AN ELEMENTARY PROOF OF EIGENVALUE PRESERVATION FOR THE CO-ROTATIONAL BERIS-EDWARDS SYSTEM

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ABSTRACT. We study the co-rotational Beris-Edwards system modeling nematic liquid crystals and revisit the eigenvalue preservation property discussed in [24]. We give an alternative but direct proof to the eigenvalue preservation of the initial data for the Q-tensor. It is noted that our proof is not only valid in the whole space case, but in the bounded domain case as well.

1. INTRODUCTION

In this paper we study the eigenvalue preservation property of solutions for a hydrodynamic system modeling the evolution of nematic liquid crystals. Mathematically speaking, this system is composed of a coupled incompressible Navier-Stokes equations with anisotropic forces and Q-tensor equations of a parabolic type that describes the evolution of the liquid crystal director field, which is called the Beris-Edwards system [4]. In the Landau-de Gennes theory [3, 10], the basic element is a symmetric, traceless tensor Q that is a tensor valued function taking values in the 5-dimensional Q-tensor space

$$\mathcal{S}_0^{(3)} \stackrel{\text{def}}{=} \big\{Q \in \mathbb{M}^{3\times 3},\, Q^t = Q,\, \mathrm{tr}(Q) = 0\big\}.$$

The simplest form of the free energy in the Landau-de Gennes theory takes the following form:

$$\mathcal{F}(Q) \stackrel{\text{def}}{=} \int_\Omega \frac{L}{2}|\nabla Q|^2 + \frac{a}{2}\,\mathrm{tr}(Q^2) - \frac{b}{3}\mathrm{tr}(Q^3) + \frac{c}{4}\,\mathrm{tr}^2(Q^2)\,dx, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth and bounded domain. Above in (1.1) we use the one constant approximation of the Oseen-Frank energy, and L, a, b, c are material dependent constants that satisfy [16, 17]

$$L > 0,\ b > 0,\ c > 0. \tag{1.2}$$

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The simplified Beris-Edwards system we study reads

(1.3)

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla P = \lambda L\nabla\cdot(Q\Delta Q - \Delta Q Q) - \lambda L\nabla\cdot(\nabla Q\odot\nabla Q),$$

(1.4)

$$\nabla\cdot\mathbf{u} = 0,$$

(1.5)

$$Q_t + \mathbf{u}\cdot\nabla Q - \omega Q + Q\omega = \Gamma\Big(L\Delta Q - aQ + b\Big[Q^2 - \frac{\mathrm{tr}(Q^2)}{3}\mathbb{I}\Big] - cQ\,\mathrm{tr}(Q^2)\Big),$$

with the following initial and boundary conditions

$$\begin{aligned}&\mathbf{u}(0,x) = \mathbf{u}_0(x) \text{ with } \nabla\cdot\mathbf{u}_0 = 0, \quad Q(0,x) = Q_0(x)\in\mathcal{S}_0^{(3)},\\ &\mathbf{u}(t,x)|_{\partial\Omega} = 0, \quad Q(t,x)|_{\partial\Omega} = Q_0(x)|_{\partial\Omega} = \tilde{Q}(x).\end{aligned} \tag{1.6}$$

Above $\mathbf{u}(t,x):(0,+\infty)\times\Omega\to\mathbb{R}^3$ stands for the incompressible fluid velocity field, $Q(t,x):(0,+\infty)\times\Omega\to\mathcal{S}_0^{(3)}$ represents the order parameter of the liquid crystal molecules and $\omega = \dfrac{\nabla\mathbf{u} - \nabla^T\mathbf{u}}{2}$ denotes the skew-symmetric part of the rate of strain tensor. The positive constants ν, λ and Γ denote the fluid viscosity, the competition between kinetic energy and elastic potential energy, and macroscopic elastic relaxation time for the molecular orientation field, respectively [24]. This simplified system is at time referred to as the "co-rotational" Beris-Edwards system [22] in the literature, whose related mathematical study can be found in [2, 6, 7, 11, 12, 22]. On the other hand, the full system is also called the "non co-rotational" Beris-Edwards system, and we refer interested readers to [1, 5, 8, 21, 25, 26] for its relevant PDE and numeric work.

