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Synchronization and Aggregation of Nonlinear Power Systems with Consideration of Bus Network Structures

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***Abstract*— We study nonlinear power systems consisting of generators, generator buses, and non-generator buses. First, looking at a generator and its bus' variables jointly, we introduce a synchronization concept for a pair of such joint generators and buses. We show that this concept is related to graph symmetry. Next, we extend, in two ways, the synchronization from a pair to a partition of all generators in the networks and show that they are related to either graph symmetry or equitable partitions. Finally, we show how an exact reduced model can be obtained by aggregating the generators and associated buses in the network when the original system is synchronized with respect to a partition, provided that the initial condition respects the partition. Additionally, the aggregation-based reduced model is again a power system.**

I. Introduction

A power system is a network of electrical generators, loads, and their associated control elements. Each of these components may be thought of as nodes of a graph, while the transmission lines connecting them can be regarded as the edges of the graph. The nodes are modeled by physical laws that typically lead to a set of differential equations. These differential equations are coupled to each other across the edges. One question that has been of interest to power engineers over many years is how do the graph-theoretic properties of these types of electrical networks impact system-theoretic properties of the grid model [1].

In this work, we study synchronization properties of power systems (see [2] for an overview) using graph-theoretic tools. Specifically, we show relations to graph symmetry and equitable partitions [3], extending the work in [4] for linear systems to nonlinear power systems. Additionally, based on our results about synchronization, we propose a structure-preserving, aggregation-based model order reduction framework for nonlinear power systems. Further, we show that for certain partitions this reduction is exact. In general, the dynamics of the reduced system can be used to approximate the dynamics of the original power system.

The motivation for model aggregation, in addition to reducing simulation time, is the possibility to simulate or control only a certain part of the grid, or a certain phenomenon that happens only over a certain time-scale. Some recent work on aggregation of linear network systems can be found in [5]–[10].

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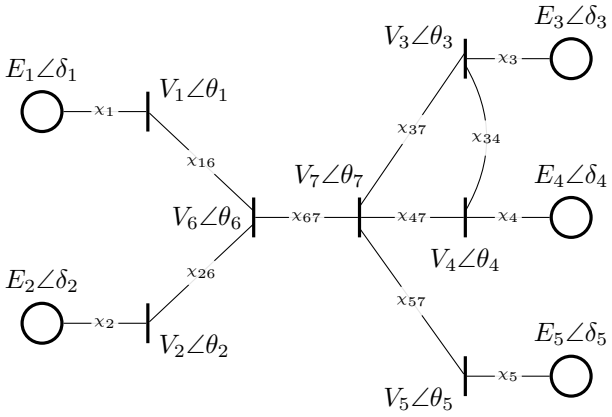

Fig. 1. Power system consisting of generators (circles) and buses (vertical bars), where the ith generator is only connected to the ith bus. See Table I for the notation.

In Section II, we describe the system we analyze. Next, we introduce synchronization for a pair of generators and prove necessary and sufficient conditions in Section III. In Section IV, we continue in a similar way with two notions of synchronization with respect to a partition. We discuss aggregation-based reduction in Section V. Finally, we give a demonstration of our results in Section VI.

II. System Description

We use the power system example in Figure 1 to introduce the type of system we analyze and to illustrate our results. As in the example in Figure 1, we consider power systems consisting of generators and buses, where each generator is connected to exactly one bus and buses can be classified into *generator buses* (those connected to one generator and some buses) and *non-generator buses* (those connected only to other buses). We follow the classical model of a synchronous generator [11], which means that the generators' voltage amplitude is constant over time t.

Let $\mathcal{G} := \{1, 2, \ldots, n\}$ and $\overline{\mathcal{G}} := \{n+1, n+2, \ldots, n+\overline{n}\}$ denote the label sets of generator and non-generator buses. In the example in Figure 1, we have $n = 5$ and $\overline{n} = 2$. The vector of currents from generators to generator buses is given as

$$\boldsymbol{I}_{\mathcal{G}}(t) = \frac{1}{\boldsymbol{i}} L_{\mathrm{D}} \left(\boldsymbol{E}_{\mathcal{G}}(t) - \boldsymbol{V}_{\mathcal{G}}(t) \right), \tag{1}$$

TABLE I
NOTATION

Symbol	Description
$\imath$	imaginary unit ($\imath^2 = -1$)
$\mathbb{R}$, $\mathbb{C}$	fields of real and complex numbers
$[a_i]_{i\in S}$	vector $(a_{i_1}, a_{i_2}, \ldots, a_{i_n})$, if $S = \{i_1, i_2, \ldots, i_n\}$
$\operatorname{diag}(a)$	diagonal matrix with a as its diagonal
$A \circ B$	Hadamard (element-wise) product of two matrices
$\mathbb{1}_n$	vector of ones of length n
$\mathbb{1}$	vector of ones with the length clear from context
e_i	the ith column of the identity matrix
e_S	matrix $[e_{i_1}\ e_{i_2}\ \cdots\ e_{i_n}]$, if $S = \{i_1, i_2, \ldots, i_n\}$
$\operatorname{Im} A$	column space of matrix $A \in \mathbb{F}^{n\times m}$
$\lvert S\rvert$	cardinality of set S
sin, cos	functions applied element-wise to a vector or a matrix
$\mathcal{G}$	label set of generator buses
$\overline{\mathcal{G}}$	label set of non-generator buses
$\boldsymbol{E}_{\mathcal{G}}(t)$	voltages of the generators at time t
E_i	voltage amplitude of the ith generator
$\delta_i(t)$	voltage phase of the ith generator at time t
$\boldsymbol{V}_{\mathcal{G}}(t)$	voltages of the generator buses at time t
$\boldsymbol{V}_{\overline{\mathcal{G}}}(t)$	voltages of the non-generator buses at time t
$V_i(t)$	voltage amplitude of the ith bus at time t
$\theta_i(t)$	voltage phase of the ith bus at time t
$\boldsymbol{I}_{\mathcal{G}}(t)$	currents from generators to generator buses at time t
χ_i	reactance between the ith generator and its bus
χ_{ij}	reactance between the ith and jth bus
L_{D}	reactance matrix, $\operatorname{diag}([\chi_i^{-1}]_{i\in\mathcal{G}})$
L	weighted graph Laplacian of the reactance network
$\delta(t)$	$[\delta_i(t)]_{i\in\mathcal{G}}$
M	diagonal matrix of inertias M_i of the generators
D	diagonal matrix of dissipativies D_i of the generators
f	vector of powers f_i to the generators
X	$(L_{\mathrm{D}} + L_{11} - L_{12}L_{22}^{-1}L_{12}^{\mathsf{T}})^{-1}L_{\mathrm{D}}$
Γ	$L_{\mathrm{D}}(L_{\mathrm{D}} + L_{11} - L_{12}L_{22}^{-1}L_{12}^{\mathsf{T}})^{-1}L_{\mathrm{D}}$
γ_{ij}	$[\Gamma]_{ij}^{-1}$
E	$[E_i]_{i\in\mathcal{G}}$
$V_{\mathcal{G}}(t)$	$[V_i(t)]_{i\in\mathcal{G}}$
$\theta_{\mathcal{G}}(t)$	$[\theta_i(t)]_{i\in\mathcal{G}}$
$\mathcal{X}_{ij}$	subspace of synchronism $\{x \in \mathbb{R}^n : x_i = x_j\}$
$\mathcal{S}_{ij}$	set of symmetrical matrices $\{A \in \mathbb{R}^{n\times n} : A\Pi_{ij} = \Pi_{ij}A\}$
$\mathcal{X}_{\mathrm{cl}}$	$\bigcap_{\ell\in\widehat{\mathcal{G}}}\bigcap_{i,j\in\mathcal{I}_\ell}\mathcal{X}_{ij}$
$\mathcal{S}_{\mathrm{cl}}$	$\bigcap_{\ell\in\widehat{\mathcal{G}}}\bigcap_{i,j\in\mathcal{I}_\ell}\mathcal{S}_{ij}$

