On tetraquarks with hidden charm and strangeness as φ-$\psi(2S)$ hadrocharmonium

Julia Yu. Panteleeva[1], Irina A. Perevalova[1], Maxim V. Polyakov[2, 3] and Peter Schweitzer[4, 5]

[1] *Physics Department, Irkutsk State University, Karl Marx str. 1, 664003 Irkutsk, Russia*
[2] *Petersburg Nuclear Physics Institute, Gatchina, 188300 St. Petersburg, Russia*
[3] *Institut für Theoretische Physik II, Ruhr-Universität Bochum, D-44780 Bochum, Germany*
[4] *Department of Physics, University of Connecticut, Storrs, CT 06269, USA*
[5] *Institute for Theoretical Physics, Tübingen University, Auf der Morgenstelle 14 72076 Tübingen, Germany*

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In the hadrocharmonium picture a $c\bar{c}$ state and a light hadron form a bound state. The effective interaction is described in terms of the chromoelectric polarizability of the $c\bar{c}$ state and energy-momentum-tensor densities of the light hadron. This picture is justified in the heavy quark limit, and may successfully account for a hidden-charm pentaquark state recently observed by LHCb. In this work we extend the formalism to the description of hidden-charm tetraquarks, and address the question of whether the resonant states observed by LHCb in the J/ψ-φ spectrum can be described as hadrocharmonia. This is a non-trivial question because nothing is known about the φ meson energy-momentum-tensor densities. With rather general assumptions about energy-momentum-tensor densities in the φ-meson we show that a $\psi(2S)$-φ bound state can exist, obtain a characteristic relation between its mass and width, and show that the tetraquark $X(4274)$ observed by LHCb in J/ψ-φ spectrum is a good candidate for a hadrocharmonium. We make predictions which will allow testing this picture. Our method can be generalized to identify other potential hadrocharmonia.

I. INTRODUCTION

Many evidences for tetraquark states with hidden charm were recently found, see Refs. [1–3] for reviews. In particular, states with hidden strangeness and charm were discovered. The most comprehensive analysis of the J/ψ-φ system was performed by the LHCb collaboration [4]. Four tetraquark states with quantum numbers $J^{PC} = 0^{++}, 1^{++}$ were observed.

Various theoretical approaches have been suggested to interpret such tetraquark states, for instance in terms of hadronic molecules formed of D-mesons or their excited states [5–7] or in the diquark picture [8–11]. It was also suggested that the observed structure at $m = 4140$ MeV is a manifestation of rescattering [12, 13]. The fit of the LHCb data on $X(4140)$ in terms of rescattering effects in the model of Ref. [12] gives a slight preference to this model over a Breit-Wigner resonance. The state $X(4274)$ with $J^{PC} = 1^{++}$ cannot be described as a molecular state or rescattering effect. In [14] it was proposed that $X(4274)$ may be a conventional $\chi_{c1}(3P)$ state. But the couplings of χ charmonia to J/ψ-φ and J/ψ-ω systems can be naturally expected to be similar, and the mass spectrum of J/ψ-ω in the decays of $B \to J/\psi\, \omega\, K$ shows no structures analog to those in the J/ψ-φ spectrum. This is a strong argument against an interpretation for any of the states $X(4140), X(4274), X(4500), X(4700)$ as conventional charmonia. For detailed discussions see the reviews [1–3].

Here we investigate the possibility of whether some of these tetraquarks can be interpreted as bound states of a φ-meson and $\psi(2S)$ in the formalism of Refs. [15–17].

This formalism provides a successful description of the pentaquark state $P_c(4450)$ observed at LHCb [18–20] as a bound state of the nucleon and $\psi(2S)$ [21, 22] if the chromoelectric polarizability of $\psi(2S)$ is $\alpha(2S) \approx 17\ \text{GeV}^{-3}$. Lattice data on the J/ψ-nucleon potential [23] support this interpretation [24]. The formalism makes also predictions for bound states of $\psi(2S)$ with Δ and hyperons [22, 25] which will allow testing this appealing approach in experiment. For studies of the J/ψ interaction with nuclear matter we refer to [26, 27].

