^2\alpha'_5}{a^2\alpha_5} - 6\frac{\mathcal{H}\mathcal{R}'_0\alpha'_5}{a^2\mathcal{R}_0\alpha_5} - 6\frac{a''\alpha'_5}{a^3\alpha_5} - 3\frac{\mathcal{H}\alpha''_5}{a^2\alpha_5} \right\} + \frac{m^2\beta'_2}{2} + \lambda\beta'_3\mathcal{R}_0^2 - 3\gamma\beta'_4\frac{a''}{a^3} \right) \delta\mathcal{R} \\
& + \frac{16\mathcal{R}_0}{a^2} \left( -\alpha_1\frac{\mathcal{R}'_0}{a\mathcal{R}_0} + \alpha_4 \left\{ -2\frac{\mathcal{H}}{a} - 2\frac{\mathcal{R}'_0}{a\mathcal{R}_0} - 2\frac{\alpha'_4}{a\alpha_4} \right\} + \alpha_5 \left\{ -2\frac{\mathcal{H}}{a} - 2\frac{\mathcal{R}'_0}{a\mathcal{R}_0} - 2\frac{\alpha'_5}{a\alpha_5} \right\} - \mu\alpha_6 \right. \\
& \left. - 2\alpha_7\frac{\mathcal{H}}{a} + 6\alpha_8\frac{\mathcal{H}}{a} + \alpha_9 \left\{ \frac{\mathcal{H}}{a} - \frac{\mathcal{R}'_0}{a\mathcal{R}_0} - \frac{\alpha'_9}{a\alpha_9} \right\} \right) \partial^2\delta\mathcal{R} - 16\alpha_1\frac{\mathcal{R}_0^2\zeta'_0}{a^3}\partial^2\delta\zeta + 8e^\alpha m_p^2\frac{\alpha'}{a^3}\partial^2[\Phi - 3\Psi] \\
& + 32\frac{\mathcal{R}_0}{a} \left( -\mu^2\alpha_2 + \mu\alpha_3 \left\{ -5\frac{\mathcal{H}}{a} - 2\frac{\mathcal{R}'_0}{a\mathcal{R}_0} - 2\frac{\alpha'_3}{a\alpha_3} \right\} + \alpha_4 \left\{ 6\mathcal{H}\frac{\mathcal{R}'_0}{a^2\mathcal{R}_0} + 2\frac{a''}{a^3} + 3\frac{\mathcal{R}''_0}{a^2\mathcal{R}_0} + 6\frac{\mathcal{H}\alpha'_4}{a^2\alpha_4} \right. \right. \\
& \left. \left. + 6\frac{\mathcal{R}'_0\alpha'_4}{a^2\mathcal{R}_0\alpha_4} + 3\frac{\alpha''_4}{a^2\alpha_4} \right\} + \alpha_5 \left\{ -3\frac{\mathcal{H}^2}{a^2} - 6\frac{\mathcal{H}\mathcal{R}'_0}{a^2\mathcal{R}_0} - 6\frac{a''}{a^3} - 6\frac{\mathcal{H}\alpha'_5}{a^2\alpha_5} \right\} \right) \delta\mathcal{R}' + \frac{e^\alpha}{a} \left( -8\Lambda_0 + 12\frac{m_p^2}{a^2} \right. \\
& \left. \times [\mathcal{H}\alpha' + \alpha'^2 + \alpha''] \right) (\Phi' + 3\Psi') - 16\frac{\mathcal{R}_0}{a^3} (2\alpha_4 + 2\alpha_5 + \alpha_9) \partial^2\delta\mathcal{R}' + 32\frac{\mathcal{R}_0}{a^2} \left( -\mu\alpha_3 + \alpha_4 \left\{ 3\frac{\mathcal{H}}{a} \right. \right. \\
& \left. \left. + 3\frac{\mathcal{R}'_0}{a\mathcal{R}_0} + 3\frac{\alpha'_4}{a\alpha_4} \right\} - 3\alpha_5\frac{\mathcal{H}}{a} \right) \delta\mathcal{R}'' + 24e^\alpha m_p^2\frac{\alpha'}{a^3}\Psi'' + 32\alpha_4\frac{\mathcal{R}_0}{a^3}\delta\mathcal{R}''' = 0. \tag{4.25}
\end{aligned}$$

We will be using these equations in the next section to compute the effective Newton's constant for dark matter in this EFT.

## 5 Effective Newton's constant for the weak equivalence principle violating dark matter

In order to compute the effective Newton's constant  $G_{\text{eff}}^{\text{DM}}$  for dark matter, we first take the Newtonian limit of the Einstein equations (4.19) and (4.20), the dark energy equation of motion (4.25), as well as the angular dark matter equation of motion (4.24) by omitting all terms that include a perturbation functions with conformal time derivatives<sup>35</sup> and dropping all background function of dimension two<sup>36</sup> in favour of  $k^2$ , where  $k$  is the wavenumber of a given mode. We also disregard all components of the stress energy tensors for the dark energy, dark matter, and interactions besides the energy densities. In addition, we make two simplifying assumptions regarding the EFT functions, namely that  $\{\alpha, \beta_1\} \rightarrow 0$  and all terms involving the EFT functions with two or more conformal time derivatives are vanishingly small. The former is assumed in order to disregard the Universal corrections to the Newton's constant<sup>37</sup> and the latter is done assuming that the EFT functions are slowly varying with time.<sup>38</sup>

<sup>35</sup>In the Newtonian limit, the gravitational potentials  $\Phi$  and  $\Psi$  are assumed to be nearly time independent. The situation for the dark matter perturbation functions is less obvious. However, a numerical study shows that terms like  $\delta\mathcal{R}''$  and  $\delta\zeta''$  are suppressed compared to terms  $k^2\delta\mathcal{R}$  and  $k^2\delta\zeta$ .

<sup>36</sup>E.g.  $\mathcal{H}^2, a''/a, \mathcal{R}''/\mathcal{R}_0, \dots$

<sup>37</sup>I.e. corrections that are not specific to dark matter.

<sup>38</sup>In consistency with approximate de Sitter invariance of the spacetime.

After implementing the above approximations, the Einstein equations (4.19) and (4.20), the dark energy equation of motion (4.25), and the angular dark matter equation of motion (4.24) reduce to (we set  $\partial^2 \rightarrow -k^2$ )

$$\begin{aligned}
m_p^2 \left[ -6 \frac{\mathcal{H}^2}{a^2} {}^N\Phi - \frac{2k^2}{a^2} {}^N\Psi \right] &\approx {}^N\delta\rho^{\text{DE}} + {}^N\delta\rho^{\text{DM}} + {}^N\delta\rho^{\text{int}}, \\
{}^N\Phi &\approx {}^N\Psi, \\
\frac{8\Lambda_0 k^2}{a^2} {}^N\tau - \frac{24\Lambda_0 \mathcal{H}}{a} ({}^N\Phi + 3{}^N\Psi) + \frac{32\mathcal{R}_0}{a} \left[ \frac{\mathcal{R}'_0 k^2}{2a^2 \mathcal{R}_0} \alpha_1 + \mu^2 \alpha_2 \left\{ -3\mathcal{H} - \frac{\mathcal{R}'_0}{\mathcal{R}_0} - \frac{\alpha'_2}{\alpha_2} \right\} + \mu \alpha_3 \right. \\
&\times \left. \left\{ -3\frac{\mathcal{H}^2}{a} - 5\frac{\mathcal{H}\mathcal{R}'_0}{a\mathcal{R}_0} - 5\frac{\mathcal{H}\alpha'_3}{a\alpha_3} - 2\frac{\mathcal{R}'_0\alpha'_3}{a\mathcal{R}_0\alpha_3} - 3\frac{a''}{a^2} - \frac{\mathcal{R}''_0}{a\mathcal{R}_0} \right\} + \frac{k^2}{a^2} \alpha_4 \left\{ \mathcal{H} + \frac{\mathcal{R}'_0}{\mathcal{R}_0} + \frac{\alpha'_4}{\alpha_4} \right\} \right. \\
&+ \frac{k^2}{a^2} \alpha_5 \left\{ \mathcal{H} + \frac{\mathcal{R}'_0}{\mathcal{R}_0} + \frac{\alpha'_5}{\alpha_5} \right\} + \frac{k^2}{2a} \mu \alpha_6 + \frac{k^2 \mathcal{H}}{a^2} \alpha_7 - 3\frac{k^2 \mathcal{H}}{a^2} \alpha_8 - \frac{k^2}{2a^2} \alpha_9 \left\{ \mathcal{H} - \frac{\mathcal{R}'_0}{\mathcal{R}_0} - \frac{\alpha'_9}{\alpha_9} \right\} \\
&\left. \pm \frac{m^2 \beta'_2}{2} + \lambda \beta'_3 \mathcal{R}_0^2 - 3\gamma \beta'_4 \frac{a''}{a^3} \right] {}^N\delta\mathcal{R} + 16\alpha_1 \frac{\mathcal{R}_0^2 \zeta'_0 k^2}{a^3} {}^N\delta\zeta &\approx 0, \\
-\frac{\mathcal{R}_0^2 k^2}{a^2} [1 - \alpha_1] {}^N\delta\zeta - 2\frac{\mathcal{R}_0}{a^2} \left( 2\mathcal{H}\zeta'_0 + \frac{\mathcal{R}'_0 \zeta'_0}{\mathcal{R}_0} + \zeta''_0 \right) {}^N\delta\mathcal{R} - \frac{\mathcal{R}_0^2 k^2 \alpha_1 \zeta'_0}{a^3} {}^N\tau &\approx 0, \tag{5.1}
\end{aligned}$$

