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INTERIOR ESTIMATES OF SEMIDISCRETE FINITE ELEMENT METHODS FOR PARABOLIC PROBLEMS WITH DISTRIBUTIONAL DATA*

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Abstract

Let $\Omega \subset \mathbb{R}^d$, $1 \le d \le 3$, be a bounded d-polytope. Consider the parabolic equation on Ω with the Dirac delta function on the right hand side. We study the well-posedness, regularity, and the interior error estimate of semidiscrete finite element approximations of the equation. In particular, we derive that the interior error is bounded by the best local approximation error, the negative norms of the error, and the negative norms of the time derivative of the error. This result implies different convergence rates for the numerical solution in different interior regions, especially when the region is close to the singular point. Numerical test results are reported to support the theoretical prediction.

Mathematics subject classification: 65M15, 65M60
Key words: Parabolic problems, Distributional data, Finite element methods, Interior estimates, Well-posedness, Singularity.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $1 \le d \le 3$, be a bounded d-polytope and $\Omega_T := \Omega \times (0, T]$. Namely, Ω is a line segment for $d = 1$, a polygon for $d = 2$, and a polyhedron for $d = 3$. Denote by $\delta_z(x) = \delta(x - z)$ the Dirac delta function at $z \in \Omega$. We consider a parabolic problem with the homogeneous Dirichlet boundary condition

$$u_t - \Delta u = f, \qquad \text{in} \quad \Omega_T, \tag{1.1a}$$
$$u = 0, \qquad \text{on} \quad \partial\Omega \times [0, T], \tag{1.1b}$$
$$u(\cdot, 0) = u_0, \qquad \text{on} \quad \Omega \times \{t = 0\}, \tag{1.1c}$$

where $u_0 \in L^2(\Omega)$ and $f = g\delta_z$ for $g \in L^2(0, T; C(\bar{\Omega}))$. The finite element approximations for parabolic equations with sufficiently smooth solutions have been well investigated in the literature (see, e.g., [3, 21, 29]). The study of numerical methods for parabolic equations with less regular data has become increasingly popular in recent years. We refer to [5, 14, 15] for equations with singular solutions due to the non-smoothness in the domain and in the coefficient

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of the differential operator. Some recent results on point-wise approximations can be found in [12,13] for fully discrete methods (finite element method in space and discontinuous Galerkin method in time). For numerical analysis on parabolic problems with Dirac delta functions, we mention [8,25] and references therein. In these works, the global convergence of the numerical scheme on the entire domain was obtained approximating the singular solution.

Partial differential equations with the δ-function sources have many applications in astrophysics and oil reservoir simulations. Especially in the latter case, an interesting model is the two-phase flow displacement in porous medium which can be described by a parabolic system. Moreover, the injection and production wells can be represented by point sources and sinks, respectively, which can be approximated by δ-singularities with different strengths. In such problems, the exact solutions are not smooth and high-order numerical schemes can yield poor convergence or strong oscillations in a vicinity (pollution region) of the singularity. In [30], Yang and Shu applied discontinuous Galerkin methods to solve linear hyperbolic equations with δ-function source terms in one space dimension and the size of the pollution region was proved to be of order $\mathcal{O}(h^{\frac{1}{2}})$, where h is the mesh size.

Note that the distributional data in Eq. (1.1) can lead to singular solutions, for which the global approximation may not be of high-order accuracy even when high-order finite element methods are used. Meanwhile, the numerical approximation in certain interior regions is often more interesting in practice. In this paper, we study the interior error estimate of the semidiscrete finite element method for Eq. (1.1). In particular, we first derive the well-posedness of the weak solution for Eq. (1.1) in suitable Sobolev spaces (Theorem 2.1). This result extends the well-posedness result in [8,16] on convex domains to general polytopal domains. Then, we show that away from the singular point z, the solution possesses higher interior regularity (Corollary 2.1), which justifies the use of the L^2 and H^1 norms of the error in such interior regions for our error analysis. The main result regarding the interior error estimate is summarized in Theorem 4.1, in which we obtain that the L^2 and H^1 norms of the error in an interior region away from z are determined by three components: the best local approximation error from the finite element space, the negative norms of the interior error, and the negative norms of the time derivative of the local error. Namely, the interior convergence may be of higher order compared with the global convergence, which is affected by the regularity of u and u_t. Applying this result to regions close to z, we further formulate an interior estimate (Corollary 4.1) that depends on the distance to the singular point. This implies that as the region gets closer to z, the interior convergence rate can slow down and eventually resemble the global convergence rate.

For elliptic boundary value problems, the finite element interior estimates have been studied in a series of papers [20, 22–24]. These results show that the error in an interior region is bounded by the best local approximation error and the error in negative norms. Thus, the interior error estimates in this paper extend these results to parabolic problems by including additional effects from the time derivative of the solution. We also mention that for parabolic equations, an interior finite element analysis was derived in [27] using the energy method on uniform meshes with specific conditions. In this paper, we use a more direct method to obtain the interior error analysis, especially when distributional data is present. In addition, our results apply to general quasi-uniform meshes.

The rest of the paper is organized as follows. In Section 2, we introduce the notation and derive the well-posedness and regularity results for the parabolic problem (1.1). In Section 3, we formulate the semidiscrete finite element approximation and recall useful properties of the numerical scheme. In Section 4, we obtain the interior error estimates for the parabolic

equation (1.1). In Section 5, we report numerical test results to verify the theory.

Throughout the paper, for two regions A and B, $A \subset B$ means that A is an *interior* proper subset of B (i.e. $\mathrm{dist}(\partial A, \partial B) > 0$), while $A \subseteq B$ means that A is a subset of B (A can be equal to B). The generic constant $C > 0$ in our analysis may be different at different occurrences. It will depend on the underlying domain, but not on the functions or the mesh size involved in the estimates.

2. Well-osedness and Regularity

In this section, we introduce the notation and derive related regularity results for Eq. (1.1).

2.1. The Notation

For an integer $m \geq 0$ and $D \subset \mathbb{R}^d$, $1 \leq d \leq 3$, let $H^m(D)$ be the Sobolev space with the norm and the seminorm

$$\|v\|_{m,D} = \left(\sum_{|\alpha| \leq m} \|\partial^\alpha v\|^2_{L^2(D)} \right)^{1/2} \quad \text{and} \quad |v|_{m,D} = \left(\sum_{|\alpha| = m} \|\partial^\alpha v\|^2_{L^2(D)} \right)^{1/2},$$

where $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{Z}^d_{\geq 0}$ is the multi-index with $|\alpha| = \sum_{1 \leq i \leq d} \alpha_i$. Denote by $L^2(D) = H^0(D)$ the standard L^2 space over D. Moreover, we denote by $H^m_0(D)$ the completion of $C^\infty_0(D)$ in $H^m(D)$. For a non-integer $s \geq 0$, the space $H^s(D)$ and $H^s_0(D)$ are defined by interpolation. More details on the Sobolev space can be found in [9, 17, 19]. Specifically, for $D \subseteq \Omega$ and $m \leq 0$ being an integer, the negative norm is defined as

$$\|v\|_{m,D} = \sup_{0 \neq \varphi \in C^\infty_0(D)} \frac{(v, \varphi)_D}{\|\varphi\|_{-m,D}},$$

where $(v, \varphi)_D = \int_D v\varphi dx$. For any negative integer m, the following two properties hold :
1. If D_1 and D_2 are two disjoint sets, then

$$\|v\|^2_{m,D_1} + \|v\|^2_{m,D_2} = \|v\|^2_{m,D_1 \cup D_2}. \tag{2.1}$$

2. If $D_1 \subset D_2$, then

$$\|v\|_{m,D_1} \leq \|v\|_{m,D_2}. \tag{2.2}$$

Meanwhile, we recall the function spaces involving time. Let X be a Banach space with norm $\|\cdot\|_X$. Then, we denote by $L^2(0,T;X)$ and $H^1(0,T;X)$ the spaces of measurable functions $v : [0,T] \to X$ such that

$$\|v\|^2_{L^2(0,T;X)} := \int_0^T \|v(t)\|^2_X dt < \infty, \quad \|v\|^2_{H^1(0,T;X)} := \int_0^T \Big(\|v(t)\|^2_X + \|v_t(t)\|_X \Big)^2 dt < \infty,$$

where $v_t = \partial_t v$; and denote by $C(0,T;X)$ the space of continuous functions $v : [0,T] \to X$ with the norm

$$\|v\|_{C(0,T;X)} := \max_{0 \leq t \leq T} \|v(t)\|_X < \infty.$$

In addition, we define the space

$$G(\Omega) := \Big\{ v,\ v \in H^1_0(\Omega),\ \Delta v \in L^2(\Omega) \Big\}.$$