In this work we investigate whether the hadrocharmonium picture can also describe some of the hidden-charm tetraquarks. We will show that the tetraquark $X(4274)$ is a good candidate for a bound state of $\psi(2S)$ with a φ-meson. We will also make predictions which will allow to test this picture.

II. THE EFFECTIVE QUARKONIUM-HADRON INTERACTION

In the heavy quark limit, when the quarkonium size is much smaller than the size of the considered hadron, here φ, the effective interaction V_{eff} of an s-wave quarkonium with the φ-meson is described in terms of the quarkonium polarizability α and the energy-momentum tensor (EMT) densities of the φ-meson,

$$V_{\text{eff}}(r) = -\,\alpha\,\frac{4\pi^2}{b}\,\frac{g_c^2}{g_s^2}\quad \nu\, T_{00}(r) - 3\, p(r)\ ,\quad \nu = 1 + \xi\,\frac{b}{8\pi^2}\,. \tag{1}$$

Here $T_{00}(r)$ and $p(r)$ are the energy density and pressure [28] inside the φ-meson which satisfy respectively (see [29] for a review on EMT form factors of hadrons and their densities)

$$\int d^3r\, T_{00}(r) = m_\varphi\,, \qquad \int d^3r\, p(r) = 0\ , \tag{2}$$

and $b = (\frac{11}{3}N_c - \frac{2}{3}N_f)$ is the leading coefficient of the Gell-Mann-Low function, g_c (g_s) is the strong coupling constant renormalized at the scale μ_c (μ_s) associated with the heavy quarkonium (φ-meson). The parameter ξ_s denotes the fraction of the hadron energy carried by gluons at the scale μ_s [30]. It is approximately $g_c \approx g_s$ and $\nu \approx 1.5$ [21]. The derivation of Eq. (1) is justified in the limit that the ratio of the quarkonium size is small compared to the effective gluon wavelength [16], and a numerically small term proportional to the current masses of the light quarks is neglected.

With the value of $\alpha(2S)$ obtained in [21, 22] and a model for EMT densities, energy density $T_{00}(r)$ and pressure $p(r)$, in the φ-meson one in principle is in the position to apply the formalism to the description of bound states of φ-mesons with $\psi(2S)$.

It should be remarked that in our situation mixing effects between $s\bar{s}$ and $c\bar{c}$ components are negligible, because the binding energy of a hadrocharmonium is small. In fact, in the heavy quark limit, $m_\phi \to \infty$ the mass of system is of $O(m_\phi)$ but its binding energy is of $O(\Lambda_{QCD})$ and hence much smaller. Thus, in the heavy quark limit, which justifies the validity of Eq. (1), mixing effects between the light- and heavy-quarkonium components can be consistently neglected.

III. EMT DENSITIES IN THE φ-MESON

Very little is known about the EMT densities in the φ-meson [31]. These densities are defined in terms of Fourier transforms of the EMT form factors $A(t)$ and $D(t)$ [28]. The energy density $T_{00}(r)$ and the pressure $p(r)$ entering the effective potential (1) are expressed in terms of form-factors $A(t)$ and $D(t)$ as follows:

$$T_{00}(r) = m_\varphi \int \frac{d^3p}{(2\pi)^3}\, e^{i\boldsymbol{pr}}\, A(-\boldsymbol{p}^2),$$
$$p(r) = \frac{1}{6\, m_\varphi}\, \frac{1}{r^2}\frac{d}{dr} r^2 \frac{d}{dr} \int \frac{d^3p}{(2\pi)^3}\, e^{i\boldsymbol{pr}}\, D(-\boldsymbol{p}^2). \quad (3)$$

Obviously the normalisation conditions (2) are satisfied automatically. We recall that the form factor $A(t)$ satisfies the constraint $A(0) = 1$, while the value of the D-term $D = D(0)$ is not fixed [29]. Almost nothing is known about the D-terms of any meson [29], except for the recent first phenomenological information on π^0 EMT form factors [32]. But π^0 is a Goldstone boson, and its D-term (see [34] and references therein) does not need to be good guideline for a vector meson like φ.