From the physical point of view, the main feature of nematic liquid crystals is the locally preferred orientation of the nematic molecule directors. To this end Q-tensors are introduced, which are considered suitably normalized second order moments of the probability distribution function. Specifically, if μ_x is a probability measure on the unit sphere $\mathbb{S}^2$ representing the orientation of liquid crystal molecules at a point x in space, then a Q-tensor denoted by $Q(x)$ is a symmetric and traceless 3×3 matrix defined by

$$Q(x) = \int_{\mathbb{S}^2}\left(\mathbf{p}\otimes\mathbf{p} - \frac{1}{3}\mathbb{I}\right)d\mu_x(\mathbf{p}). \tag{1.7}$$

Indeed it is a crude measure (from the viewpoint of statistical theory) of how the second-moment tensor associated with a given probability measure deviates from its isotropic value [18, 23]. It is noted that (1.7) imposes a constraint such that (see [18])

$$-\frac{1}{3}\le\lambda_i(Q)\le\frac{2}{3},\quad\forall\, 1\le i\le 3.$$

Hence it is easy to check that not every symmetric and traceless 3×3 matrix is a *physical* Q-tensor but only those whose eigenvalues range in $[-\frac{1}{3},\frac{2}{3}]$.

Motivated by the physical interpretation of the Q-tensors, it seems to be of great importance to understand how the fluid dynamics would affect the behavior of eigenvalues of the Q-tensors as time evolves. Partially motivated by this question, in [24], the authors proved that certain eigenvalue constraints of the initial data Q_0 are preserved by the evolution problem (1.3)-(1.6) when the domain is either the entire Euclidean space or a periodic box. Inspired by the idea in [24], in this paper we give an alternative but direct proof whose argument works well both in the whole space case and in the bounded domain case.

Our main result is stated as follows.

Theorem 1.1. *For any $\mathbf{u}_0 \in H_0^1(\Omega)$, $\nabla \cdot \mathbf{u}_0 = 0$, $Q_0 \in H^2(\Omega; \mathcal{S}_0^{(3)})$ and $\tilde{Q} \in H^{\frac{5}{2}}(\partial\Omega)$, let $(\mathbf{u}(t,x), Q(t,x))$ be the unique local strong solution to the evolution problem* (1.3)-(1.6) *on $[0,T]$. We assume*

$$0 \le a \le \frac{b^2}{24c}, \tag{1.8}$$

and the initial data Q_0 and the boundary data $\tilde{Q}$ satisfy

(1.9)

$$\lambda_i(Q_0(x)) \in \Big[-\frac{b+\sqrt{b^2-24ac}}{12c}, \frac{b+\sqrt{b^2-24ac}}{6c} \Big], \qquad \forall x \in \Omega,\ 1 \le i \le 3.$$

Then for any $t \in (0,T]$ and $x \in \Omega$, the eigenvalues of $Q(t,x)$ stay in the same interval.

Remark 1.1. *By Theorem 1.1 in [15], the existence and uniqueness of local strong solutions to the evolution problem* (1.3)-(1.6) *is ensured, and satisfies*

$$\mathbf{u} \in H^1(0,T; H^1(\Omega)) \cap L^\infty(0,T; H^2(\Omega)), \quad \nabla \cdot \mathbf{u} = 0,$$

$$Q \in H^2\big(0,T; L^2(\Omega; \mathcal{S}_0^{(3)})\big) \cap H^1\big(0,T; H^2(\Omega; \mathcal{S}_0^{(3)})\big) \cap L^\infty\big(0,T; H^3(\Omega; \mathcal{S}_0^{(3)})\big).$$

Remark 1.2. *Compared to [24], one extra assumption in Theorem 1.1 is $a \ge 0$ which captures a regime of physical interest but not the deep nematic regime [9]. This assumption is only used to get the same lower bound $-\frac{b+\sqrt{b^2-24ac}}{12c}$, but not needed to achieve the upper bound $\frac{b+\sqrt{b^2-24ac}}{6c}$. We also want to point out that this assumption* (1.8) *is different from its counterpart imposed in [24] because the bulk part are dealt with in different ways.*

The idea of the proof is to proceed by contradiction and to exploit the variational characterization of the eigenvalues in relation to the evolution problem (1.3)– (1.6), which works for the solution Q with $C^{1,2}$ regularity. If we were able to show the solutions to the Beris-Edwards system were regular enough we would be done, however this seems out of reach at the moment, though an interesting problem on its own. Fortunately, we can bypass this difficulty by using a regularization argument discussed in [24] that preserves the eigenvalue constraints (the eigenvalues converge pointwise in fact in the whole domain).

