TWO-SCALE METHOD FOR THE MONGE-AMPÈRE EQUATION: POINTWISE ERROR ESTIMATES

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ABSTRACT. In this paper we continue the analysis of the two-scale method for the Monge-Ampère equation for dimension $d \geq 2$ introduced in [12]. We prove continuous dependence of discrete solutions on data that in turn hinges on a discrete version of the Alexandroff estimate. They are both instrumental to prove pointwise error estimates for classical solutions with Hölder and Sobolev regularity. We also derive convergence rates for viscosity solutions with bounded Hessians which may be piecewise smooth or degenerate.

Key words. Monge-Ampère, two-scale method, monotone, continuous dependence, error estimates, classical and viscosity solutions, degenerate.

AMS subject classifications. 65N30, 65N15, 65N12, 65N06, 35J96

1. Introduction

We consider the Monge-Ampère equation with Dirichlet boundary condition

{E:MA} $$\begin{cases} \det D^2 u = f & \text{in } \Omega \subset \mathbb{R}^d, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $f \geq 0$ is uniformly continuous, Ω is a uniformly convex domain and g is a continuous function. We seek a *convex* solution u of (1.1), which is critical for (1.1) to be elliptic and have a unique viscosity solution [8].

The Monge-Ampère equation has a wide spectrum of applications, which has led to an increasing interest in the investigation of efficient numerical methods. There are several existing methods for the Monge-Ampère equation, as described in [12]. Error estimates in $H^1(\Omega)$ are established in [3, 4] for solutions with $H^3(\Omega)$ regularity or more. Awanou [1] also proved a linear rate of convergence for classical solutions for the wide-stencil method, when applied to a perturbed Monge-Ampère equation with an extra lower order term δu; the parameter $\delta > 0$ is independent of the mesh and appears in reciprocal form in the rate.

On the other hand, Nochetto and Zhang followed an approach based on the discrete Alexandroff estimate developed in [13] and established pointwise error estimates in [14] for the method of Oliker and Prussner [15]. In this paper we follow a similar approach and derive pointwise rates of convergence for classical solutions

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of (1.1) that have Hölder or Sobolev regularity and for viscosity solutions with bounded Hessians which may be piecewise smooth or degenerate.

It is worth mentioning a rather strong connection between the semi-Lagrangian method of Feng and Jensen [5] and our two-scale approach introduced in [12]. In a forthcoming paper we explore this connection and derive optimal error estimates in special cases via enhanced techniques for pointwise error analysis.

1.1. **Our contribution.** The two-scale method was introduced in [12] and hinges on the following formula for the determinant of the semi-positive Hessian D^2w of a smooth function w, first suggested by Froese and Oberman [6]:

{E:Det}

$$\det D^2 w(x) = \min_{(v_1,\dots,v_d)\in\mathbb{S}^\perp} \prod_{j=1}^d v_j^T D^2 w(x)\ v_j, \tag{1.2}$$

where $\mathbb{S}^\perp$ is the set of all $d-$orthonormal bases in $\mathbb{R}^d$. To discretize this expression, we impose our discrete solutions to lie on a space of continuous piecewise linear functions over an unstructured quasi-uniform mesh $\mathcal{T}_h$ of size h; this defines the fine scale. The mesh also defines the computational domain Ω_h, which we describe in more detail in Section 2. The coarser scale δ corresponds to the length of directions used to approximate the directional derivatives that appear in (1.2), namely

$$\nabla^2_\delta w(x;v) := \frac{w(x+\delta v) - 2w(x) + w(x-\delta v)}{\delta^2} \quad \text{and} \quad |v| = 1,$$

for any $w \in C^0(\overline{\Omega})$; To render the method practical, we introduce a discretization $\mathbb{S}^\perp_\theta$ of the set $\mathbb{S}^\perp$ governed by the parameter θ and denote our discrete solution by u_ε, where $\varepsilon = (h,\delta,\theta)$ represents the scales of the method and the parameter θ. We define the discrete Monge-Ampère operator to be

$$T_\varepsilon[u_\varepsilon](x_i) := \min_{\mathbf{v}\in\mathbb{S}^\perp_\theta} \left(\prod_{j=1}^d \nabla^{2,+}_\delta u_\varepsilon(x_i;v_j) - \sum_{j=1}^d \nabla^{2,-}_\delta u_\varepsilon(x_i;v_j) \right),$$

where $\nabla^{2,\pm}_\delta$ are the positive and negative parts of ∇^2_δ. In Section 2 we review briefly the role of each term in the operator T_ε and recall some key properties of T_ε.

The merit of this definition of T_ε is that it leads to a clear separation of scales, which is a key theoretical advantage over the original wide stencil method of [6]. This also yields continuous dependence of discrete solutions on data, namely Proposition 4.6, which allows us to prove rates of convergence in $L^\infty(\Omega)$ for our method depending on the regularity of u; this is not clear for the wide stencil method of [6]. Moreover, the two-scale method is formulated over unstructured meshes $\mathcal{T}_h$, which adds flexibility to partition arbitrary uniformly convex domains Ω. This is achieved at the expense of points $x_i \pm \delta v_j$ no longer being nodes of $\mathcal{T}_h$, which is responsible for an additional interpolation error in the consistency estimate of T_ε. To locate such points and evaluate $\nabla^2_\delta u_\varepsilon(x_i;v_j)$, we resort to fast search techniques within [16, 17] and thus render the two-scale method practical. Compared with the error analysis of the Oliker-Prussner method [13], we do not require $\mathcal{T}_h$ to be cartesian.

In [12] we prove existence and uniqueness of a discrete solution for our method, and convergence to the viscosity solution of (1.1), using the discrete comparison principle. In this paper we prove rates of convergence for classical solutions with either Hölder or Sobolev regularity and for a special class of viscosity solutions.

The first important tool for proving pointwise rates of convergence is the discrete Alexandroff estimate introduced in [13]: if w_h is an arbitrary continuous piecewise linear function, $w_h \geq 0$ on $\partial\Omega_h$, and Γw_h stands for its convex envelope, then

$$\max_{x_i \in \mathcal{N}_h} w_h^-(x_i) \leq C \left(\sum_{x_i \in C_-(w_h)} |\partial \Gamma w_h(x_i)| \right)^{1/d}$$

where $\partial\Gamma w_h$ is the subdifferential of Γw_h and $C_-(w_h)$ represents the lower contact set of w_h, i.e. the set of interior nodes $x_i \in \mathcal{N}_h^0$ such that $\Gamma w_h(x_i) = w_h(x_i)$; hereafter we write $w_h^-(x_i) := -\min\{w_h(x_i), 0\}$. To control the measure of the subdifferential at each node, we show the following estimate

$$|\partial w_h(x_i)| \leq \delta^d \min_{(v_1,\ldots,v_d)\in\mathbb{S}^\perp} \prod_{j=1}^{d} \nabla_\delta^2 w_h(x_i; v_j) \quad \forall\, x_i \in \mathcal{N}_h^0,$$

such that the ball centered at x_i and of radius δ is contained in Ω_h. Combining both estimates, we derive the following continuous dependence estimate

$$\max_{\Omega_h} (u_h - w_h)^- \leq C\delta \left(\sum_{x_i \in C_-(u_h - w_h)} \left(T_\varepsilon[u_h](x_i)^{1/d} - T_\varepsilon[w_h](x_i)^{1/d} \right)^d \right)^{1/d}$$

for all continuous piecewise linear functions u_h and w_h such that $T_\varepsilon[u_h](x_i) \geq 0$ and $T_\varepsilon[w_h](x_i) \geq 0$ for all $x_i \in \mathcal{N}_h^0$. This result is instrumental and, combined with operator consistency and a discrete barrier argument close to the boundary, eventually leads to the following pointwise error estimates

$$\|u_\varepsilon - u\|_{L^\infty(\Omega_h)} \leq C(d, \Omega, f, u)\ h^{\frac{\alpha+k}{\alpha+k+2}}$$

provided $u \in C^{2+k,\alpha}(\overline{\Omega})$ with $0 < \alpha \leq 1$ and $k = 0, 1$, as well as

$$\|u_\varepsilon - u\|_{L^\infty(\Omega_h)} \leq C(d, \Omega, f, u)\ h^{1-\frac{2}{s}}$$

provided $u \in W_p^s(\Omega)$ with $2 + d/p < s \leq 4$ and $p > d$, and δ is suitably chosen in terms of h; see Theorems 5.3 and 5.4. We also consider a special case of viscosity solutions with bounded but discontinuous Hessians, and manage to prove a rate of convergence (see Theorem 5.5). Since these theorems are proven under the nondegeneracy assumption $f > 0$, we examine in Theorem 5.6 the effect of degeneracy $f \geq 0$. In [12] we explore numerically both classical and viscosity solutions and observe linear rates with respect to h for both cases, which are better than predicted by this theory.

1.2. **Outline.** We start by briefly presenting the operator T_ε in Section 2 and recalling some important results from [12]. In Section 3 we mention the discrete Alexandroff estimate and combine it in Section 4 with some geometric estimates to obtain the continuous dependence of the discrete solution on data. This is much stronger than stability, and is critical to prove rates of convergence for fully nonlinear PDEs. Lastly, in Section 5 we combine this result with operator consistency and a discrete barrier argument close to the boundary to derive rates of convergence upon making judicious choices of δ and θ in terms of h.

S:KeyProperties

2. Key Properties of the Discrete Operator

We recall briefly some of the key properties of operator T_ε, as proven in [12].

S:MonotoneDefinition

2.1. **Definition of T_ε.** Let $\mathcal{T}_h$ be a shape-regular and quasi-uniform triangulation with meshsize h. The computational domain Ω_h is the union of elements of $\mathcal{T}_h$ and $\Omega_h \neq \Omega$. If $\mathcal{N}_h$ denotes the nodes of $\mathcal{T}_h$, then $\mathcal{N}_h^b := \{x_i \in \mathcal{N}_h : x_i \in \partial\Omega_h\}$ are the boundary nodes and $\mathcal{N}_h^0 := \mathcal{N}_h \setminus \mathcal{N}_h^b$ are the interior nodes. We require that $\mathcal{N}_h^b \subset \partial\Omega$, which in view of the convexity of Ω implies that Ω_h is also convex and $\Omega_h \subset \Omega$. We denote by $\mathbb{V}_h$ the space of continuous piecewise linear functions over $\mathcal{T}_h$. We let $\mathbb{S}^\perp$ be the collection of all d-tuples of orthonormal bases and $\mathbf{v} := (v_1, \ldots, v_d) \in \mathbb{S}^\perp$ be a generic element, whence each component $v_i \in \mathbb{S}$, the unit sphere $\mathbb{S}$ of $\mathbb{R}^d$. We next introduce a finite subset $\mathbb{S}_\theta$ of $\mathbb{S}$ governed by the angular parameter $\theta > 0$: given $v \in \mathbb{S}$, there exists $v^\theta \in \mathbb{S}_\theta$ such that

$$|v - v^\theta| \leq \theta.$$

Likewise, we let $\mathbb{S}_\theta^\perp \subset \mathbb{S}^d$ be a finite approximation of $\mathbb{S}^\perp$: for any $\mathbf{v} = (v_j)_{j=1}^d \in \mathbb{S}^\perp$ there exists $\mathbf{v}^\theta = (v_j^\theta)_{j=1}^d \in \mathbb{S}_\theta^\perp$ such that $v_j^\theta \in \mathbb{S}_\theta$ and $|v_j - v_j^\theta| \leq \theta$ for all $1 \leq j \leq d$ and conversely. Note that $\mathbb{S}_\theta^\perp$ is not a subset of $\mathbb{S}^\perp$, which means that the vectors $(v_j^\theta)_{j=1}^d$ may not be orthogonal.