In a very simple description one may assume simple generic forms, e.g. dipole and quadrupole Ansätze. In this case we describe the EMT densities in the φ-meson in terms of 3 parameters:

$$A(t) = \frac{1}{(1 - t/M_1^2)^2}, \qquad D(t) = \frac{D}{(1 - t/M_2^2)^3}, \quad (4)$$

where M_1 is the dipole mass of $A(t)$, D is the value of the D-term, and M_2 is the quadrupole mass of $D(t)$. The mass parameter M_1 can be related to the mean square radius of the energy density in the φ-meson, $\langle r^2 \rangle = 12/M_1^2$, whereas the mass parameter M_2 is related to the mechanical mean square radius of the φ-meson (for the definition and discussion of the mechanical radius see Ref. [29]) as $r^2_{\rm mech} = 12/M_2^2$.

The radii and D-term of the φ-meson are not known (see e.g. [31]). Therefore here we shall assume wide ranges of values for these parameters (with $i = 1, 2$):

$$0.05\ \text{fm}^2 < r_i^2 < 1\ \text{fm}^2, \quad -15 < D < 0. \quad (5)$$

The D-term is expected to be negative (see e.g. the discussion in [29]). The interval of D in (5) includes the value of $D = -1$ which corresponds to the D-term for a non-interacting point-like vector particle [33]. In the parameter space (5) we include on purpose realistic as well as rather exotic values.

With the parameters in above mentioned intervals we obtain a set of effective potentials whose form varies considerably. For illustrative purposes we plot in Fig. 1 examples of the resulting effective potentials. Due to the normalisation conditions (2) all effective potentials in the set are normalised by the condition:

$$\int d^3r\, V_{\rm eff}(r) = -\,\alpha \frac{4\pi^2}{b}\,\frac{g_c^2}{g_s^2}\,\nu\, m_\phi. \quad (6)$$

In the next sections we study the possible $\psi(2S)$-φ bound states and their partial decay width to φ and J/ψ.

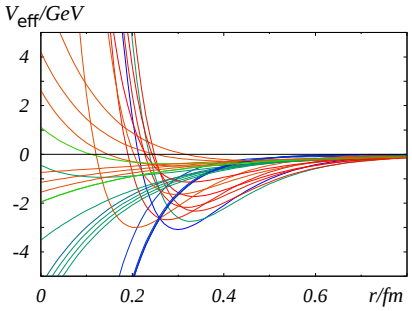

FIG. 1: Examples of the effective potentials obtained from different values of the parameters in the intervals (5) in our Ansätze for φ-meson EMT densities.

IV. MASS AND PARTIAL DECAY WIDTH OF THE $\psi(2S)$-φ HADROCHARMONIUM

Let m_ψ, m_J, m_φ denote the masses of $\psi(2S)$, J/ψ, φ-meson. The mass of the tetraquark state is defined as $M = m_\psi + m_\varphi + E_{\rm bind}$. The binding energy $E_{\rm bind} < 0$

[1] We chose the quadrupole Ansatz for $D(t)$ in order to avoid a divergent pressure at the origin. However, we checked that our results are only moderately affected if one uses a singular at the origin pressure $p(r)$.

is obtained from solving the non-relativistic Schrödinger equation with the effective potential defined in terms of the $\psi(2S)$ chromoelectric polarizability $\alpha(2S)$ [21]

$$-\frac{\boldsymbol{\nabla}^2}{2\mu_2} + V_{\rm eff}(r) - E_{\rm bind} \quad \Psi(\boldsymbol{r}) = 0, \qquad (7)$$

where μ_2 is the reduced mass $\bar{2}^{-1}\mu = m_\psi^{-1} + m_\phi^{-1}$ of the bound particles.