For simplicity we set the eigenvalues of matrix Q

$$\lambda_i(t,x) \overset{\text{def}}{=} \lambda_i(Q(t,x)), \quad 1 \le i \le 3.$$

Without loss of generality we assume

$$\lambda_1(t,x) \ge \lambda_2(t,x) \ge \lambda_3(t,x), \quad \forall (t,x) \in \bar{\Omega} \times [0,T]$$

As a matter of fact, we may establish the following more general result based on Theorem 1.1.

Corollary 1.1. *For any given $\mathbf{u}_0 \in H_0^1(\Omega)$, $\nabla \cdot \mathbf{u}_0 = 0$, $Q_0 \in H^2(\Omega; \mathcal{S}_0^{(3)})$ and $\tilde{Q} \in H^{\frac{5}{2}}(\partial\Omega)$, the unique local strong solution $(\mathbf{u}(t,x), Q(t,x))$ to the evolution problem (1.3)-(1.6) on $[0,T]$ satisfies*

$$\lambda_1(t,x) \le \max\Big[\frac{b+\sqrt{b^2-24ac}}{6c}, \max_{\bar{\Omega}} \lambda_1(Q_0)\Big], \tag{1.10}$$

$$\lambda_3(t,x) \ge \min\Big[-\frac{b+\sqrt{b^2-24ac}}{12c}, \min_{\bar{\Omega}} \lambda_3(Q_0)\Big], \tag{1.11}$$

for any $t \in (0,T]$ and $x \in \Omega$.

Remark 1.3. *Analogously, Theorem 1.1 and Corollary 1.1 also valid in the static case, provided the corresponding solution $Q \in C(\bar{\Omega}) \cap C^2(\Omega)$. However, this regularity issue cannot be solved directly by following the approximation argument in the appendix part, and henceforth is beyond the scope of our paper.*

The proof of Theorem 1.1 and Corollary 1.1 is given in Section 2, while a related technical regularization lemma is presented in the appendix.

2. Proof of Theorem 1.1

Here and after, we let $|Q|$ denote the Frobenius norm of $Q \in \mathcal{S}_0^{(3)}$, that is, $|Q| = \sqrt{tr(Q^tQ)}$ where tr denotes the trace of a matrix. Also, because of the traceless property of Q-tensors, for all $(t,x) \in \bar{\Omega} \times [0,T]$ one has

$$\lambda_1(t,x) + \lambda_2(t,x) + \lambda_3(t,x) = 0. \tag{2.1}$$

To begin with, we see from [19, 20] that

Lemma 2.1. *For $1 \le i \le 3$, $\lambda_i(t,x) \in C(\bar{\Omega} \times [0,T])$*

Now we are ready to prove Theorem 1.1.

Proof. Due to Lemma 3.1 in the Appendix, we may assume

$$Q(t,x) \in C^{1,2}((0,T) \times \Omega) \cap C([0,T] \times \bar{\Omega}).$$

Step 1. Let

$$\lambda_1(t_0,x_0) = \max_{(t,x) \in [0,T] \times \bar{\Omega}} \lambda_1(t,x). \tag{2.2}$$

We shall show that $\lambda_1(t_0,x_0) \le \dfrac{b+\sqrt{b^2-24ac}}{6c}$. We prove by a contradiction argument. Suppose

$$(t_0,x_0) \in (0,T]\times\Omega, \text{ and } \lambda_1(t_0,x_0) > \frac{b+\sqrt{b^2-24ac}}{6c}. \tag{2.3}$$

Let $\vec{v} \in \mathbb{S}^2$ be the corresponding unit eigenvector, such that $Q(t_0,x_0)\vec{v} = \lambda_1(t_0,x_0)\vec{v}$. Meanwhile, we denote

$$f(t,x) = \langle Q(t,x)\vec{v},\vec{v}\rangle_{\mathbb{R}^3},$$

then it is easy to check from (1.9) and (2.2) that

$$f(t_0,x_0) = \max_{(t,x)\in[0,T]\times\bar{\Omega}} f(t,x) \tag{2.4}$$

Next, we take the matrix inner product of equation (1.5) with $\vec{v}\vec{v}^t$ and evaluate the resultant at (t_0,x_0). Note that $\mathbf{u}\cdot\nabla f = 0$

$$\omega^{ik}Q^{kj}\vec{v}^i\vec{v}^j = \omega^{ik}\big(Q^{kj}\vec{v}^j\big)\vec{v}^i = \lambda_1\omega^{ik}\vec{v}^k\vec{v}^i = 0,$$