where the vectors of voltages of generators and generator buses are denoted as

$$\boldsymbol{E}_{\mathcal{G}}(t) := [E_i(\cos\delta_i(t) + \imath\sin\delta_i(t))]_{i\in\mathcal{G}} \in \mathbb{C}^n,$$
$$\boldsymbol{V}_{\mathcal{G}}(t) := [V_i(t)(\cos\theta_i(t) + \imath\sin\theta_i(t))]_{i\in\mathcal{G}} \in \mathbb{C}^n,$$

and L_{D} is a positive diagonal reactance matrix given as

$$L_{\mathrm{D}} := \operatorname{diag}\left(\left[\chi_i^{-1}\right]_{i\in\mathcal{G}}\right),$$

where χ_i is the reactance between the ith generator and its bus (see Figure 1). We assume the generator voltage amplitudes E_i and reactances χ_i are given constants. Additionally, we assume the line resistances to be negligible.

The relation between the currents and voltages is given as

$$\begin{bmatrix}\boldsymbol{I}_{\mathcal{G}}(t)\\ 0\end{bmatrix} = \frac{1}{\imath}\begin{bmatrix}L_{11} & L_{12}\\ L_{12}^{\mathsf{T}} & L_{22}\end{bmatrix}\begin{bmatrix}\boldsymbol{V}_{\mathcal{G}}(t)\\ \boldsymbol{V}_{\overline{\mathcal{G}}}(t)\end{bmatrix}, \tag{2}$$

where the voltage vector of non-generator buses is denoted as

$$\boldsymbol{V}_{\overline{\mathcal{G}}}(t) := [V_i(t)(\cos\theta_i(t) + \imath\sin\theta_i(t))]_{i\in\overline{\mathcal{G}}} \in \mathbb{C}^{\overline{n}}$$

and $L = [L_{ij}] \in \mathbb{R}^{(n+\overline{n})\times(n+\overline{n})}$ denotes the weighted graph Laplacian of the reactance network. In particular, the (i,j)-th element of L is $-\chi_{ij}^{-1}$ if the ith and jth buses are connected (see Figure 1) and the ith diagonal element is $\sum_{j\neq i}\chi_{ij}^{-1}$. In the following, we assume that the reactance network is connected, i.e. L is irreducible. This assumption can be made without loss of generality because the same arguments can be applied to each connected component. For the example in Figure 1 with $\chi_{ij} = 1$ for all i, j, we have

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 2 & -1 & 0 & 0 & -1\\ 0 & 0 & -1 & 2 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & 1 & 0 & -1\\ -1 & -1 & 0 & 0 & 0 & 3 & -1\\ 0 & 0 & -1 & -1 & -1 & -1 & 4\end{bmatrix}.$$

The dynamics of generators is given by

$$M\ddot{\delta}(t) + D\dot{\delta}(t) = f - \left[\frac{E_iV_i(t)}{\chi_i}\sin(\delta_i(t) - \theta_i(t))\right]_{i\in\mathcal{G}}, \tag{3a}$$

with voltage phases $\delta(t) := [\delta_i(t)]_{i\in\mathcal{G}}$, inertia constants $M := \operatorname{diag}\big([M_i]_{i\in\mathcal{G}}\big)$, $M_i > 0$, damping constants $D := \operatorname{diag}\big([D_i]_{i\in\mathcal{G}}\big)$, $D_i \geq 0$, and input powers $f \in \mathbb{R}^n$ [11]. Eliminating $\boldsymbol{I}_{\mathcal{G}}(t)$ from (1) and (2), we obtain

$$\begin{bmatrix}L_{\mathrm{D}}\left(\boldsymbol{E}_{\mathcal{G}}(t) - \boldsymbol{V}_{\mathcal{G}}(t)\right)\\ 0\end{bmatrix} = \begin{bmatrix}L_{11} & L_{12}\\ L_{12}^{\mathsf{T}} & L_{22}\end{bmatrix}\begin{bmatrix}\boldsymbol{V}_{\mathcal{G}}(t)\\ \boldsymbol{V}_{\overline{\mathcal{G}}}(t)\end{bmatrix}, \tag{3b}$$

The set of equations (3) forms a differential-algebraic system. We can remove the algebraic constraints to find an equivalent set of differential equations using Kron reduction [12]. First, from (3b), we find

$$\begin{aligned}\boldsymbol{V}_{\overline{\mathcal{G}}}(t) &= -L_{22}^{-1}L_{12}^{\mathsf{T}}\boldsymbol{V}_{\mathcal{G}}(t),\\ \boldsymbol{V}_{\mathcal{G}}(t) &= X\boldsymbol{E}_{\mathcal{G}}(t),\end{aligned} \tag{4}$$

where

$$X := \left(L_{\mathrm{D}} + L_{11} - L_{12}L_{22}^{-1}L_{12}^{\mathsf{T}}\right)^{-1}L_{\mathrm{D}}. \tag{5}$$