Remark 2.1. Since the domain Ω in Eq. (1.1) is a d-polytope, we here briefly recall the regularity property for the associated elliptic problem. For $q \in L^2(\Omega)$, the elliptic problem

$$-\Delta v = q \quad \text{in} \quad \Omega, \qquad v = 0 \quad \text{on} \quad \partial\Omega, \tag{2.3}$$

has a unique solution in $H_0^1(\Omega)$. The regularity of the solution, however, depends on both the regularity of q and the geometry of the domain Ω. For a sufficiently smooth function q, we usually have $v \in H^{1+\alpha}(\Omega)$ when Ω has a non-smooth boundary ($d = 2, 3$), where $\alpha > 1/2$ is the regularity index determined by the domain. In particular, the index $\alpha < 1$ in the following cases. (I) Ω contains a reentrant corner ($d = 2$). (II) Ω contains part of an edge ($d = 3$) with dihedral angle $> \pi$. (III) Ω contains a vertex ($d = 3$) with certain opening angle such that the solution is singular ($v \notin H^2$) in the neighborhood of the vertex. Unlike the two-dimensional case, the relation between the vertex angle and the solution regularity does not follow a simple formula, but can be computed numerically. Consequently, for the global regularity, if Ω is convex, $G(\Omega) = H^2(\Omega) \cap H_0^1(\Omega)$; and if Ω is non-convex, $G(\Omega) \subset H^{1+\alpha}(\Omega)$ for some $\alpha > 1/2$ [10]. Meanwhile, in an interior region $D \subset \Omega$, the solution of (2.3) possesses higher regularity $\|v\|_{H^{k+2}(D)} \le C\|q\|_{H^k(\Omega)}$, for $k \ge 0$.

2.2. The Equation

Consider the forward problem associated with Eq. (1.1)

$$v_t - \Delta v = q, \qquad \text{in} \quad \Omega_T, \tag{2.4a}$$
$$v = 0, \qquad \text{on} \quad \partial\Omega \times [0, T], \tag{2.4b}$$
$$v = v_0, \qquad \text{on} \quad \Omega \times \{t = 0\}, \tag{2.4c}$$

and the backward problem

$$v_t + \Delta v = q, \qquad \text{in} \quad \Omega \times [0, T), \tag{2.5a}$$
$$v = 0, \qquad \text{on} \quad \partial\Omega \times [0, T], \tag{2.5b}$$
$$v = 0, \qquad \text{on} \quad \Omega \times \{t = T\}. \tag{2.5c}$$

Then, we have the following well-posedness and regularity results.

Proposition 2.1. *Suppose $q \in L^2(0, T; L^2(\Omega))$ and $v_0 \in H_0^1(\Omega)$. Then, each of the problems* (2.4) *and* (2.5) *admits a unique solution $v \in L^2(0, T; G(\Omega))$, such that $v_t \in L^2(0, T; L^2(\Omega))$.*

Proof. For the forward problem (2.4), the conclusion follows from the regularity estimates (Theorem 5.1.1, [10]). Meanwhile, with the change of variable $\tau = T - t$ and $\tilde{v}(\tau) := v(t)$, Eq. (2.5) becomes

$$\tilde{v}_\tau - \Delta \tilde{v} = -\tilde{q}, \qquad \text{in} \quad \Omega_T, \tag{2.6a}$$
$$\tilde{v} = 0, \qquad \text{on} \quad \partial\Omega \times [0, T], \tag{2.6b}$$
$$\tilde{v} = 0, \qquad \text{on} \quad \Omega \times \{\tau = 0\}, \tag{2.6c}$$

which is a well-posed forward problem. Therefore, the same argument for (2.4) applies to (2.6), and consequently leads to the desired regularity estimates for the backward problem (2.5).

Now, we show Eq. (1.1) has a well-posed solution.

Theorem 2.1. *Suppose $u_0 \in L^2(\Omega)$ and $f = g\delta_z$, where $g \in L^2(0,T;C(\bar{\Omega}))$. Then, the parabolic problem* (1.1) *admits a unique weak solution $u \in L^2(0,T;L^2(\Omega))$ that satisfies*

$$-(u, v_t)_{\Omega_T} - (u, \Delta v)_{\Omega_T} = \int_\Omega u_0 v(x,0)dx + \int_0^T \int_\Omega f v dx dt, \quad \forall v \in X, \tag{2.7}$$

where $(u,v)_{\Omega_T} := \int_{\Omega_T} uv dx dt$ and $X := \{v,\ v \in L^2(0,T;G(\Omega)) \cap H^1(0,T;L^2(\Omega))$ and $v(T) = 0\}$.

Proof. Case I. We first consider the case when $f \in L^2(0,T;L^2(\Omega))$. Let $V := \{v,\ v \in L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$ and $v(T) = 0\}$. Then, equation (1.1) has a unique solution $u \in L^2(0,T;H_0^1(\Omega))$ (Theorems 2.1 and 4.2, Chapter 3 of [16]) satisfying

$$-(u, \Delta v + v_t)_{\Omega_T} = \int_\Omega u_0 v(x,0)dx + (f,v)_{\Omega_T}, \quad \forall v \in V. \tag{2.8}$$

Since $X \subset V$, this solution $u \in L^2(0,T;H_0^1(\Omega)) \subset L^2(0,T;L^2(\Omega))$ satisfies (2.7) for $v \in X$. Now, we show the uniqueness. Assume there is another solution $u_1 \in L^2(0,T;L^2(\Omega))$ of Eq. (1.1) satisfying (2.7). Consider $w = u - u_1 \in L^2(0,T;L^2(\Omega))$. Let v be the solution of Eq. (2.5) with $q = w$. Then, by Proposition 2.1, $v \in X$; and Eq. (2.7) gives rise to $(w,w)_{\Omega_T} = 0$. Thus, $L^2(0,T;L^2(\Omega)) \ni w = 0$ and therefore the solution $u \in L^2(0,T;L^2(\Omega))$ is unique.