For $x_i \in \mathcal{N}_h^0$, we use centered second differences with a coarse scale δ

{E:2Sc2Dif}

$$\nabla_\delta^2 w(x_i; v_j) := \frac{w(x_i + \hat{\delta} v_j) - 2w(x_i) + w(x_i - \hat{\delta} v_j)}{\hat{\delta}^2} \tag{2.1}$$

where $\hat{\delta} := \rho\delta$ with $0 < \rho \leq 1$ the biggest number such that the ball centered at x_i of radius $\hat{\delta}$ is contained in Ω_h. This is well defined for any $w \in C^0(\overline{\Omega})$, in particular for $w \in \mathbb{V}_h$. We define $\varepsilon := (h, \delta, \theta)$ and we seek $u_\varepsilon \in \mathbb{V}_h$ such that $u^\varepsilon(x_i) = g(x_i)$ for $x_i \in \mathcal{N}_h^b$ and for $x_i \in \mathcal{N}_h^0$

{E:2ScOp}

$$T_\varepsilon[u_\varepsilon](x_i) := \min_{\mathbf{v} \in \mathbb{S}_\theta^\perp} \left(\prod_{j=1}^d \nabla_\delta^{2,+} u_\varepsilon(x_i; v_j) - \sum_{j=1}^d \nabla_\delta^{2,-} u_\varepsilon(x_i; v_j) \right) = f(x_i), \tag{2.2}$$

where we use the notation

$$\nabla_\delta^{2,+} u_\varepsilon(x_i; v_j) = \max\left(\nabla_\delta^2 u_\varepsilon(x_i; v_j), 0\right), \quad \nabla_\delta^{2,-} u_\varepsilon(x_i; v_j) = -\min\left(\nabla_\delta^2 u_\varepsilon(x_i; v_j), 0\right)$$

to indicate positive and negative parts of the centered second differences.

S:PropertiesMonotone

2.2. **Key Properties of T_ε.** One of the critical properties of the Monge-Ampère equation is the convexity of the solution u. The following notion mimics this at the discrete level.

D:discrete-convexity

Definition 2.1 (discrete convexity)**.** We say that $w_h \in \mathbb{V}_h$ is discretely convex if

$$\nabla_\delta^2 w_h(x_i; v_j) \geq 0 \qquad \forall x_i \in \mathcal{N}_h^0, \quad \forall v_j \in \mathbb{S}_\theta.$$

The following lemma guarantees the discrete convexity of subsolutions of (2.2) [12, Lemma 2.2].

L:DisConv

Lemma 2.2 (discrete convexity)**.** *If $w_h \in \mathbb{V}_h$ satisfies*

{E:Oper}

$$T_\varepsilon[w_h](x_i) \geq 0 \quad \forall x_i \in \mathcal{N}_h^0, \tag{2.3}$$

then w_h is discretely convex and as a consequence

{E:simpler-def}

$$T_\varepsilon[w_h](x_i) = \min_{\mathbf{v} \in \mathbb{S}_\theta^\perp} \prod_{j=1}^d \nabla_\delta^2 w_h(x_i; v_j), \tag{2.4}$$

namely

$$\nabla_\delta^{2,+} w_h(x_i; v_j) = \nabla_\delta^2 w_h(x_i; v_j), \quad \nabla_\delta^{2,-} w_h(x_i; v_j) = 0 \quad \forall x_i \in \mathcal{N}_h^0, \quad \forall v_j \in \mathbb{S}_\theta.$$

Conversely, if w_h is discretely convex, then (2.3) is valid.

Another important property of operator T_ε that relies on its monotonicity is the following discrete comparison principle [12, Lemma 2.4].

L:DCP **Lemma 2.3** (discrete comparison principle). *Let $u_h, w_h \in \mathbb{V}_h$ with $u_h \le w_h$ on the discrete boundary $\partial\Omega_h$ be such that*

{E:comparison}
$$T_\varepsilon[u_h](x_i) \ge T_\varepsilon[w_h](x_i) \ge 0 \quad \forall x_i \in \mathcal{N}_h^0. \tag{2.5}$$

Then, $u_h \le w_h$ everywhere.

We now state a consistency estimate, proved in [12, Lemma 4.1], that leads to pointwise rates of convergence. To this end, given a node $x_i \in \mathcal{N}_h^0$, we denote by

{E:Bi}
$$B_i := \cup\{\overline{T} : T \in \mathcal{T}_h, \text{ dist } (x_i, T) \le \hat{\delta}\} \tag{2.6}$$

where $\hat{\delta}$ is defined in (2.1). We also define the δ-interior region

{Omega-delta}
$$\Omega_{h,\delta} = \{T \in \mathcal{T}_h \ : \ \text{dist}(x, \partial\Omega_h) \ge \delta \ \forall x \in T\}, \tag{2.7}$$

and the δ-boundary region:

$$\omega_{h,\delta} = \Omega \setminus \Omega_{h,\delta}.$$

L:FullConsistency **Lemma 2.4** (consistency of $T_\varepsilon[\mathcal{I}_h u]$). *Let $x_i \in \mathcal{N}_h^0 \cap \Omega_{h,\delta}$ and B_i be defined as in (2.6). If $u \in C^{2+k,\alpha}(B_i)$ with $0 < \alpha \le 1$ and $k = 0, 1$ is convex, and $\mathcal{I}_h u$ is its piecewise linear interpolant, then*

{E:FullConsistency}
$$\left|\det D^2 u(x_i) - T_\varepsilon[\mathcal{I}_h u](x_i)\right| \le C_1(d, \Omega, u)\delta^{k+\alpha} + C_2(d, \Omega, u)\left(\frac{h^2}{\delta^2} + \theta^2\right), \tag{2.8}$$

where

{E:C1-C2}
$$C_1(d, \Omega, u) = C|u|_{C^{2+k,\alpha}(B_i)}|u|_{W^2_\infty(B_i)}^{d-1}, \quad C_2(d, \Omega, u) = C|u|_{W^2_\infty(B_i)}^d. \tag{2.9}$$

If $x_i \in \mathcal{N}_h^0$ and $u \in W^2_\infty(B_i)$, then (2.8) remains valid with $\alpha = k = 0$ and $C^{2+k,\alpha}(B_i)$ replaced by $W^2_\infty(B_i)$.

3. Discrete Alexandroff Estimate

S:dAle

In this section, we review several concepts related to convexity as well as the discrete Alexandroff estimate of [13]. We first recall several definitions.

D:ConvEnv **Definition 3.1** (subdifferential).

(*i*) The subdifferential of a function w at a point $x_0 \in \Omega_h$ is the set

$$\partial w(x_0) := \left\{p \in \mathbb{R}^d : \ w(x) \ge w(x_0) + p \cdot (x - x_0), \ \ \forall x \in \Omega_h\right\}.$$

(*ii*) The subdifferential of a function w on set $E \subset \Omega_h$ is $\partial u(E) := \cup_{x \in E} \partial w(x)$.

def:CEandLCS **Definition 3.2** (convex envelope and discrete lower contact set).

(*i*) The convex envelope Γu of a function w is defined to be

$$\Gamma w(x) := \sup_L \{L(x), \ L(y) \le w(y) \text{ for all } y \in \Omega_h \text{ and } L \text{ is affine}\}.$$

(*ii*) The discrete lower contact set $C_-(w_h)$ of a function $w_h \in \mathbb{V}_h$ is the set of nodes where the function coincides with its convex envelope, i.e.

$$\mathcal{C}_-(w_h) := \left\{x_i \in \mathcal{N}_h^0 : \Gamma w_h(x_i) = w_h(x_i)\right\}.$$

R:2DifConvEnv **Remark 3.3** (w_h dominates Γw_h)**.** Since $w_h \ge \Gamma w_h$, at a contact node $x_i \in \mathcal{C}_-(w_h)$ we have

$$\nabla_\delta^2 \Gamma w_h(x_i; v_j) \le \nabla_\delta^2 w_h(x_i; v_j)(x_i) \qquad \forall v_j \in \mathbb{S}_\theta.$$

R:MinConvEnv **Remark 3.4** (minima of w_h and Γw_h)**.** A consequence of Definition 3.2 (convex envelope and discrete lower contact set) is that the minima of $w_h \in \mathbb{V}_h$ and Γw_h are attained at the same contact nodes and are equal.

We can now present the discrete Alexandroff estimate from [13], which states that the minimum of a discrete function is controlled by the measure of the subdifferential of its convex envelope in the discrete contact set.

P:DAE **Proposition 3.5** (discrete Alexandroff estimate [13])**.** *Let v_h be a continuous piecewise linear function that satisfies $v_h \ge 0$ on $\partial\Omega_h$. Then,*

$$\max_{x_i \in \mathcal{N}_h^0} v_h(x_i)^- \le C \left(\sum_{x_i \in \mathcal{C}_-(v_h)} |\partial \Gamma v_h(x_i)| \right)^{1/d}$$

where $C = C(d, \Omega)$ depends only on the dimension d and the domain Ω.

S:CoDeDa

4. Continuous Dependence on Data

We derive the continuous dependence of the discrete solution on data in Section 4.3, which is essential to prove rates of convergence. To this end, we first prove a stability estimate in the max norm in Section 4.1 and the concavity of the discrete operator in Section 4.2.

S:stab

4.1. Stability of the Two-Scale Method.

We start with some geometric estimates. The first and second lemmas connect the discrete Alexandroff estimate with the 2-scale method. They allow us to estimate the measure of the subdifferential of a discrete function w_h in terms of our discrete operator $T_\varepsilon[w_h]$, defined in (2.2).