where the left superscript “N” on the perturbation functions stands for “Newtonian”. In this limit, the perturbed energy densities for the dark energy, dark matter, and interactions become

$$\begin{aligned}
{}^N\delta\rho^{\text{DE}} &\approx -\Lambda_0 {}^N\Phi, \\
{}^N\delta\rho^{\text{DM}} = {}^N\mathcal{R}\delta\rho^{\text{DM}} + {}^N\zeta\delta\rho^{\text{DM}} &\approx 2\mathcal{R}_0 {}^N\delta\mathcal{R} \left[ \pm m^2 + 2\lambda\mathcal{R}_0^2 + \frac{\zeta'^2_0}{a^2} \right] - 2 \left[ \frac{\mathcal{R}'_0^2}{a^2} + \mathcal{R}_0^2 \frac{\zeta'^2_0}{a^2} \right] {}^N\Phi, \\
{}^N\delta\rho^{\text{int}} &\approx 2\mathcal{R}_0 \left[ \pm m^2 \beta_2 + 2\lambda \beta_3 \mathcal{R}_0^2 - \frac{2k^2}{a^2} \tilde{\gamma} \right] {}^N\delta\mathcal{R} + \mathcal{R}_0^2 \left[ \frac{12}{a^2} \tilde{\gamma} \left( \mathcal{H}^2 + 2\frac{\mathcal{H}\mathcal{R}'_0}{\mathcal{R}_0} + \frac{\mathcal{H}\tilde{\gamma}'}{\tilde{\gamma}} \right) - 2\mu^2 \alpha_2 \right. \\
&+ 2\frac{k^2}{a^2} \alpha_5 - \frac{6}{a} \mu \mathcal{H} \alpha_6 + \alpha_7 \left( -\frac{6\mathcal{H}^2}{a^2} + \frac{3a''}{a^3} \right) + \frac{6\mathcal{H}^2}{a^2} (3\alpha_8 + \alpha_9) + 6\alpha_{10} \left( \frac{\mathcal{H}^2}{a^2} + \frac{a''}{a^3} \right) \left. \right] {}^N\Phi \\
&+ 2\frac{\mathcal{R}_0^2 k^2}{a^2} (2\tilde{\gamma} + 3\alpha_{10}) {}^N\Psi + \frac{\mathcal{R}_0^2}{a} \left[ \pm m^2 \beta'_2 + \lambda \beta'_3 \mathcal{R}_0^2 - \frac{k^2}{a^2} \left\{ 2\tilde{\gamma}' + 4\alpha_7 \left( \frac{\mathcal{R}'_0}{\mathcal{R}_0} + \mathcal{H} + \frac{\alpha'_7}{\alpha_7} \right) \right. \right. \\
&\left. \left. - 2\mathcal{H}(3\alpha_8 + \alpha_9) \right\} \right] {}^N\tau, \tag{5.2}
\end{aligned}$$

where we used eqs. (B.5), (B.6), (B.8), and (B.9).

Using the above equations we find the following Poisson equation for the dark matter perturbations,

$$-\frac{k^2}{a^2} {}^N\Psi = 4\pi G_{\text{eff}}^{\text{DM}} {}^N\delta\rho^{\text{DM}}, \tag{5.3}$$

where the effective Newton’s constant  $G_{\text{eff}}^{\text{DM}}$  is given by

$$\frac{G_{\text{eff}}^{\text{DM}}}{G} = 1 + \frac{\Delta_\gamma + \Delta_{\gamma=0}}{m^2 + \frac{\zeta'^2_0}{a^2} + 2\lambda\mathcal{R}_0^2} + \mathcal{O}(2). \tag{5.4}$$

Here  $\mathcal{O}(2)$  stands for terms that are second order or higher in the EFT functions, and we defined

$$\begin{aligned}\Delta_\gamma &= -2\frac{k^2}{a^2}\tilde{\gamma} + \mathcal{O}(k^0), \\ \Delta_{\gamma=0} &= m^2\beta_2 + 2\mathcal{R}_0^2\lambda\beta_3 - 8\pi G\mathcal{R}_0^2\left(m^2 + \frac{\zeta_0'^2}{a^2} + 2\lambda\mathcal{R}_0^2\right)(\alpha_5 + 3\alpha_{10}) + \mathcal{O}(k^{-2}),\end{aligned}\quad (5.5)$$

where  $\Delta_\gamma$  vanishes when  $\gamma \rightarrow 0$  and  $\Delta_{\gamma=0}$  has no dependence on  $\gamma$ . The above linearization in the EFT functions is based on the assumption that the violation of the WEP and the non-minimal coupling between the spacetime curvature and the dark matter fields are small effects that can be incorporated as perturbations to the standard minimally coupled dark matter. This assumption is reasonable if dark matter has similar gravitational/dark energy interactions to baryonic matter.

As can be seen from eq. (5.5), only five of the nearly dozen EFT operators introduced in section 3.2 contribute significantly in the Newtonian limit. We now use the results of the analysis performed in ref. [38] to derive individual constraints on the five EFT functions  $\tilde{\gamma}, \beta_2, \beta_3, \alpha_5, \alpha_{10}$ .<sup>39</sup> From ref. [38] we have

$$\frac{G_{\text{eff}}^{\text{DM}}}{G} - 1 \lesssim 10^{-5}, \quad (5.6)$$

which gives

$$\begin{aligned}|\tilde{\gamma}| &\lesssim \frac{m^2}{k_{\text{phys}}^2} 10^{-5}, \\ |\beta_2| &\lesssim 10^{-5}, \quad |\beta_3| \lesssim \frac{m^4}{\Lambda_0 \lambda} 10^{-5}, \\ |\alpha_5|, |\alpha_{10}| &\lesssim \frac{m^2}{H_0^2} 10^{-5}.\end{aligned}\quad (5.7)$$

Here  $k_{\text{phys}}$  denotes the current value of the wavenumber  $k$  and we used the approximations  $m^2 + \zeta'^2/a^2 + 2\lambda\mathcal{R}_0^2 \sim m^2$  and  $m^2\mathcal{R}_0^2 \sim \Lambda_0$ . Given that  $m \gtrsim 10^{-24}$  eV and  $k_{\text{phys}} \lesssim 10^{-30}$  eV for the CMB modes, the functions  $\tilde{\gamma}$ ,  $\alpha_5$ , and  $\alpha_{10}$  are not strongly constrained. On the other hand, the above constraint on  $\beta_2$  is quite strong. Likewise, unless  $m^4/(\lambda\Lambda_0) \gtrsim 10^5$ ,  $\beta_3$  is tightly bounded. Note that the constraints given in eq. (5.7) on  $\tilde{\gamma}$ ,  $\beta_2$ ,  $\beta_3$ , and  $\alpha_5$  are far more stringent than the ones obtained from eq. (4.17).