It follows that

$$\Gamma := L_{\mathrm{D}}\left(L_{\mathrm{D}} + L_{11} - L_{12}L_{22}^{-1}L_{12}^{\mathsf{T}}\right)^{-1}L_{\mathrm{D}} = L_{\mathrm{D}}X$$

is a positive definite matrix with positive elements, since $L_{\mathrm{D}} + L_{11} - L_{12}L_{22}^{-1}L_{12}^{\mathsf{T}}$ is positive definite and an M-matrix (i.e., its eigenvalues have positive real parts and its off-diagonal elements are nonpositive, which implies that the elements of its inverse are positive). We denote its elements by $\gamma_{ij}^{-1} := [\Gamma]_{ij}$. Then, multiplying (4) from the left by L_{D}, we find

$$\left[\frac{V_i(t)}{\chi_i}\cos\theta_i(t)\right]_{i\in\mathcal{G}} = \Gamma\,[E_i\cos\delta_i(t)]_{i\in\mathcal{G}},$$
$$\left[\frac{V_i(t)}{\chi_i}\sin\theta_i(t)\right]_{i\in\mathcal{G}} = \Gamma\,[E_i\sin\delta_i(t)]_{i\in\mathcal{G}},$$

which together with (3a) and the trigonometric identity $\sin(\delta_i(t) - \theta_i(t)) = \sin\delta_i(t)\cos\theta_i(t) - \cos\delta_i(t)\sin\theta_i(t)$ gives us

$$\begin{aligned}&M\ddot{\delta}(t) + D\dot{\delta}(t)\\ &= f - \big(\operatorname{diag}\big([E_i\sin\delta_i(t)]_{i\in\mathcal{G}}\big)\,\Gamma\,[E_i\cos\delta_i(t)]_{i\in\mathcal{G}}\\ &\qquad - \operatorname{diag}\big([E_i\cos\delta_i(t)]_{i\in\mathcal{G}}\big)\,\Gamma\,[E_i\sin\delta_i(t)]_{i\in\mathcal{G}}\big).\end{aligned}$$

Thus, now by using $\sin\delta_i(t)\cos\delta_j(t) - \cos\delta_i(t)\sin\delta_j(t) = \sin(\delta_i(t) - \delta_j(t))$, the Kron-reduced system of (3) is given as

$$M_i\ddot{\delta}_i(t) + D_i\dot{\delta}_i(t) = f_i - \sum_{k=1}^{n} \frac{E_i E_k}{\gamma_{ik}} \sin(\delta_i(t) - \delta_k(t)), \tag{6a}$$

with generator buses' voltages and phases satisfying

$$L_{\mathrm{D}}\boldsymbol{V}_{\mathcal{G}}(t) = \Gamma\boldsymbol{E}_{\mathcal{G}}(t). \tag{6b}$$

Denoting $E := [E_i]_{i\in\mathcal{G}}$, $V_{\mathcal{G}}(t) := [V_i(t)]_{i\in\mathcal{G}}$, and $\theta_{\mathcal{G}}(t) := [\theta_i(t)]_{i\in\mathcal{G}}$, we can write (3a) and (6a) more compactly as

$$M\ddot{\delta}(t) + D\dot{\delta}(t) = f - L_{\mathrm{D}}\left(E \circ V_{\mathcal{G}}(t) \circ \sin(\delta(t) - \theta_{\mathcal{G}}(t))\right), \tag{7}$$

and

$$\begin{aligned} &M\ddot{\delta}(t) + D\dot{\delta}(t) \\ &= f - \left(\Gamma \circ EE^{\mathsf{T}} \circ \sin\left(\delta(t)\mathbb{1}_n^{\mathsf{T}} - \mathbb{1}_n\delta(t)^{\mathsf{T}}\right)\right)\mathbb{1}_n. \end{aligned}$$

III. Synchronization of Generator Pair

Let us denote the subspace of the synchronism between the ith and jth elements by

$$\mathcal{X}_{ij} := \{x \in \mathbb{R}^n : x_i = x_j\}.$$

In this notation, we introduce the following notion of synchronism for the power system (3).

Definition 1. Consider the power system (3). The ith and jth generators are said to be *synchronized* if

$$\delta(t) \in \mathcal{X}_{ij} \text{ and } \boldsymbol{V}_{\mathcal{G}}(t) \in \mathcal{X}_{ij}, \text{ for all } t \geq 0$$

and for any initial condition $\delta(0), \dot{\delta}(0) \in \mathcal{X}_{ij}$.

To characterize this generator synchronism in an algebraic manner, let us define a set of symmetrical matrices with respect to the permutation of the ith and jth columns and rows by

$$\mathcal{S}_{ij} := \{A \in \mathbb{R}^{n\times n} : A\Pi_{ij} = \Pi_{ij}A\}, \tag{8}$$

where Π_{ij} denotes the permutation matrix associated with the ith and jth elements, i.e., all diagonal elements of Π_{ij} other than the ith and jth elements are 1, the (i,j)-th and (j,i)-th elements are 1, and the others are zero. Note that $\mathcal{S}_{ij}$ is not the set of usual symmetric (Hermitian) matrices; the condition in (8) represents the invariance with respect to the permutation of the ith and jth columns and rows, i.e., $\Pi_{ij}^{\mathsf{T}}A\Pi_{ij} = A$. See Lemma 13 for equivalent conditions.

We state the main result about synchronization of a pair of generators and prove it in the remainder of this Section.

Theorem 2. *Consider the power system (3). The following two statements hold.*

1) *Let $n = 2$ and $M_1 = M_2$. Then the two generators are synchronized if and only if $D_1 = D_2$, $f_1 = f_2$, and $E_1 = E_2$.*
2) *Let $n \geq 3$ and $M \in \mathcal{S}_{ij}$. Then the ith and jth generators are synchronized if and only if $D \in \mathcal{S}_{ij}$, $f \in \mathcal{X}_{ij}$, $E \in \mathcal{X}_{ij}$, and $\Gamma \in \mathcal{S}_{ij}$.*

Remark 3. Essentially, this result shows that the ith and jth generators are synchronized when the system equation are invariant under swapping the ith and jth label.