L:SubDifBound **Lemma 4.1** (subdifferential vs hyper-rectangle)**.** *Let $w \in C^0(\overline{\Omega}_h)$ be convex and $x_i \in \mathcal{N}_h^0$ be so that $x_i \pm \hat{\delta} v \in \overline{\Omega}_h$ for all $v \in \mathbb{S}_\theta$ with $\hat{\delta} \le \delta$. If $\mathbf{v} = (v_j)_{j=1}^d \in \mathbb{S}_\theta^\perp$ and*

$$\alpha_{i,j}^\pm := \frac{w(x_i \pm \hat{\delta} v_j) - w(x_i)}{\hat{\delta}} \quad \forall\, 1 \le j \le d,$$

then

$$\partial w(x_i) \subset \left\{p \in \mathbb{R}^d : \ \alpha_{i,j}^- \le p \cdot v_j \le \alpha_{i,j}^+ \ 1 \le j \le d\right\}.$$

Proof. Take $p \in \partial w(x_i)$ and write

$$w(x) \ge w(x_i) + p \cdot (x - x_i) \ \ \forall x \in \overline{\Omega}_h.$$

Consequently, for any $1 \le j \le d$ we infer that

$$w(x_i + \hat{\delta} v_j) \ge w(x_i) + \hat{\delta}\ p \cdot v_j, \quad w(x_i - \hat{\delta} v_j) \ge w(x_i) - \hat{\delta}\ p \cdot v_j,$$

or equivalently

$$\frac{w(x_i) - w(x_i - \hat{\delta} v_j)}{\hat{\delta}} \le p \cdot v_j \le \frac{w(x_i + \hat{\delta} v_j) - w(x_i)}{\hat{\delta}}.$$

This implies that p belongs to the desired set. ☐

L:HypRectVol

Lemma 4.2 (hyper-rectangle volume). *For any d-tuple $\mathbf{v} = (v_j)_{j=1}^d \in \mathbb{S}^\perp$ the volume of the set*

$$K = \left\{ p \in \mathbb{R}^d : \; a_j \le p \cdot v_j \le b_j, \quad j = 1, \dots, d \right\}$$

is given by

$$|K| = \prod_{j=1}^{d} (b_i - a_i).$$

Proof. Let $V = [v_1, \cdots, v_d] \in \mathbb{R}^{d\times d}$ be the orthogonal matrix whose columns are the elements of $\mathbf{v}$; hence $v_j = V e_j$ where $\{e_j\}_{j=1}^d$ is the canonical basis in $\mathbb{R}^d$. We now seek a more convenient representation of K

$$\begin{aligned} K &= \left\{ p \in \mathbb{R}^d : \; a_j \le p \cdot (V e_j) \le b_j, \quad j = 1, \dots, d \right\} \\ &= V^{-T} \left\{ x \in \mathbb{R}^d : \; a_j \le x \cdot e_j \le b_j, \quad j = 1, \dots, d \right\} = V^{-T} \widetilde{K}, \end{aligned}$$

whence

$$|K| = |\det V^{-T}| \; |\widetilde{K}| = |\widetilde{K}| = \prod_{j=1}^{d} (b_j - a_j),$$

because $\widetilde{K}$ is an orthogonal hyper-rectangle. ☐

The following result is an immediate consequence of Lemma 4.1 for $\mathbf{v} \in \mathbb{S}_\theta^\perp$.

C:HypRectVol

Corollary 4.1 (approximate hyper-rectangle volume). *For any d-tuple $\mathbf{v} = (v_j)_{j=1}^d \in \mathbb{S}_\theta^\perp$ the volume of the set*

$$K_\theta = \left\{ p \in \mathbb{R}^d : \; a_j \le p \cdot v_j \le b_j, \quad j = 1, \dots, d \right\}$$

is given by

$$|K_\theta| = \prod_{j=1}^{d} (b_i - a_i) + O(\theta^2).$$

R:ignore-theta

Remark 4.3 (lack of orthogonality of $\mathbb{S}_\theta^\perp$). Since the extra term in Corollary 4.1 is of order θ^2, which already occurs in Lemma 2.4 (consistency of $T_\varepsilon[\mathcal{I}_h u_\varepsilon]$), it does not affect the error estimates of Theorems 5.3–5.6 and it will be ignored from now on. Therefore, we will invoke Lemma 4.2 (hyper-rectangle volume) for $\mathbf{v} \in \mathbb{S}_\theta^\perp$ for simplicity, even though $\mathbf{v} = (v_j)_{j=1}^d$ might not be an orthogonal basis.

Combining Lemmas 4.1 and 4.2 with Remark 4.3, we get the following corollary.

C:SubDifOper

Corollary 4.2 (subdifferential vs discrete operator). *For every $x_i \in \mathcal{N}_h^0 \cap \Omega_{h,\delta}$ and a convex function w we have that*

$$|\partial w(x_i)| \le \left(\min_{\mathbf{v} \in \mathbb{S}_\theta^\perp} \prod_{j=1}^{d} \nabla_\delta^2 w(x_i; v_j) \right) \delta^d.$$

L:Stability

Lemma 4.4 (stability). *If $w_h \in \mathbb{V}_h$ is $w_h \ge 0$ on $\partial\Omega_h$, then*

$$\max_{x_i \in \mathcal{N}_h^0} w_h(x_i)^- \le C\delta \left(\sum_{x_i \in \mathcal{C}_-(w_h)} T_\varepsilon[w_h](x_i) \right)^{1/d}.$$

Proof. Since the function $w_h \geq 0$ on $\partial\Omega_h$, we invoke Proposition 3.5 (discrete Alexandroff estimate) for w_h to obtain

$$\max_{x_i \in \mathcal{N}_h^0} w_h(x_i)^- \leq C \left(\sum_{x_i \in \mathcal{C}_-(w_h)} |\partial \Gamma w_h(x_i)| \right)^{1/d}$$

Applying Corollary 4.2 (subdifferential vs discrete operator) to the convex function $\Gamma w_h(x_i)$ at a contact point $x_i \in \mathcal{C}_-(w_h)$ and recalling Remark 3.3, we have

$$|\partial \Gamma w_h(x_i)| \leq \delta^d \min_{\mathbf{v} \in \mathbb{S}_\theta^\perp} \prod_{j=1}^d \nabla_\delta^2 \Gamma w_h(x_i; v_j) \leq \delta^d \min_{\mathbf{v} \in \mathbb{S}_\theta^\perp} \prod_{j=1}^d \nabla_\delta^2 w_h(x_i; v_j) = \delta^d T_\varepsilon[w_h](x_i),$$

where the last equality follows from Lemma 2.2 (discrete convexity). ☐

S:concavity

4.2. **Concavity of the Discrete Operator.** We recall concavity properties of $(\det A)^{1/d}$ for symmetric positive semi-definite matrices A and extend them to T_ε. The results can be traced back to [9, 11], but we present them here for completeness.

L:Concavity

Lemma 4.5 (concavity of determinant)**.** *The following two statements are valid.*

(*i*) *For every symmetric positive semi-definite (SPSD) matrix A we have that*

$$(\det A)^{1/d} = \frac{1}{d} \inf \left\{ \operatorname{tr}(AB) \;\middle|\; B \text{ is SPD and } \det B = 1 \right\}$$

(*ii*) *The function $A \mapsto (\det A)^{1/d}$ is concave on SPSD matrices.*

Proof. We proceed in three steps.

Step 1: Proof of (i) for A invertible. Let B be SPD with $\det B = 1$. Then $B^{1/2}$ is well defined, $\det(B^{1/2}) = 1$ and we obtain

$$\det A = \det(B^{1/2} A B^{1/2}).$$

Let P be an orthogonal matrix that converts $B^{1/2}AB^{1/2}$ into a diagonal matrix D, namely $D = PB^{1/2}AB^{1/2}P^T$. Applying the geometric mean inequality yields

$$\det(B^{1/2}AB^{1/2})^{1/d} = (\det D)^{1/d} \leq \frac{1}{d}\operatorname{tr} D = \frac{1}{d}\operatorname{tr}(B^{1/2}AB^{1/2}) = \frac{1}{d}\operatorname{tr}(AB),$$

where we have used the invariance of the trace under cyclic permutations of the factor to write the last two equalities. This shows that

$$(\det A)^{1/d} \leq \frac{1}{d} \inf \left\{ \operatorname{tr}(AB) \;\middle|\; B \text{ is SPD and } \det B = 1 \right\}$$

This inequality is actually equality provided A is invertible. In fact, we can take $B = (\det A)^{1/d} A^{-1}$, which is SPD and $\det B = 1$. This proves (i) for A nonsingular.

Step 2: Proof of (i) for A singular. Given the singular value decomposition of A

$$A = \sum_{i=1}^d \lambda_i v_i \otimes v_i, \quad \lambda_1 \geq \cdots \lambda_k > \lambda_{k+1} = \cdots = \lambda_d = 0,$$

with orthogonal vectors $(v_i)_{i=1}^d$, we can assume that $k > 0$ for otherwise $A = 0$ and the assertion is trivial. Given a parameter $\sigma > 0$, let B be defined by

$$B := \sum_{i=1}^k \sigma v_i \otimes v_i + \sum_{i=k+1}^d \sigma^{-\beta} v_i \otimes v_i$$

and $\beta = k/(d-k)$ because then $\det B = \sigma^k \sigma^{-\beta(d-k)} = 1$. Therefore,

$$AB = \sigma \sum_{i=1}^{k} \lambda_i v_i \otimes v_i \quad \Rightarrow \quad \mathrm{tr}(AB) = \sigma \sum_{i=1}^{k} \lambda_i \to 0 \quad \text{as } \sigma \to 0,$$

which proves (i) for A singular since B is SPD.

Step 3: Proof of (ii). Let A and B be SPSD matrices and $0 \le \lambda \le 1$. Then $\lambda A + (1-\lambda)B$ is also SPSD and we can apply (i) to

$$\begin{aligned}(\det[\lambda A + (1-\lambda)B])^{1/d} &= \frac{1}{d} \inf \left\{ \mathrm{tr}[(\lambda A + (1-\lambda)B)C] \,\Big|\, C \text{ is SPD and} \det C = 1 \right\} \\ &\ge \frac{\lambda}{d} \inf \left\{ \mathrm{tr}(AC) \,\Big|\, C \text{ is SPD and } \det C = 1 \right\} \\ &+ \frac{1-\lambda}{d} \inf \left\{ \mathrm{tr}(BC) \,\Big|\, C \text{ is SPD and } \det C = 1 \right\} \\ &= \lambda (\det A)^{1/d} + (1-\lambda)(\det B)^{1/d}.\end{aligned}$$

This completes the proof. ☐

Upon relabeling $\widehat{A} = \lambda A$ and $\widehat{B} = (1-\lambda)B$, which are still SPSD, we can write Lemma 4.5 (ii) as follows:

{E:concavity}

$$(\det \widehat{A})^{1/d} + (\det \widehat{B})^{1/d} \le \left(\det(\widehat{A} + \widehat{B}) \right)^{1/d}. \tag{4.1}$$

We now show that our discrete operator $T_\varepsilon[\cdot]$ possesses a similar property.