## 6 Concluding remarks

Our main objective in this paper was to incorporate dark matter in the EFT of dark energy. Unfortunately, our current understanding of dark matter physics does not provide us with a preferred model for dark matter. We chose to model dark matter using a complex scalar field, the coherent excitations of which corresponds to the usual notion of dark matter particles. In this respect, the dark matter model under study overlaps with axion dark matter models.

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<sup>39</sup>In ref. [38], the authors use the induced changes to the location of the acoustic peaks of the CMB anisotropy spectra to constrain the difference between the inertial and gravitational masses of dark matter. Note that the constraints are valid for the modes with physical wavelengths  $\lambda_{\text{phys}} \gtrsim \mathcal{O}(1)$  Mpc.

The formulated EFT has a total of fourteen relevant and marginally relevant operators in the dark matter sector, ten of which only contribute to the linear perturbation theory while the other four contribute to the background dynamics as well. The former set of operators necessarily violate the WEP in the dark matter sector. We used the results of two cosmological studies to constrain the coefficients of eight of these operators. However, the most meaningful constraint only applied to one of these coefficients, namely  $\beta_2(\phi)$  which enters the dark matter effective action via  $m^2\beta_2(\phi)\Pi^\dagger\Pi$ . The most stringent bound on this quantity was obtained from a study of the CMB acoustic peaks [38], which places an upper bound of  $10^{-5}$  on its magnitude. This is the best observational evidence constraining the WEP violation in the dark matter sector. It should be noted, however, that the majority of the EFT operators are not constrained or at least well constrained by the current observational data.

It was noted that the astrophysical and cosmological signatures of the WEP violation, primarily the effective Newton's constant  $G_{\text{eff}}^{\text{DM}}$  for dark matter, is degenerate with the effects of non-minimal coupling between dark matter and gravity. This coupling is negligible for background cosmology as  $R \sim H_0^2 \ll m^2$ . However, its contribution to  $G_{\text{eff}}^{\text{DM}}$  is proportional to  $k_{\text{phys}}^2/m^2$ , which becomes significant on scales shorter than the Compton wavelength of the dark matter particles. Such a coupling could significantly modify structure formation on sub-galactic scales, particularly for ultralight dark matter fields. Moreover, since the dark matter perturbations in this model are expected to have a significant sound speed on scales shorter than their Compton wavelengths [42], it would be interesting to explore through N-body simulations how non-minimal couplings to gravity alter structure formation on small scales as predicted by this model.

Finally, as we noted in section 3.4, the EFT constructed here requires a certain degree of fine tuning to both successfully reproduce the background  $\Lambda\text{CDM}$  cosmology and to be sensible as a radiatively stable EFT. Discovering any connections with UV theories from which such an EFT can result would be of primary interest.

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## A Some background geometric tensors

We find it useful to reminder the reader of the values of some of the geometric tensors evaluated on the spatially flat FRW background that we used in the construction of the EFT given in section 3. We gather this information in table 3.

## B The stress energy tensor for imperfect relativistic fluids

Here we review the basic formalism and definitions for the stress energy tensor of imperfect relativistic fluids. For a more comprehensive discussion, one may consult [73].

		Background Value
1	$\bar{\phi}$	$t$
2	$\vec{n}$	$-\frac{1}{a}\vec{\partial}_\eta$
3	$\nabla_c n^c$	$-3\frac{\mathcal{H}}{a}$
4	$n^c \nabla_c n^d$	0
5	$h_{cd}$	$a^2 \delta_{cd}$
6	$K_{cd}$	$-a' \delta_{cd}$
7	$R_{cdd} dx^c dx^d$	$3\left(\mathcal{H}^2 - \frac{a''}{a}\right) d\eta^2 + \left(\mathcal{H}^2 + \frac{a''}{a}\right) dx^i dx_i$
8	$R$	$6\frac{a''}{a^3}$

**Table 3.** The background values for some of the geometric tensors used in the EFT formalism.

For imperfect fluids, one typically expresses the stress energy tensor using the following ansatz

$$T_{ab} = \epsilon u_a u_b - \mathfrak{C}_{(a} u_{b)} + \mathfrak{p} h_{ab} + \Sigma_{ab}, \quad (\text{B.1})$$

where  $u_a$  is the unit normal vector field associated with the fluid's comoving frame,  $\epsilon \equiv T_{ab} u^a u^b$  is the locally measured energy density in the fluid's comoving frame,  $\mathfrak{C}_a \equiv T_{cd} h^{(c} u_a u^{b)}$  is regarded as the heat flow covector for fluids in thermal equilibrium,  $h_{ab} \equiv g_{ab} + u_a u_b$  is the spatial metric intrinsic to the surfaces of constant time in the fluid's comoving frame,  $\mathfrak{p} \equiv T_{ab} h^{ab}/3$  is the generalized pressure for the fluid, and  $\Sigma_{ab} \equiv T_{cd} h_a^c h_b^d - T_{cd} h^{cd} h_{ab}/3$  is the anisotropic stress tensor for the fluid. The generalized pressure  $\mathfrak{p}$  is typically decomposed in the following way

$$\mathfrak{p} \equiv p - \nu \nabla_a u^a, \quad (\text{B.2})$$

where  $p$  is regarded as the pressure and  $\nu > 0$  is called the bulk viscosity. For fluids in thermal equilibrium,  $\mathfrak{C}_a$  is proportional to the temperature gradient, though its interpretation for fluids that are not in thermal equilibrium is somewhat obscure. Also, note that  $\Sigma_{ab}$  is symmetric and traceless by definition.

### B.1 Stress energy tensors in comoving frames

Here we compute the dark energy, dark matter, and interactions stress energy tensors up to linear order in the scalar perturbations in the Newtonian gauge defined in eq. (2.36). To begin, we express each stress energy tensor in terms of its components in the corresponding comoving frame by writing<sup>40</sup>

$$\begin{aligned} T_{ab}^{\text{DE}} &= \epsilon^{\text{DE}} n_a n_b - n_{(a} \mathfrak{C}_{b)}^{\text{DE}} + \mathfrak{p}^{\text{DE}} h_{ab} + \Sigma_{ab}^{\text{DE}}, \\ \mathcal{R} T_{ab}^{\text{DM}} &= \mathcal{R} \epsilon^{\text{DM}} \mathcal{R} u_a \mathcal{R} u_b - \mathcal{R} u_{(a} \mathcal{R} \mathfrak{C}_{b)}^{\text{DM}} + \mathcal{R} \mathfrak{p}^{\text{DM}} \mathcal{R} H_{ab} + \mathcal{R} \Sigma_{ab}^{\text{DM}}, \\ \mathcal{R} T_{ab}^{\text{int}} &= \mathcal{R} \epsilon^{\text{int}} \mathcal{R} u_a \mathcal{R} u_b - \mathcal{R} u_{(a} \mathcal{R} \mathfrak{C}_{b)}^{\text{int}} + \mathcal{R} \mathfrak{p}^{\text{int}} \mathcal{R} H_{ab} + \mathcal{R} \Sigma_{ab}^{\text{int}}, \\ \zeta T_{ab}^{\text{DM}} &= \zeta \epsilon^{\text{DM}} \zeta u_a \zeta u_b - \zeta u_{(a} \zeta \mathfrak{C}_{b)}^{\text{DM}} + \zeta \mathfrak{p}^{\text{DM}} \zeta H_{ab} + \zeta \Sigma_{ab}^{\text{DM}}, \\ \zeta T_{ab}^{\text{int}} &= \zeta \epsilon^{\text{int}} \zeta u_a \zeta u_b - \zeta u_{(a} \zeta \mathfrak{C}_{b)}^{\text{int}} + \zeta \mathfrak{p}^{\text{int}} \zeta H_{ab} + \zeta \Sigma_{ab}^{\text{int}}. \end{aligned} \quad (\text{B.3})$$

<sup>40</sup>The stress energy tensors are analogous to that of an imperfect relativistic fluid. The analogy, however, is superficial in that a general interacting field theory does not admit a fluid description.

	energy density	generalized pressure	heat flow	anisotropic stress
$\beta_1$	†	†		
$\beta_2$	✓	✓		
$\beta_3$	✓	✓		
$\tilde{\gamma}$	✓	✓	✓	✓
$\alpha$	†	†		
$\alpha_1$				
$\alpha_2$	✓	✓	✓	
$\alpha_3$	✓		✓	
$\alpha_4$	✓	✓	✓	
$\alpha_5$	✓	✓	✓	
$\alpha_6$	✓	✓		
$\alpha_7$	✓	✓	✓	
$\alpha_8$	✓	✓	✓	
$\alpha_9$	✓	✓	✓	✓
$\alpha_{10}$	✓	✓	✓	✓

**Table 4.** Above is a summary of the contribution of the EFT functions and their associated operators to the dark energy (denoted with †) and the radial interactions (denoted with ✓) stress energy tensors.