$$Q^{ik}\omega^{kj}\vec{v}^i\vec{v}^j = \big(Q^{ik}\vec{v}^i\big)\omega^{kj}\vec{v}^j = \lambda_1\omega^{kj}\vec{v}^k\vec{v}^j = 0$$

hence we get

$$\begin{aligned}\partial_t f &= \Delta f - a\lambda_1 - c\,\mathrm{tr}(Q^2)\lambda_1 + b\Big[\lambda_1^2 - \frac{\mathrm{tr}(Q^2)}{3}\Big]\\ &= \Delta f - \lambda_1\big[a + c(\lambda_1^2+\lambda_2^2+\lambda_3^2)\big] + b\Big(\lambda_1^2 - \frac{\lambda_1^2+\lambda_2^2+\lambda_3^2}{3}\Big) \quad \text{at } (t_0,x_0).\end{aligned} \tag{2.5}$$

Using Cauchy-Schwarz inequality and $Q \in \mathcal{S}_0^{(3)}$, we get

$$\lambda_1^2+\lambda_2^2+\lambda_3^2 \ge \lambda_1^2 + \frac{(\lambda_2+\lambda_3)^2}{2} = \frac{3}{2}\lambda_1^2,$$

which combined with (2.5) at (t_0,x_0) gives

$$\begin{aligned}\partial_t f|_{(t_0,x_0)} &\le \Delta f - a\lambda_1 - \frac{3c}{2}\lambda_1^3 + \frac{b}{2}\lambda_1^2\Big|_{(t_0,x_0)} \le -\frac{3c}{2}\lambda_1\Big(\lambda_1^2 - \frac{b}{3c}\lambda_1 + \frac{2a}{3c}\Big)\Big|_{(t_0,x_0)}\\ &< 0.\end{aligned}$$

Above in the last inequality we used (2.3). However, (2.4) indicates that

$$\partial_t f|_{(t_0,x_0)} \ge 0,$$

which is a contradiction. Therefore, we conclude that

$$\lambda_1(t,x) \le \frac{b+\sqrt{b^2-24ac}}{6c}, \quad \forall (t,x)\in(0,T]\times\Omega. \tag{2.6}$$

Step 2. Let

$$\lambda_3(\tilde{t},\tilde{x}) = \min_{(t,x)\in[0,T]\times\bar{\Omega}} \lambda_3(t,x). \tag{2.7}$$

We shall again show that $\lambda_3(\tilde{t},\tilde{x}) \geq \dfrac{-b-\sqrt{b^2-24ac}}{12c}$, by contradiction. Suppose

$$(\tilde{t},\tilde{x}) \in (0,T]\times\Omega, \text{ and } \lambda_3(\tilde{t},\tilde{x}) < \frac{-b-\sqrt{b^2-24ac}}{12c}. \tag{2.8}$$

Let $\vec{w} \in \mathbb{S}^2$ be the corresponding unit eigenvector, such that $Q(\tilde{t},\tilde{x})\vec{w} = \lambda_3(\tilde{t},\tilde{x})\vec{w}$. Meanwhile, we denote

$$g(t,x) = \langle Q(t,x)\vec{w},\vec{w}\rangle_{\mathbb{R}^3},$$

then we see from (1.9) and (2.7) that

$$g(\tilde{t},\tilde{x}) = \min_{(t,x)\in[0,T]\times\bar{\Omega}} g(t,x) \tag{2.9}$$

After taking the matrix inner product of equation (1.5) with $\vec{w}\vec{w}^t$, and evaluating at $(\tilde{t},\tilde{x})$, it gives

$$\begin{aligned}
\mathbf{u}\cdot\nabla g &= 0,\\
\omega^{ik}Q^{kj}\vec{w}^i\vec{w}^j = \omega^{ik}\big(Q^{kj}\vec{w}^j\big)\vec{w}^i &= \lambda_1\omega^{ik}\vec{w}^k\vec{w}^j = 0,\\
Q^{ik}\omega^{kj}\vec{w}^i\vec{w}^j = \big(Q^{ik}\vec{w}^i\big)\omega^{kj}\vec{w}^j &= \lambda_1\omega^{kj}\vec{w}^k\vec{w}^j = 0.
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
\partial_t g &= \Delta g - a\lambda_3 - c\,\mathrm{tr}(Q^2)\lambda_3 + b\Big[\lambda_3^2 - \frac{\mathrm{tr}(Q^2)}{3}\Big]\\
&= \Delta g - \lambda_3\big[a + c(\lambda_1^2+\lambda_2^2+\lambda_3^2)\big] + b\Big(\lambda_3^2 - \frac{\lambda_1^2+\lambda_2^2+\lambda_3^2}{3}\Big) \quad \text{at } (\tilde{t},\tilde{x}).
\end{aligned} \tag{2.10}$$