C:OperIneq

Corollary 4.3 (concavity of discrete operator)**.** *Given two functions $u_h, w_h \in \mathbb{V}_h$, we have*

$$\left(T_\epsilon[u_h](x_i)\right)^{1/d} + \left(T_\epsilon[w_h](x_i)\right)^{1/d} \le \left(T_\epsilon[u_h + w_h](x_i)\right)^{1/d},$$

for all nodes $x_i \in \mathcal{N}_h^0$ such that $\nabla_\delta^2 u_h(x_i; v_j) \ge 0,\ \nabla_\delta^2 w_h(x_i; v_j) \ge 0$ for all $v_j \in \mathbb{S}_\theta$.

Proof. We argue in two steps.

Step 1. For $a = (a_j)_{j=1}^d \in \mathbb{R}^d$ with $a_j \ge 0,\ j = 1, \ldots, d$ we consider the function

$$f(a) := \left(\prod_{j=1}^{d} a_j \right)^{1/d},$$

which can be conceived as the determinant of a diagonal (and thus symmetric) positive semi-definite matrix with diagonal elements $(a_j)_{j=1}^d$, i.e.

$$f(a) = \left(\det \mathrm{diag}\{a_1, \ldots, a_d\} \right)^{1/d}.$$

Applying (4.1) to $\widehat{A} = \mathrm{diag}\{a_1, \ldots, a_d\}, \widehat{B} = \mathrm{diag}\{b_1, \ldots, b_d\}$ with $a = (a_j)_{j=1}^d, b = (b_j)_{j=1}^d \ge 0$ componentise, we deduce

$$f(a) + f(b) \le f(a+b).$$

Step 2. We now apply this formula to the discrete operator. Since both u_h, w_h are discretely convex at $x_i \in \mathcal{N}_h^0$, so is $u_h + w_h$, and we can apply Lemma 2.2 (discrete convexity) to write

$$T_\varepsilon[u_h + w_h](x_i) = \prod_{j=1}^{d} \nabla_\delta^2 [u_h + w_h](x_i; v_j)$$

for a suitable $\mathbf{v} = (v_j)_{j=1}^d \in \mathbb{S}_\theta^\perp$. Making use again of (2.4), this time for u_h and w_h and for the specific set of directions $\mathbf{v}$ just found, we obtain

$$\big(T_\varepsilon[u_h](x_i)\big)^{\frac{1}{d}} + \big(T_\varepsilon[w_h](x_i)\big)^{\frac{1}{d}} \le \left(\prod_{j=1}^d \nabla_\delta^2 u_h(x_i;v_j)\right)^{\frac{1}{d}} + \left(\prod_{j=1}^d \nabla_\delta^2 w_h(x_i;v_j)\right)^{\frac{1}{d}}$$
$$\le \left(\prod_{j=1}^d \nabla_\delta^2 u_h(x_i;v_j) + \nabla_\delta^2 w_h(x_i;v_j)\right)^{\frac{1}{d}} = \big(T_\varepsilon[u_h + w_h](x_i)\big)^{\frac{1}{d}},$$

where the second inequality is given by Step 1 for $a = (\nabla_\delta^2 u_h(x_i;v_j))_{j=1}^d$ and $b = (\nabla_\delta^2 w_h(x_i;v_j))_{j=1}^d$. This is the asserted estimate. □

S:cont-depend

4.3. **Continuous Dependence of the Two-Scale Method on Data.**

We are now ready to prove the continuous dependence of discrete solutions on data. This will be instrumental later for deriving rates of convergence for the two-scale method.

P:ContDep

Proposition 4.6 (continuous dependence on data). *Given two functions $u_h, w_h \in \mathbb{V}_h$ such that $u_h \ge w_h$ on $\partial\Omega_h$ and*

$$T_\varepsilon[u_h](x_i) = f_1(x_i) \ge 0 \quad \text{and} \quad T_\varepsilon[w_h](x_i) = f_2(x_i) \ge 0$$

at all interior nodes $x_i \in \mathcal{N}_h^0$, we have that

$$\max_{\Omega_h}(u_h - w_h)^- \le C\ \delta \left(\sum_{x_i \in \mathcal{C}_-(u_h - w_h)} \left(f_1(x_i)^{1/d} - f_2(x_i)^{1/d}\right)^d\right)^{1/d}.$$

Proof. Since $u_h - w_h \in \mathbb{V}_h$ and $u_h - w_h \ge 0$ on $\partial\Omega_h$, Lemma 4.4 (stability) yields

$$\max_{x_i \in \mathcal{N}_h^0}(u_h - w_h)(x_i)^- \le C\delta \left(\sum_{x_i \in \mathcal{C}_-(u_h - w_h)} T_\epsilon[u_h - w_h](x_i)\right)^{1/d}.$$

Since $x_i \in \mathcal{C}_-(u_h - w_h)$, we have that $\nabla_\delta^2(u_h - w_h)(x_i;v_j) \ge 0$, whence

$$\nabla_\delta^2 u_h(x_i;v_j) \ge \nabla_\delta^2 w_h(x_i;v_j) \ge 0 \quad \forall v_j \in \mathbb{S}_\theta,$$

where we have made use of Lemma 2.2 (discrete convexity). Invoking Corollary 4.3 (concavity of discrete operator) for $u_h - w_h$ and w_h, we deduce

$$\big(T_\epsilon[u_h - w_h](x_i)\big)^{1/d} \le \big(T_\epsilon[u_h](x_i)\big)^{1/d} - \big(T_\epsilon[w_h](x_i)\big)^{1/d},$$

whence

$$\max_{x_i \in \mathcal{N}_h^0}(u_h - w_h)(x_i)^- \le C\delta \left(\sum_{x_i \in \mathcal{C}_-(u_h - w_h)} \left(T_\epsilon[u_h](x_i)^{1/d} - T_\epsilon[w_h](x_i)^{1/d}\right)^d\right)^{1/d}$$
$$= C\delta \left(\sum_{x_i \in \mathcal{C}_-(u_h - w_h)} \left(f_1(x_i)^{1/d} - f_2(x_i)^{1/d}\right)^d\right)^{1/d}.$$

This completes the proof. □

S:RoC

5. Rates of Convergence

We now combine the preceding estimates to prove pointwise convergence rates for solutions with continuous Hessians, and either Hölder or Sobolev regularity, and later for a special case of viscosity solutions with discontinuous Hessians; these results require the nondegeneracy assumption $f \geq f_0 > 0$. We also deal with the degenerate case $f \geq 0$ and derive error estimates of reduced order. We state all error estimates over the computational domain $\Omega_h \subset \Omega$.

S:Barrier

5.1. Barrier Function.

We recall here the two discrete barrier functions introduced in [12, Lemmas 5.1, 5.2]. The first one is critical in order to control the behavior of u_ε close to the boundary of Ω_h and prove the convergence to the unique viscosity solution u of (1.1). We now use the same barrier function to control the pointwise error of u_ε and u close to the boundary. The second barrier allows us to treat the degenerate case $f \geq 0$, using techniques similar to the case $f > 0$.

L:Barrier

Lemma 5.1 (discrete boundary barrier)**.** *Let Ω be uniformly convex and $E > 0$ be arbitrary. For each node $z \in \mathcal{N}_h^0$ with $\operatorname{dist}(z, \partial\Omega_h) \leq \delta$, there exists a function $p_h \in \mathbb{V}_h$ such that $T_\varepsilon[p_h](x_i) \geq E$ for all $x_i \in \mathcal{N}_h^0$, $p_h \leq 0$ on $\partial\Omega_h$ and*

$$|p_h(z)| \leq C E^{1/d} \delta$$

with C depending on Ω.

L:BarrierInterior

Lemma 5.2 (discrete interior barrier)**.** *Let Ω be contained in the ball $B(x_0, R)$ of center x_0 and radius R. If $q(x) := \frac{1}{2}\big(|x - x_0|^2 - R^2\big)$, then its interpolant $q_h := \mathcal{I}_h q \in \mathbb{V}_h$ satisfies*

$$T_\varepsilon[q_h](x_i) \geq 1 \quad \forall x_i \in \mathcal{N}_h^0, \qquad q_h(x_i) \leq 0 \quad \forall x_i \in \mathcal{N}_h^b.$$

S:RatesHolder

5.2. Error Estimates for Solutions with Hölder Regularity.

We now deal with classical solutions u of (1.1) of class $C^{2+k,\alpha}(\overline{\Omega})$, with $k = 0, 1$ and $0 < \alpha \leq 1$, and derive pointwise error estimates. We proceed as follows. We first use Lemma 5.1 (discrete boundary barrier) to control $u_\varepsilon - \mathcal{I}_h u$ in the δ-neighborhood $\omega_{h,\delta}$ of $\partial\Omega_h$, where the consistency error of $T_\varepsilon[\mathcal{I}_h u]$ is of order one according to Lemma 2.4 (consistency of $T_\varepsilon[\mathcal{I}_h u]$). In the δ-interior region $\Omega_{h,\delta}$ we combine the interior consistency error of $T_\varepsilon[\mathcal{I}_h u]$ from Lemma 2.4 and Proposition 4.6 (continuous dependence on data). Judicious choices of δ and θ in terms of h conclude the argument.

