Published for SISSA by Springer

Received: *December 29, 2018*
Accepted: *May 7, 2019*
Published: *May 15, 2019*

Exact generalized partition function of 2D CFTs at large central charge

Anatoly Dymarsky[a,b] **and Kirill Pavlenko**[b,c]

[a] *Department of Physics and Astronomy, University of Kentucky, Lexington, KY, 40506, U.S.A.*
[b] *Skolkovo Institute of Science and Technology, Skolkovo Innovation Center, Bolshoy Boulevard 30, bld. 1, Moscow 121205, Russia*
[c] *Moscow Institute of Physics and Technology, Institutsky per. 9, Dolgoprudny 141700, Russia*

E-mail: a.dymarsky@uky.edu, kirill.pavlenko@skoltech.ru

Abstract: We discuss generalized partition function of 2d CFTs on thermal cylinder decorated by higher qKdV charges. We propose that in the large central charge limit qKdV charges factorize such that generalized partition function can be rewritten in terms of auxiliary non-interacting bosons. The explicit expression for the generalized free energy is readily available in terms of the boson spectrum, which can be deduced from the conventional thermal expectation values of qKdV charges. In other words, the picture of the auxiliary non-interacting bosons allows extending thermal one-point functions to the full non-perturbative generalized partition function. We verify this conjecture for the first seven qKdV charges using recently obtained pertrubative results and find corresponding contributions to the auxiliary boson masses. We further extend these results by conjecturing the full spectrum of bosons and find an exact expression for the generalized partition function as a function of infinite tower of chemical potentials in the limit of large central charge.

Keywords: Conformal and W Symmetry, Conformal Field Theory, Integrable Field Theories

ArXiv ePrint: 1812.05108

https://doi.org/10.1007/JHEP05(2019)077

JHEP05(2019)077

Contents

1 Introduction

Generalized partition function of 2d CFTs decorated by higher qKdV charges [1–3], the so-called Generalized Gibbs Ensemble,

$$Z = \mathrm{Tr} \exp\left\{-\sum_{k=1}^{\infty} \mu_{2k-1} Q_{2k-1}\right\}, \quad \mu_1 \equiv \beta, \quad Q_1 \equiv H, \tag{1.1}$$

has been in the focus of attention recently in the context of thermalization of large c 2d conformal theories [4–15]. In this work we assume thermodynamic limit, when the size of the spatial circle goes to infinity $\ell \to \infty$ and (1.1) describes theory on a thermal cylinder.

In a recent work [15] we observed that in the large central charge limit first two non-trivial qKdV charges Q_3, Q_5 admit simple structure. Namely,

$$\ell^{2k-1} Q_{2k-1} = P_k(L_0) + \ell^{2k-1} \tilde{Q}_{2k-1}, \tag{1.2}$$

where P_k is a polynomial of degree k, $P_k(L_0) = L_0^k + \dots$, and the operator $\tilde{Q}_{2k-1}$ accounts for the rest. Written in the conventional basis of conformal theory (sets $\{m_i\}$, $m_1 \geq m_2, \dots, \geq m_k$, are arranged in dominance order),

$$|m_i, \Delta\rangle = L_{-m_1} \dots L_{-m_k} |\Delta\rangle, \tag{1.3}$$

$\tilde{Q}_{2k-1}$ is a polynomial in Δ and c. In what follows we assume Δ scales linearly with c. This scaling is associated with the saddle point contribution to (1.1) in the limit, when ℓ goes to infinity, see [15]. Written in the basis (1.3), leading scaling of $\tilde{Q}_{2k-1}$ with c is c^{k-1}. At this order matrix of $\tilde{Q}_{2k-1}$ is lower-triangular. At first two leading orders in $1/c$ the eigenvalues of Q_{2k-1} are (in the following expression we suppress terms which do not contribute in the thermodynamic limit — see section 2.1 below)

$$\ell^{2k-1} Q_{2k-1}|\lambda\rangle = \lambda|\lambda\rangle, \quad \lambda = \Delta^k + \sum_{p=0}^{k-1} \sum_i m_i^{2p+1} c^p \Delta^{k-1-p}\, \xi_k^p + O(c^{k-2}), \tag{1.4}$$

where ξ_k^p are some numerical coefficients. Remarkably, (1.4) are linear in the occupation numbers n_r, provided the sets$\{m_i\}$ are rewritten in terms of free boson representation,

$$\sum_i m_i^p = \sum_r r^p n_r. \tag{1.5}$$

where each set $\{m_i\}$ is parametrized by the set of integer n_r counting the number of times natural number r appears in the set $\{m_i\}$. The linearity of λ in n_r is crucial for what follows. Technically it is due to the fact that (1.4) includes only a single sum over m_i.