We arrange the proof of Theorem 2 into a sequence of Propositions in this Section, with some technical Lemmas in the Appendix. We begin by analyzing the equations of the system (3) without assumptions on n and M.

Proposition 4. *The ith and jth generators are synchronized if and only if*

$$\frac{D_i}{M_i} = \frac{D_j}{M_j}, \tag{9}$$

$$\frac{f_i}{M_i} = \frac{f_j}{M_j}, \tag{10}$$

$$\frac{E_i}{M_i\gamma_{ik}} = \frac{E_j}{M_j\gamma_{jk}}, \text{ for } k \neq i,j, \tag{11}$$

$$\frac{\chi_i}{\gamma_{ik}} = \frac{\chi_j}{\gamma_{jk}}, \text{ for } k \neq i,j, \text{ and} \tag{12}$$

$$\frac{\chi_i E_i}{\gamma_{ii}} + \frac{\chi_i E_j}{\gamma_{ij}} = \frac{\chi_j E_i}{\gamma_{ji}} + \frac{\chi_j E_j}{\gamma_{jj}}. \tag{13}$$

Proof. The proof can be found in [13]. □

Let us now assume that $E_i \neq E_j$ and see what follows from conditions of Proposition 4. From (12) and Lemma 12, it follows that $\frac{\chi_i}{\gamma_{ii}} + \frac{\chi_i}{\gamma_{ij}} = \frac{\chi_j}{\gamma_{ji}} + \frac{\chi_j}{\gamma_{jj}}$. Then, by (13) and $E_i \neq E_j$, it is necessary that $\frac{\chi_i}{\gamma_{ii}} = \frac{\chi_j}{\gamma_{ji}}$ and $\frac{\chi_i}{\gamma_{ij}} = \frac{\chi_j}{\gamma_{jj}}$. This, together with (12), means that the ith and jth rows in X are equal, which is a contradiction with X being invertible. Therefore, for ith and jth generators to be synchronized, it is necessary that $E_i = E_j$. This allows us to simplify the statement of Proposition 4. We can simplify it further by assuming $M_i = M_j$, which gives us the following Corollary.

Corollary 5. *Let $M_i = M_j$. Then the ith and jth generators are synchronized if and only if*

$$D_i = D_j,$$

$$f_i = f_j,$$

$$E_i = E_j,$$

$$\gamma_{ik} = \gamma_{jk}, \text{ for } k \neq i,j, \tag{14}$$

$$\frac{\chi_i}{\gamma_{ik}} = \frac{\chi_j}{\gamma_{jk}}, \text{ for } k \neq i,j, \text{ and} \tag{15}$$

$$\frac{\chi_i}{\gamma_{ii}} + \frac{\chi_i}{\gamma_{ij}} = \frac{\chi_j}{\gamma_{ji}} + \frac{\chi_j}{\gamma_{jj}}. \tag{16}$$

In the following, we separate the $n = 2$ and $n \geq 3$ cases. First, we use Corollary 5 to prove part 1 of Theorem 2.

Proof of Theorem 2, part 1. This is true since (14) and (15) are empty statements, while (16) follows immediately from Lemma 12. □

Finally, to prove part 2 of Theorem 2, we simplify the statement of Corollary 5 for the case of $n \geq 3$. This gives us the following Corollary.

Corollary 6. *Let $n \geq 3$ and $M_i = M_j$. Then the ith and jth generators are synchronized if and only if*

$$\begin{aligned} D_i &= D_j, \\ f_i &= f_j, \\ E_i &= E_j, \\ \gamma_{ik} &= \gamma_{jk}, \text{ for } k \neq i, j, \qquad (17) \\ \chi_i &= \chi_j, \text{ and} \qquad (18) \\ \gamma_{ii} &= \gamma_{jj}. \qquad (19) \end{aligned}$$

Proof. From (14) and (15) follows (18), using that there are at least three generators. Then, from (16), (18), and symmetry $\gamma_{ij} = \gamma_{ji}$ follows (19). □

Corollary 6, together with two Lemmas in the Appendix, allows us to complete the proof of Theorem 2.

Proof of Theorem 2, part 2. Conditions (17) and (19), by Lemma 13, are equivalent to $\Gamma \in \mathcal{S}_{ij}$, which, by Lemma 15, is in turn equivalent to $L_{\mathrm{D}} \in \mathcal{S}_{ij}$ and $L_{11} - L_{12}L_{22}^{-1}L_{12}^{\mathsf{T}} \in \mathcal{S}_{ij}$. Therefore, (17) and (19) imply (18). □

IV. Synchronization of Generator Partition

Let $\mathcal{I} = \{\mathcal{I}_\ell\}_{\ell \in \widehat{\mathcal{G}}}$ be a partition of the set $\mathcal{G}$, where $\widehat{\mathcal{G}} = \{1, 2, \ldots, \widehat{n}\}$ and $\widehat{n} \leq n$. In particular, the clusters $\mathcal{I}_\ell$ satisfy

1) $\mathcal{I}_\ell \neq \emptyset$, for all $\ell \in \widehat{\mathcal{G}}$,
2) $\mathcal{I}_{\ell_1} \cap \mathcal{I}_{\ell_2} = \emptyset$, for all $\ell_1, \ell_2 \in \widehat{\mathcal{G}}$ such that $\ell_1 \neq \ell_2$, and
3) $\bigcup_{\ell \in \widehat{\mathcal{G}}} \mathcal{I}_\ell = \mathcal{G}$.

Let us denote

$$\mathcal{X}_{\mathrm{cl}} := \bigcap_{\ell \in \widehat{\mathcal{G}}} \bigcap_{i,j \in \mathcal{I}_\ell} \mathcal{X}_{ij}, \quad \mathcal{S}_{\mathrm{cl}} := \bigcap_{\ell \in \widehat{\mathcal{G}}} \bigcap_{i,j \in \mathcal{I}_\ell} \mathcal{S}_{ij}.$$

We define the aggregation matrix as

$$P = \begin{bmatrix} e_{\mathcal{I}_1} \mathbb{1}_{|\mathcal{I}_1|} & e_{\mathcal{I}_2} \mathbb{1}_{|\mathcal{I}_2|} & \cdots & e_{\mathcal{I}_{\widehat{n}}} \mathbb{1}_{|\mathcal{I}_{\widehat{n}}|} \end{bmatrix} \in \mathbb{R}^{n \times \widehat{n}}.$$

Notice that $\mathcal{X}_{\mathrm{cl}} = \operatorname{Im} P$.