We claim

$$\partial_t g\big|_{(\tilde{t},\tilde{x})} \geq -\frac{3c}{2}\lambda_3\Big(\lambda_3^2 + \frac{b}{6c}\lambda_3 + \frac{a}{6c}\Big)\Big|_{(\tilde{t},\tilde{x})}. \tag{2.11}$$

Here, we focus on the proof of the theorem and the proof of the claim will be postponed to the end of the section. Combining (2.8), and the claim (2.11), we get

$$\partial_t g|_{(\tilde{t},\tilde{x})} > 0.$$

On the other hand, however, based on (2.9) one can deduce that

$$\partial_t g|_{(\tilde{t},\tilde{x})} \leq 0,$$

which is again a contradiction. Thus

$$\lambda_3(t,x) \geq -\frac{b+\sqrt{b^2-24ac}}{12c}, \quad \forall (t,x)\in(0,T]\times\Omega. \tag{2.12}$$

The proof is complete by combining (2.6) and (2.12). □

It remains to prove the claim (2.11).

Proof. We divide the proof of (2.11) into three cases. By (2.1) and (2.7), we know that $|\lambda_2| \le |\lambda_3|$.

Case 1: $\lambda_2|_{(\tilde{t},\tilde{x})} \ge 0$

First of all, note that

$$\frac{3}{2}\lambda_3^2 = \frac{(\lambda_1+\lambda_2)^2}{2} + \lambda_3^2 \le \lambda_1^2+\lambda_2^2+\lambda_3^2 \le (\lambda_1+\lambda_2)^2+\lambda_3^2 = 2\lambda_3^2,$$

which together with (2.10) and the fact that g attains minimum at $(\tilde{t},\tilde{x})$ yields

$$\begin{aligned}\partial_t g|_{(\tilde{t},\tilde{x})} \ge \Delta g - a\lambda_3 - \frac{3c}{2}\lambda_3^3 + \frac{b}{3}\lambda_3^2\Big|_{(\tilde{t},\tilde{x})} &\ge -a\lambda_3 - \frac{3c}{2}\lambda_3^3 + \frac{b}{3}\lambda_3^2\Big|_{(\tilde{t},\tilde{x})} \\ &\ge -\frac{3c}{2}\lambda_3\Big(\lambda_3^2 + \frac{b}{6c}\lambda_3 + \frac{a}{6c}\Big)\Big|_{(\tilde{t},\tilde{x})}\end{aligned}$$

Case 2: $\dfrac{(\sqrt{5}-1)}{2}\lambda_3\big|_{(\tilde{t},\tilde{x})} \le \lambda_2|_{(\tilde{t},\tilde{x})} < 0$

In this case, again thanks to (2.1) $|\lambda_1| > |\lambda_3|$ at $(\tilde{t},\tilde{x})$ we have

$$\begin{aligned}2\lambda_3^2 < \lambda_1^2+\lambda_2^2+\lambda_3^2 &= (\lambda_2+\lambda_3)^2+\lambda_2^2+\lambda_3^2 = 2(\lambda_2^2+\lambda_3^2+\lambda_2\lambda_3) \\ &= 2(\lambda_2^2+\lambda_2\lambda_3-\lambda_3^2)+4\lambda_3^2 \le 4\lambda_3^2.\end{aligned}$$

Hence

$$\begin{aligned}\partial_t g|_{(\tilde{t},\tilde{x})} \ge \Delta g - a\lambda_3 - 2c\lambda_3^3 - \frac{b}{3}\lambda_3^2\Big|_{(\tilde{t},\tilde{x})} &= -2c\lambda_3\Big(\lambda_3^2 + \frac{b}{6c}\lambda_3 + \frac{a}{2c}\Big)\Big|_{(\tilde{t},\tilde{x})} \\ &\ge -\frac{3c}{2}\lambda_3\Big(\lambda_3^2 + \frac{b}{6c}\lambda_3 + \frac{a}{6c}\Big)\Big|_{(\tilde{t},\tilde{x})}\end{aligned}$$

Case 3: $\lambda_3|_{(\tilde{t},\tilde{x})} \le \lambda_2|_{(\tilde{t},\tilde{x})} < \dfrac{(\sqrt{5}-1)}{2}\lambda_3\big|_{(\tilde{t},\tilde{x})} < 0$