The decay of the tetraquark into φ and J/ψ requires that $M > m_J + m_\varphi$ and is governed by the same effective potential but rescaled, since now the $\alpha(2S \to 1S)$ polarizability is relevant. The formula for the decay width is given by [21, 25]

$$\Gamma = \frac{\mu_1 |\boldsymbol{q}|}{\pi} \left(\frac{\alpha(2S \to 1S)}{\alpha(2S)} \right)^2 \left| \int d^3r \, \Psi(\boldsymbol{r}) \, V_{\rm eff}(r) \, e^{i\boldsymbol{q}\boldsymbol{r}} \right|^2 \qquad (8)$$

where μ_1 is the reduced mass $\bar{1}^{-1}\mu = m_J^{-1} + m_\varphi^{-1}$ of the decay products, and $|\boldsymbol{q}| = \sqrt{2\mu_1(M - m_J - m_\varphi)}$ corresponds to the center-of-mass frame momentum of the decay products. The bound-state wave function $\Psi(\boldsymbol{r})$ corresponding to the binding energy $E_{\rm bind} = M - m_\psi - m_\phi$ is normalised to unity $\int d^3r \, |\Psi(\boldsymbol{r})|^2 = 1$.

To evaluate the binding energy and width in Eqs. (7, 8) we use the value $\alpha(2S) \approx 17$ GeV^{-3} which was shown to yield a robust description of the pentaquark state $P_c(4450)$ interpreted as a N-$\psi(2S)$ bound state under varying assumptions on different chiral models for nucleon EMT densities [21,22]. In a recent study [38] a wide range of values was estimated 18 GeV^{-3} $\alpha(2S)$ 270 GeV^{-3} by inferring $\alpha(1S)$ from available results for nucleon-J/ψ scattering lengths and exploring the relation $\alpha(2S)/\alpha(1S) = 502/7$ derived in the heavy-quark and large-N_c limit by treating quarkonia as Coulomb systems [39]. Interestingly, the lowest value of this range is compatible with $\alpha(2S) \approx 17$ GeV^{-3} from [21,22]. For the transitional chromoelectric polarizability we use $|\alpha(2S \to 1S)| \approx 2$GeV^{-3} from Ref. [16]. The φ-meson EMT densities are modeled as described in Sec. III with parameters varied in the wide intervals of Eq. (5).

Not surprisingly, we obtain a wide range of masses M for the corresponding tetraquarks. Practically every M in the allowed range $2m_\varphi < M < m_\psi + m_\varphi$ is realized for some choices of parameters M_1, M_2, D in the range (5). Also the results for Γ vary considerably.

The mass and width are functions $\Gamma(M_1, M_2, D)$ and $M(M_1, M_2, D)$ of parameters M_1, M_2, D which are varied randomly in the ranges (5). At first glance one would expect a scatter plot of $\Gamma(M_1, M_2, D)$ versus $M(M_1, M_2, D)$ to yield a random Γ-M -distribution filling out the whole M –Γ plane. But surprisingly we find that the points lie more or less on one curve, see Fig. 2. This is remarkable: even though we know nothing about the structure of the φ-meson, we can predict that M and Γ of candidate $\psi(2S)$-φ tetraquarks are systematically correlated. This is not a feature of a particular parametrization (dipole and quadrupole). We checked that it is also the case for other form factor parametrizations, e.g. higher multipoles. Very similar results are obtained also with EMT densities of a "smeared out" point-like boson [34] or with a simple square well potential. Notice that the same values for (M, Γ) can be obtained from different combinations of the parameters in the intervals (5).

In the remainder of this section we will clarify the question why M and Γ are correlated in this characteristic way. In the next section we will address the implications of this finding.

The bound state problem and the width can be conveniently solved and evaluated in position space. To understand the Γ-M -relation it is convenient to work in momentum space. Assuming that the bound state problem is solved (in position space) and the wave function $\Psi(\boldsymbol{r})$ is known, we define the momentum-space wave function as

$$\Psi(\boldsymbol{p}) = \int d^3r \, e^{-i\boldsymbol{p}\boldsymbol{r}} \, \Psi(\boldsymbol{r}) \,, \qquad (9)$$

and introduce the form factor $F_{\rm eff}(\boldsymbol{p}) \equiv F_{\rm eff}(-\boldsymbol{p}^2)$ as the