The form of the spectrum (1.4) was previously established only for Q_3, Q_5. In this paper we verify that (1.4) applies to higher charges $Q_{2k-1}, k \leq 7$ as well. If (1.4) applies to all charges, at first two orders in $1/c$ generalized partition function (1.1) reduces to that one of non-interacting auxiliary bosons with the spectrum given in terms of μ_{2k-1} and ξ_k^p.

In principle the coefficients ξ_k^p can be deduced directly from the explicit form of Q_{2k-1} in terms of Virasoro generators L_n, as was done for Q_3, Q_5 in [15]. Extending this strategy to higher charges is difficult because their explicit form is not known and difficult to calculate. A much simpler way to obtain ξ_k^p follows from the expression for thermal average of Q_{2k-1} over a particular Verma module,

$$\langle Q_{2k-1}\rangle_{\beta,\Delta} = \mathrm{Tr}_\Delta(q^{L_0} Q_{2k-1}), \qquad q = e^{-\beta/\ell}, \tag{1.6}$$

where the sum in (1.6) goes over all states of the form (1.3) with a fixed Δ. This one-point function was calculated recently for the first seven qKdV charges Q_{2k-1}, $k \leq 7$, in [14]. Using this result we confirm the proposed form of the eigenvalues (1.4) and obtain corresponding coefficients ξ_k^p. We notice these coefficients admit a simple form, which can be easily generalized to all k,

$$\xi_k^p = 24^{-p} \frac{(2k-1)\Gamma(k+1)\Gamma(1/2)}{2\,\Gamma(p+3/2)\Gamma(k-p)}. \tag{1.7}$$

Assuming that (1.4) and (1.7) apply to all higher Q_{2k-1}, generalized partition function at first two orders in $1/c$ expansion reduces to that one of non-interacting auxiliary bosons, yielding

$$
\begin{aligned}
Z &= e^F, & F &= \frac{\pi^2\ell}{6\,\beta}\left(c' f_0 + f_1 + O(1/c')\right), \qquad (1.8)\\
f_0 &= \sum_{k=1}^{\infty} t_{2k-1}\sigma^k(2k-1), & \sqrt{\sigma} &= \sum_{k=1}^{\infty} t_{2k-1}\sigma^k k,\\
f_1 &= -\frac{12}{\pi}\int_0^\infty d\kappa \log\left(1-e^{-2\pi\kappa\gamma}\right),\\
\gamma &= \sum_{k=1}^{\infty} t_{2k-1}\sigma^{k-1}k(2k-1)_2F_1(1,1-k,3/2,-\kappa^2/\sigma),\\
c' &= c-1, & t_{2k-1} &= \left(\frac{\pi^2 c'}{6\beta^2}\right)^{k-1}\frac{\mu_{2k-1}}{\beta}, \qquad t_1 \equiv 1.
\end{aligned}
$$

Here $\sigma(t_1, t_3, \ldots)$ is a function which satisfies $\sqrt{\sigma} = \sum_{k=1}^{\infty} t_{2k-1}\sigma^k k$. It can be expressed explicitly in terms of an infinite power series in t_{2k-1}, see (3.11). The conjectural expression for f_1 is the main result of this paper.

This paper is organized as follows: in the next section we discuss first seven qKdV charges Q_{2k-1}, $k \leq 7$, and verify they are consistent with (1.4). We also calculate corresponding coefficients ξ_k^p and conclude that (1.7) describes all of them. In section three we assume (1.4) and (1.7) are valid beyond $k \leq 7$ for all Q_{2k-1} and calculate generalized partition function (1.8) assuming $\beta \neq 0$. The case of $\beta = 0$ is discussed in the appendix A. The relation between $1/c$ and $1/c'$ expansion is discussed in the appendix B.