We define two notions generalizing the synchronization of two generators to a partition of generators.

Definition 7. The system (3) is said to be *strongly synchronized with respect to partition* $\mathcal{I}$ if the ith and jth generators are synchronized for all $i, j \in \mathcal{I}_\ell$ and all $\ell \in \widehat{\mathcal{G}}$, i.e. $\delta(t) \in \mathcal{X}_{ij}$ and $\boldsymbol{V}_{\mathcal{G}}(t) \in \mathcal{X}_{ij}$ for all $t \geq 0$ and for any $\delta(0), \dot{\delta}(0) \in \mathcal{X}_{ij}$, $i, j \in \mathcal{I}_\ell$, and $\ell \in \widehat{\mathcal{G}}$.

The system (3) is said to be *weakly synchronized with respect to partition* $\mathcal{I}$ if, for arbitrary $\delta(0), \dot{\delta}(0) \in \mathcal{X}_{\mathrm{cl}}$, there exist functions $\widehat{\delta} : [0, \infty) \to \mathbb{R}^{\widehat{n}}$ and $\widehat{\boldsymbol{V}}_{\widehat{\mathcal{G}}} : [0, \infty) \to \mathbb{C}^{\widehat{n}}$ such that $\delta(t) = P\widehat{\delta}(t)$ and $\boldsymbol{V}_{\mathcal{G}}(t) = P\widehat{\boldsymbol{V}}_{\widehat{\mathcal{G}}}(t)$, i.e. $\delta(t) \in \mathcal{X}_{\mathrm{cl}}$ and $\boldsymbol{V}_{\mathcal{G}}(t) \in \mathcal{X}_{\mathrm{cl}}$ for all $t \geq 0$ and for any $\delta(0), \dot{\delta}(0) \in \mathcal{X}_{\mathrm{cl}}$.

Remark 8. Notice that strong synchronization is equivalent to $\mathcal{X}_{ij} \times \mathcal{X}_{ij} \times \mathcal{X}_{ij}$ being an invariant set for $(\delta, \dot{\delta}, \widehat{\boldsymbol{V}}_{\widehat{\mathcal{G}}})$ for any $i, j \in \mathcal{I}_\ell$ and $\ell \in \widehat{\mathcal{G}}$, while weak synchronization is equivalent to an invariant set being $\mathcal{X}_{\mathrm{cl}} \times \mathcal{X}_{\mathrm{cl}} \times \mathcal{X}_{\mathrm{cl}}$. This means that, if the power system is strongly synchronized, when two generators and their buses in the same cluster have equal state, they will remain equal. If the power system is weakly synchronized, then when the states of every generator and its bus are equal to all others in the same cluster, they will stay equal. From this, we see that that if the system (3) is strongly synchronized with respect to $\mathcal{I}$, then it is also weakly synchronized with respect to $\mathcal{I}$, since $\mathcal{X}_{\mathrm{cl}} \times \mathcal{X}_{\mathrm{cl}} \times \mathcal{X}_{\mathrm{cl}} \subseteq \mathcal{X}_{ij} \times \mathcal{X}_{ij} \times \mathcal{X}_{ij}$, for all $i, j \in \mathcal{I}_\ell$ and all $\ell \in \widehat{\mathcal{G}}$.

Further, the ith and jth generators are synchronized if and only if (3) is either strongly or weakly synchronized with respect to $\{\{i, j\}\} \cup \{\{k\} : k \neq i, j\}$.

Finally, notice that (3) is always both strongly and weakly synchronized with respect to $\{\{i\} : i \in \mathcal{G}\}$.

In the following, we show necessary and sufficient conditions for the two synchronization notions. To start, in the next Proposition, we present cases when the structure of Γ has no influence. It also illustrates the relation between strong and weak synchronization.

Proposition 9. *Let $\mathcal{I} = \{\mathcal{G}\}$, $M, D \in \mathcal{S}_{\mathrm{cl}}$, and $f, E \in \mathcal{X}_{\mathrm{cl}}$. Then the system (3) is weakly synchronized with respect to $\{\mathcal{G}\}$. If additionally $n = 2$, then (3) is also strongly synchronized with respect to $\{\mathcal{G}\}$.*

Proof. From the assumptions, it follows that $M = \widehat{m}I$, $D = \widehat{d}I$, $f = \widehat{f}\mathbb{1}$, and $E = \widehat{E}\mathbb{1}$, for some $\widehat{m} > 0$, $\widehat{d} \geq 0$, and $\widehat{f}, \widehat{E} \in \mathbb{R}$. Notice that for $\mathcal{I} = \{\mathcal{G}\}$, we have $P = \mathbb{1}$.

Let us assume that $\delta(0), \dot{\delta}(0) \in \operatorname{Im} \mathbb{1}$. To prove weak synchronization, we need to show that $\delta(t) \in \operatorname{Im} \mathbb{1}$ and $\boldsymbol{V}_{\mathcal{G}}(t) \in \operatorname{Im} \mathbb{1}$. For the former, it is enough to show that $\ddot{\delta}(t) \in \operatorname{Im} \mathbb{1}$ if $\delta(t), \dot{\delta}(t) \in \operatorname{Im} \mathbb{1}$, which is clear, since then $\ddot{\delta}(t) = -M^{-1}D\dot{\delta}(t) + M^{-1}f = -\frac{\widehat{d}}{\widehat{m}}\dot{\delta}(t) + \frac{\widehat{f}}{\widehat{m}}\mathbb{1}$. For the latter, we see that $\boldsymbol{V}_{\mathcal{G}} = L_{\mathrm{D}}^{-1}\Gamma \boldsymbol{E}_{\mathcal{G}} \in \operatorname{Im} \mathbb{1}$ whenever $\boldsymbol{E}_{\mathcal{G}} \in \operatorname{Im} \mathbb{1}$, which is equivalent to $\delta \in \operatorname{Im} \mathbb{1}$.

The second part follows from part 1 of Theorem 2. □

We continue with the first main result of this Section—the necessary and sufficient conditions for strong synchronization. Here, symmetrical conditions for Γ are relevant.