For each stress energy tensor above,  $\epsilon$  is the locally measured energy density in the comoving frame of the given fluid component,  $p$  is the generalized pressure, and  $\mathfrak{C}^a$  and  $\Sigma_{ab}$  are the analogous heat flow vector and anisotropic shear stress tensor for an imperfect fluid. We also define the four velocities and the spatial metrics associated with the surfaces of constant  $\mathcal{R}$  and  $\zeta$  using

$$\begin{aligned} {}^{\mathcal{R}}u_a &\equiv \frac{\nabla_a \mathcal{R}}{\sqrt{-g^{cd}\nabla_c \mathcal{R}\nabla_d \mathcal{R}}}, & {}^{\zeta}u_a &\equiv \frac{\nabla_a \zeta}{\sqrt{-g^{cd}\nabla_c \zeta\nabla_d \zeta}}, \\ {}^{\mathcal{R}}H_{ab} &\equiv g_{ab} + {}^{\mathcal{R}}u_a {}^{\mathcal{R}}u_b, & {}^{\zeta}H_{ab} &\equiv g_{ab} + {}^{\zeta}u_a {}^{\zeta}u_b. \end{aligned} \quad (\text{B.4})$$

Our objective then comes down to computing  $\epsilon$ ,  $p$ ,  $\mathfrak{C}_a$  and  $\Sigma_{ab}$  for each of the above stress energy tensors up to linear order in the scalar perturbations. A summary of the contribution of the EFT operators to the stress energy tensors is given in table. 4.

For the dark energy stress energy tensor we find

$$\begin{aligned} \epsilon^{\text{DE}} &= \rho_0^{\text{DE}} + \delta\rho^{\text{DE}} = e^{2\alpha} \left[ \frac{e^{-\alpha}}{2} \Lambda_0 + \tilde{\Lambda} \right] + e^\alpha \left[ \Lambda_0 \left( -\Phi + \frac{\alpha'}{2a} \tau + \frac{\tau'}{a} \right) + \frac{e^\alpha}{a} \tilde{\Lambda} \tau \left( 2\alpha' + \frac{\tilde{\Lambda}'}{\tilde{\Lambda}} \right) \right], \\ p^{\text{DE}} &= p_0^{\text{DE}} + \delta p^{\text{DE}} = e^{2\alpha} \left[ \frac{e^{-\alpha}}{2} \Lambda_0 - \tilde{\Lambda} \right] + e^\alpha \left[ \Lambda_0 \left( -\Phi + \frac{\alpha'}{2a} \tau + \frac{\tau'}{a} \right) - \frac{e^\alpha}{a} \tilde{\Lambda} \tau \left( 2\alpha' + \frac{\tilde{\Lambda}'}{\tilde{\Lambda}} \right) \right], \\ \mathfrak{C}_i^{\text{DE}} &= \Sigma_{ab}^{\text{DE}} = 0, \end{aligned} \quad (\text{B.5})$$

and for the angular dark matter and interactions stress energy tensors we find

$$\begin{aligned} \zeta p^{\text{DM}} &= \zeta \epsilon^{\text{DM}} = \zeta \rho_0^{\text{DM}} + \zeta \delta\rho^{\text{DM}} = \mathcal{R}_0^2 \frac{\zeta_0'^2}{a^2} + 2\mathcal{R}_0 \left[ (\delta\mathcal{R} - \mathcal{R}_0 \Phi) \frac{\zeta_0'^2}{a^2} + \mathcal{R}_0 \frac{\zeta_0' \delta\zeta'}{a^2} \right], \\ \zeta \mathfrak{C}_i^{\text{DM}} &= \zeta \Sigma_{ab}^{\text{DM}} = 0, \end{aligned} \quad (\text{B.6})$$

and

$$\zeta \mathfrak{p}^{\text{int}} = \zeta \mathfrak{e}^{\text{int}} = \zeta \mathfrak{C}_a^{\text{int}} = \zeta \Sigma_{ab}^{\text{int}} = 0, \quad (\text{B.7})$$

where  $i$  denotes a spatial index. For the radial dark matter stress energy tensor we find

$$\begin{aligned} \mathcal{R}_{\mathfrak{e}}^{\text{DM}} &= \mathcal{R}_{\rho_0^{\text{DM}}} + \mathcal{R}_{\delta\rho^{\text{DM}}} = \frac{\mathcal{R}_0'^2}{a^2} + \mathcal{R}_0^2 \left[ \pm m^2 + \lambda \mathcal{R}_0^2 \right] + 2\mathcal{R}_0 \delta \mathcal{R} \left[ \pm m^2 + 2\lambda \mathcal{R}_0^2 \right] \\ &\quad - \frac{2\mathcal{R}_0'^2}{a^2} \Phi + \frac{2\mathcal{R}_0' \delta \mathcal{R}'}{a^2}, \\ \mathcal{R}_{\mathfrak{p}}^{\text{DM}} &= \mathcal{R}_{p_0^{\text{DM}}} + \mathcal{R}_{\delta p^{\text{DM}}} = \frac{\mathcal{R}_0'^2}{a^2} + \mathcal{R}_0^2 \left[ \mp m^2 - \lambda \mathcal{R}_0^2 \right] + 2\mathcal{R}_0 \delta \mathcal{R} \left[ \mp m^2 - 2\lambda \mathcal{R}_0^2 \right] \\ &\quad - \frac{2\mathcal{R}_0'^2}{a^2} \Phi + \frac{2\mathcal{R}_0' \delta \mathcal{R}'}{a^2}, \\ \mathcal{R}_{\mathfrak{C}_i^{\text{DM}}} &= \mathcal{R}_{\Sigma_{ab}^{\text{DM}}} = 0. \end{aligned} \quad (\text{B.8})$$

As expected, the dark matter stress energy tensor is that of a perfect fluid.