Case II. We consider the case $f = g\delta_z$ with $g \in L^2(0,T;C(\bar{\Omega}))$. Denote by $M(\Omega)$ the dual space of $C(\bar{\Omega})$, such that

$$\|v\|_{M(\Omega)} = \sup_{\|w\|_{C(\bar{\Omega})}=1} \int_\Omega wv dx.$$

Let $\{\delta_n\}$, $\delta_n \in C(\bar{\Omega})$, be a sequence converging weakly to δ_z in $M(\Omega)$ such that

$$\|\delta_n\|_{L^1(\Omega)} \le C\|\delta_z\|_{M(\Omega)}. \tag{2.9}$$

Such sequence $\{\delta_n\}$ can be constructed, for example, using mollifiers as in [6]. Let u_n be the solution of Eq. (1.1) with δ_z replaced by δ_n (namely the right hand side $g\delta_n \in L^2(0,T;L^2(\Omega))$). Therefore, by the standard well-posedness result, $u_n \in L^2(0,T;H_0^1(\Omega)) \subset L^2(0,T;L^2(\Omega))$. For any $q \in L^2(0,T;L^2(\Omega))$, let $v \in X$ be the solution of Eq. (2.5). Then, by the argument in Case I, we have

$$\begin{aligned}(q,u_n)_{\Omega_T} &= (v_t + \Delta v, u_n)_{\Omega_T} = -\int_\Omega u_0 v(x,0)dx - (g\delta_n, v)_{\Omega_T} \\ &\le \|u_0\|_{L^2(\Omega)}\|v(x,0)\|_{L^2(\Omega)} + C\|g\|_{L^2(0,T;L^\infty(\Omega))}\|\delta_n\|_{L^1(\Omega)}\|v\|_{L^2(0,T;L^\infty(\Omega))}.\end{aligned}$$

Note that $v \in X$ implies $v \in C(0,T;L^2(\Omega))$. Thus, by Proposition 2.1,

$$\|v(x,0)\|_{L^2(\Omega)} \le \|v\|_{C(0,T;L^2(\Omega))} \le C\|v\|_{H^1(0,T;L^2(\Omega))} \le C\|q\|_{L^2(0,T;L^2(\Omega))}.$$

Meanwhile, by the fact $G(\Omega) \subset H^{1+\alpha}(\Omega)$ for $\alpha > 1/2$ (Remark 2.1), the Sobolev Embedding Theorem, and Proposition 2.1, we have

$$\|v\|_{L^2(0,T;L^\infty(\Omega))} \le C\|v\|_{L^2(0,T;G(\Omega))} \le C\|q\|_{L^2(0,T;L^2(\Omega))}.$$

Therefore, using (2.9), we deduce

$$(q,u_n)_{\Omega_T} \le C\|q\|_{L^2(0,T;L^2(\Omega))}\Big(\|u_0\|_{L^2(\Omega)} + \|g\|_{L^2(0,T;L^\infty(\Omega))}\|\delta_z\|_{M(\Omega)}\Big).$$

This implies that $\{u_n\}$ is a bounded sequence in $L^2(0,T;L^2(\Omega))$, and therefore contains a subsequence $\{u_{n_k}\}$ that is weakly convergent to a function in $L^2(0,T;L^2(\Omega))$. Denote this function by u. Then, a standard calculation shows that u satisfies Eq. (2.7) with $f = g\delta_z$. The proof is hence completed. □

Then, we have the interior regularity estimate for Eq. (1.1).

Corollary 2.1. *Let $D \subset D_L \subset \Omega$ be two d-dimensional concentric balls with increasing radii, such that $z \notin \bar{D}_L$. For the parabolic problem* (1.1)*, suppose that $u_0 \in H^2(D_L) \cap L^2(\Omega)$. Then, we have $u \in C(0,T;H^1(D))$ and $u_t \in C(0,T;L^2(D))$.*

Proof. First, we let $D \subset D_S \subset D'' \subset D' \subset D_L$ be five d-dimensional concentric balls with increasing radii. Let $\omega_L \in C_0^\infty(D_L)$ be such that $\omega_L = 1$ on D'. Similarly, let $\omega'' \in C_0^\infty(D'')$ be such that $\omega'' = 1$ on D_S.

By Theorem 2.1 and equation (1.1), $\tilde{u} := \omega_L u \in L^2(0,T;L^2(D_L))$ and it satisfies

$$\tilde{u}_t - \Delta\tilde{u} = f_1 = \omega_L f - u\Delta\omega_L - 2\nabla\omega_L \cdot \nabla u, \qquad \text{in} \quad D_L \times (0,T], \tag{2.10a}$$

$$\tilde{u} = 0, \qquad \text{on} \quad \partial D_L \times [0,T], \tag{2.10b}$$

$$\tilde{u} = \omega_L u_0, \qquad \text{on} \quad D_L \times \{t = 0\}. \tag{2.10c}$$

Since $u \in L^2(0,T;L^2(D_L))$, $\omega_L g\delta_z = 0$ ($z \notin \operatorname{supp}(\omega_L)$), and $u_0 \in H^2(D_L)$, we have

$$f_1 = -u\Delta\omega_L - 2\nabla\omega_L \cdot \nabla u \in L^2(0,T;H^{-1}(D_L)) \quad \text{and} \quad \omega_L u_0 \in H_0^1(D_L). \tag{2.11}$$

Therefore, the standard well-posedness result for Eq. (2.10) gives rise to $\tilde{u} \in L^2(0,T;H_0^1(D_L))$, which in turn implies

$$u \in L^2(0,T;H^1(D')). \tag{2.12}$$

Now, we study the regularity of u on the annulus $R := D' \setminus D$. Let $\omega_R \in C_0^\infty(R)$ be such that $\omega_R = 1$ on $D'' \setminus D_S$. Consider Eq. (2.10) with D_L and ω_L being replaced by R and ω_R, respectively. Then, similar to (2.11), using (2.12), we have in this case

$$f_1 = -u\Delta\omega_R - 2\nabla\omega_R \cdot \nabla u \in L^2(0,T;L^2(R)) \quad \text{and} \quad \omega_R u_0 \in H_0^1(R).$$

Then, by Proposition 2.1 (the argument for the forward problem), we obtain

$$\omega_R u \in L^2(0,T;H^2(R)) \quad \text{and} \quad \omega_R u_t \in L^2(0,T;L^2(R)). \tag{2.13}$$

Consider Eq. (2.10) with D_L and ω_L being replaced by D'' and ω'', respectively. Then, similar to (2.11), we have in this case

$$f_1 = -u\Delta\omega'' - 2\nabla\omega'' \cdot \nabla u \quad \text{and} \quad \omega'' u_0 \in H_0^1(D''). \tag{2.14}$$

Since the derivatives of ω'' are zero on D_S, by (2.13) and (2.14), we have $f_1 \in L^2(0,T;H^1(D''))$ and $(f_1)_t \in L^2(0,T;H^{-1}(D''))$. Thus, $f_1 \in C(0,T;L^2(D''))$ (Theorem 3.1, Chapter 1 in [16]) and therefore $f_1(0)$ is well defined in $L^2(D'')$. Note that $\Delta(\omega'' u_0) + f_1(0) = \omega''\Delta u_0 \in L^2(D'')$. Then, $\tilde{u}_t = \omega'' u_t$ is the weak solution of the parabolic problem

$$\tilde{u}_{tt} - \Delta\tilde{u}_t = (f_1)_t, \qquad \text{in} \quad D'' \times (0,T]. \tag{2.15a}$$

$$\tilde{u}_t = 0, \qquad \text{on} \quad \partial D'' \times [0,T], \tag{2.15b}$$

$$\tilde{u}_t = \Delta(\omega'' u_0) + f_1(0), \qquad \text{on} \quad D'' \times \{t = 0\}. \tag{2.15c}$$

Thus, by the standard well-posedness result for Eq. (2.15), we conclude $\tilde{u}_t \in L^2(0,T;H^1(D''))$ and $\tilde{u}_{tt} \in L^2(0,T;H^{-1}(D''))$. This, together with the estimate in (2.12), leads to $u \in C(0,T;H^1(D))$ and $u_t \in C(0,T;L^2(D))$, which completes the proof. □

3. The Finite Element Approximation

In this section, we define the semidiscrete finite element approximation of Eq. (1.1) and derive estimates that will be useful to further carry out the interior error analysis for the numerical solution.