T:RatesHolder

Theorem 5.3 (rates of convergence for classical solutions)**.** *Let $f(x) \geq f_0 > 0$ for all $x \in \Omega$. Let u be the classical solution of (1.1) and u_ε be the discrete solution of (2.2). If $u \in C^{2,\alpha}(\overline{\Omega})$ for $0 < \alpha \leq 1$ and*

$$\delta = \left(\frac{|u|_{W^2_\infty(\Omega)}}{|u|_{C^{2,\alpha}(\overline{\Omega})}} \right)^{\frac{1}{2+\alpha}} h^{\frac{2}{2+\alpha}}, \quad \theta = \left(\frac{|u|_{C^{2,\alpha}(\overline{\Omega})}}{|u|_{W^2_\infty(\Omega)}} \right)^{\frac{1}{2+\alpha}} h^{\frac{\alpha}{2+\alpha}},$$

then

$$\|u - u_\varepsilon\|_{L^\infty(\Omega_h)} \leq C(\Omega, d, f_0)\ |u|_{C^{2,\alpha}(\overline{\Omega})}^{\frac{1}{2+\alpha}}\ |u|_{W^2_\infty(\Omega)}^{\frac{2d-1+d\alpha}{2+\alpha}}\ h^{\frac{\alpha}{2+\alpha}}$$

Otherwise, if $u \in C^{3,\alpha}(\overline{\Omega})$ for $0 < \alpha \leq 1$ and

$$\delta = \left(\frac{|u|_{W^2_\infty(\Omega)}}{|u|_{C^{3,\alpha}(\overline{\Omega})}} \right)^{\frac{1}{3+\alpha}} h^{\frac{2}{3+\alpha}}, \quad \theta = \left(\frac{|u|_{C^{3,\alpha}(\overline{\Omega})}}{|u|_{W^2_\infty(\Omega)}} \right)^{\frac{1}{3+\alpha}} h^{\frac{1+\alpha}{3+\alpha}},$$

then

$$\|u-u_\varepsilon\|_{L^\infty(\Omega_h)} \le C(\Omega,d,f_0)\ |u|_{C^{3,\alpha}(\overline{\Omega})}^{\frac{1}{3+\alpha}}\ |u|_{W^2_\infty(\Omega)}^{\frac{3d-1+d\alpha}{3+\alpha}}\ h^{\frac{1+\alpha}{3+\alpha}}$$

Proof. Since the interpolation error $\|u-\mathcal{I}_h u\|_{L^\infty(\Omega_h)} \le Ch^2|u|_{W^2_\infty(\Omega)}$ is of higher order than the asserted rates, we replace u by $\mathcal{I}_h u$ and limit ourselves to proving the asserted estimates for $\mathcal{I}_h u - u_\varepsilon$. In fact, we only prove

$$\max_{\Omega_h}(u_\varepsilon-\mathcal{I}_h u) \le C(\Omega,d,f_0)\ |u|_{C^{2+k,\alpha}(\overline{\Omega})}^{\frac{1}{2+k+\alpha}}\ |u|_{W^2_\infty(\Omega)}^{\frac{2d-1+d(k+\alpha)}{2+k+\alpha}}\ h^{\frac{k+\alpha}{2+k+\alpha}}$$

depending on the regularity $C^{2+k,\alpha}(\overline{\Omega})$ of u, $k=0,1$, because the estimates for $\max_{\Omega_h}(\mathcal{I}_h u - u_\varepsilon)$ are similar. We proceed in three steps.

Step 1: Boundary estimate. We show that for $z \in \mathcal{N}_h^0$ so that $\operatorname{dist}(z,\partial\Omega_h) \le \delta$

$$u_\varepsilon - \mathcal{I}_h u(z) \le C|u|_{W^2_\infty(\Omega)}\delta.$$

Let p_h be the function of Lemma 5.1 (discrete boundary barrier), for z fixed, and examine the behavior of $u_\varepsilon + p_h$. For any interior node $x_i \in \mathcal{N}_h^0$, we have

$$\begin{aligned}\prod_{j=1}^d \nabla^2_\delta(u_\varepsilon+p_h)(x_i;v_j) &= \prod_{j=1}^d\left(\nabla^2_\delta u_\varepsilon(x_i;v_j)+\nabla^2_\delta p_h(x_i;v_j)\right)\\ &\ge \prod_{j=1}^d \nabla^2_\delta u_\varepsilon(x_i;v_j) + \prod_{j=1}^d \nabla^2_\delta p_h(x_i;v_j) \quad \forall \mathbf{v}=(v_j)_{j=1}^d \in \mathbb{S}_\theta^\perp,\end{aligned}$$

because $\nabla^2_\delta u_\varepsilon(x_i;v_j)\ge 0$ and $\nabla^2_\delta p_h(x_i;v_j) \ge 0$. We apply Lemma 2.4 (consistency of $T_\varepsilon[\mathcal{I}_h u]$) to obtain

$$\begin{aligned}T_\varepsilon[u_\varepsilon+p_h](x_i) &\ge T_\varepsilon[u_\varepsilon](x_i)+T_\varepsilon[p_h](x_i)\\ &\ge f(x_i)+E\\ &\ge T_\varepsilon[\mathcal{I}_h u](x_i) - C|u|^d_{W^2_\infty(\Omega)} + E \ge T_\varepsilon[\mathcal{I}_h u](x_i),\end{aligned}$$

provided $E \ge C|u|^d_{W^2_\infty(\Omega)}$. Since $\mathcal{I}_h u = u_\varepsilon$ and $p_h \le 0$ on $\partial\Omega_h$, we deduce from Lemma 2.3 (discrete comparison principle) that

$$u_\varepsilon(z)+p_h(z) \le \mathcal{I}_h u(z),$$

whence,

$$u_\varepsilon(z) - \mathcal{I}_h u(z) \le C|u|_{W^2_\infty(\Omega)}\delta.$$

Step 2: Interior estimate. We show that for all $x_i \in \mathcal{N}_h^0$ so that $\operatorname{dist}(x_i,\partial\Omega_h)\ge\delta$

$$T_\varepsilon[u_\varepsilon](x_i) - T_\varepsilon[\mathcal{I}_h u](x_i) \le C_1(u)\delta^{\alpha+k} + C_2(u)\left(\frac{h^2}{\delta^2}+\theta^2\right)$$

with $k=0,1$ and

$$C_1(u) = C|u|_{C^{2+k,\alpha}(\overline{\Omega})}\ |u|^{d-1}_{W^2_\infty(\Omega)}, \quad C_2(u) = C|u|^d_{W^2_\infty(\Omega)}$$

dictated by Lemma 2.4. Step 1 guarantees that

$$u_\varepsilon - \mathcal{I}_h u \le C|u|_{W^2_\infty(\Omega)}\delta \quad \text{on } \partial\Omega_{h,\delta},$$

where $\Omega_{h,\delta}$ is defined in (2.7). Let $d_\varepsilon := \mathcal{I}_h u - u_\varepsilon + C|u|_{W^2_\infty(\Omega)}\delta$ and note that $d_\varepsilon \geq 0$ on $\partial\Omega_{h,\delta}$. We then apply Proposition 4.6 (continuous dependence on data) to d_ε in $\Omega_{h,\delta}$, in conjunction with Lemma 2.4 (consistency of $T_\varepsilon[\mathcal{I}_h u]$), to obtain

$$\max_{\Omega_{h,\delta}} d_\varepsilon^- \leq \delta \left(\sum_{x_i \in \mathcal{C}_-(d_\varepsilon)} \left((f(x_i) + e)^{1/d} - f(x_i)^{1/d} \right)^d \right)^{1/d}$$

with $e := C_1(u)\delta^{\alpha+k} + C_2(u)\left(\frac{h^2}{\delta^2} + \theta^2\right)$. We now use that the function $t \mapsto t^{1/d}$ is concave with derivative $\frac{1}{d}t^{1/d-1}$ and $f(x_i) \geq f_0 > 0$ to estimate

$$(f(x_i) + e)^{1/d} - f(x_i)^{1/d} \leq \frac{e}{d f_0^{\frac{d-1}{d}}},$$

whence

$$\max_{\Omega_{h,\delta}} d_\varepsilon^- \leq C\delta \left(\sum_{x_i \in \mathcal{C}_-(d_\varepsilon)} \left(C_1(u)\delta^{\alpha+k} + C_2(u)\left(\frac{h^2}{\delta^2} + \theta^2\right) \right)^d \right)^{1/d}.$$

Since the cardinality of $\mathcal{C}_-(d_\varepsilon)$ is bounded by that of $\mathcal{N}_h$, which in turn is bounded by Ch^{-d} with C depending on shape regularity, we end up with

$$\max_{\Omega_h} (u_\varepsilon - \mathcal{I}_h u) \leq C|u|_{W^2_\infty(\Omega)}\delta + C\frac{\delta}{h}\left(C_1(u)\delta^{\alpha+k} + C_2(u)\left(\frac{h^2}{\delta^2} + \theta^2\right)\right).$$

Step 3: Choice of δ and θ. To find an optimal choice of δ and θ in terms of h, we minimize the right-hand side of the preceding estimate. We first set $\theta^2 = \frac{h^2}{\delta^2}$ and realize that the error is smallest when

$$C_1(u)\frac{\delta^{1+k+\alpha}}{h} = C_2(u)\frac{h}{\delta} \quad \Longrightarrow \quad \delta = \left(\frac{C_2(u)}{C_1(u)}h^2\right)^{\frac{1}{2+k+\alpha}}$$

Consequently,

$$\max_{\Omega_h} (u_\varepsilon - \mathcal{I}_h u) \leq C|u|_{W^2_\infty(\Omega)} \left(\frac{C_2(u)}{C_1(u)}h^2\right)^{\frac{1}{2+k+\alpha}} + \left(C_2(u)^{1+k+\alpha}\; C_1(u)\; h^{k+\alpha}\right)^{\frac{1}{2+k+\alpha}}$$

and we see that the boundary term is always of higher order, since $k+\alpha \leq 2$. This leads readily to the desired estimate upon writing the constants $C_1(u)$ and $C_2(u)$ in terms of $|u|_{C^{2+k,\alpha}(\overline{\Omega})}$ and $|u|_{W^2_\infty(\Omega)}$, and completes the proof. □

S:RatesSobolev

5.3. **Error Estimates for Solutions with Sobolev Regularity.** We now derive error estimates for solutions $u \in W^s_p(\Omega)$ with $s > 2 + \frac{d}{p}$ so that $W^s_p(\Omega) \subset C^2(\overline{\Omega})$. We exploit the structure of the estimate of Proposition 4.6 (continuous dependence on data) which shows that its right-hand side accumulates in l^d rather than l^∞.

T:RatesSobolev

Theorem 5.4 (convergence rate for W^s_p solutions)**.** *Let $f \geq f_0 > 0$ in Ω and let the viscosity solution u of (1.1) be of class $W^s_p(\Omega)$ with $\frac{d}{p} < s - 2 - k \leq 1,\ k = 0, 1$. If u_ε is the discrete solution of (2.2),*

$$\delta = \left(|u|_{W^2_\infty(\Omega)}|u|^{-1}_{W^s_p(\Omega)}\right)^{\frac{1}{s}} h^{\frac{2}{s}}, \quad \theta = \left(|u|_{W^2_\infty(\Omega)}|u|^{-1}_{W^s_p(\Omega)}\right)^{-\frac{1}{s}} h^{1-\frac{2}{s}},$$

then

$$\|u - u_\varepsilon\|_{L^\infty(\Omega_h)} \leq C(d,\Omega,f_0)\; |u|^{\frac{1}{s}}_{W^s_p(\Omega)}\; |u|^{d-\frac{1}{s}}_{W^2_\infty(\Omega)}\; h^{1-\frac{2}{s}},$$

where the constant $C(d,\Omega,f_0)$ depends only on d,Ω and f_0.