Theorem 10. *Let $n \geq 3$, $\mathcal{I}$ arbitrary, and $M \in \mathcal{S}_{\mathrm{cl}}$. Then the system (3) is strongly synchronized with respect to $\mathcal{I}$ if and only if $D \in \mathcal{S}_{\mathrm{cl}}$, $f \in \mathcal{X}_{\mathrm{cl}}$, $E \in \mathcal{X}_{\mathrm{cl}}$, and $\Gamma \in \mathcal{S}_{\mathrm{cl}}$.*

Proof. It follows from applying part 2 of Theorem 2 for every ith and jth generator where $i, j \in \mathcal{I}_\ell$ and $\ell \in \widehat{\mathcal{G}}$. □

We conclude this Section with the second main result—the necessary and sufficient conditions for weak synchronization. Instead of symmetrical conditions, $\mathcal{X}_{\mathrm{cl}}$ being Γ-invariant is one of the conditions. Since $\mathcal{X}_{\mathrm{cl}} = \operatorname{Im} P$, this actually means that $\mathcal{I}$ is an equitable partition for a graph whose adjacency matrix is Γ [14].

Theorem 11. *Let $|\mathcal{I}| \geq 2$, $M, D \in \mathcal{S}_{\mathrm{cl}}$, and $f, E \in \mathcal{X}_{\mathrm{cl}}$. Then the system (3) is weakly synchronized with respect to $\mathcal{I}$ if and only if*

$$L_{\mathrm{D}} \in \mathcal{S}_{\mathrm{cl}} \quad \text{and} \quad \mathcal{X}_{\mathrm{cl}} \text{ is } \Gamma\text{-invariant}. \qquad (20)$$

Proof. The proof can be found in [13]. □

V. Aggregation of Power Systems

Let us assume that the system (3) is weakly synchronized with respect to a partition $\mathcal{I}$. Let also the initial condition satisfy $\delta(0), \dot{\delta}(0) \in \mathcal{X}_{\mathrm{cl}}$. Then there exist $\widehat{\delta}$ and $\widehat{\boldsymbol{V}}_{\widehat{\mathcal{G}}}$ such that $\delta(t) = P\widehat{\delta}(t)$ and $\boldsymbol{V}_{\mathcal{G}}(t) = P\widehat{\boldsymbol{V}}_{\widehat{\mathcal{G}}}(t)$, which also gives us $V_{\mathcal{G}}(t) = P\widehat{V}_{\widehat{\mathcal{G}}}(t)$ and $\theta_{\mathcal{G}}(t) = P\widehat{\theta}_{\widehat{\mathcal{G}}}(t)$. Inserting this into (3) with dynamics rewritten as in (7), we find

$$\begin{aligned} &MP\ddot{\widehat{\delta}}(t) + DP\dot{\widehat{\delta}}(t) \\ &= f - L_{\mathrm{D}}\left(E \circ P\widehat{V}_{\widehat{\mathcal{G}}}(t) \circ \sin\left(P\widehat{\delta}(t) - P\widehat{\theta}_{\widehat{\mathcal{G}}}(t)\right)\right), \\ &\begin{bmatrix} L_{\mathrm{D}}\left(\boldsymbol{E}_{\mathcal{G}}(t) - P\widehat{\boldsymbol{V}}_{\widehat{\mathcal{G}}}(t)\right) \\ 0 \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{12}^{\mathsf{T}} & L_{22} \end{bmatrix} \begin{bmatrix} P\widehat{\boldsymbol{V}}_{\widehat{\mathcal{G}}}(t) \\ \boldsymbol{V}_{\overline{\mathcal{G}}}(t) \end{bmatrix}. \end{aligned}$$

Assuming additionally that $E \in \mathcal{X}_{\mathrm{cl}}$, i.e. $E = P\widehat{E}$ for some $\widehat{E} \in \mathbb{R}^{\widehat{n}}$, and pre-multiplying the above dynamics and first block-row of the constraint by P^{T}, we obtain

$$\begin{aligned} &\widehat{M}\ddot{\widehat{\delta}}(t) + \widehat{D}\dot{\widehat{\delta}}(t) \\ &= \widehat{f} - \widehat{L}_{\mathrm{D}}\left(\widehat{E} \circ \widehat{V}_{\widehat{\mathcal{G}}}(t) \circ \sin\left(\widehat{\delta}(t) - \widehat{\theta}_{\widehat{\mathcal{G}}}(t)\right)\right), \end{aligned} \tag{21a}$$

$$\begin{bmatrix} \widehat{L}_{\mathrm{D}}\left(\widehat{\boldsymbol{E}}_{\widehat{\mathcal{G}}}(t) - \widehat{\boldsymbol{V}}_{\widehat{\mathcal{G}}}(t)\right) \\ 0 \end{bmatrix} = \begin{bmatrix} \widehat{L}_{11} & \widehat{L}_{12} \\ \widehat{L}_{12}^{\mathsf{T}} & L_{22} \end{bmatrix} \begin{bmatrix} \widehat{\boldsymbol{V}}_{\widehat{\mathcal{G}}}(t) \\ \boldsymbol{V}_{\overline{\mathcal{G}}}(t) \end{bmatrix}, \tag{21b}$$

where $\widehat{M} = P^{\mathsf{T}}MP$, $\widehat{D} = P^{\mathsf{T}}DP$, $\widehat{f} = P^{\mathsf{T}}f$, $\widehat{L}_{\mathrm{D}} = P^{\mathsf{T}}L_{\mathrm{D}}P$, $\widehat{L}_{11} = P^{\mathsf{T}}L_{11}P$, $\widehat{L}_{12} = P^{\mathsf{T}}L_{12}$. Moreover, from $\delta(t) = P\widehat{\delta}(t)$ follows that $\widehat{\delta}(0) = (P^{\mathsf{T}}P)^{-1}P^{\mathsf{T}}\delta(0)$ and $\dot{\widehat{\delta}}(0) = (P^{\mathsf{T}}P)^{-1}P^{\mathsf{T}}\dot{\delta}(0)$.

Notice that the reduced model (21) is again a power system of the same form as (3). In particular, we have that $\widehat{M}$, $\widehat{D}$, and $\widehat{L}_{\mathrm{D}}$ are positive definite diagonal matrices and that $\widehat{L}$ is a Laplacian matrix. Additionally, note that this projection-based reduction can be done for arbitrary power system and arbitrary partition. In general, we can take (21) with $\widehat{\delta}(0) = (P^{\mathsf{T}}P)^{-1}P^{\mathsf{T}}\delta(0)$, $\dot{\widehat{\delta}}(0) = (P^{\mathsf{T}}P)^{-1}P^{\mathsf{T}}\dot{\delta}(0)$, and $\widehat{E} = (P^{\mathsf{T}}P)^{-1}P^{\mathsf{T}}E$. We can also apply Kron reduction to this reduced model.