Let $\mathcal{T}_h = \{T_i\}$ be a quasi-uniform triangulation of Ω with shape-regular d-dimensional simplexes T_i, where $h = \max_{T\in\mathcal{T}}(\mathrm{diam}(T))$ is the mesh parameter. Denote by $S_h = S_h(\Omega) \subset H_0^1(\Omega)$ the continuous Lagrange finite element space of degree $m \geq 1$ associated with $\mathcal{T}_h$. Then, the semidiscrete finite element approximation of Eq. (1.1) is to find $u_h(t) = u_h(\cdot,t) \in H^1(0,T;S_h)$, such that

$$(u_{h,t},v) + (\nabla u_h, \nabla v) = (f,v), \quad \forall v \in S_h,\ t>0, \quad \text{and} \quad u_h(0) = u_{0,h}, \tag{3.1}$$

where $u_{h,t} = \partial_t u_h$, $(v,w) = \int_\Omega vw dx$, and $u_{0,h}$ is some approximation of u_0 in S_h.

Remark 3.1. The finite element solution u_h in (3.1) is well defined. When Eq. (1.1) possesses a sufficiently smooth solution and initial condition, it can be shown that for $0 \leq t \leq T$, the finite element approximation globally converges in the optimal rates [28]

$$\|u(t) - u_h(t)\|_{L^2(\Omega)} \leq Ch^{m+1} \quad \text{and} \quad |u(t) - u_h(t)|_{H^1(\Omega)} \leq Ch^m.$$

When the solution of Eq. (1.1) is singular due to the non-smoothness of the domain, the convergence of u_h deteriorates, similar to the behavior of the finite element solution for the associated elliptic problem (2.3) [5]. The singular source term (the δ-function in f) can also give rise to singular solutions in Eq. (1.1). In this case, the results in [25] imply the reduced global approximation rate,

$$\|u(t) - u_h(t)\|_{L^2(\Omega)} \leq Ch^{2-d/2-\epsilon} \qquad \text{for any } \epsilon > 0,$$

where the regularity estimate $u \in C(0,T;L^2(\Omega))$ was implicitly assumed. In this paper, we are interested in the error bounds for $\|u(t) - u_h(t)\|_{L^2}$ and $|u(t) - u_h(t)|_{H^1}$ in an interior subregion D_0 away from z. According to Corollary 2.1, the solution belongs to $C(0,T;L^2(D_0))$ and further to $C(0,T;H^1(D_0))$ with proper initial conditions. Therefore, these norms of the error are well defined for Eq. (1.1).

In addition, we summarize useful interior properties of the finite element space. For any interior domain $D \subset \Omega$, we define $S_h(D)$ to be the restriction of $S_h(\Omega)$ on D and

$$\mathring{S}_h(D) = \{\chi \in S_h(D) : \text{supp } \chi \subseteq \bar{D}\},$$

where $\bar{D}$ is the closure of D.

Then, there exist $k_0 > 0$ and $0 < h_0 < 1$, such that for $h \in (0,h_0]$ and for $D_0 \subset D$ with $\mathrm{dist}(D_0, D) \geq k_0 h$, the following approximating properties hold (see [4, 11, 18–20, 23, 26] and reference therein).

1. (Interpolation Approximation) For any $u \in H^{k+1}(D)$, $0 \le k \le m$, there exists a function $\chi \in S_h(D)$, such that

$$\|u-\chi\|_{0,D} + h\|u-\chi\|_{1,D} \le Ch^{k+1}\|u\|_{k+1,D}. \tag{3.2}$$

2. (Super-approximation) Let $\omega \in C_0^\infty(D_0)$. Then, for any $\chi \in S_h(D)$, there exists $\eta \in \mathring{S}_h(D)$, such that

$$\|\omega\chi - \eta\|_{1,D} \le Ch\|\chi\|_{1,D_0}. \tag{3.3}$$

3. (Inverse Inequality) For any nonnegative integer p and $\chi \in S_h(D)$, we have

$$\|\chi\|_{1,D_0} \le Ch^{-(p+1)}\|\chi\|_{-p,D}. \tag{3.4}$$

Note that the constant C involved in (3.2) – (3.4) depends on the regions of interest D_0 and D. Let $B(x_0, r)$ be the d-dimensional ball with center x_0 and radius r. Using the scaling argument, one can formulate the following estimates on the reference region.

Lemma 3.1. *Let $D := B(x_0, r) \subset \Omega$ be an interior region. The dilation $\hat{x} = (x - x_0)/r$ translates D into a unit size domain $\hat{D}$ and $S_h(D)$ into a new finite dimensional space $S_{h/r}(\hat{D})$. Then, $S_{h/r}(\hat{D})$ satisfies the estimates* (3.2)–(3.4) *with h replaced by h/r, where the constants in the estimates are independent of r.*

Proof. The proof follows from a straightforward calculation. □

We end this section by considering the elliptic problem in an interior subregion $D := B(x_0, r) \subset \Omega$

$$-\Delta\psi = g \quad \text{in} \quad D, \tag{3.5a}$$
$$\psi = 0 \quad \text{on} \quad \partial D. \tag{3.5b}$$

Since D has a smooth boundary, for any $k \ge 0$, the full regularity estimate [1, 2, 6, 7, 16] holds

$$\|\psi\|_{k+2,D} \le C\|g\|_{k,D}, \tag{3.6}$$

where the constant C depends on k and the region involved.

4. The Interior Error Analysis

In this section, we obtain the interior estimates for the semidiscrete finite element approximation (3.1) to the parabolic problem (1.1).

4.1. Some lemmas

To simplify the presentation, we let $D_0 \subset D \subset D_L \subset \Omega$ be three d-dimensional concentric balls $B(x_0, r_0)$, $B(x_0, r)$ and $B(x_0, r_L)$, such that $z \notin \bar{D}_L$. Suppose the initial conditions in equation (1.1) satisfy the conditions in Corollary 2.1. Thus, $u \in C(0, T; H^1(D))$ and $u_t \in C(0, T; L^2(D))$. Therefore, for $0 \le t \le T$, $u(t) \in H^1(D)$. Using integration by parts on Eq. (1.1), we have

$$(u_t, v) + (\nabla u, \nabla v) = (f, v) \quad \forall v \in \mathring{S}_h(D),\ t > 0. \tag{4.1}$$

Thus, by (3.1) and (4.1), we arrive at the error equation

$$(e_t, v) + (\nabla e, \nabla v) = 0, \quad \forall v \in \mathring{S}_h(D),\ t > 0, \tag{4.2}$$

where $e := u - u_h$.

To better present our analysis, instead of (4.2), we consider a more general form of the error equation. Let $\mathcal{E} \in H^1(D)$ satisfy

$$(e_t, v) + (\nabla \mathcal{E}, \nabla v) = 0, \quad \forall v \in \mathring{S}_h(D),\ t > 0. \tag{4.3}$$

Then, we first have the interior estimate on $\mathcal{E}$ in negative norms.

Lemma 4.1. *Recall the concentric balls $D_0 \subset D \subset D_L \subset \Omega$ above. Let $0 \le k \le m-1$ be an integer. Then, for h sufficiently small, there exists a constant C independent of h, such that*

$$\|\mathcal{E}\|_{-k,D_0} \le C\Big(h^{k+1}\|\mathcal{E}\|_{1,D} + \|\mathcal{E}\|_{-k-1,D} + h^{k+1}\|e_t\|_{-1,D} + \|e_t\|_{-k-2,D}\Big). \tag{4.4}$$