Note that

$$\begin{aligned}4\lambda_3^2 &< 2(\lambda_2^2+\lambda_2\lambda_3-\lambda_3^2)+4\lambda_3^2 = 2(\lambda_2^2+\lambda_3^2+\lambda_2\lambda_3) \\ &= (\lambda_2+\lambda_3)^2+\lambda_2^2+\lambda_3^2 = \lambda_1^2+\lambda_2^2+\lambda_3^2 \quad \text{at } (\tilde{t},\tilde{x}),\end{aligned}$$

which gives

$$\begin{aligned}&-\lambda_3\big[a+c(\lambda_1^2+\lambda_2^2+\lambda_3^2)\big] + b\Big(\lambda_3^2 - \frac{\lambda_1^2+\lambda_2^2+\lambda_3^2}{3}\Big) \\ &= -2c\lambda_3\left[(\lambda_2^2+\lambda_3^2+\lambda_2\lambda_3) + \frac{b}{6c}\frac{(2\lambda_2^2+2\lambda_2\lambda_3-\lambda_3^2)}{\lambda_3} + \frac{a}{2c}\right] \\ &\overset{\text{def}}{=} -2c\lambda_3 H(\lambda_2,\lambda_3) \quad \text{at } (\tilde{t},\tilde{x}).\end{aligned}$$

We proceed to show that

$$H(\lambda_2,\lambda_3) \ge \lambda_3^2 + \frac{b}{6c}\lambda_3 + \frac{a}{6c} \quad \text{at } (\tilde{t},\tilde{x}), \tag{2.13}$$

which is equivalent to

$$\lambda_2^2+\lambda_2\lambda_3+\frac{b}{3c}\frac{(\lambda_2^2+\lambda_2\lambda_3-\lambda_3^2)}{\lambda_3}+\frac{a}{3c}\geq 0 \quad \text{at } (\tilde{t},\tilde{x}). \tag{2.14}$$

Let us denote $\mu=\lambda_2^2+\lambda_2\lambda_3-\lambda_3^2\big|_{(\tilde{t},\tilde{x})}$, then $0<\mu\leq\lambda_3^2$ due to the assumption in **Case 3** and (2.14) is reduced to

$$\Big(1+\frac{b}{3c}\frac{1}{\lambda_3}\Big)\mu+\lambda_3^2+\frac{a}{3c}\geq 0 \quad \text{at } (\tilde{t},\tilde{x}). \tag{2.15}$$

By (2.8), we have

$$-3\leq 1-\frac{4b}{b+\sqrt{b^2-24ac}}\leq\Big(1+\frac{b}{3c}\frac{1}{\lambda_3}\Big)_{(\tilde{t},\tilde{x})}\leq 1 \tag{2.16}$$

If $0\leq\Big(1+\frac{b}{3c}\frac{1}{\lambda_3}\Big)_{(\tilde{t},\tilde{x})}\leq 1$, then (2.15) is automatically true. Otherwise since μ is a monotone decreasing, nonnegative function of λ_2 on the given interval, we have

$$\begin{aligned}\Big(1+\frac{b}{3c}\frac{1}{\lambda_3}\Big)\mu+\lambda_3^2+\frac{a}{3c} &\geq \Big(1+\frac{b}{3c}\frac{1}{\lambda_3}\Big)\lambda_3^2+\lambda_3^2+\frac{a}{3c}\\ &= 2\Big(\lambda_3^2+\frac{b}{6c}\lambda_3+\frac{a}{6c}\Big)>0 \quad \text{at } (\tilde{t},\tilde{x}),\end{aligned}$$

where we used (2.8) in the last inequality above. In all, (2.15) is valid, and so is (2.13). As a consequence,

$$\partial_t g|_{(\tilde{t},\tilde{x})}\geq -2c\lambda_3\Big(\lambda_3^2+\frac{b}{6c}\lambda_3+\frac{a}{6c}\Big)\Big|_{(\tilde{t},\tilde{x})}\geq-\frac{3c}{2}\lambda_3\Big(\lambda_3^2+\frac{b}{6c}\lambda_3+\frac{a}{6c}\Big)\Big|_{(\tilde{t},\tilde{x})}.$$

The proof of the claim 2.11 is complete. □

After this, Corollary 1.1 can be easily established.