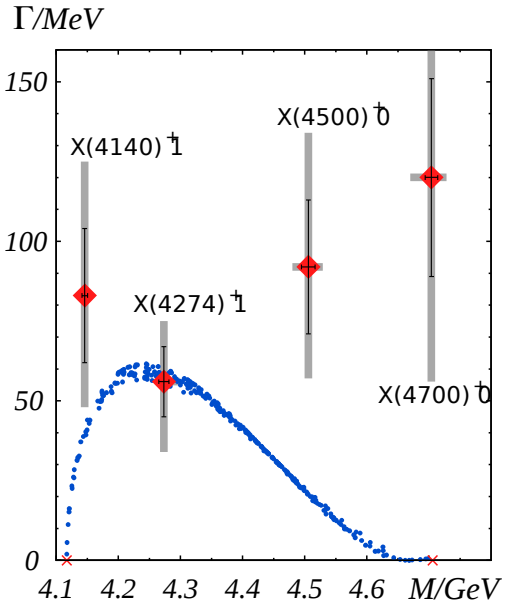

FIG. 2: The scatter plot of the decay width $\Gamma(M_1, M_2, D)$ vs mass $M(M_1, M_2, D)$ of tetraquarks obtained from varying the parameters M_1, M_2, D, which describe the unknown φ-meson EMT form factors (4), within a wide range of the values (5). In this plot 310 different points are shown. Remarkably, even though we randomly scan a large parameter space, the Γ-M -values lie approximately on a characteristic curve, see text. The crosses on the M -axis indicate the bounds $2m_\varphi < M < m_\psi + m_\varphi$. For comparison we show the four tetraquarks in the J/ψ-φ resonance region with their statistical (thin lines) and systematic (shaded areas) uncertainties and spin parity assignments [4]. The state $X(4274)$ emerges as a candidate for the description as a hadrocharmonium. This method can be used to identify other possible hadroquarkonia.

Fourier transform of the effective potential as

$$V_{\rm eff}(r) = \int \frac{{\rm d}^3p}{(2\pi)^3}\, F_{\rm eff}(\boldsymbol{p})\, e^{i\boldsymbol{p}\boldsymbol{r}}\,. \qquad (10)$$

If we take the Schrödinger equation in momentum space

$$\left(\frac{\boldsymbol{p}^2}{2\mu_2} - E_{\rm bind}\right)\Phi(\boldsymbol{p}) = -\int \frac{{\rm d}^3p'}{(2\pi)^3}\, F_{\rm eff}(\boldsymbol{p}-\boldsymbol{p}')\Phi(\boldsymbol{p}') \qquad (11)$$

and multiply it by its complex conjugate, we obtain

$$\left(\frac{\boldsymbol{p}^2}{2\mu_2} - E_{\rm bind}\right)^2 |\Phi(\boldsymbol{p})|^2 = \left|\int \frac{{\rm d}^3p'}{(2\pi)^3}\, F_{\rm eff}(\boldsymbol{p}-\boldsymbol{p}')\Phi(\boldsymbol{p}')\right|^2 . \qquad (12)$$

At the same time, the formula for the decay width can be expressed as

$$\Gamma = \frac{\alpha(2S\to 1S)^2}{\alpha(2S)}\,\frac{\mu_1|\boldsymbol{q}|}{\pi}\left|\int \frac{{\rm d}^3p'}{(2\pi)^3}\, F_{\rm eff}(\boldsymbol{q}-\boldsymbol{p}')\Phi(\boldsymbol{p}')\right|^2 . \qquad (13)$$

Thus we see that the binding energy and the partial width of the hadrocharmonium are related as

$$\Gamma = \frac{\alpha(2S\to 1S)^2}{\alpha(2S)}\,\frac{\mu_1|\boldsymbol{q}|}{\pi}\left(\frac{\boldsymbol{q}^2}{2\mu_2} - E_{\rm bind}\right)^2 |\Phi(\boldsymbol{q})|^2, \qquad (14)$$

with $|\boldsymbol{q}| = \sqrt{2\mu_1(E_{\rm bind} + m_\psi - m_J)}$. Notice that the center-of-mass momentum of the decay products is bound as $0 < \boldsymbol{q}^2 < 2\mu_1(m_\psi - m_J)$.

Consider a class of potentials obtained from continuously-differentiable (adiabatic) variations of certain parameters. Then $\Phi(\boldsymbol{p})$, and hence also $|\Phi(\boldsymbol{q})|^2$, will vary in a continuously differentiable manner as the parameter space is scanned. If we varied a single parameter in a potential, we would obtain a unique Γ-M -curve. In our case we vary multiple parameters in the potential, and obtain families of Γ-M -curves. Notice, however, that only those deformations of $V_{\rm eff}(r)$ are possible which preserve the normalization condition (6). This explains why the results for (Γ, M) all occupy a relatively narrow region in the Γ-M plane.