Proof. Let D' be a d-dimensional ball with the same center as D_0 and D, such that $D_0 \subset D' \subset D$. For h sufficiently small, we have $\operatorname{dist}(\partial D_0, \partial D')$, $\operatorname{dist}(\partial D', \partial D) > k_0 h$. Let $\omega \in C_0^\infty(D')$ with $\omega = 1$ on D_0. Then for any $k \ge 0$, we have

$$\|\mathcal{E}\|_{-k,D_0} \le \|\omega\mathcal{E}\|_{-k,D} = \sup_{0 \ne g \in C_0^\infty(D)} \frac{(\omega\mathcal{E}, g)_D}{\|g\|_{k,D}} \le C \sup_{\psi} \frac{(\nabla(\omega\mathcal{E}), \nabla\psi)_D}{\|\psi\|_{k+2,D}}, \tag{4.5}$$

where ψ is the solution of Eq. (3.5) and the last inequality is due to the estimate (3.6). For any $\chi \in \mathring{S}_h(D)$, by the error Eq. (4.3), we have

$$\begin{aligned}
&(\nabla(\omega\mathcal{E}), \nabla\psi)_D = (\nabla\mathcal{E}, \nabla(\omega\psi))_D + (\mathcal{E}, \nabla\cdot(\psi\nabla\omega))_D + (\mathcal{E}, \nabla\omega\cdot\nabla\psi)_D \\
=&(\nabla\mathcal{E}, \nabla(\omega\psi - \chi))_D + (\mathcal{E}, \nabla\cdot(\psi\nabla\omega))_D + (\mathcal{E}, \nabla\omega\cdot\nabla\psi)_D - (e_t, \chi)_D \\
=&(\nabla\mathcal{E}, \nabla(\omega\psi - \chi))_D + (\mathcal{E}, \nabla\cdot(\psi\nabla\omega))_D + (\mathcal{E}, \nabla\omega\cdot\nabla\psi)_D - (e_t, \omega\psi)_D - (e_t, \chi - \omega\psi)_D.
\end{aligned}$$

Choosing a suitable χ and using (3.2), we have for $0 \le k \le m-1$,

$$\begin{aligned}
&(\nabla(\omega\mathcal{E}), \nabla\psi)_D \\
\le& C\Big(h^{k+1}\|\mathcal{E}\|_{1,D} + \|\mathcal{E}\|_{-k-1,D} + \|e_t\|_{-k-2,D} + h^{k+1}\|e_t\|_{-1,D}\Big)\|\psi\|_{k+2,D}.
\end{aligned} \tag{4.6}$$

Combining (4.5) and (4.6), we obtain the estimate (4.4). □

In particular, using Lemma 4.1, we obtain the interior L^2 error estimate.

Lemma 4.2. *Recall the concentric balls $D_0 \subset D \subset D_L \subset \Omega$ in Lemma 4.1 and $\mathcal{E}$ from (4.3). Let $0 \le k \le m-1$ be an integer. Then, for h sufficiently small, there exists a constant C independent of h, such that*

$$\|\mathcal{E}\|_{0,D_0} \le C\Big(h\|\mathcal{E}\|_{1,D} + \|\mathcal{E}\|_{-k-1,D} + h\|e_t\|_{-1,D} + \|e_t\|_{-2,D}\Big). \tag{4.7}$$

Proof. Let $D_0 \subset D_1 \subset \cdots \subset D_{k+1} = D$ be $k+2$ d-dimensional concentric balls with increasing radii. Setting $k = 0$ in (4.4), we have

$$\|\mathcal{E}\|_{0,D_0} \le C\Big(h\|\mathcal{E}\|_{1,D_1} + \|\mathcal{E}\|_{-1,D_1} + h\|e_t\|_{-1,D_1} + \|e_t\|_{-2,D_1}\Big). \tag{4.8}$$

We apply (4.4) again to $\|\mathcal{E}\|_{-1,D_1}$. By $h \le 1$ and (2.2), we have

$$\|\mathcal{E}\|_{0,D_0} \le C\bigg(h\|\mathcal{E}\|_{1,D_2} + \|\mathcal{E}\|_{-2,D_2} + h\|e_t\|_{-1,D_2} + \sum_{l=2}^{3}\|e_t\|_{-l,D_2}\bigg). \tag{4.9}$$

Continuing this process up to D_{k+1} and using $\|e_t\|_{-p-2,D} \le C\|e_t\|_{-2,D}$ for any $p \ge 0$, we obtain the desired result (4.7). □

Now, we consider a discrete version of $\mathcal{E}$. Suppose that $\mathcal{E}_h \in S_h(D)$ is such that the equation holds

$$(\nabla\mathcal{E}_h, \nabla v) + (e_t, v) = 0, \quad \forall v \in \mathring{S}_h(D).$$

Namely. $\mathcal{E}_h$ satisfies the error Eq. (4.3). Then, we have the following estimate regarding $\mathcal{E}_h$.

Lemma 4.3. *Recall the concentric balls $D_0 \subset D \subset D_L \subset \Omega$ in Lemma 4.1 and $\mathcal{E}_h \in S_h(D)$ defined above. Let $0 \le k \le m-1$ be an integer. Then, for h sufficiently small, we have*

$$\|\mathcal{E}_h\|_{1,D_0} \le C\bigg(h\|\mathcal{E}_h\|_{1,D} + \|\mathcal{E}_h\|_{-k-1,D} + \|e_t\|_{-1,D}\bigg). \tag{4.10}$$

Proof. Let $D_0 \subset D' \subset D$ be three d-dimensional concentric balls with increasing radii. For h sufficiently small, we have $\operatorname{dist}(\partial D_0, \partial D')$, $\operatorname{dist}(\partial D', \partial D) \ge k_0 h$ with a constant $k_0 > 0$. Define the Ritz projection $P : H^1(D') \to \mathring{S}_h(D)$, such that for any $\varphi \in H^1(D')$,

$$(\nabla(\varphi - P\varphi), \nabla v) = 0, \quad \forall v \in \mathring{S}_h(D).$$

Let $\omega \in C_0^\infty(D')$ with $\omega = 1$ on D_0. Then, we have

$$\|\mathcal{E}_h\|_{1,D_0} \le \|\omega\mathcal{E}_h\|_{1,D'} \le \|\omega\mathcal{E}_h - P(\omega\mathcal{E}_h)\|_{1,D'} + \|P(\omega\mathcal{E}_h)\|_{1,D'}. \tag{4.11}$$

Now, for the first term on the right hand side, by (3.3), we obtain

$$\|\omega\mathcal{E}_h - P(\omega\mathcal{E}_h)\|_{1,D'} \le \inf_{\zeta\in\mathring{S}_h(D)}\|\omega\mathcal{E}_h - \zeta\| \le Ch\|\mathcal{E}_h\|_{1,D}. \tag{4.12}$$

For the second term on the right hand side of (4.11), we have

$$\|P(\omega\mathcal{E}_h)\|_{1,D'} \le \|P(\omega\mathcal{E}_h)\|_{1,D} \le C(\nabla P(\omega\mathcal{E}_h), \frac{\nabla P(\omega\mathcal{E}_h)}{\|P(\omega\mathcal{E}_h)\|_{1,D}})_D = C(\nabla(\omega\mathcal{E}_h), \nabla\phi)_D, \tag{4.13}$$

where $\phi = P(\omega\mathcal{E}_h)/\|P(\omega\mathcal{E}_h)\|_{1,D} \in \mathring{S}_h(D)$ and $\|\phi\|_{1,D} = 1$. Therefore, using integration by parts and (4.3), we have

$$\begin{aligned}(\nabla(\omega\mathcal{E}_h), \nabla\phi)_D &= (\nabla\mathcal{E}_h, \nabla(\omega\phi))_D + (\mathcal{E}_h, \nabla\cdot(\phi\nabla\omega))_D + (\mathcal{E}_h, \nabla\omega\cdot\nabla\phi)_D\\ &= (\nabla\mathcal{E}_h, \nabla(\omega\phi - \chi))_D + (\mathcal{E}_h, \nabla\cdot(\phi\nabla\omega))_D + (\mathcal{E}_h, \nabla\omega\cdot\nabla\phi)_D - (e_t, \chi)_D,\end{aligned} \tag{4.14}$$

where $\chi \in \mathring{S}_h(D)$ is arbitrary. Choose $\chi = P(\omega\phi)$ and therefore $\|\chi\|_{1,D} \le C\|\phi\|_{1,D}$. Then, by (4.14) and (3.3), we obtain

$$(\nabla\omega\mathcal{E}_h, \nabla\phi)_D \le C(h\|\mathcal{E}_h\|_{1,D} + \|e_t\|_{-1,D} + \|\mathcal{E}_h\|_{0,D'})\|\phi\|_{1,D}. \tag{4.15}$$

For the term $\|\mathcal{E}_h\|_{0,D'}$ in (4.15), we can further apply the estimate (4.7) in Lemma 4.2 with $\mathcal{E}$ and D_0 replaced by $\mathcal{E}_h$ and D', respectively. Then, the estimate (4.10) follows from (4.11), (4.12) and (4.15). □

Using Lemma 4.3, we next show that the H^1 norm of $\mathcal{E}_h$ in an interior region is bounded by the negative norms of $\mathcal{E}_h$ and of e_t in a slightly larger interior region.