Proof. We proceed as in Theorem 5.3. The boundary estimate of Step 1 remains intact, namely

$$u_\varepsilon(z) - \mathcal{I}_h u(z) \le C\ |u|_{W^2_\infty(\Omega)}\ \delta$$

for all $z \in \mathcal{N}_h^0$ such that $\mathrm{dist}(z,\partial\Omega_h) \le \delta$. On the other hand, Step 2 yields

$$\max_{\Omega_{h,\delta}}(u_\varepsilon - \mathcal{I}_h u) \lesssim \delta |u|_{W^2_\infty(\Omega)} + \delta \left(\sum_{x_i \in \mathcal{N}_h^0} C_1(u)^d \delta^{(k+\alpha)d} + C_2(u)^d \left(\frac{h^2}{\delta^2} + \theta^2\right)^d \right)^{1/d},$$

where $C_1(u)$ and $C_2(u)$ are defined in Lemma 2.4 (consistency of $T_\varepsilon[\mathcal{I}_h u]$) and $0 < \alpha = s-2-k-\frac{d}{p} \le 1$ corresponds to the Sobolev embedding $W^s_p(B_i) \subset C^{2+k,\alpha}(B_i)$. In the following calculations we resort to the Sobolev inequality [7, Theorem 2.9]

$$|u|_{C^{2+k,\alpha}(B_i)} \le C |u|_{W^s_p(B_i)},$$

involving only semi-norms. We stress that $C>0$ depends on the Lipschitz constant of B_i but not on its size. The latter is due to the fact that the Sobolev numbers of $W^{s-2-k}_p(B_i)$ and $C^{0,\alpha}(B_i)$ coincide: $0 < s-k-2-d/p = \alpha \le 1$. We refer to [7, Theorem 2.9] for a proof for $0<s<1$. We now use the Hölder inequality with exponent $\frac{p}{d} > 1$ to obtain

$$\begin{aligned}\left(\sum_{x_i \in \mathcal{N}_h^0} C_1(u)^d\right)^{\frac{1}{d}} &\lesssim \left(\sum_{x_i \in \mathcal{N}_h^0} |u|^d_{W^s_p(B_i)} |u|^{d(d-1)}_{W^2_\infty(B_i)}\right)^{\frac{1}{d}} \\ &\lesssim \left(\sum_{x_i \in \mathcal{N}_h^0} |u|^{d\frac{p}{d}}_{W^s_p(B_i)}\right)^{\frac{1}{d}\frac{d}{p}} \left(\sum_{x_i \in \mathcal{N}_h^0} |u|^{d(d-1)\frac{p}{p-d}}_{W^2_\infty(B_i)}\right)^{\frac{1}{d}\frac{p-d}{p}}.\end{aligned}$$

Since the cardinality of the set of balls B_i containing an arbitrarily given $x \in \Omega$ is proportional to $\left(\frac{\delta}{h}\right)^d$, while the cardinality of $\mathcal{N}_h^0$ is proportional to h^{-d}, we get

$$\begin{aligned}\left(\sum_{x_i \in \mathcal{N}_h^0} C_1(u)^d\right)^{\frac{1}{d}} &\lesssim \left(\frac{\delta}{h}\right)^{\frac{d}{p}} |u|_{W^s_p(\Omega)} \left(h^{-d} |u|^{\frac{d(d-1)p}{p-d}}_{W^2_\infty(\Omega)}\right)^{\frac{p-d}{pd}} \\ &\lesssim \frac{\delta^{\frac{d}{p}}}{h}\ |u|_{W^s_p(\Omega)}\ |u|^{d-1}_{W^2_\infty(\Omega)}.\end{aligned}$$

Exploiting that $\alpha + k + \frac{d}{p} + 1 = s - 1$, we readily arrive at

$$\delta \left(\sum_{x_i \in \mathcal{N}_h^0} C_1(u)^d\ \delta^{(k+\alpha)d}\right)^{\frac{1}{d}} \lesssim \frac{\delta^{s-1}}{h} |u|_{W^s_p(\Omega)}\ |u|^{d-1}_{W^2_\infty(\Omega)}.$$

In addition, we have

$$\left(\sum_{x_i \in \mathcal{N}_h^0} C_2(u)^d\right)^{\frac{1}{d}} \lesssim |u|^d_{W^2_\infty(\Omega)}\ \frac{1}{h},$$

whence

$$\delta \left(\sum_{x_i \in \mathcal{N}_h^0} C_2(u)^d \left(\frac{h^2}{\delta^2} + \theta^2 \right)^d \right)^{\frac{1}{d}} \lesssim |u|^d_{W^2_\infty(\Omega)} \, \frac{\delta}{h} \left(\frac{h^2}{\delta^2} + \theta^2 \right).$$

Collecting the previous estimates, we end up with

$$\max_{\Omega_h} (u_\varepsilon - \mathcal{I}_h u) \lesssim \delta |u|_{W^2_\infty(\Omega)} + |u|^{d-1}_{W^2_\infty(\Omega)} \frac{\delta}{h} \left(|u|_{W^s_p(\Omega)} \delta^{s-1} + |u|_{W^2_\infty(\Omega)} \left(\frac{h^2}{\delta^2} + \theta^2 \right) \right).$$

To find an optimal relation among h, δ and θ, we first choose $\theta^2 = \frac{h^2}{\delta^2}$ and next equate the two terms in the second summand, which we call I_2. We obtain

$$\delta = \left(\frac{|u|_{W^2_\infty(\Omega)}}{|u|_{W^s_p(\Omega)}} \right)^{\frac{1}{s}} h^{\frac{2}{s}}, \quad \theta = \left(\frac{|u|_{W^2_\infty(\Omega)}}{|u|_{W^s_p(\Omega)}} \right)^{-\frac{1}{s}} h^{1-\frac{2}{s}},$$

whence

$$I_2 \lesssim |u|^{\frac{1}{s}}_{W^s_p(\Omega)} \, |u|^{d-\frac{1}{s}}_{W^2_\infty(\Omega)} \, h^{1-\frac{2}{s}}.$$

Since $\delta |u|_{W^2_\infty(\Omega)} \lesssim h^{\frac{2}{s}} \le h^{1-\frac{2}{s}}$ for the range $2 < s \le 4$, we conclude that

$$\max_{\Omega_h} (u_\varepsilon - \mathcal{I}_h u) \lesssim |u|^{\frac{1}{s}}_{W^s_p(\Omega)} \, |u|^{d-\frac{1}{s}}_{W^2_\infty(\Omega)} \, h^{1-\frac{2}{s}},$$

which is the asserted estimate. □

The error estimate of Theorem 5.4 (convergence rate for W^s_p-solutions) is of order $\frac{1}{2}$ for $s = 4$ and $u \in W^4_p(\Omega)$ with $p > d$. This rate requires much weaker regularity than the corresponding error estimate in Theorem 5.3, namely $u \in C^{3,1}(\overline{\Omega}) = W^4_\infty(\Omega)$. In both cases, the relation between δ and h is $\delta \approx h^{\frac{1}{2}}$.

S:RatesPW

5.4. **Error Estimates for Piecewise Smooth Solutions.** We now derive pointwise rates of convergence for a larger class of solutions than in Section 5.3. These are viscosity solutions which are piecewise W^s_p but have discontinuous Hessians across a Lipschitz $(d-1)$-dimensional manifold $\mathcal{S}$; we refer to the second numerical example in [12]. Since $T_\varepsilon[\mathcal{I}_h u]$ has a consistency error of order one in a δ-region around $\mathcal{S}$, due to the discontinuity of $D^2 u$, we exploit the fact that the measure of this region is proportional to $\delta|\mathcal{S}|$. We are thus able to adapt the argument of Theorem 5.4 (convergence rate for W^s_p solutions), and accumulate such consistency error in l^d, at the expense of an extra additive term of order $h^{-1}\delta^{1+\frac{1}{d}}$. This yields a convergence rate depending on the dimension d.

T:RatesPW

Theorem 5.5 (convergence rate for piecewise smooth solutions). *Let $\mathcal{S}$ denote a $(d-1)$-dimensional Lipschitz manifold that divides Ω into two disjoint subdomains Ω_1, Ω_2 so that $S = \overline{\Omega}_1 \cap \overline{\Omega}_2$ Let $f \ge f_0 > 0$ in Ω and let $u \in W^s_p(\Omega_i) \cap W^2_\infty(\Omega)$, for $i = 1, 2$ and $\frac{d}{p} < s - 2 \le 1$, be the viscosity solution of (1.1). If u_ε denotes the discrete solution of (2.2), then for $\beta = \min(s, 2 + \frac{1}{d})$ we have*

$$\|u - u_\varepsilon\|_{L^\infty(\Omega_h)} \le C(d, \Omega) \, |u|^{\frac{1}{\beta}}_{W^s_p(\Omega \setminus \mathcal{S})} \, |u|^{d-\frac{1}{\beta}}_{W^2_\infty(\Omega)} \, h^{1-\frac{2}{\beta}}$$

with $|u|_{W^s_p(\Omega \setminus \mathcal{S})} := \max_i |u|_{W^s_p(\Omega_i)}$, provided

$$\delta = \left(|u|_{W^2_\infty(\Omega)} |u|^{-1}_{W^s_p(\Omega \setminus \mathcal{S})} \right)^{\frac{1}{\beta}} h^{\frac{2}{\beta}}, \quad \theta = \left(|u|_{W^2_\infty(\Omega)} |u|^{-1}_{W^s_p(\Omega \setminus \mathcal{S})} \right)^{-\frac{1}{\beta}} h^{1-\frac{2}{\beta}}.$$

Proof. We proceed as in Theorems 5.3 and 5.4. The boundary layer estimate relies on the regularity $u \in W^2_\infty(\Omega)$ which is still valid, whence for all $x \in \Omega_h$ such that $\operatorname{dist}(x, \partial\Omega_h) \le \delta$ we obtain

$$u_\varepsilon(x) - \mathcal{I}_h u(x) \le C|u|_{W^2_\infty(\Omega)}\delta.$$

Consider now the internal layer

$$\mathcal{S}_h^\delta := \{x \in \Omega_h : \ \operatorname{dist}(x, \mathcal{S}) \le \delta\},$$

which is the region affected by the discontinuity of the Hessian D^2u. Recall the auxiliary function $d_\varepsilon = \mathcal{I}_h u - u_\varepsilon + C|u|_{W^2_\infty(\Omega)}\delta$ of Theorem 5.3 (rates of convergence for classical solutions) and split the contact set $\mathcal{C}_-^\delta(d_\varepsilon) := \mathcal{C}_-(d_\varepsilon) \cap \Omega_{h,\delta}$ as follows:

$$\mathcal{S}_{h,1}^\delta := \mathcal{C}_-^\delta(d_\varepsilon) \cap \mathcal{S}_h^\delta, \quad \mathcal{S}_{h,2}^\delta := \mathcal{C}_-^\delta(d_\varepsilon) \setminus \mathcal{S}_h^\delta.$$