Finally, for the radial interactions stress energy we find

$$\begin{aligned} \mathcal{R}_{\mathfrak{e}}^{\text{int}} &= \mathcal{R}_{\rho_0^{\text{int}}} + \mathcal{R}_{\delta\rho^{\text{int}}} = \mathcal{R}_0^2 \left[ \pm m^2 \beta_2 + \lambda \beta_3 \mathcal{R}_0^2 - \frac{6}{a^2} \tilde{\gamma} \left( \mathcal{H}^2 + \frac{\mathcal{H}\tilde{\gamma}'}{\tilde{\gamma}} + 2 \frac{\mathcal{H}\mathcal{R}_0'}{\mathcal{R}_0} \right) \right] + 2\mathcal{R}_0 \delta \mathcal{R} \\ &\quad \times \left[ \pm m^2 \beta_2 + 2\lambda \beta_3 \mathcal{R}_0^2 - \frac{6}{a^2} \tilde{\gamma} \left( \mathcal{H}^2 + \frac{\mathcal{H}\mathcal{R}_0'}{\mathcal{R}_0} + \frac{\mathcal{H}\tilde{\gamma}'}{\tilde{\gamma}} \right) \right] + \mathcal{R}_0^2 \Phi \left[ \frac{12}{a^2} \tilde{\gamma} \left( \mathcal{H}^2 + 2 \frac{\mathcal{H}\mathcal{R}_0'}{\mathcal{R}_0} + \frac{\mathcal{H}\tilde{\gamma}'}{\tilde{\gamma}} \right) \right. \\ &\quad \left. - 2\mu^2 \alpha_2 - \frac{6}{a} \mu \mathcal{H} \alpha_6 + \alpha_7 \left( - \frac{6\mathcal{H}^2}{a^2} + \frac{3a''}{a^3} \right) + \frac{6\mathcal{H}^2}{a^2} (3\alpha_8 + \alpha_9) + 6\alpha_{10} \left( \frac{\mathcal{H}^2}{a^2} + \frac{a''}{a^3} \right) \right] - 6 \frac{\mathcal{R}_0^2}{a^2} \alpha_{10} \Psi \\ &\quad \times \left( 2\mathcal{H}^2 + \frac{2\mathcal{H}\mathcal{R}_0'}{\mathcal{R}_0} + \frac{\mathcal{H}\alpha_{10}'}{\alpha_{10}} \right) + \frac{\mathcal{R}_0^2}{a} \tau \left[ \pm m^2 \beta_2' + \lambda \beta_3' \mathcal{R}_0^2 - \frac{6}{a^2} \tilde{\gamma}' \left( 2 \frac{\mathcal{H}\mathcal{R}_0'}{\mathcal{R}_0} + \frac{\mathcal{H}\tilde{\gamma}''}{\tilde{\gamma}'} \right) \right] + \frac{4\tilde{\gamma} \mathcal{R}_0}{a^2} \partial^2 \delta \mathcal{R} \\ &\quad + \frac{\mathcal{R}_0^2}{a^3} \partial^2 \tau \left[ 2\tilde{\gamma}' + 4\alpha_7 \left( \frac{\mathcal{R}_0'}{\mathcal{R}_0} + \mathcal{H} + \frac{\alpha_7'}{\alpha_7} \right) - 2\mathcal{H} (3\alpha_8 + \alpha_9) \right] - 2 \frac{\mathcal{R}_0^2}{a^2} \left[ \alpha_5 \partial^2 \Phi + (2\tilde{\gamma} + 3\alpha_{10}) \partial^2 \Psi \right] \\ &\quad - \frac{12\tilde{\gamma} \mathcal{H} \mathcal{R}_0}{a^2} \delta \mathcal{R}' - \frac{2\mathcal{R}_0^2}{a} \tau' \left[ \frac{3}{a^2} \tilde{\gamma}' \mathcal{H} + \mu^2 \alpha_2 + \frac{2}{a} \mu \alpha_3 \left( 2 \frac{\mathcal{R}_0'}{\mathcal{R}_0} + 3\mathcal{H} + \frac{\alpha_3'}{\alpha_3} \right) - \frac{2}{a^2} \alpha_4 \left( 2 \frac{\mathcal{R}_0'^2}{\mathcal{R}_0^2} + 2 \frac{\mathcal{R}_0''}{\mathcal{R}_0} \right. \right. \\ &\quad \left. \left. + 5 \frac{\mathcal{H}\mathcal{R}_0'}{\mathcal{R}_0} + \frac{\alpha_4'}{\alpha_4} \left\{ \frac{5\mathcal{H}}{2} + \frac{4\mathcal{R}_0'}{\mathcal{R}_0} \right\} + \frac{\alpha_4''}{\alpha_4} \right) + \frac{3}{a^2} \alpha_5 \left( 3\mathcal{H}^2 + \frac{4\mathcal{H}\mathcal{R}_0'}{\mathcal{R}_0} + \frac{2a''}{a} + \frac{2\mathcal{H}\alpha_5'}{\alpha_5} \right) \right] + \frac{\mathcal{R}_0^2}{a^2} \left[ \Phi' \right. \\ &\quad \times \left( 2\alpha_4 \left\{ \frac{2\mathcal{R}_0'}{\mathcal{R}_0} + \frac{\alpha_4'}{\alpha_4} \right\} + 3\mathcal{H} (-2\alpha_5 + \alpha_7) \right) + 3\Psi' \left( 2\tilde{\gamma} \left\{ 2\mathcal{H} + 2 \frac{\mathcal{R}_0'}{\mathcal{R}_0} + \frac{\tilde{\gamma}'}{\tilde{\gamma}} \right\} + \alpha_7 \left\{ \frac{4\mathcal{R}_0'}{\mathcal{R}_0} + 5\mathcal{H} \right. \right. \\ &\quad \left. \left. + 2 \frac{\alpha_7'}{\alpha_7} \right\} - 2\mathcal{H} (3\alpha_8 + \alpha_9) - \alpha_{10} \left\{ 2\mathcal{H} + \frac{2\mathcal{R}_0'}{\mathcal{R}_0} + \frac{\alpha_{10}'}{\alpha_{10}} \right\} \right) + \frac{\mathcal{R}_0^2}{a^3} (2\alpha_5 + \alpha_7) \partial^2 \tau' + \frac{\mathcal{R}_0^2}{a^2} \left[ \frac{\tau''}{a^3} \left\{ 6\mathcal{H} \alpha_5 \right. \right. \\ &\quad \left. \left. - 2\alpha_4 \left( 2 \frac{\mathcal{R}_0'}{\mathcal{R}_0} + \frac{\alpha_4'}{\alpha_4} \right) \right\} + 3\alpha_7 \Psi'' \right], \end{aligned} \quad (\text{B.9})$$

$$\begin{aligned} \mathcal{R}_{\mathfrak{p}}^{\text{int}} &= \mathcal{R}_{p_0^{\text{int}}} + \mathcal{R}_{\delta p^{\text{int}}} = \mathcal{R}_0^2 \left[ \mp m^2 \beta_2 - \lambda \beta_3 \mathcal{R}_0^2 + \frac{2}{a^2} \tilde{\gamma} \left( - \mathcal{H}^2 + 2 \frac{\mathcal{H}\mathcal{R}_0'}{\mathcal{R}_0} + 2 \frac{\mathcal{R}_0'^2}{\mathcal{R}_0^2} + \frac{\mathcal{H}\tilde{\gamma}'}{\tilde{\gamma}} \right. \right. \\ &\quad \left. \left. + 4 \frac{\mathcal{R}_0' \tilde{\gamma}'}{\mathcal{R}_0 \tilde{\gamma}} + 2 \frac{a''}{a} + 2 \frac{\mathcal{R}_0''}{\mathcal{R}_0} + \frac{\tilde{\gamma}''}{\tilde{\gamma}} \right) \right] + 2\mathcal{R}_0 \delta \mathcal{R} \left[ \mp m^2 \beta_2 - 2\lambda \beta_3 \mathcal{R}_0^2 + \frac{2}{a^2} \tilde{\gamma} \left( - \mathcal{H}^2 + \frac{\mathcal{H}\mathcal{R}_0'}{\mathcal{R}_0} + \frac{\mathcal{H}\tilde{\gamma}'}{\tilde{\gamma}} \right. \right. \\ &\quad \left. \left. + 2 \frac{\mathcal{R}_0' \tilde{\gamma}'}{\mathcal{R}_0 \tilde{\gamma}} + 2 \frac{a''}{a} + \frac{\mathcal{R}_0''}{\mathcal{R}_0} + \frac{\tilde{\gamma}''}{\tilde{\gamma}} \right) \right] - \frac{8\tilde{\gamma} \mathcal{R}_0}{3a^2} \partial^2 \delta \mathcal{R} + 4 \frac{\tilde{\gamma} \mathcal{R}_0}{a^2} \delta \mathcal{R}'' + \frac{4}{a^2} \tilde{\gamma} \mathcal{R}_0 \delta \mathcal{R}' \left[ \mathcal{H} + 2 \frac{\mathcal{R}_0'}{\mathcal{R}_0} + 2 \frac{\tilde{\gamma}'}{\tilde{\gamma}} \right] \\ &\quad + \mathcal{R}_0^2 \Phi \left[ \frac{4}{a^2} \tilde{\gamma} \left( \mathcal{H}^2 - 2 \frac{\mathcal{H}\mathcal{R}_0'}{\mathcal{R}_0} - 2 \frac{\mathcal{R}_0'^2}{\mathcal{R}_0^2} - \frac{\mathcal{H}\tilde{\gamma}'}{\tilde{\gamma}} - 4 \frac{\mathcal{R}_0' \tilde{\gamma}'}{\mathcal{R}_0 \tilde{\gamma}} - 2 \frac{a''}{a} - 2 \frac{\mathcal{R}_0''}{\mathcal{R}_0} - \frac{\tilde{\gamma}''}{\tilde{\gamma}} \right) + 2\mu^2 \alpha_2 - \mu \alpha_6 \left( \frac{2\mathcal{R}_0'}{a\mathcal{R}_0} \right. \right. \\ &\quad \left. \left. + \frac{\alpha_6'}{a\alpha_6} \right) - \alpha_7 \left( 19 \frac{\mathcal{H}\mathcal{R}_0'}{a^2 \mathcal{R}_0} + 9 \frac{a''}{a^3} + 2 \frac{\mathcal{R}_0'^2}{a^2 \mathcal{R}_0^2} - \frac{\mathcal{R}_0''}{a^2 \mathcal{R}_0} + \left\{ \frac{8\mathcal{H}}{a^2} + 4 \frac{\mathcal{R}_0'}{a^2 \mathcal{R}_0} \right\} \frac{\alpha_7'}{\alpha_7} + \frac{\alpha_7''}{a^2 \alpha_7} \right) + 3\alpha_8 \left( - \frac{2\mathcal{H}^2}{a^2} \right. \right. \end{aligned}$$