Proof of Corollary 1.1. Without loss of generality we may assume

$$Q(t,x)\in C^{1,2}((0,T)\times\Omega)\cap C([0,T]\times\bar{\Omega}),$$

and let

$$\lambda_1(t_0,x_0)=\max_{(t,x)\in[0,T]\times\bar{\Omega}}\lambda_1(t,x). \tag{2.17}$$

If

$$\lambda_1(t_0,x_0)>\max\Big[\frac{b+\sqrt{b^2-24ac}}{6c},\max_{\bar{\Omega}}\lambda_1(Q_0)\Big],$$

then $t_0\in(0,T]$ and $x_0\in\Omega$. As a consequence, it follows from the same argument as in the proof of Theorem 1.1 that

$$0\leq-\frac{3c}{2}\lambda_1\Big(\lambda_1^2-\frac{b}{3c}\lambda_1+\frac{2a}{3c}\Big)\Big|_{(t_0,x_0)}<0$$

due to the assumption that $\lambda_1(t_0,x_0)>\frac{b+\sqrt{b^2-24ac}}{6c}$, which is a contradiction. The corresponding lower bound for $\lambda_3(t,x)$ can be proved in a similar manner.

□

Remark 2.1. *With minor modifications, one may check easily that the above arguments are also valid for the whole space case that is shown in [24].*

3. Appendix

In this appendix section using the same idea as [24], we prove

Lemma 3.1. *Let* $\mathbf{u} \in H^1(0,T;H^1(\Omega)) \cap L^\infty(0,T;H^2(\Omega))$, $\mathbf{u}_\delta \in C^\infty([0,T]\times\bar{\Omega})$, $\nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{u}_\delta = \mathbf{u}|_{\partial\Omega} = \mathbf{u}_\delta|_{\partial\Omega} = 0$ *be such that* $\mathbf{u}_\delta \to \mathbf{u}$ *as* $\delta \to 0$ *strongly in* $H^1(0,T;H^1(\Omega)) \cap L^\infty(0,T;H^2(\Omega))$. *Let* Q^δ *be the unique classical solution in* $C^{1,2}((0,T)\times\Omega) \cap C([0,T]\times\bar{\Omega})$ *of the system*

$$
\begin{aligned}
(3.18)\qquad & Q^\delta_t + \mathbf{u}_\delta\cdot\nabla Q^\delta - \omega_\delta Q^\delta + Q^\delta\omega_\delta \\
& = \Gamma\left(L\Delta Q^\delta - aQ^\delta + b\Big[(Q^\delta)^2 - \frac{\operatorname{tr}(Q^\delta)^2}{3}\mathbb{I}\Big] - cQ^\delta \operatorname{tr}(Q^\delta)^2\right),
\end{aligned}
$$

$$
(3.19)\qquad Q^\delta(t,x)|_{\partial\Omega} = \tilde{Q}(x),
$$

where $\omega_\delta = \frac{\nabla\mathbf{u}^\delta - \nabla^T\mathbf{u}^\delta}{2}$. *Assume that*

$$
(3.20)\qquad \tilde{m} \le \lambda_i(Q^\delta(t,x)) \le \tilde{M}, \qquad \forall\, 1\le i\le 3, \quad (t,x)\in[0,T]\times\Omega.
$$

Then $Q^{(\delta)}(t,x) \to Q(t,x)$ *as* $\delta\to 0$, $\forall t\ge 0, x\in\Omega$, *where* Q *is the unique solution in* $H^1(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^3(\Omega))$ *of*

(3.21)

$$
Q_t + \mathbf{u}\cdot\nabla Q - \omega Q + Q\omega = \Gamma\Big(L\Delta Q - aQ + b\Big[Q^2 - \frac{\operatorname{tr}(Q^2)}{3}\mathbb{I}\Big] - cQ\operatorname{tr}(Q^2)\Big),
$$

$$
(3.22)\qquad Q(t,x)|_{\partial\Omega} = \tilde{Q}(x).
$$

Furthermore, we have an eigenvalue constraint on Q *such that*

$$
(3.23)\qquad \tilde{m} \le \lambda_i(Q(t,x)) \le \tilde{M}, \qquad \forall\, 1\le i\le 3, \quad (t,x)\in(0,T]\times\Omega
$$

provided the initial data Q_0 *and boundary data* $\tilde{Q}$ *have the same constraint.*

Remark 3.1. *Theorem 1.1 in [15] is only one step away from the existence of local classical solutions to the evolution problem* (1.3)-(1.6)*, which nevertheless cannot be improved using the method therein. However, it remains to be an interesting question to study.*