Lemma 4.4. *With the conditions in Lemma* 4.3*, there exists a constant C independent of h, such that*

$$\|\mathcal{E}_h\|_{1,D_0} \le C(\|\mathcal{E}_h\|_{-k-1,D} + \|e_t\|_{-1,D}). \tag{4.16}$$

Proof. Let $D_0 \subset D_1 \subset \cdots \subset D_{k+3} = D$ be d-dimensional concentric balls with increasing radii. Applying Lemma 4.3 with D_0 and D replaced by D_j, D_{j+1}, $0 \le j \le k+1$, we obtain

$$\|\mathcal{E}_h\|_{1,D_j} \le C\Big(h\|\mathcal{E}_h\|_{1,D_{j+1}} + \|\mathcal{E}_h\|_{-k-1,D_{j+1}} + \|e_t\|_{-1,D_{j+1}}\Big).$$

Starting with $j = 0$ and iterating $k+2$ times, one gets

$$\|\mathcal{E}_h\|_{1,D_0} \le C\Big(h^{k+2}\|\mathcal{E}_h\|_{1,D_{k+2}} + \|\mathcal{E}_h\|_{-k-1,D} + \|e_t\|_{-1,D}\Big). \tag{4.17}$$

By the inverse inequality (3.4), we have

$$h^{k+1}\|\mathcal{E}_h\|_{1,D_{k+2}} \le C\|\mathcal{E}_h\|_{-k-1,D}. \tag{4.18}$$

The inequality (4.16) then follows from (4.17) and (4.18). □

4.2. Interior error estimates

Now, we proceed to derive the estimates on the H^1 and L^2 norms of the finite element approximation error (4.2) in interior regions.

Theorem 4.1. *Let $G_0 \subset G \subset G_L \subset \Omega$ be interior subregions of Ω. Suppose $z \notin \bar{G}_L$ and h sufficiently small. Let $0 \le k \le m$ be an integer. Recall the error $e = u - u_h$ in* (4.2)*. Then, for any $\chi \in S_h(G)$, there exists a constant C independent of h, such that*

$$\|e\|_{1,G_0} \le C\Big(\|u-\chi\|_{1,G} + \|e\|_{-k,G} + \|e_t\|_{-1,G}\Big), \tag{4.19}$$

$$\|e\|_{0,G_0} \le C\Big(h\|u-\chi\|_{1,G} + \|e\|_{-k,G} + h\|e_t\|_{-1,G} + \|e_t\|_{-2,G}\Big). \tag{4.20}$$

Proof. Using a covering argument, it suffices to show the estimates (4.19) and (4.20) for two interior d-dimensional concentric balls $D_0 \subset D \subset \Omega$. In what follows, we let $D_0 \subset D_0' \subset D' \subset D$ be four concentric balls with increasing radii. Let $\omega \in C_0^\infty(D')$ and $\omega = 1$ on D_0'.

Let P be the Ritz projection onto $\mathring{S}_h(D)$ as in Lemma 4.3. Then, we first have

$$\|e\|_{1,D_0} \le \|\omega u - P(\omega u)\|_{1,D_0} + \|P(\omega u) - u_h\|_{1,D_0}. \tag{4.21}$$

Note that for any $v \in \mathring{S}_h(D_0')$, we have

$$(\nabla(P(\omega u) - u_h), \nabla v)_{D_0'} = (\nabla(\omega u - u_h), \nabla v)_{D_0'} = (\nabla(u - u_h), \nabla v)_{D_0'} = (-e_t, v)_{D_0'}.$$

Therefore, $P(\omega u) - u_h$ satisfies the error Eq. (4.3) with D replaced by D_0'. Thus, applying Lemma 4.4 with $\mathcal{E}_h = P(\omega u) - u_h$ and D replaced by D_0', we have

$$\begin{aligned}
\|P(\omega u) - u_h\|_{1,D_0} \le& C\Big(\|P(\omega u) - u_h\|_{-k,D_0'} + \|e_t\|_{-1,D_0'}\Big)\\
\le& C\Big(\|u - u_h\|_{-k,D_0'} + \|\omega u - P(\omega u)\|_{-k,D_0'} + \|e_t\|_{-1,D_0'}\Big)\\
\le& C\Big(\|u - u_h\|_{-k,D} + \|\omega u - P(\omega u)\|_{1,D'} + \|e_t\|_{-1,D}\Big)\\
\le& C\Big(\|u - u_h\|_{-k,D} + \|u\|_{1,D} + \|e_t\|_{-1,D}\Big).
\end{aligned} \tag{4.22}$$

Meanwhile, for the first term of the right hand side in (4.21), we have

$$\|\omega u - P(\omega u)\|_{1,D_0} \le C\|u\|_{1,D}. \tag{4.23}$$

Then, by (4.21), (4.22) and (4.23), we obtain

$$\|e\|_{1,D_0} \le C(\|u\|_{1,D} + \|e\|_{-k,D} + \|e_t\|_{-1,D}).$$

For any $\chi \in S_h(D)$, if we rewrite $u - u_h = (u - \chi) - (u_h - \chi)$, we obtain

$$\|e\|_{1,D_0} \le C(\|u - \chi\|_{1,D} + \|e\|_{-k,D} + \|e_t\|_{-1,D}). \tag{4.24}$$

This proves the estimate (4.19). The inequality (4.20) follows from Lemma 4.2 and from similar calculations as in (4.21)-(4.24) with a suitable modification of the subdomains involved. □

Remark 4.1. In Theorem 4.1, the constant C depends on the regions G_0 and G that are arbitrary but fixed. In practical computations, it is also important to quantify such dependence when $\mathrm{dist}(\partial G_0, \partial G)$ is close to h. This shall give rise to local error estimates for the finite element approximation in interior regions near the singular point z.

Corollary 4.1. *Let $G_0 \subset G \subset G_L \subset \Omega$ be interior subregions of Ω. Suppose $z \notin \bar{G}_L$. For $k_0 > 0$, suppose $k_0 h \le r := dist(\partial G_0, \partial G) \le 1$. Let $0 \le k \le m$ be an integer. Recall the error $e = u - u_h$ in (4.2). Then, there exists a constant C independent of h and r, such that for any $\chi \in S_h(G)$, we have*

$$\|e\|_{1,G_0} \le C\Big(|u - \chi|_{1,G} + r^{-1}\|u - \chi\|_{0,G} + r^{-k-1}\|e\|_{-k,G} + \|e_t\|_{-1,G}\Big), \tag{4.25}$$

$$\|e\|_{0,G_0} \le C\Big(h|u - \chi|_{1,G} + hr^{-1}\|u - \chi\|_{0,G} + r^{-k}\|e\|_{-k,G} + h\|e_t\|_{-1,G} + \|e_t\|_{-2,G}\Big). \tag{4.26}$$

Proof. Let $D_0(x_i) \subset D(x_i) \subset \Omega$ be two d-dimensional concentric balls centered at x_i with radii $r/2$ and r, respectively. Note that $\bar{G}_0$ can be covered by a finite number of balls $D_0(x_i)$ such that $x_i \in \bar{G}_0$, and $\cup_i D(x_i)$ is a subset of G. Thus, it suffices to show the estimates (4.25) and (4.26) for the two balls $D_0(x_i)$ and $D(x_i)$. To simplify the notation, we let $D_0 = D_0(x_i)$ and $D = D(x_i)$ below.