An argument similar to Step 2 (interior estimate) of Theorem 5.3, based on combining Proposition 4.6 (continuous dependence on data) and Lemma 2.4 (consistency of $T_\varepsilon[\mathcal{I}_h u]$) with assumption $f \ge f_0 > 0$, yields

$$\max_{\Omega_{h,\delta}} d_\varepsilon^- \lesssim \delta \left(\sum_{x_i \in \mathcal{S}_{h,1}^\delta} C_2(u)^d \right)^{1/d} + \delta \left(\sum_{x_i \in \mathcal{S}_{h,2}^\delta} C_1(u)^d \delta^{(k+\alpha)d} + C_2(u)^d \left(\frac{h^2}{\delta^2} + \theta^2 \right)^d \right)^{1/d} =: I_1 + I_2,$$

because the consistency error in $\mathcal{S}_{h,1}^\delta$ is bounded by $C_2(u) = C|u|^d_{W^2_\infty(B_i)}$. As in Theorem 5.4 (convergence rate for W^s_p solutions), $C_1(u)$ satisfies

$$C_1(u) \lesssim |u|_{W^s_p(B_i)} |u|^{d-1}_{W^2_\infty(B_i)}.$$

Since the number of nodes $x_i \in \mathcal{S}_{h,1}^\delta$ is bounded by $C|\mathcal{S}|\delta h^{-d}$, we deduce

$$I_1 \lesssim \delta \left(\sum_{x_i \in \mathcal{S}_{h,1}^\delta} C_2(u)^d \right)^{1/d} \lesssim |u|^d_{W^2_\infty(\Omega)} \frac{\delta^{1+\frac{1}{d}}}{h}.$$

For I_2 we distinguish whether x_i belongs to Ω_1 or Ω_2 and accumulate $C_1(u)$ in ℓ^p, exactly as in Theorem 5.4, to obtain

$$I_2 \lesssim |u|^{d-1}_{W^2_\infty(\Omega)} \left(|u|_{W^s_p(\Omega\setminus\mathcal{S})} \frac{\delta^{s-1}}{h} + |u|_{W^2_\infty(\Omega)} \frac{\delta}{h} \left(\frac{h^2}{\delta^2} + \theta^2 \right) \right).$$

Collecting the previous estimates yields

$$\max_{\Omega_h}(u_\varepsilon - \mathcal{I}_h u) \lesssim |u|_{W^2_\infty(\Omega)}\delta + |u|^{d-1}_{W^2_\infty(\Omega)} \frac{\delta}{h} \left(|u|_{W^s_p(\Omega\setminus\mathcal{S})}\delta^{s-2} + |u|_{W^2_\infty(\Omega)} \left(\frac{h^2}{\delta^2} + \theta^2 + \delta^{\frac{1}{d}} \right) \right).$$

We finally realize that this estimate is similar to that in the proof of Theorem 5.4 except for the extra additive term $|u|^d_{W^2_\infty(\Omega)}\delta^{1+\frac{1}{d}}h^{-1}$, which dominates for $\frac{1}{d} \le s-2$. Therefore, upon setting $\beta = \min(s, 2+\frac{1}{d})$, the desired estimate and relations between δ, θ and h follow as in Theorem 5.4. This concludes the proof. □

S:RatesDegen

5.5. Error Estimates for Piecewise Smooth Solutions with Degenerate f. We observe that in all three preceding theorems we assume that $f \geq f_0 > 0$. This is an important assumption in the proofs, since it allows us to use the concavity of $t \mapsto t^{1/d}$ and Proposition 4.6 (continuous dependence on data) to obtain

{E:fconcavity}

$$(f(x_i)+e)^{1/d} - f(x_i)^{1/d} \leq \frac{e}{d f_0^{\frac{d-1}{d}}}, \tag{5.1}$$

where e is related to the consistency of the operator in Lemma 2.4 (consistency of $T_\varepsilon[\mathcal{I}_h u]$). We see that this is only possible if $f_0 > 0$. If we allow f to touch zero, then (5.1) reduces to

{E:fconcavitydegen}

$$(f(x_i)+e)^{1/d} - f(x_i)^{1/d} \leq e^{1/d}, \tag{5.2}$$

with equality for $f(x_i) = 0$. This leads to a rate of order $\left(\frac{\delta}{h}\right)^{1-\frac{2}{d}} \geq 1$ for $d \geq 2$. To circumvent this obstruction, we use Lemma 5.2 (interior barrier function) which allows us to introduce an extra parameter $\sigma > 0$ that compensates for the lack of lower bound $f_0 > 0$ and yields pointwise error estimates of reduced order.

T:RatesDegen

Theorem 5.6 (degenerate forcing $f \geq 0$)**.** *Let $\mathcal{S}$ denote a $(d-1)$-dimensional Lipschitz manifold that divides Ω into two disjoint subdomains Ω_1, Ω_2 such that $\mathcal{S} = \overline{\Omega}_1 \cap \overline{\Omega}_2$. Let $f \geq 0$ in Ω and let $u \in W_p^s(\Omega_i) \cap W_\infty^2(\Omega)$, for $i = 1, 2$ and $\frac{d}{p} < s - 2 \leq 1$, be the viscosity solution of (1.1). If u_ε denotes the discrete solution of (2.2), then for $\beta = \min\left(s, 2 + \frac{1}{d}\right)$ we have*

$$\|u - u_\varepsilon\|_{L^\infty(\Omega_h)} \leq C(d,\Omega)\ |u|_{W_p^s(\Omega\setminus\mathcal{S})}^{\frac{1}{d\beta}}\ |u|_{W_\infty^2(\Omega)}^{1-\frac{1}{d\beta}}\ h^{\frac{1}{d}\left(1-\frac{2}{\beta}\right)}$$

with $|u|_{W_p^s(\Omega\setminus\mathcal{S})} := \max_i |u|_{W_p^s(\Omega_i)}$, provided

$$\delta = \left(|u|_{W_\infty^2(\Omega)} |u|_{W_p^s(\Omega\setminus\mathcal{S})}^{-1}\right)^{1/\beta} h^{2/\beta}, \quad \theta = \left(|u|_{W_\infty^2(\Omega)} |u|_{W_p^s(\Omega\setminus\mathcal{S})}^{-1}\right)^{-1/\beta} h^{1-\frac{2}{\beta}}.$$

Proof. We employ the interior barrier function q_h of Lemma 5.2 scaled by a parameter $\sigma > 0$ to control $u_\varepsilon - \mathcal{I}_h u$ and $\mathcal{I}_h u - u_\varepsilon$ in two steps. The parameter σ allows us to mimic the calculation in (5.1). In the third step we choose σ optimally with respect to the scales of our scheme.

Step 1: Upper bound for $u_\varepsilon - \mathcal{I}_h u$. Let $w_h := u_\varepsilon + \sigma q_h$ and $v_h := \mathcal{I}_h u + C|u|_{W_\infty^2(\Omega)}\delta$ and for $z \in \mathcal{N}_h^0$ such that $\operatorname{dist}(z, \partial\Omega_h) \leq \delta$, let p_h be the discrete barrier function of Lemma 5.1 associated with z. We show that

$$w_h(z) \leq v_h(z).$$

Since, $p_h, q_h \leq 0$ on $\partial\Omega_h$, we have $w_h + p_h \leq \mathcal{I}_h u$ on $\partial\Omega_h$. Using Lemma 2.4 (consistency of $T_\varepsilon[\mathcal{I}_h u]$) we also see that

$$\begin{aligned} T_\varepsilon[w_h + p_h](x_i) &\geq f(x_i) + \sigma^d + E \\ &\geq T_\varepsilon[\mathcal{I}_h u](x_i) - C|u|_{W_\infty^2(\Omega)}^d + \sigma^d + E \geq T_\varepsilon[\mathcal{I}_h u](x_i) \quad \forall\ x_i \in \mathcal{N}_h^0 \end{aligned}$$

for $E = C|u|_{W_\infty^2(\Omega)}$, whence Lemma 2.3 (discrete comparison principle) yields

$$w_h(z) - C|u|_{W_\infty^2(\Omega)}\delta \leq w_h(z) + p_h(z) \leq \mathcal{I}_h u(z) \quad \Rightarrow \quad w_h(z) \leq v_h(z).$$

We now focus on $\Omega_{h,\delta}$ and consider the internal layer

$$\mathcal{S}_h^\delta := \{x \in \Omega_{h,\delta} : \ \operatorname{dist}(x, \mathcal{S}) \leq \delta\},$$

which is the region affected by the discontinuity of the Hessian D^2u. We also define the auxiliary function $d_\varepsilon := v_h - w_h$ and split $\mathcal{C}_-^\delta(d_\varepsilon) = \mathcal{C}_-(d_\varepsilon) \cap \Omega_{h,\delta}$ as follows:

$$\mathcal{S}_{h,1}^\delta := \mathcal{C}_-^\delta(d_\varepsilon) \cap \mathcal{S}_h^\delta, \quad \mathcal{S}_{h,2}^\delta := \mathcal{C}_-^\delta(d_\varepsilon) \setminus \mathcal{S}_h^\delta.$$

Since the previous argument guarantees that $d_\varepsilon \geq 0$ on $\partial\Omega_{h,\delta}$, Proposition 4.6 (continuous dependence on data) gives

$$\max_{\Omega_{h,\delta}} d_\varepsilon^- \leq \delta \left(\sum_{x_i \in \mathcal{C}_-^\delta(d_\varepsilon)} \left(\big(T_\varepsilon[v_h](x_i)\big)^{1/d} - \big(T_\varepsilon[w_h](x_i)\big)^{1/d} \right)^d \right)^{1/d}.$$

In order to split the right-hand side, we further note that

$$T_\varepsilon[w_h](x_i) \geq T_\varepsilon[u_\varepsilon](x_i) + T_\varepsilon[\sigma q_h](x_i) \geq f(x_i) + \sigma^d,$$

whence

$$\begin{aligned}\max_{\Omega_{h,\delta}} d_\varepsilon^- \leq \delta & \left(\sum_{x_i \in \mathcal{S}_{h,1}^\delta} \left(\big(f(x_i) + e(x_i)\big)^{1/d} - \big(f(x_i) + \sigma^d\big)^{1/d} \right)^d \right)^{1/d} \\ & + \delta \left(\sum_{x_i \in \mathcal{S}_{h,2}^\delta} \left(\big(f(x_i) + e(x_i)\big)^{1/d} - \big(f(x_i) + \sigma^d\big)^{1/d} \right)^d \right)^{1/d} =: I_1 + I_2,\end{aligned}$$

where $e(x_i)$ stands for an appropriate local bound for the consistency error of Lemma 2.4. We now observe that $e(x_i) \geq \sigma^d$ for all x_i's that belong to the contact set $\mathcal{C}_-^\delta(d_\varepsilon)$ of d_ε because all terms in the above sum are non-negative. If there is no such x_i, then the above bound implies that $d_\varepsilon^- = 0$ and $w_h \leq v_h$, whence $u_\varepsilon \leq \mathcal{I}_h u + C\sigma + C|u|_{W^2_\infty(\Omega)}\delta$. Otherwise, the above observation and $f(x_i) \geq 0$ imply that, for both I_1 and I_2, we can use the bound

$$\begin{aligned}(f(x_i) + e(x_i))^{1/d} &- (f(x_i) + \sigma^d)^{1/d} \\ &= (f(x_i) + \sigma^d + (e(x_i) - \sigma^d)^{1/d} - (f(x_i) + \sigma^d)^{1/d} \\ &\leq \frac{e(x_i) - \sigma^d}{d\sigma^{d\frac{d-1}{d}}} \leq d^{-1}\sigma^{1-d} e(x_i).\end{aligned}$$