$$\begin{aligned}
& + 8 \frac{\mathcal{H} \mathcal{R}'_0}{a^2 \mathcal{R}_0} + 4 \frac{a''}{a^3} + 4 \frac{\mathcal{H} \alpha'_8}{a^2 \alpha_8} \Big) + 2 \alpha_9 \left( - \frac{\mathcal{H}^2}{a^2} + 4 \frac{\mathcal{H} \mathcal{R}'_0}{a^2 \mathcal{R}_0} + 2 \frac{a''}{a^3} + 2 \frac{\mathcal{H} \alpha'_9}{a^2 \alpha_9} \right) + 2 \alpha_{10} \left( 2 \frac{\mathcal{H} \mathcal{R}'_0}{a^2 \mathcal{R}_0} + \frac{\mathcal{H} \alpha'_{10}}{a^2 \alpha_{10}} \right) \Big] \\
& - 2 \mathcal{R}_0^2 \Psi \left[ \mu \alpha_6 \left( 3 \frac{\mathcal{H}}{a} + 2 \frac{\mathcal{R}'_0}{a \mathcal{R}_0} + \frac{\alpha'_6}{a \alpha_6} \right) + \alpha_7 \left( 3 \frac{\mathcal{H}^2}{a^2} + 10 \frac{\mathcal{H} \mathcal{R}'_0}{a^2 \mathcal{R}_0} + 3 \frac{a''}{a^3} + 2 \frac{\mathcal{R}'_0^2}{a^2 \mathcal{R}_0^2} + 2 \frac{\mathcal{R}_0''}{a^2 \mathcal{R}_0} + 5 \frac{\mathcal{H} \alpha'_7}{a^2 \alpha_7} \right. \right. \\
& \left. \left. + 4 \frac{\mathcal{R}'_0 \alpha'_7}{a^2 \mathcal{R}_0 \alpha_7} + \frac{\alpha''_7}{a^2 \alpha_7} \right) - 6 \alpha_8 \left( \frac{\mathcal{H}^2}{a^2} + 2 \frac{\mathcal{H} \mathcal{R}'_0}{a^2 \mathcal{R}_0} + \frac{a''}{a^3} + \frac{\mathcal{H} \alpha'_8}{a^2 \alpha_8} \right) - 2 \alpha_9 \left( \frac{\mathcal{H}^2}{a^2} + 2 \frac{\mathcal{H} \mathcal{R}'_0}{a^2 \mathcal{R}_0} + \frac{a''}{a^3} + \frac{\mathcal{H} \alpha'_9}{a^2 \alpha_9} \right) \right. \\
& \left. - \alpha_{10} \left( \frac{\mathcal{H}^2}{a^2} + 8 \frac{\mathcal{H} \mathcal{R}'_0}{a^2 \mathcal{R}_0} + 3 \frac{a''}{a^3} + 4 \frac{\mathcal{H} \alpha'_{10}}{a^2 \alpha_{10}} \right) \right] + \frac{\mathcal{R}_0^2}{a} \tau \left[ \mp m^2 \beta'_2 - \lambda \beta'_3 \mathcal{R}_0^2 + 2 \tilde{\gamma}' \left( - 2 \frac{\mathcal{H} \mathcal{R}'_0}{a^2 \mathcal{R}_0} + 2 \frac{\mathcal{R}'_0^2}{a^2 \mathcal{R}_0^2} \right. \right. \\
& \left. \left. + \frac{a''}{a^3} + 2 \frac{\mathcal{R}_0''}{a^2 \mathcal{R}_0} - \frac{\mathcal{H} \tilde{\gamma}''}{a^2 \tilde{\gamma}'} + 4 \frac{\mathcal{R}'_0 \tilde{\gamma}''}{a^2 \mathcal{R}_0 \tilde{\gamma}'} + \frac{\tilde{\gamma}'''}{a^2 \tilde{\gamma}'} \right) \right] + \frac{\mathcal{R}_0^2}{3a^3} \partial^2 \tau \left[ - 4 \tilde{\gamma}' - 6 \alpha_7 \left( 2 \frac{\mathcal{R}'_0}{\mathcal{R}_0} + \mathcal{H} + \frac{\alpha'_7}{\alpha_7} \right) + 6 \alpha_8 \right. \\
& \times \left( 2 \frac{\mathcal{R}'_0}{\mathcal{R}_0} + \mathcal{H} + \frac{\alpha'_8}{\alpha_8} \right) + 2 \alpha_9 \left( 2 \frac{\mathcal{R}'_0}{\mathcal{R}_0} + \mathcal{H} + \frac{\alpha'_9}{\alpha_9} \right) \Big] - \frac{4 \mathcal{R}_0^2}{3a^2} (\tilde{\gamma} + \alpha_{10}) \partial^2 [\Phi - \Psi] + 2 \frac{\mathcal{R}_0^2}{a} \tau' \left[ \frac{\tilde{\gamma}'}{a^2} \left( - \mathcal{H} \right. \right. \\
& \left. \left. + 4 \frac{\mathcal{R}'_0}{\mathcal{R}_0} + 2 \frac{\tilde{\gamma}''}{\tilde{\gamma}'} \right) - \mu^2 \alpha_2 + \alpha_4 \left( \frac{2 \mathcal{H} \mathcal{R}'_0}{a^2 \mathcal{R}_0} + \frac{\mathcal{H} \alpha'_4}{a^2 \alpha_4} \right) + \alpha_5 \left( - 3 \frac{\mathcal{H}^2}{a^2} + 2 \frac{\mathcal{H} \mathcal{R}'_0}{a^2 \mathcal{R}_0} + \frac{\mathcal{H} \alpha'_5}{a^2 \alpha_5} + \frac{a''}{a^3} \right) \right] + 2 \frac{\mathcal{R}_0^2}{a^2} \Phi' \\
& \times \left[ - \tilde{\gamma} \left( 2 \mathcal{H} + 2 \frac{\mathcal{R}'_0}{\mathcal{R}_0} + \frac{\tilde{\gamma}'}{\tilde{\gamma}} \right) + \alpha_4 \left( 2 \frac{\mathcal{R}'_0}{\mathcal{R}_0} + \frac{\alpha'_4}{\alpha_4} \right) + \alpha_5 \left( 2 \frac{\mathcal{R}'_0}{\mathcal{R}_0} - \mathcal{H} + \frac{\alpha'_5}{\alpha_5} \right) - 3 \mathcal{H} \alpha_7 + 3 \mathcal{H} \alpha_8 + \mathcal{H} \alpha_9 \right] \\
& + \frac{\mathcal{R}_0^2}{a^2} \Psi' \left[ 4 \tilde{\gamma} \left( 2 \mathcal{H} + 2 \frac{\mathcal{R}'_0}{\mathcal{R}_0} + \frac{\tilde{\gamma}'}{\tilde{\gamma}} \right) - 6 \alpha_7 \left( 2 \frac{\mathcal{R}'_0}{\mathcal{R}_0} + 2 \mathcal{H} + \frac{\alpha'_7}{\alpha_7} \right) + 6 \alpha_8 \left( 2 \frac{\mathcal{R}'_0}{\mathcal{R}_0} + 2 \mathcal{H} + \frac{\alpha'_8}{\alpha_8} \right) + 2 \alpha_9 \left( 2 \frac{\mathcal{R}'_0}{\mathcal{R}_0} \right. \right. \\
& \left. \left. + 2 \mathcal{H} + \frac{\alpha'_9}{\alpha_9} \right) + \alpha_{10} \left( 10 \mathcal{H} + 6 \frac{\mathcal{R}'_0}{\mathcal{R}_0} + 3 \frac{\alpha'_{10}}{\alpha_{10}} \right) \right] - 2 \frac{\mathcal{R}_0^2}{a^3} \tau'' \left[ - \tilde{\gamma} \alpha_4 \left( 2 \frac{\mathcal{R}'_0}{\mathcal{R}_0} + \frac{\alpha'_4}{\alpha_4} \right) + \alpha_5 \left( 2 \frac{\mathcal{R}'_0}{\mathcal{R}_0} - 3 \mathcal{H} + \frac{\alpha'_5}{\alpha_5} \right) \right] \\
& + 2 \frac{\mathcal{R}_0^2 \alpha_5}{a^2} \Phi'' + 2 \frac{\mathcal{R}_0^2}{a^2} \Psi'' (-2 \gamma \beta_4 - 3 \alpha_7 + 3 \alpha_8 + \alpha_9 + \alpha_{10}) - 2 \frac{\mathcal{R}_0^2}{3a^3} \partial^2 \tau' (3 \alpha_7 - 3 \alpha_8 - \alpha_9) - 2 \frac{\mathcal{R}_0^2 \alpha_5}{a^3} \tau''', \tag{B.10}
\end{aligned}$$