Proof. Without loss of generality, we set $\Gamma = L = 1$. To begin with, we show apriori L^∞ bound on Q. Since $c>0$, there exists $\eta_0 > 0$, such that

$$
-a|M|^2 + b\operatorname{tr}(M)^3 - c|M|^4 \le 0, \qquad \forall\, |M|\ge\eta_0,\ M\in\mathcal{S}^3_0
$$

Let $\eta \overset{\text{def}}{=} \max\{\|Q_0\|_{L^\infty(\Omega)}, \|\tilde{Q}\|_{L^\infty(\Omega)}, \eta_0\}$. Multiplying (3.21) with $Q(|Q|^2 - \eta)_+$, then integrating over Ω and using integration by parts, we obtain

$$
\frac{1}{2}\frac{d}{dt}\int_\Omega (|Q|^2-\eta)_+^2 = -\int_\Omega |\nabla Q|^2(|Q|^2-\eta)_+\,dx - \int_\Omega |\nabla(|Q|^2-\eta)_+|^2\,dx
$$

$$+\int_\Omega(-a|Q|^2+b\,\mathrm{tr}(Q^3)-c|Q|^4)(|Q|^2-\eta)_+\le 0$$

Thus

$$\|Q(t,\cdot)\|_{L^\infty(\Omega)}\le\eta,\quad\forall\,0\le t\le T. \tag{3.24}$$

In the same way, we conclude

$$\|Q^\delta(t,\cdot)\|_{L^\infty(\mathbb{R}^3)}\le\eta,\quad\forall\,0\le t\le T,\,\forall\delta>0. \tag{3.25}$$

Let $R^\delta\overset{\text{def}}{=}Q^\delta-Q\in\mathcal{S}_0^3$. Then it is easy to see

$$\begin{aligned}&\partial_t R^\delta+\mathbf{u}_\delta\nabla R_\delta-\omega_\delta R^\delta+R^\delta\omega_\delta\\&=\Delta R^\delta-aR^\delta+(\mathbf{u}-\mathbf{u}_\delta)\nabla Q-(\omega-\omega_\delta)Q+Q(\omega-\omega_\delta)\\&+b\Big[R^\delta Q^\delta+QR^\delta-\frac{\mathrm{tr}(R^\delta Q^\delta+QR^\delta)}{3}\mathbb{I}\Big]\\&-c\Big[|Q^\delta|^2R^\delta+\mathrm{tr}(R^\delta Q^\delta+QR^\delta)Q\Big],\end{aligned}$$

with initial and boundary datum $R_0^\delta=R^\delta|_{\partial\Omega}\equiv 0$. Multiplying the above equation with R^δ, integrating over Ω, by (3.24) and (3.25), we get

$$\begin{aligned}&\frac{d}{dt}\int_\Omega|R^\delta|^2\,dx\\&\le C\int_\Omega|R^\delta|^2\,dx+C\int_\Omega\big[(\mathbf{u}-\mathbf{u}_\delta)\nabla Q-(\omega-\omega_\delta)Q+Q(\omega-\omega_\delta)\big]R^\delta\,dx\\&\le C\|R^\delta\|_{L^2}^2+C\big(\|\nabla Q\|_{L^2}\|\mathbf{u}-\mathbf{u}_\delta\|_{L^2}+\|Q\|_{L^2}\|\omega(s)-\omega_\delta(s)\|_{L^2}\big)\\&\le C\|R^\delta\|_{L^2}^2+C\big(\|\mathbf{u}-\mathbf{u}_\delta\|_{L^2}+\|\omega(s)-\omega_\delta(s)\|_{L^2}\big).\end{aligned} \tag{3.26}$$

Hence Gronwall's inequality gives

$$\|R^\delta(t,\cdot)\|_{L^2}^2\le Ce^{Ct}\int_0^t\|\mathbf{u}-\mathbf{u}_\delta\|_{H^1}(s)\,ds\to 0,\quad\text{as }\delta\to 0$$

due to the assumption that $\mathbf{u}_\delta\to\mathbf{u}$ strongly in $L^\infty(0,T;H^2(\Omega))$. Therefore, combined with the fact that $Q^\delta,Q\in C([0,T]\times\bar\Omega)$ we get

$$Q^\delta(t,x)\to Q(t,x),\quad\forall\,(t,x)\in[0,T]\times\Omega \tag{3.27}$$

Moreover, (3.23) follows from (3.27) and Lemma 2.1

□

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