We now examine the two terms I_1 and I_2 separately. In the set $\mathcal{S}_{h,1}^\delta$, $e(x_i)$ is bounded by $C_2(u) = C|u|^d_{W^2_\infty(\Omega)}$ according to Lemma 2.4 (consistency $T_\varepsilon[\mathcal{I}_h u]$). We combine this with the fact that the number of nodes x_i that belong to $\mathcal{S}_{h,1}^\delta$ is bounded by $C|\mathcal{S}|\delta h^{-d}$ to deduce that

$$I_1 = C\delta \left(\sum_{x_i \in \mathcal{S}_{h,1}^\delta} C_2(u)^d \right)^{1/d} \lesssim \sigma^{1-d} \, |u|^d_{W^2_\infty(\Omega)} \frac{\delta^{1+\frac{1}{d}}}{h};$$

this resembles a similar bound in Theorem 5.5 (convergence rate for piecewise smooth solutions) except for the factor σ^{1-d}. In the set $\mathcal{S}_{h,2}^\delta$, the same bound derived in Theorem 5.5 holds for each Ω_i again with the additional factor σ^{1-d}

$$I_2 \leq C\sigma^{1-d} \, |u|^{d-1}_{W^2_\infty(\Omega)} \left(|u|_{W^s_p(\Omega\setminus\mathcal{S})} \frac{\delta^{s-1}}{h} + |u|_{W^2_\infty(\Omega)} \, \frac{\delta}{h} \, \left(\frac{h^2}{\delta^2} + \theta^2 \right) \right).$$

Combining the bounds of I_1 and I_2 with the definition of d_ε, we obtain

$$u_\varepsilon \le \mathcal{I}_h u + C\left(|u|_{W^2_\infty(\Omega)}\delta + \sigma + \sigma^{1-d}\,\frac{\delta}{h}\left(C_1(u)\delta^{s-2} + C_2(u)\;\left(\delta^{1/d} + \frac{h^2}{\delta^2} + \theta^2\right)\right)\right),$$

where $C_1(u) = C|u|_{W^s_p(\Omega\setminus\mathcal{S})}|u|^{d-1}_{W^2_\infty(\Omega)}$ and $C_2(u) = C|u|^d_{W^2_\infty(\Omega)}$ as in Theorem 5.5.

Step 2: Lower bound for $u_\varepsilon - \mathcal{I}_h u$. To prove the reverse inequality, we proceed as in Step 1, except that this time we define $v_h := u_\varepsilon + C|u|_{W^2_\infty(\Omega)}\delta$ and $w_h := \mathcal{I}_h u + \sigma q_h$. An argument similar to Step 1 yields $w_h \le v_h$ in $\omega_{h,\delta}$ which, combined with Proposition 4.6 (continuous dependence on data) in $\Omega_{h,\delta}$, gives

$$\mathcal{I}_h u \le u_\varepsilon + C\left(|u|_{W^2_\infty(\Omega)}\delta + \sigma + \sigma^{1-d}\,\frac{\delta}{h}\left(C_1(u)\delta^{s-2} + C_2(u)\;\left(\delta^{1/d} + \frac{h^2}{\delta^2} + \theta^2\right)\right)\right).$$

Step 3: Choice of δ, θ and σ. Combining Steps 1 and 2 yields

$$\|u_\varepsilon - \mathcal{I}_h u\|_{L^\infty(\Omega_h)} \lesssim |u|_{W^2_\infty(\Omega)}\delta + \sigma + \sigma^{1-d}\,\frac{\delta}{h}\left(C_1(u)\delta^{s-2} + C_2(u)\;\left(\delta^{1/d} + \frac{h^2}{\delta^2} + \theta^2\right)\right).$$

We now minimize the right-hand side upon choosing δ, θ and σ suitably with respect to h. We see that for $s \le 2 + \frac{1}{d}$, we obtain, similarly to Theorem 5.5

$$\delta = \left(\frac{|u|_{W^2_\infty(\Omega)}}{|u|_{W^s_p(\Omega\setminus\mathcal{S})}}\right)^{\frac{1}{s}} h^{\frac{2}{s}}.$$

At this stage it remains to optimize σ, namely

$$\sigma = \sigma^{1-d}|u|^{\frac{1}{s}}_{W^s_p(\Omega\setminus S)}\;|u|^{d-\frac{1}{s}}_{W^2_\infty(\Omega)}\;h^{1-\frac{2}{s}},$$

which leads to

$$\sigma = |u|^{\frac{1}{ds}}_{W^s_p(\Omega\setminus S)}\;|u|^{1-\frac{1}{ds}}_{W^2_\infty(\Omega)}\;h^{\frac{1}{d}\left(1-\frac{2}{s}\right)}.$$

For any higher value of s the rate is dictated by the limiting case $s = 2 + \frac{1}{d}$. Since $\|u - \mathcal{I}_h u\|_{L^\infty(\Omega_h)} \le C|u|_{W^2_\infty(\Omega)}h^2$ is of higher order, the proof is complete. □

Theorem 5.6 is an extension of Theorem 5.5 to the degenerate case $f \ge 0$, but the same techniques and estimates extend as well to Theorems 5.3 and 5.4.

6. Conclusions

In this paper we extend the analysis of the two-scale method introduced in [12]. We derive continuous dependence of discrete solutions on data and use it to prove rates of convergence in the L^∞ norm in the computational domain Ω_h for four different cases. We first prove rates of order up to $h^{1/2}$ for smooth classical solutions with Hölder regularity. We then exploit the structure of the continuous dependence estimate of discrete solutions on data to derive error estimates for classical solutions with Sobolev regularity, thereby achieving the same rates under weaker regularity assumptions. In a more general scenario, we derive error estimates for viscosity solutions with discontinuous Hessian across a surface with appropriate smoothness, but otherwise possessing piecewise Sobolev regularity. Lastly, we use an interior barrier function that allows us to remove the nondegeneracy assumption $f > 0$ at the cost of a reduced rate that depends on dimension. Our theoretical predictions are sub-optimal with respect to the linear rates observed experimentally in [12] for a smooth classical solution and a piecewise smooth viscosity solution with degenerate right-hand side $f \geq 0$. This can be attributed to the fact that the continuous dependence estimate of discrete solutions on data introduces a factor $\frac{\delta}{h} \gg 1$ in the error estimates. This feature is similar to the discrete ABP estimate developed in [10] and is the result of using sets of measure $\approx \delta^d$ instead of $\approx h^d$ to approximate subdifferentials. In a forthcoming paper we will tackle this issue and connect our two-scale method with that of Feng and Jensen [5].

References

Aw 1. Awanou G., 2016, *Convergence rate of a stable, monotone and consistent scheme for the Monge-Ampère equation*, Symmetry, 8 (18):1-7.

BrenScott 2. Brenner S. C., Scott R., 2008, *The Mathematical Theory of Finite Element Methods*, Springer.

BrenNeil 3. Brenner S. C., Gudi T., Neilan M., Sung L-Y., 2011, *C^0 penalty methods for the fully nonlinear Monge-Ampère equation*, Math. Comp., 80(276):19791995.

BrenNeil2 4. Brenner S. C., Neilan M., 2012, *Finite element approximations of the three dimensional Monge-Ampère equation*, ESAIM Math. Model. Numer. Anal., 46(5):9791001.

FeJe:16 5. Feng X., Jensen M., 2016, *Convergent semi-Lagrangian methods for the Monge-Ampère equation on unstructured grids*, arXiv:1602.04758v2.

FrOb1 6. Froese B., Oberman A., 2012, *Convergent finite difference solvers for viscosity solutions of the elliptic Monge-Ampre equation in dimensions two and higher*, SIAM J. Numer. Anal., 49(4):1692-1714.

Giusti 7. Giusti E., 2003, *Direct Methods in the Calculus of Variation*, Singapore: World Scientific.

Gut 8. Gutiérrez C., 2001, *The Monge-Ampère Equation*, Birkhäuser.

Krylov 9. N. V. Krylov, 1987, *Nonlinear Elliptic and Parabolic Equations of the Second Order*, Springer Netherlands.

KuoTru 10. Kuo H-J., Trudinger N. S., 2000, *A note on the discrete Aleksandrov-Bakelman maximum principle*, In Proceedings of 1999 International Conference on Nonlinear Analysis (Taipei), (4),55-64.

Lions 11. P. L. Lions, 1983, *Hamilton-Jacobi-Bellman equations and the optimal control of stochastic systems, Proc. Int. Congress of Math., Warsaw.*

NoNtZh 12. Nochetto R. H., Ntogkas D., Zhang W., 2017, *Two-scale method for the Monge-Ampère equation: convergence to the viscosity solution*, arXiv:1706.06193.

NoZh 13. Nochetto R. H., Zhang W., 2014, *Discrete ABP estimate and convergence rates for linear elliptic equations in non-divergence form*, Found. Comp. Math. (to appear).

NoZh2 14. Nochetto R. H., Zhang W., 2016, *Pointwise rates of convergence for the Oliker-Prussner method for the Monge-Ampère equation*, arXiv:1611.02786.

OlPr 15. V. I. Oliker, L. D. Prussner, *On the numerical solution of the equation $(\partial^2 z/\partial x^2)(\partial^2 z/\partial y^2) - (\partial^2 z/\partial x \partial y)^2 = f$ and its discretizations*, I. Numer. Math., 54(3):271-293, 1988.

WalkerPaper 16. S. W. Walker, *FELICITY: A Matlab/C++ Toolbox for Developing Finite Element Methods and Simulation Modeling*, submitted.

WalkerWeb 17. S. W. Walker, *FELICITY: Finite ELement Implementation and Computational Interface Tool for You*. http://www.mathworks.com/matlabcentral/fileexchange/31141-felicity.

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