$$\begin{aligned}
^{\mathcal{R}} \mathfrak{C}_i^{\text{int}} &= \mathcal{R}_0^2 \left[ \frac{2 \tilde{\gamma}}{a \mathcal{R}_0} \partial_i \delta \mathcal{R} \left( - 4 \mathcal{H}^2 - 2 \mathcal{H} \frac{\mathcal{R}'_0}{\mathcal{R}_0} - 2 \mathcal{H} \frac{\tilde{\gamma}'}{\tilde{\gamma}} + 2 \frac{\mathcal{R}'_0 \tilde{\gamma}'}{\mathcal{R}_0 \tilde{\gamma}} + 2 \frac{a''}{a} + 2 \frac{\mathcal{R}_0''}{\mathcal{R}_0} + \frac{\tilde{\gamma}''}{\tilde{\gamma}} \right) - \frac{4 \tilde{\gamma}}{a \mathcal{R}_0} \partial_i \delta \mathcal{R}' \right. \\
& + \partial_i \tau \left( \frac{2 \tilde{\gamma}}{a^2} \left\{ - 2 \frac{\mathcal{R}'_0 \tilde{\gamma}'}{\mathcal{R}_0 \tilde{\gamma}} + 2 \frac{\tilde{\gamma}'}{\tilde{\gamma}} \mathcal{H} - \frac{\tilde{\gamma}''}{\tilde{\gamma}} \right\} + 2 \mu^2 \alpha_2 + \frac{2}{a} \mu \alpha_3 \left\{ 3 \mathcal{H} + 2 \frac{\mathcal{R}'_0}{\mathcal{R}_0} + \frac{\alpha'_3}{\alpha_3} \right\} + 2 \alpha_4 \left\{ - \frac{4 \mathcal{H} \mathcal{R}'_0}{a^2 \mathcal{R}_0} \right. \right. \\
& \left. \left. - \frac{2 \mathcal{R}'_0^2}{a^2 \mathcal{R}_0^2} - \frac{2 \mathcal{R}_0''}{a^2 \mathcal{R}_0} - \frac{2 \mathcal{H} \alpha'_4}{a^2 \alpha_4} - \frac{4 \mathcal{R}'_0 \alpha'_4}{a^2 \mathcal{R}_0 \alpha_4} - \frac{\alpha''_4}{a^2 \alpha_4} \right\} + 2 \alpha_5 \left\{ 3 \frac{\mathcal{H}^2}{a^2} + 3 \frac{a''}{a^3} + 6 \frac{\mathcal{H} \mathcal{R}'_0}{a^2 \mathcal{R}_0} + 3 \frac{\mathcal{H} \alpha'_5}{a^2 \alpha_5} \right\} \right) \\
& + \frac{2}{a} \partial_i \tau' \left( - \frac{\tilde{\gamma}'}{a} - \mu \alpha_3 + \alpha_4 \left\{ 2 \frac{\mathcal{R}'_0}{a \mathcal{R}_0} + \frac{\alpha'_4}{a \alpha_4} \right\} - 4 \frac{\mathcal{H}}{a} \alpha_5 \right) + \frac{1}{a^2} \partial_i \partial^2 \tau (\alpha_7 - 2 \alpha_8 - 2 \alpha_9) + \frac{2}{a^2} \alpha_5 \partial_i \tau'' \\
& + \partial_i \Phi \left( \frac{2 \tilde{\gamma}}{a} \left\{ 2 \frac{\mathcal{R}'_0}{\mathcal{R}_0} + 2 \mathcal{H} + \frac{\tilde{\gamma}'}{\tilde{\gamma}} \right\} + 2 \mu \alpha_3 - 2 \alpha_4 \left\{ 2 \frac{\mathcal{R}'_0}{a \mathcal{R}_0} + \frac{\alpha'_4}{a \alpha_4} \right\} + \frac{\mathcal{H}}{a} (6 \alpha_5 + 3 \alpha_7 - 6 \alpha_8 - 2 \alpha_9) \right) \\
& \left. - \frac{4 \mathcal{H} \mathcal{R}_0^2 \alpha_{10}}{a} \partial_i \Psi - 2 \frac{\alpha_5}{a} \partial_i \Phi' + \frac{1}{a} (4 \tilde{\gamma} + 3 \alpha_7 - 6 \alpha_8 - 2 \alpha_9) \partial_i \Psi' \right], 
\end{aligned}$$

$$\begin{aligned}
^{\mathcal{R}} \Sigma_{ij, (i \neq j)}^{\text{int}} &= 4 \tilde{\gamma} \mathcal{R}_0 \partial_i \partial_j \delta \mathcal{R} + \frac{2}{a} \mathcal{R}_0^2 \left( \tilde{\gamma}' + \alpha_9 \left[ \mathcal{H} + 2 \frac{\mathcal{R}'_0}{\mathcal{R}_0} + \frac{\alpha'_9}{\alpha_9} \right] \right) \partial_i \partial_j \tau + 2 \mathcal{R}_0^2 (\tilde{\gamma} + \alpha_{10}) \partial_i \partial_j [\Phi - \Psi] \\
& + 2 \alpha_9 \frac{\mathcal{R}_0^2}{a} \partial_i \partial_j \tau', \\
^{\mathcal{R}} \Sigma_{ii}^{\text{int}} &= - \frac{2}{3a} \mathcal{R}_0^2 \left( \tilde{\gamma}' + \alpha_9 \left[ \mathcal{H} + 2 \frac{\mathcal{R}'_0}{\mathcal{R}_0} + \frac{\alpha'_9}{\alpha_9} \right] \right) \left[ \partial^2 \tau - 3 \partial_i^2 \tau \right] - \frac{4}{3} \tilde{\gamma} \mathcal{R}_0 \left[ \partial^2 \delta \mathcal{R} - 3 \partial_i^2 \delta \mathcal{R} \right] \\
& - \frac{2}{3} \mathcal{R}_0^2 (\tilde{\gamma} + \alpha_{10}) \left[ \partial^2 (\Phi - \Psi) - 3 \partial_i^2 (\Phi - \Psi) \right] - \frac{2 \mathcal{R}_0^2}{3a} \alpha_9 \left[ \partial^2 \tau' - 3 \partial_i^2 \tau' \right], \tag{B.11}
\end{aligned}$$

where  $\partial^2 \equiv \delta^{ij} \partial_i \partial_j$ .

		dimension
1	$\delta\Pi^\dagger\delta\Pi$	2
2	$\nabla_a\delta\Pi^\dagger\delta\Pi$ , c.c	3
3	$\nabla_a\delta\Pi^\dagger\nabla_b\delta\Pi$	4

**Table 5.** Operators that are quadratic in dark matter perturbations.

## C The effective field theory dark matter operators

The effective action for the dark matter sector expressed in eq. (3.24) consists of all the relevant and marginally relevant dark matter operators that independently contribute to the equations of motion at linear order in scalar perturbations. Some of the operators are expressed in their perturbed forms since their contributions to the background equations of motion are degenerate with those of the unperturbed operators. Our intention in this section is to show that any effective dark matter operator that is quadratic in perturbations can be expressed in terms of the operators listed in table 1. To this end, we first analyse the operators that are quadratic in the dark matter perturbations and then move on to the ones that are linear in the dark matter perturbations.