VI. Illustrative Example

For the example in Figure 1, let $\chi_i = 1$ and $\chi_{ij} = 1$ for all i, j. Then we have

$$\Gamma = \frac{1}{32}\begin{bmatrix} 21 & 5 & 2 & 2 & 2 \\ 5 & 21 & 2 & 2 & 2 \\ 2 & 2 & 16 & 8 & 4 \\ 2 & 2 & 8 & 16 & 4 \\ 2 & 2 & 4 & 4 & 20 \end{bmatrix}.$$

Additionally, let $M = D = I_5$, $f = 0$, and $E = \mathbb{1}_5$. Then, using Theorem 2, we see that the first and second generators are synchronized, and that the same is true for the third and fourth. By definition, this implies that the system is strongly synchronized with respect to $\{\{1,2\},\{3,4\},\{5\}\}$. On the other hand, from Theorem 11 and

$$\Gamma\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 13 & 3 \\ 13 & 3 \\ 2 & 14 \\ 2 & 14 \\ 2 & 14 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}\left(\frac{1}{16}\begin{bmatrix} 13 & 3 \\ 2 & 14 \end{bmatrix}\right),$$

we see that the system is weakly synchronized with respect to $\{\{1,2\},\{3,4,5\}\}$, but not strongly. Using the partition

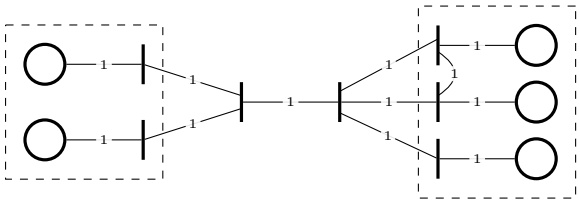

Fig. 2. Partition $\{\{1,2\},\{3,4,5\}\}$ applied to the original power system in Figure 1 with $\chi_i = \chi_{ij} = 1$ for all i, j.

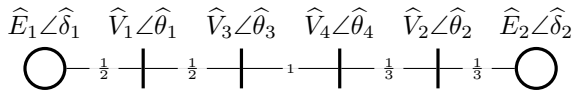

Fig. 3. Reduced power system obtained by aggregating the system in Figure 2 with $M = D = I_5$, $f = 0$, and $E = \mathbb{1}_5$.

$\mathcal{I} = \{\{1,2\},\{3,4,5\}\}$ for aggregation, we find the following reduced quantities: $\widehat{M} = \widehat{D} = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right]$, $\widehat{f} = 0$, $\widehat{E} = \mathbb{1}_2$, $\widehat{L}_{\mathrm{D}} = \widehat{L}_{11} = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right]$, $\widehat{L}_{12} = \left[\begin{smallmatrix} -2 & 0 \\ 0 & -3 \end{smallmatrix}\right]$, $\widehat{\Gamma} = \frac{1}{8}\left[\begin{smallmatrix} 13 & 3 \\ 3 & 21 \end{smallmatrix}\right]$. The Figure 2 shows the partition and Figure 3 the associated reduced power system. From the definition of weak synchronization, we know that this reduced power system exactly reproduces the initial value response of the original system for any initial condition $\delta(0), \dot{\delta}(0) \in \mathcal{X}_{\mathrm{cl}}$, taking the initial condition of the reduced model to be $\widehat{\delta}(0) = (P^{\mathsf{T}}P)^{-1}P^{\mathsf{T}}\delta(0)$ and $\dot{\widehat{\delta}}(0) = (P^{\mathsf{T}}P)^{-1}P^{\mathsf{T}}\dot{\delta}(0)$.

To demonstrate the possibility to aggregate using any partition, including those with respect to which the power system is not weakly synchronized, and any initial condition, we show simulation result for partition $\{\{1,2,3\},\{4,5\}\}$ in Figure 4. We see that, in this case, the reduced model matches the steady state and approximates the transient behavior. Finding sufficient conditions for matching the steady state and deriving error bounds is a possible topic of future research.

VII. Conclusions

We analyzed power systems consisting of generators and buses. We introduced a notion of synchronization for a pair of generators and two for a partition of the set of generators. We proved equivalent conditions depending on the Kron-reduced system being symmetrical or equitable. This additionally gives a relation between symmetrical matrices and equitable partitions. We showed how a synchronized power systems can be exactly approximated with a reduced system by aggregating generators and their buses. Furthermore, this provides an aggregation-based reduction method for arbitrary power systems, although finding bounds for the approximation error remains an open problem.

Appendix

Lemma 12. *For X as in (5), we have $X\mathbb{1} = \mathbb{1}$.*

Proof. After some algebraic manipulation, it is clear $X\mathbb{1} = \mathbb{1}$ is equivalent to $\left(L_{11} - L_{12}L_{22}^{-1}L_{12}^{\mathsf{T}}\right)\mathbb{1} = 0$, which follows from $L\mathbb{1} = 0$. □

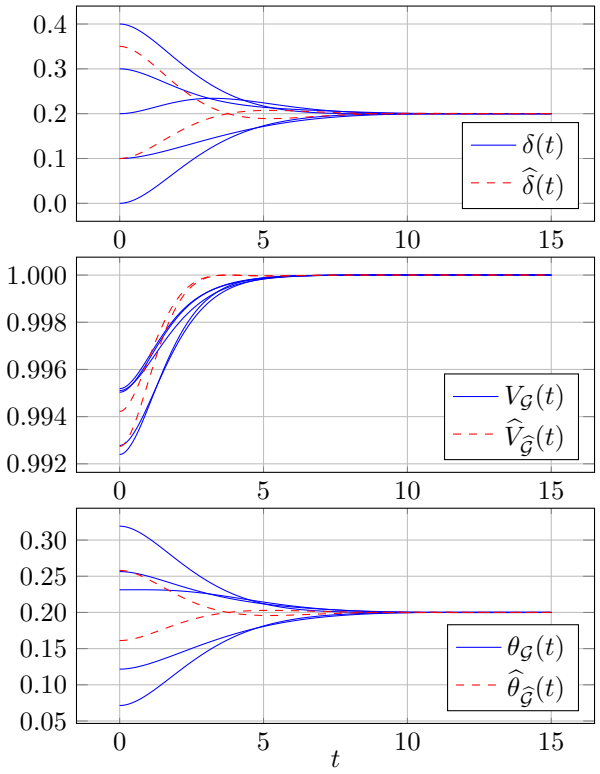

Fig. 4. Initial value response of the original power system from Figure 1 and a reduced system obtained by aggregating with partition $\{\{1,2,3\},\{4,5\}\}$. Original system's parameters are $\chi_i = \chi_{ij} = 1$ for all i,j, $M = D = I_5$, $f = 0$, and $E = \mathbb{1}_5$. The initial value is $\delta(0) = (0, 0.1, 0.2, 0.3, 0.4)$ and $\dot{\delta}(0) = 0$.