We use a local coordinate system on D, such that x_i is the origin. Then, the same dilation $\hat{x} = (x - x_i)/r$ as in Lemma 3.1 translates D_0 and D to $\hat{D}_0$ and $\hat{D}$ with $\mathrm{dist}(\partial\hat{D}_0, \partial\hat{D}) = 1/2$. Meanwhile, for a function v on D, we define $\hat{v}(\hat{x}) = v(x)$. Therefore, by the scaling argument, we have

$$\|\hat{e}\|_{0,\hat{D}_0} = r^{-d/2}\|e\|_{0,D_0} \quad \text{and} \quad |\hat{e}|_{1,\hat{D}_0} = r^{-d/2+1}|e|_{1,D_0}, \tag{4.27}$$

and the error Eq. (4.2) becomes

$$r^2(\hat{e}_t, \hat{v}) + (\nabla\hat{e}, \nabla\hat{v}) = 0, \tag{4.28}$$

where $\hat{v} \in \mathring{S}_{h/r}(\hat{D})$. Now, using the error equation (4.28) and following the same lines in the proof of Theorem 4.1, we obtain for any $\hat{\chi} \in S_{h/r}(\hat{D})$,

$$\|\hat{e}\|_{1,\hat{D}_0} \le C\Big(\|\hat{u} - \hat{\chi}\|_{1,\hat{D}} + \|\hat{e}\|_{-k,\hat{D}} + r^2\|\hat{e}_t\|_{-1,\hat{D}}\Big), \tag{4.29}$$

$$\|\hat{e}\|_{0,\hat{D}_0} \le C\Big(hr^{-1}\|\hat{u} - \hat{\chi}\|_{1,\hat{D}} + \|\hat{e}\|_{-k,\hat{D}} + hr\|\hat{e}_t\|_{-1,\hat{D}} + r^2\|\hat{e}_t\|_{-2,\hat{D}}\Big). \tag{4.30}$$

In addition, by the definition of the H^{-k} norm and the fact $r \leq 1$, we have for $0 \leq j \leq k$,

$$\|\hat{v}\|_{-j,\hat{D}} \leq r^{-(d/2+j)}\|v\|_{-j,D}. \tag{4.31}$$

Then, by (4.27) and (4.29)-(4.31), we obtain

$$\begin{aligned}
r^{-d/2+1}\|e\|_{1,D_0} \leq C\Big(&r^{-d/2+1}|u-\chi|_{1,D} + r^{-d/2}\|u-\chi\|_{0,D} \\
&+r^{-d/2-k}\|e\|_{-k,D} + r^{2-d/2-1}\|e_t\|_{-1,D}\Big), \\
r^{-d/2}\|e\|_{0,D_0} \leq C\Big(&hr^{-d/2}|u-\chi|_{1,D} + hr^{-d/2-1}\|u-\chi\|_{0,D} \\
&+r^{-d/2-k}\|e\|_{-k,D} + hr^{-d/2}\|e_t\|_{-1,D} + r^{-d/2}\|e_t\|_{-2,D}\Big).
\end{aligned}$$

The proof is thus completed. □

Remark 4.2. According to Theorem 4.1, the H^1 norm of the error in an interior region away from the singular point z is bounded by the combination of the best local approximation error in the finite element space, a negative norm of the interior error, and a negative norm of the time derivative of the error. Note that the term $\|e\|_{-k,G}$ in general is determined by the smoothness of u, u_t, and the adjoint problem (2.3) with $q \in C_0^\infty(\Omega)$ [28]. The first two terms of the upper bound in (4.19) also occur in the interior H^1 norm error estimate for elliptic problems [20]. Meanwhile, $\|e_t\|_{-1,G}$ is an extra term that represents the impact of the time derivative of the error in the approximation of the parabolic problem. It can also be the dominant term in the upper bound. The interior L^2 estimate (4.20) has a similar flavor as the interior H^1 estimate (4.19).

Corollary 4.1 provides the interior error estimates when the boundary distance between the two interior regions is small. An implication of these estimates is that if the interior region of interest G_0 is close to the singular point z ($\text{dist}(\partial G_0, z) = O(h)$), we have $r = O(h)$. In this case, the additional factors (functions of r) in the estimates (4.25) and (4.26) can override the high-order convergence in $\|e\|_{-k,G}$, and consequently make the upper bounds of the local error in G_0 comparable to the upper bounds of the global error.

5. Numerical Experiments

In this section, we provide numerical test results to verify the estimates in Theorem 4.1. To simplify the calculations, we choose in the tests the following parabolic problem with the distributional data defined in a one-dimensional spacial domain $\Omega = (-\pi, \pi)$, such that the singular point $z = 0$ is at the origin:

$$\begin{aligned}
&u_t = u_{xx} + 2\delta_z(x), \quad (x,t) \in (-\pi,\pi) \times (0,T], \\
&u(-\pi,t) = u(\pi,t) = 0, \quad t \in [0,T], \\
&u(x,0) = u_0(x), \quad x \in (-\pi,\pi),
\end{aligned} \tag{5.1}$$

where u_0 is the initial condition that we will specify later. As in the semidiscrete scheme (3.1), we discretize the spacial derivatives by using the continuous finite element methods on a uniform partition with mesh size h on Ω. We choose the approximation $u_{0,h} \in S_h$ of the initial condition as the L^2-projection of u_0 in the finite element space. Without loss of generality, in

each test, we use meshes such that the singular point $z = 0$ is in the interior of an element. Then, we report numerical results from the semidiscrete approximations using piecewise linear ($m = 1$), quadratic ($m = 2$) and cubic ($m = 3$) finite element methods.

5.1. Test I

We consider the parabolic equation (5.1) with the initial condition

$$u_0(x) = \begin{cases} \pi + x + \sin(x), & x \in (-\pi, 0), \\ \pi - x + \sin(x), & x \in [0, \pi). \end{cases} \tag{5.2}$$

Therefore, the exact solution is

$$u(x,t) = \begin{cases} \pi + x + e^{-t}\sin(x), & x \in (-\pi, 0), \\ \pi - x + e^{-t}\sin(x), & x \in [0, \pi). \end{cases}$$

For any $t \geq 0$, we see that

$$u(t) \in H^{3/2-\epsilon}(\Omega) \quad \text{and} \quad u_t \in C^\infty(\Omega), \tag{5.3}$$

where $\epsilon > 0$ is arbitrarily small. Then, using the semidiscrete scheme discussed above, we expect to see the following convergence rates in the global L^2 and H^1 norms [25, 28]

$$\|u(t) - u_h(t)\|_{L^2(\Omega)} \leq Ch^{1.5-\epsilon} \quad \text{and} \quad \|u(t) - u_h(t)\|_{H^1(\Omega)} \leq Ch^{0.5-\epsilon}. \tag{5.4}$$

Meanwhile, for an interior region $G_0 = (-1, -0.5) \cup (0.5, 1) \subset \Omega$ that is away from the singular point $z = 0$, note that $u(t)$ is smooth in G_0. Since the adjoint elliptic problem (2.3) with $q \in C_0^\infty(\Omega)$ is smooth, in view of (5.3), according to the negative norm estimates in [28], Theorem 4.1 shall give rise to

$$\|e\|_{1,G_0} \leq Ch^m \quad \text{and} \quad \|e\|_{0,G_0} \leq Ch^{m+1}, \tag{5.5}$$

where m is the degree of the polynomial in the finite element method.