2 Thermal average of Q_{2k-1}

In this section we discuss how the form of the eigenvalues (1.4) can be verified and the coefficients ξ_k^p can be fixed from the explicit form of thermal one-point averages (1.6) obtained in [14]. Because of the lower-triangular form of $\tilde{Q}_{2k-1}$, leading terms of λ contribute to the thermal average (1.6) as a linear combination of Eisenstein series, or functions σ_r, with the coefficients polynomially dependent on c and Δ,

$$
\sum_{\{m_i\}}\sum_i q^{\Delta+n} m_i^r = \sigma_r \chi, \qquad n \equiv \sum_i m_i, \qquad \chi = \frac{q^\Delta}{\prod_i (1-q^i)}, \tag{2.1}
$$

where $q \equiv e^{-\beta/\ell}$. Functions σ_k are related to Eisenstein series via

$$
\sigma_p = \sum_{k=0}^{\infty}\frac{k^p q^k}{1-q^k}, \qquad E_{2p} = 1 + \frac{2}{\zeta(1-2p)}\sigma_{2p-1}. \tag{2.2}
$$

In other words, to fix ξ_k^p we need to find coefficients in front of $\sigma_{2p+1}c^p\Delta^{k-1-p}$.

2.1 Thermodynamic limit

In this paper we are concerned with the extensive part of free energy, i.e. we are taking thermodynamic limit by taking the size of the spatial circle to infinity $\ell \to \infty$, while c is

JHEP05(2019)077

large, but fixed. In this limit the only relevant contributions to (1.1) are those when Q_{2k-1} contribute extensively, i.e. scale linearly with ℓ. We therefore neglect all terms in (1.4) which are suppressed in comparison with ℓ^{2k}. Using saddle point approximation for (1.1) it is easy to see that in the limit $\ell \to \infty$ relevant scaling of Δ is $\Delta \sim c\,\ell^2$, while the occupation numbers $n_r \sim \ell$ for $r \sim \ell$, see [15] for details. Thus, all terms in (1.4) indeed scale as ℓ^{2k}. Scaling of the modular functions $\sigma_p(q)$ (not to be confused with $\sigma(t_{2k-1})$ introduced later in (3.3)) with ℓ in thermodynamic limit can be obtained by replacing the summation with integration, yielding $\sigma_p \sim \ell^{p+1}$.

2.2 Q_1

As a warm-up we start our analysis with

$$\ell Q_1 = L_0 - \frac{c}{24}. \tag{2.3}$$

The constant term $-c/24$ does not contribute in the thermodynamic limit and therefore the structure (1.2) is manifest with $\tilde{Q}_1 = 0$. The eigenvalues of $L_0 = \Delta + n$, $n \equiv \sum_i m_i$, have the form (1.4) with $\xi_1^0 = 1$. Although this is straightforward we want to derive the same result in a slightly different way,

$$\mathrm{Tr}_\Delta(q^{L_0} L_0) = \partial\chi = (\Delta + \sigma_1)\chi, \qquad \partial \equiv q\partial_q. \tag{2.4}$$

Hence $\xi_1^0 = 1$ is simply the coefficient in front of σ_1.

2.3 Q_3

The explicit expression for Q_3 is bulky,

$$\ell^3 Q_3 = L_0^2 - \frac{c+2}{12} L_0 + \frac{c(5c+22)}{2880} + 2\sum_{i=1}^{\infty} L_{-i}L_i, \tag{2.5}$$

but only first and last terms contribute in the thermodynamic limit yielding (1.2) with $\ell^3\tilde{Q}_3 = 2\sum_{i=1}^{\infty} L_{-i}L_i$. Thermal average (1.6) can be calculated using trace cyclicity [16], yielding [14, 15]

$$\ell^3\,\mathrm{Tr}_\Delta(q^{L_0} Q_3) = \left(D^2 + \frac{c}{1440}E_4\right)\chi, \tag{2.6}$$

where here and below

$$D^k = \left(\partial - \frac{k-1}{6}E_2\right)\left(\partial - \frac{k-2}{6}E_2\right)\ldots\partial. \tag{2.7}$$