Lemma 13. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix and $i,j \in \{1,2,\dots,n\}$ such that $i \neq j$. Then $A \in \mathcal{S}_{ij}$ if and only if $a_{ii} = a_{jj}$ and $a_{ik} = a_{jk}$ for all $k \neq i,j$.*

Proof. From the definition, it can be seen that $A \in \mathcal{S}_{ij}$ is equivalent to $a_{ii} = a_{jj}$, $a_{ij} = a_{ji}$, $a_{ik} = a_{jk}$, and $a_{ki} = a_{kj}$ for all $k \neq i,j$. Using that A is symmetric, the conditions of the Lemma follow. □

Lemma 14. *Let $A, B \in \mathcal{S}_{ij}$ for some $i,j \in \{1,2,\dots,n\}$ such that $i \neq j$ and $\alpha,\beta \in \mathbb{R}$. Then,*

1) *$\alpha A + \beta B \in \mathcal{S}_{ij}$,*
2) *$AB \in \mathcal{S}_{ij}$, and*
3) *if A is nonsingular, then $A^{-1} \in \mathcal{S}_{ij}$.*

Proof. Follows directly from the definition of $\mathcal{S}_{ij}$ in (8). □

Lemma 15. *Let $i,j \in \{1,2,\dots,n\}$ be such that $i \neq j$. We have $\Gamma \in \mathcal{S}_{ij}$ if and only if $L_{\mathrm{D}} \in \mathcal{S}_{ij}$ and $L_{11} - L_{12}L_{22}^{-1}L_{12}^{\mathsf{T}} \in \mathcal{S}_{ij}$.*

Proof. $\boxed{\Leftarrow}$ Follows from Lemma 14.

$\boxed{\Rightarrow}$ First we show that $L_{\mathrm{D}} \in \mathcal{S}_{ij}$. Using $\Gamma = L_{\mathrm{D}}X$, $\Pi_{ij}\mathbb{1} = \mathbb{1}$, and $X\mathbb{1} = \mathbb{1}$, from $\Gamma\Pi_{ij}\mathbb{1} = \Pi_{ij}\Gamma\mathbb{1}$ it follows that $L_{\mathrm{D}}\mathbb{1} = \Pi_{ij}L_{\mathrm{D}}\mathbb{1}$. Since L_{D} is a diagonal matrix, from this we see that $L_{\mathrm{D}} \in \mathcal{S}_{ij}$. Now $L_{11} - L_{12}L_{22}^{-1}L_{12}^{\mathsf{T}} \in \mathcal{S}_{ij}$ follows from Lemma 14. □

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References

[1] A. M. Annaswamy and M. Amin, "IEEE vision for smart grid controls: 2030 and beyond," pp. 1–168, June 2013.
[2] F. Dörfler and F. Bullo, "Synchronization in complex networks of phase oscillators: A survey," *Automatica*, vol. 50, no. 6, pp. 1539–1564, 2014.
[3] A. Rahmani, M. Ji, M. Mesbahi, and M. Egerstedt, "Controllability of multi-agent systems from a graph-theoretic perspective," *SIAM J. Cont. Optim.*, vol. 48, no. 1, pp. 162–186, 2009.
[4] T. Ishizaki, R. Ku, and J.-i. Imura, "Eigenstructure analysis from symmetrical graph motives with application to aggregated controller design," in *55th IEEE Conference on Decision and Control (CDC), Las Vegas, USA*, 2016, pp. 5744–5749.
[5] T. Ishizaki, K. Kashima, J.-i. Imura, and K. Aihara, "Model reduction and clusterization of large-scale bidirectional networks," *IEEE Trans. Autom. Control*, vol. 59, no. 1, pp. 48–63, Jan. 2014.
[6] T. Ishizaki, K. Kashima, A. Girard, J.-i. Imura, L. Chen, and K. Aihara, "Clustered model reduction of positive directed networks," *Automatica*, vol. 59, pp. 238–247, 2015.
[7] P. Mlinarić, S. Grundel, and P. Benner, "Efficient model order reduction for multi-agent systems using QR decomposition-based clustering," in *54th IEEE Conference on Decision and Control (CDC), Osaka, Japan*, Dec. 2015, pp. 4794–4799.
[8] X. Cheng, Y. Kawano, and J. M. A. Scherpen, "Graph structure-preserving model reduction of linear network systems," in *European Control Conference (ECC)*, 2016, pp. 1970–1975.
[9] N. Xue and A. Chakrabortty, "Optimal control of large-scale networks using clustering based projections," Cornell University, arXiv e-print 1609.05265, 2016. [Online]. Available: http://arxiv.org/abs/1609.05265
[10] X. Cheng, Y. Kawano, and J. M. A. Scherpen, "Reduction of second-order network systems with structure preservation," *IEEE Trans. Autom. Control*, vol. 62, no. 10, pp. 5026–5038, 2017.
[11] P. Kundur, *Power System Stability and Control*. McGraw-Hill, 1994.
[12] G. Kron, *Tensor analysis of networks*. New York: Wiley, 1939.
[13] P. Mlinarić, T. Ishizaki, A. Chakrabortty, S. Grundel, P. Benner, and J.-i. Imura, "Synchronization and aggregation of nonlinear power systems with consideration of bus network structures," Cornell University, arXiv e-prints 1803.00439, 2018, cs.SY, math.DS. [Online]. Available: http://arxiv.org/abs/1803.00439
[14] C. Godsil and G. Royle, *Algebraic graph theory*, ser. Graduate Texts in Mathematics. Springer-Verlag, New York, 2001, vol. 207.