The specific shape of the Γ-M -curves can be understood as follows. For $M \to m_J + m_\varphi$ we have $|\boldsymbol{q}| \to 0$, i.e. the phase space of the decay naturally suppresses the decay width as $\Gamma = c_1|\boldsymbol{q}|$ for small $|\boldsymbol{q}|$. The dimensionless coefficient c_1 is of order unity and weakly dependent on the details of the wave functions, see App. A. In the opposite limit $M \to m_\psi + m_\varphi$ we deal with a bound state problem in the threshold limit, $E_{\rm bind} \to 0$. In a weakly bound case many properties of a quantum system are largely insensitive to the details of the specific potential, see e.g. the pioneering work of Wigner on deuteron [35]. This implies a suppression of the momentum-space wave-function in the limit of $E_{\rm bind} \to 0$, such that Γ approaches zero, see Appendix A.

In summary, in the hadrocharmonium picture the mass and partial width Γ of a tetraquark decaying into J/ψ and φ are correlated in a characteristic way.

V. DISCUSSION OF RESULTS AND CONCLUSIONS

The EMT densities in the φ-meson are not known. This prevents us from making explicit predictions for the mass of the $\psi(2S)$-φ bound state in the hadrocharmonium picture. With physically very broad assumptions about the EMT densities in the φ-meson and taking the value of the chromoelectric polarizability $\alpha(2S)$ to be $\alpha(2S) \approx 17$ GeV^{-3} as needed to describe $P_c(4450)$ pentaquark as a bound state of the nucleon and $\psi(2S)$ [21,22], we obtained that a $\psi(2S)$-φ bound state can form. Although we cannot make precise predictions for the mass of such state, we obtained a characteristic relation between mass of the state and its partial decay width to J/ψ and φ.

In our approach the s-wave bound state of the two vector mesons $\psi(2S)$ and φ with $J^{PC} = 1^{--}$ has positive parity and positive C-parity and corresponds to a mass-degenerate multiplet $J^{PC} = 0^{++}$, 1^{++}, 2^{++}. The degeneracy is lifted by the hyperfine interaction which is suppressed by the inverse of the heavy quark mass and expected to be small. Recent lattice studies of the J/ψ-N effective potentials [23] showed that the hyperfine interaction is very small.

Interestingly, the state $X(4274)$ observed in the $J/\psi\,\varphi$ channel has a width of $\Gamma = 56 \pm {}^{18}_{11}$ MeV [4] exactly in the range predicted by our scatter plot, see Fig. 2. The LHCb collaboration obtained for this state the quantum numbers $J^{PC} = 1^{++}$. If one interprets this state as a $\psi(2S)$-φ bound state, one should expect two further nearly mass-degenerate resonances with spin 0 and 2 in this energy region. It would be interesting to check this hypothesis in partial wave analysis.

It is important to stress that adopting this interpretation for $X(4274)$ implies that the $X(4140)$, $X(4500)$, $X(4700)$ cannot be s-wave $\psi(2S)$-φ bound states. These states could be other hadrocharmonium states, possibly with $l \geq 1$ which might be possible in specific regions of the parameter space. Or their explanation may require different binding mechanisms. Addressing this question goes beyond the scope of this work.

Assuming that the state $X(4274)$ is a hadrocharmonium allows us to gain some (very vague) information on the EMT densities of the φ-meson. The $\psi(2S)$-φ bound state with the mass around $X(4274)$ appears for the following range of parameters: $r_E^2 \in [0.1,\, 0.55]$ fm^2, $r^2_{\rm mech} \in [0.08,\, 0.5]$ fm^2 and $D \in [-5,\, 0]$, the smaller radii correspond the larger values of $|D|$. This is a very reasonable range of parameters for EMT densities in the φ-meson, for example, in the AdS/QCD model one finds $r_E^2 = 0.21$ fm^2 for the ρ-meson [37]. This approach would yield similar results for other vector mesons such as φ.