### C.1 Second order operators in the dark matter sector

We list all such operators up to dimension four in table 5. The indices are contracted using the background  $n^a$ ,  $h^{ab}$ , and  $g^{ab}$ .

All operators listed in the table above, except the second one, directly result from perturbing the operators that are listed in table 1. The second operator can enter the effective Lagrangian density (3.24) in the following form

$$\bar{\mu}n^a(\kappa\nabla_a\delta\Pi^\dagger\delta\Pi + \kappa^*\delta\Pi^\dagger\nabla_a\delta\Pi), \quad (\text{C.1})$$

for some complex function  $\kappa$  and some constant  $\bar{\mu}$  of dimension one. It is not difficult to see that a rescaling of the dark matter fields  $\Pi$  and  $\Pi^\dagger$  by some complex function, together with a redefinition of the functions  $\alpha$ ,  $\alpha_1$ ,  $\tilde{m}^2$ , and  $\tilde{\lambda}$  removes this extra term.

### C.2 First order operators in the dark matter sector

Up to integration by parts, these operators enter the effective action in the form

$$\mathcal{O} \times (\delta\Pi^\dagger\Pi_0 + \text{c.c.}), \quad ^{41} \quad (\text{C.2})$$

where  $\mathcal{O}$  is first order in the gravitational / dark energy perturbations and up to dimension two. All such operators must be perturbations to the operators that transform covariantly under spacetime diffeomorphisms, and when expressed in the dark energy uniform density gauge they must transform covariantly under the foliation preserving diffeomorphisms. They

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<sup>41</sup>One might consider the more general linearly perturbed dark matter operator

$$\kappa\delta\Pi^\dagger\Pi_0 + \text{c.c.},$$

for some complex function  $\kappa$ . This term is consistent with the global U(1) symmetry. However, such a term would have to result from perturbing an operator in the form of  $(\Pi^\dagger\Pi)^m$ , which then implies that  $\kappa$  has to be a real function.

		dimension
1	$\delta g_{ab}, \delta n_a, \delta f$	0
2	$\nabla_c \delta g_{ab}, \nabla_c \delta n_a, \nabla_c \delta f, \delta K_{ab}$	1
3	$\nabla_d \nabla_c \delta g_{ab}, \nabla_d \nabla_c \delta n_a, \nabla_d \nabla_c \delta f, \nabla_d \delta K_{ab}, \delta R_{abcd}^{(3)}, \delta R_{abcd}$	2

**Table 6.** Operators that are linear in gravitational / dark energy perturbations. Recall that  $f \equiv g^{ab} \bar{\phi}_{,a} \bar{\phi}_{,b}$ .  $R_{abcd}^{(3)}$  is the Riemann curvature tensor on surfaces of constant  $\bar{\phi}$ .

are listed in table 6. The indices on the operators listed above are contracted using the background values of  $g^{ab}$  and  $h^{ab}$  (or alternatively  $g^{ab}$  and  $n^a n^b$ ). Also note that the covariant derivative is evaluated using the background FRW metric.

We begin by noting that the operators associated with  $\delta g_{ab}$  are generated in the first line of action (3.24). For instance, the operator

$$g^{ab} \delta g_{ab} (\delta \Pi^\dagger \Pi_0 + \text{c.c}) \quad (\text{C.3})$$

comes from the term  $\sqrt{-g} \Pi^\dagger \Pi$ . Through simple algebraic operations, such as integration by parts, the same can be seen to be true for  $\nabla_c \delta g_{ab}$  and  $\nabla_d \nabla_c \delta g_{ab}$ .

The operator  $\delta n_a$ , which should appear in the effective action as  $n^a \delta n_a$ , and  $\delta f$  provide a similar set of information about the spacetime foliation function  $\bar{\phi}$ , with both being proportional to the lapse function  $\Phi$  defined in eq. (2.36) in the dark energy uniform density gauge.

As for  $\nabla_b \delta n_a$  and  $\nabla_c \nabla_b \delta n_a$ , they encode information about the changes to the unit normal vector field to the surfaces of  $\bar{\phi}$ . This information is provided by the extrinsic curvature and changes to the extrinsic curvature along the unit normal to surfaces of  $\bar{\phi}$ , as well as the operators  $\nabla_a \delta f$  and  $\nabla_a \nabla_b \delta f$ . Therefore we regard the operators associated with  $\delta n_a$  as being redundant.

The operators associated with  $\delta K_{ab}$  appear in the effective action in one of the two following forms

$$n^a n^b \delta K_{ab}, \quad h^{ab} \delta K_{ab}. \quad (\text{C.4})$$

Note that  $h^{ab} \delta K_{ab}$  is generated by  $\delta K$ , and  $n^a n^b \delta K_{ab}$  vanishes. This latter results from

$$n^a n^b K_{ab} = 0 \rightarrow \delta(n^a n^b K_{ab}) = (\delta n^a n^b + \delta n^b n^a) K_{ab} + n^a n^b \delta K_{ab} = 0, \quad (\text{C.5})$$

together with  $\vec{n} = -\vec{\partial}_\eta/a$  and the fact that the background extrinsic curvature is purely spatial. As for the ones corresponding to  $\nabla_a \delta K_{bc}$ , they should enter the action in the following forms

$$n^a h^{bc} \nabla_a \delta K_{bc}, \quad n^a n^b n^c \nabla_a \delta K_{bc}, \quad n^b h^{ac} \nabla_a \delta K_{bc}. \quad (\text{C.6})$$

The middle term in eq. (C.6) vanishes by simple algebraic operations and using eq. (C.5). The first term can be written as

$$\begin{aligned} n^a h^{bc} \nabla_a \delta K_{bc} &= n^a \nabla_a [h^{bc} \delta K_{bc}] - n^a \delta K_{bc} \nabla_a h^{bc} = n^a \nabla_a [\delta K - \delta h^{bc} K_{bc}] - n^a \delta K_{bc} \nabla_a h^{bc} \\ &= n^a \nabla_a \delta K - \delta K_{bc} n^a \nabla_a [n^b n^c] - n^a \nabla_a [\delta h^{bc} K_{bc}] = n^a \nabla_a \delta K - n^a \nabla_a [\delta g^{bc} K_{bc}], \end{aligned} \quad (\text{C.7})$$

where we used  $n^a \nabla_a n^b = 0$  and  $n^a K_{ab} = 0$ . It is evident from the last expression in eq. (C.7) that this operator is already accounted for in the effective action. Similarly, we

write  $n^b h^{ac} \nabla_a \delta K_{bc}$  as

$$\begin{aligned} n^b h^{ac} \nabla_a \delta K_{bc} &= h^{ac} \nabla_a [n^b \delta K_{bc}] - h^{ac} \delta K_{bc} \nabla_a n^b = h^{ac} \nabla_a [n^b \delta K_{bc}] - \delta K_{bc} K^{bc} \\ &= -h^{ac} \nabla_a [\delta n^b K_{bc}] - \delta K_{bc} K^{bc}. \end{aligned} \quad (\text{C.8})$$

This term can be generated from  $\delta[K_{ab} K^{ab}]$  along with operators associated with  $\delta f$  in the effective action.

Finally, the remaining two geometric objects are the Riemann curvature tensors for the leaves of foliation and for the spacetime. It follows from

$$R_{abcd}^{(3)} = h_a{}^e h_b{}^f h_c{}^g h_d{}^h R_{efgh} - K_{ac} K_{bd} + K_{bc} K_{ad} \quad (\text{C.9})$$

that the perturbations to the Riemann curvature tensor of the leaves of foliation can be expressed in terms of  $\delta n_a$  (or  $\delta f$ ),  $\delta K_{ab}$ , and  $\delta R_{abcd}$ . As for the spacetime curvature tensor, the following terms

$$\delta R_{abcd} n^a n^c g^{bd}, \quad \delta R_{abcd} g^{ac} g^{bd} \quad (\text{C.10})$$

along with their various indices-permuted counterparts should be included in the effective action. We instead choose the perturbations to the Ricci tensor  $\delta R_{ab}$  for the spacetime. The difference is absorbed in the terms involving the operators  $\delta f$  and  $\delta g^{ab}$ .

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