Leading term Δ^2 follows from ∂^2. Using (2.4), we calculate the coefficients in front of Δ and c

$$\ell^3\,\mathrm{Tr}_\Delta(q^{L_0} Q_3) = \Delta^2 + \Delta\left(6\sigma_1 - \frac{1}{6}\right) + \frac{c}{6}\left(\sigma_3 + \frac{1}{240}\right) + \partial\sigma_1. \tag{2.8}$$

JHEP05(2019)077

To express E_{2p} in terms of σ_{2p-1} we need the numerical values of zeta-function, which we write down here for reader's convenience,

$$\zeta(-1) = -\frac{1}{12}, \qquad \zeta(-3) = \frac{1}{120}, \qquad \zeta(-5) = -\frac{1}{252}, \qquad \zeta(-7) = \frac{1}{240},$$
$$\zeta(-9) = -\frac{1}{132}, \qquad \zeta(-11) = -\frac{691}{32760}, \qquad \zeta(-13) = -\frac{1}{12}. \tag{2.9}$$

We are only interested in the first two terms of $1/c$ expansion (Δ is assumed to scale linearly with c), hence the term $\partial\sigma_1$ from (2.8) can be neglected. Next, we only consider the terms which contribute extensively in the thermodynamic limit $\ell \to \infty$. We assume that Δ scales as ℓ^2 while the scaling of $\sigma_r \propto \ell^{r+1}$ follows from its explicit form. There is another more intuitive way to understand that directly from (2.1). Main contribution to the thermal average comes from the partitions $\{m_i\}$ which consist of approximately $n^{1/2}$ terms and each term $m_i \sim n^{1/2}$, while typical $n = \sum_i m_i$ scales as ℓ^2. Keeping only the terms scaling as ℓ^4 in (2.8) we obtain

$$\ell^3 \,\mathrm{Tr}_\Delta(q^{L_0} Q_3) = \Delta^2 + 6\Delta\,\sigma_1 + \frac{c}{6}\sigma_3 + O(1/c), \tag{2.10}$$

in full consistency with (1.4). This result agrees with the calculation of [15], which utilizes the explicit form of Q_3 in terms of Virasoro algebra generators. First term $L_0^2 = (\Delta + n)^2$ yields $\Delta^2 + 2\Delta n$, (n^2 can be neglected because it contributes as c^0), while the eigenvalue of $\ell^3 \tilde{Q}_3 = \frac{c}{6}(\sum_i m_i^3 - n) + 4\Delta n$ completes it to (2.10), or (1.4) with $\xi_2^2 = 1/6$ and $\xi_2^1 = 4$.

2.4 Q_5

The calculation for Q_3 reveals the pattern how the terms of interest enter the full expression for the thermal average. The leading term Δ^k of the eigenvalue of Q_{2k-1} follows from $D^k\chi$, as well as $\xi_{k-1}^0 \Delta^{k-1}\sigma_1$. The term $\xi_{k-1}^1 c\Delta^{k-2}\sigma_3$ follows from $cE_4 D^{k-2}\chi$, and so on. In case of Q_5 we have for the thermal average [14],

$$\ell^5 \,\mathrm{Tr}_\Delta(q^{L_0} Q_5) = \left(D^3 + \frac{c+4}{288} E_4 D - \frac{c(c+14)}{36288} E_6\right)\chi. \tag{2.11}$$

This yields in the limit of interest

$$\ell^5 \,\mathrm{Tr}_\Delta(q^{L_0} Q_5) = \left(\Delta^3 + 15\Delta^2\sigma_1 + \frac{5}{6} c\Delta\sigma_3 + \frac{1}{72} c^2\sigma_5\right)\chi, \tag{2.12}$$

where the last term came from $c^2 E_6 D^{k-3}\chi$, $k = 3$. This result is in full agreement with the explicit calculation of [15].