In Table 5.1, we display the numerical results for the semidiscrete approximations of equation (5.1) with the initial condition (5.2). Here, we denote by M the number of sub-intervals in the partition. We see in the table that the convergence in the global L^2 (resp. H^1) norm is like $h^{1.5}$ (resp. $h^{0.5}$) for both $m = 1, 2, 3$. The local error in G_0 is optimal in terms of the degree of polynomials used in the tests. It is clear that these numerical results are consistent with our theoretical predictions in (5.4) and in (5.5), and therefore verify the theory.

Table 5.1: Test I: the convergence of the semidiscrete scheme using $m = 1, 2, 3$ piecewise polynomials at time $t = 0.1$.

		Global Domain Ω				Interior Domain G_0			
	M	$\|e\|_{0,\Omega}$	order	$\|e\|_{1,\Omega}$	order	$\|e\|_{0,G_0}$	order	$\|e\|_{1,G_0}$	order
$m = 1$	51	8.01E-03	–	3.73E-01	–	3.73E-04	–	7.70E-02	–
	101	2.87E-03	1.48	2.60E-01	0.52	8.77E-05	2.08	3.90E-02	0.98
	201	1.02E-03	1.49	1.83E-01	0.51	2.15E-05	2.02	1.96E-02	0.94
$m = 2$	51	5.01E-03	–	1.65E-01	–	1.45E-04	–	3.77E-03	–
	101	1.76E-03	1.51	1.17E-01	0.49	1.65E-05	3.15	9.59E-04	1.97
	201	6.28E-04	1.49	8.29E-02	0.50	2.05E-06	3.01	2.42E-04	1.99
$m = 3$	51	2.03E-03	–	1.03E-01	–	8.79E-05	–	1.24E-03	–
	101	7.19E-04	1.49	7.34E-02	0.49	5.51E-06	3.99	1.53E-04	3.02
	201	2.56E-04	1.49	5.12E-02	0.52	3.46E-07	3.99	1.89E-05	3.01

5.2. Test II

In this test, we demonstrate the impact of the lack of regularity in u_t on the interior estimates in (4.19) and (4.20). Consider the parabolic Eq. (5.1) with the initial condition.

$$u_0(x) = 0, \qquad x \in (-\pi, \pi). \tag{5.6}$$

Then, based on the standard regularity estimate, we have for almost everywhere $t \geq 0$,

$$u(t) \in H^{3/2-\epsilon}(\Omega), \tag{5.7}$$

where $\epsilon > 0$ is arbitrarily small. In addition, using a local argument similar to the one in Corollary 2.1, we can see that $u(t)$ is smooth in the interior region $G_0 = (-1, -0.5) \cup (0.5, 1) \subset \Omega$. Therefore, the first term $\|u - \chi\|_{1,G}$ in (4.19) and (4.20) is of optimal rate. Meanwhile, the function $\tilde{u} = u_t$ satisfies the following parabolic equation with non-smooth initial data

$$\begin{aligned} &\tilde{u}_t = \tilde{u}_{xx} && (x,t) \in (-\pi, \pi) \times (0, T], \\ &\tilde{u}(-\pi, t) = \tilde{u}(\pi, t) = 0, && t \in [0, T], \\ &\tilde{u}(x, 0) = 2\delta_z(x), && x \in (-\pi, \pi). \end{aligned} \tag{5.8}$$

Then, the error estimates in [28] for non-smooth initial data imply that when t is relatively large in comparison to the mesh size h, due to the smoothing property of the solution operator of the parabolic problem, the terms in Theorem 4.1 that involve negative norms of e and e_t can also be bounded by the best local approximation error. Consequently, the local convergence of the semidiscrete scheme in G_0 should resemble the optimal local convergence in Test I. However, when t is small, the non-smooth initial data in u_t can disturb the convergence in $\|e\|_{-k,G}$, $\|e_t\|_{-1,G}$, and $\|e_t\|_{-2,G}$ [28]. Thus, if t is small, we expect to see poor convergence for the interior errors $\|e\|_{1,G_0}$ and $\|e\|_{0,G_0}$, even though u is smooth in G_0. In contrast, for the equation in Test I, u_t is always smooth, and therefore the interior estimates (5.5) were seen for any $t > 0$ numerically.

We report numerical results solving (5.1) with the initial condition (5.6) in Tables 5.2 – 5.3. In these tests, since the exact solution is unknown, we use the numerical approximations with sufficiently refined meshes ($M = 1601$) as the reference solution to compute the errors. In Table 5.2, we list the global and interior convergence rates at $t = 1$. These results are similar to those in Table 5.1, which are aligned with our theoretical prediction. Namely, at $t = 1$, with the smoothing property of the solution operator, the global convergence rates are determined by the regularity of the solution in (5.7); and the convergence rates in G_0 are determined by the best approximation error. In Table 5.3, we compare Test I and Test II for the convergence results in the interior region G_0 when t is small ($t = 10^{-4}$). It is clear that the interior convergence rates in Test I are optimal when t is small, while no convergence is seen in Test II. This confirms our discussion above: due to the non-smooth initial data in (5.8), when t is small, the negative norms of e and e_t in our estimates (4.19) and (4.20) can be the dominant terms in Test II; while the interior convergence in Test I is optimal for any $t > 0$ due to the regularity estimates for u and u_t in (5.3).

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Table 5.2: Test II: the convergence of the semidiscrete scheme using $m = 1, 2, 3$ piecewise polynomials at time $t = 1$.

		Global Domain Ω				Interior Domain G_0			
	M	$\|e\|_{0,\Omega}$	order	$\|e\|_{1,\Omega}$	order	$\|e\|_{0,G_0}$	order	$\|e\|_{1,G_0}$	order
$m = 1$	51	7.13E-03	–	6.67E-01	–	1.24E-04	–	7.26E-03	–
	101	2.56E-03	1.48	4.74E-01	0.49	2.57E-05	2.27	3.68E-03	0.98
	201	9.12E-04	1.49	3.37E-01	0.49	5.28E-06	2.10	1.85E-03	0.99
$m = 2$	51	2.68E-03	–	2.58E-01	–	8.15E-05	–	2.14E-04	–
	101	9.65E-04	1.47	1.85E-01	0.48	9.91E-06	3.04	5.23E-05	2.03
	201	3.46E-04	1.48	1.32E-01	0.49	1.26E-06	2.98	1.28E-05	2.01
$m = 3$	51	1.22E-03	–	3.61E-01	–	4.83E-05	–	7.15E-05	–
	101	4.16E-04	1.55	2.55E-01	0.50	3.36E-06	4.01	8.58E-06	3.05
	201	1.37E-04	1.60	1.80E-01	0.50	2.21E-07	3.93	1.08E-06	2.99

Table 5.3: The comparison between Test I and Test II: the convergence in the interior domain G_0 using linear ($m = 1$) and cubic ($m = 3$) piecewise polynomials $t = 10^{-4}$.

		$m = 1$				$m = 3$			
	M	$\|e\|_{0,G_0}$	order	$\|e\|_{1,G_0}$	order	$\|e\|_{0,G_0}$	order	$\|e\|_{1,G_0}$	order
Test I	51	1.44E-03	–	7.63E-02	–	1.19E-04	–	1.34E-03	–
	101	2.77E-04	2.38	3.86E-02	0.98	7.54E-06	3.98	1.69E-04	2.98
	201	5.82E-05	2.03	1.94E-02	0.99	4.73E-07	3.99	2.08E-05	3.02
Test II	51	9.61E-05	–	3.79E-03	–	1.35E-04	–	1.38E-02	–
	101	8.57E-05	0.15	1.01E-02	-1.41	4.18E-05	1.69	7.59E-03	0.86
	201	5.45E-04	-2.67	1.95E-02	-0.95	6.03E-05	-0.53	3.57E-02	-2.23

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