We also note that if we consider the chromoelectric polarizability $\alpha(2S)$ as a free parameter, the $\psi(2S)$-φ bound state appears for $\alpha(2S) > \alpha_{\rm crit}(2S) \in [2,\, 4]$ GeV^{-3} if we vary the parameters of EMT densities in above mentioned range. Note that this range of critical values for the chromoelectric polarizability is just slightly above the polarizability $\alpha(1S) = 1.5 \pm 0.6$ GeV^{-3} of J/ψ deter-

mined in Ref. [24] from the lattice data of Ref. [23]. We remark that $\alpha_{crit}(1S)$ is larger (for the same potential) than $\alpha_{crit}(2S)$ due to $\mu_1 < \mu_2$. Thus, bound states of J/ψ and φ most probably are not possible in the hadrocharmonium picture. This is in line with lattice QCD studies, where the J/ψ-φ potential was found too weak to form bound states [36].

Using the example of the $\psi(2S)$-φ hadrocharmonium we demonstrated that the partial $J/\psi\varphi$ decay width is correlated in a characteristic way with the mass of the state. This interesting "approximate universality" of the Γ-M dependence is a generic feature of the approach and can be expected to hold also for other hadroquarkonia. The implications of this observation will be studied elsewhere.

Other interesting questions concern whether also other J/ψ-φ resonances can be described as bound or resonant states in the hadrocharmonium picture, and whether hadroquarkonia with the heavier $b\bar{b}$ states can exist. The chromoelectric polarizabilities of bottomonia are smaller than for charmonia [37], and the corresponding V_{eff} is in general weaker. The formation of hidden-bottom tetraquarks in the hadrocharmonium picture may therefore be more difficult. But these interesting topics deserve dedicated studies and will be addressed elsewhere.

VI. ACKNOWLEDGMENTS

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Appendix A: The partial decay width Γ in extreme limits

In the limit $|\boldsymbol{q}| \to 0$, where $E_{bind} \to m_J - m_\psi$ approaches its maximal value, we obtain from (14)

$$\Gamma = c_1 |\boldsymbol{q}| + O(|\boldsymbol{q}|^3),$$
$$c_1 = \frac{\alpha(2S \to 1S)}{\alpha(2S)} 4\mu_1 (m_\psi - m_J)^2 \langle r^{3/2} \rangle^2. \quad \text{(A1)}$$

In Eq. (A1) we defined $\langle r^{3/2} \rangle = \int_0^\infty dr\, r\, u(r)$. Here $u(r)$ is the radial part $u(r)$ of the (s-wave) ground-state wave function $\Psi(\boldsymbol{r}) = u(r)/r\, Y_{00}$. We define $u(r)$ to be real, positive, and normalized as $\int_0^\infty dr\, u(r)^2 = 1$. Notice that $u(r)$ has dimension (length)$^{-1/2}$. One has naturally $\langle r^{3/2} \rangle^2 = a_0 R_h^3$. Here R_h is the characteristic hadronic radius of the problem associated with the range of the potential $V_{eff}(r)$ and set by the radius of the φ-meson, and a_0 is a numerical factor of order unity. Quark models indicate that the φ-meson is about the size of the proton or somewhat smaller. If we use this as a guideline and assume for the characteristic radius $R_h \sim 0.8$ fm, we find for the slope $c_1 \sim 1$.

In the opposite limit when $E_{bind} \to 0$ the size of the bound-state wave function in coordinate space grows as $\sim 1/\sqrt{2\mu_2 |E_{bind}|}$. This implies that the momentum-space wave function $\Phi(\boldsymbol{p})$ becomes more and more narrow and hence $|\Phi(\boldsymbol{q})|^2$ in Eq. (14) goes to zero for fixed $\boldsymbol{q}$. One can show on general grounds (see e.g. Ref. [40]) that in the limit $E_{bind} \to 0$ and $\boldsymbol{q}$ fixed, the wave function (squared) in the momentum space $|\Phi(\boldsymbol{q})|^2 \propto \sqrt{-E_{bind}}$ and hence $\Gamma \propto \sqrt{-E_{bind}}$.

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