2.5 Q_7

The original expression for $\mathrm{Tr}_\Delta(q^{L_0} Q_7)$ calculated in [14] is quadratic in E_4, but using the identify $E_4^2 = E_8$ it can be written as follows

$$\ell^7 \,\mathrm{Tr}_\Delta(q^{L_0} Q_7) = \left(D^4 + \frac{(7c+64)}{720} E_4 D^2 - \frac{c^2 + 24c + 74}{6480} E_6 D + \frac{c\left(c^2 + \frac{103c}{4} + 175\right)}{518400} E_8\right)\chi.$$

This immediately gives

$$\ell^7\,\mathrm{Tr}_\Delta(q^{L_0}Q_7) = \left(\Delta^4 + 28\Delta^3\sigma_1 + \frac{7}{3}c\Delta^2\sigma_3 + \frac{7}{90}c^2\Delta\sigma_5 + \frac{1}{1080}c^3\sigma_7\right)\chi. \tag{2.13}$$

Corresponding values of ξ_3^p are easy to obtain using numerical values (2.9).

2.6 Q_9

The expression for Q_9 is too bulky and here we only write relevant terms using $E_4^2 = E_8$ and $E_4E_6 = E_{10}$,

$$\begin{aligned}\ell^9\,\mathrm{Tr}_\Delta(q^{L_0}Q_9) = \Bigg(&D^5 + \left(\frac{7c}{720} + O(c^0)\right)E_4D^3 + \left(-\frac{c^2}{2016} + O(c^1)\right)E_2D^2 \\ &+ \left(-\frac{c^3}{80640} + O(c^2)\right)E_8D + \left(-\frac{c^4}{4790016} + O(c^3)\right)E_{10}\Bigg)\chi. \end{aligned}\tag{2.14}$$

Corresponding values of ξ_4^p immediately follow from here.

2.7 Q_{11}, Q_{13}, and beyond

Calculation of the eigenvalues of Q_{11} and Q_{13} is completely analogous, but to rewrite the leading part of $\mathrm{Tr}_\Delta(q^{L_0}Q_{2k-1})$ as a linear combination of D^k and terms of the form $c^{k-1-p}E_{2(k-p)}D^p$, $p = 0,\ldots,k-2$, we need to use the identities

$$E_{12} = \frac{441}{691}E_4^3 + \frac{250}{691}E_6^2, \qquad E_{14} = E_4^2E_6. \tag{2.15}$$

Resulting values of the coefficients ξ_k^p for $k = 1,\ldots,7$, are summarized in the table below

$$\xi_k^p = \begin{pmatrix} 1 & & & & & & \\ 6 & \frac{1}{6} & & & & & \\ 15 & \frac{5}{6} & \frac{1}{72} & & & & \\ 28 & \frac{7}{3} & \frac{7}{90} & \frac{1}{1080} & & & \\ 45 & 5 & \frac{1}{4} & \frac{1}{168} & \frac{1}{18144} & & \\ 66 & \frac{55}{6} & \frac{11}{18} & \frac{11}{504} & \frac{11}{27216} & \frac{1}{326592} & \\ 91 & \frac{91}{6} & \frac{91}{72} & \frac{13}{216} & \frac{13}{7776} & \frac{13}{513216} & \frac{1}{6158592} \end{pmatrix}, \quad p = 0,\ldots,k-1, \tag{2.16}$$

here p indexes rows and k indexes columns. These values can be concisely written as

$$\xi_k^p = 24^{-p}\frac{(2k-1)\Gamma(k+1)\Gamma(1/2)}{2\,\Gamma(p+3/2)\Gamma(k-p)}, \tag{2.17}$$

which extends this result for all k.

3 Generalized partition function

From now on we assume that (1.4) applies to all qKdV charges with the coefficients ξ_k^p given by (2.17). Given that all Q_{2k-1} mutually commute, the generalized partition function (1.1)

JHEP05(2019)077

is given by the sum over primaries Δ and sets (Young tables) $\{m_i\}$, parameterizing descendants via (1.3),

$$Z = \sum_{\Delta} \sum_{\{m_i\}} \exp\left(-\sum_{k=1}^{\infty} \frac{\mu_{2k-1}}{\ell^{2k-1}} \left(\Delta^k + \sum_{p=0}^{k-1} \sum_i m_i^{2p+1} c^p \Delta^{k-1-p} \, \xi_k^p + O(c^{k-2}) \right) \right). \quad (3.1)$$

At large central charge sum over Δ can be substituted by an integral

$$\sum_{\Delta} \to \int d\Delta \, e^{\pi\sqrt{2c'\Delta/3}}, \qquad c' \equiv c-1, \quad (3.2)$$

where the density of primaries follows from Cardy formula [17, 18]. It is convenient to introduce σ via

$$\Delta = \frac{c'\pi^2\ell^2}{6\,\beta^2}\sigma. \quad (3.3)$$

So far we were discussing $1/c$ expansion, but the results look more elegant if we do an expansion in $1/c'$. Since at leading order $c = c' + O(1)$, the structure of λ remains the same: Δ^k contributes as $(c')^k$ while $c^p\Delta^{k-1-p}$ terms contribute as $(c')^{k-1}$. Going from the sets $\{m_i\}$ to free boson representation (1.5), the partition function reduces to that one of non-interacting auxiliary bosons

$$Z(\beta,t) = \int d\sigma \exp\left\{ \frac{c'\pi^2\ell}{6\beta} \left(2\sqrt{\sigma} - \sum_{k=1}^{\infty} t_{2k-1}\sigma^k \right) \right\} \sum_{n_1,n_2,\dots} e^{-\sum_{r=1}^{\infty} n_r M_r + O(1/c')}, \quad (3.4)$$

$$\log Z \equiv F = \frac{\pi^2\ell}{6\beta}\left(c' f_0(t) + f_1(t) + O(1/c')\right), \quad (3.5)$$

$$t_{2k-1} = \left(\frac{\pi^2 c'}{6\beta^2}\right)^{k-1} \frac{\mu_{2k-1}}{\beta}, \qquad t_1 \equiv 1, \quad (3.6)$$

where the spectrum of bosons is given by

$$M_r = \sum_{k=1}^{\infty} t_{2k-1}\sigma^{k-1} \sum_{p=0}^{k-1} \xi_k^p \left(\frac{6}{\pi^2\sigma}\right)^p \left(\frac{\beta r}{\ell}\right)^{2p+1} \quad (3.7)$$

$$= \frac{\beta r}{\ell} \sum_{k=1}^{\infty} t_{2k-1}\sigma^{k-1} k(2k-1)\, {}_2F_1\left(1, 1-k, 3/2, -\frac{1}{\sigma}\left(\frac{\beta r}{2\pi\ell}\right)^2\right). \quad (3.8)$$

In (3.4) we write the partition function as a function of β, t_{2k-1}. For the given fixed β, t_{2k-1} the terms contributing as $(c')^{k-2}$ to eigenvalues of Q_{2k-1} contribute to free energy as $1/c'$. Our scope is to calculate free energy up to the first two orders in $1/c'$ expansion, i.e. only keep the terms which survive in the $c' \to \infty$ limit. Hence $O(1/c')$ terms can be neglected.

Up to $1/c'$ corrections the value of σ is determined via saddle point approximation of

$$Z_0(\beta,t) = \exp\left\{\frac{c'\pi^2\ell}{6\beta} f_0\right\} = \int d\sigma \exp\left\{ \frac{c'\pi^2\ell}{6\beta} \left(2\sqrt{\sigma} - \sum_{k=1}^{\infty} t_{2k-1}\sigma^k \right) \right\}, \quad (3.9)$$

while the remaining sum over the boson occupation numbers n_r in (3.4) "takes" saddle point value of σ as an input. The saddle point equation

$$\sqrt{\sigma} = \sum_{k=1}^{\infty} t_{2k-1}\sigma^k k, \tag{3.10}$$

can be solved in terms of an infinite series

$$\sigma = 1 + \sum_{n=1}^{\infty} \sum_{k_1,\ldots,k_n=2}^{\infty} 2\frac{(-1)^n}{n!}\frac{(2K-n+1)!}{(2K-2n+2)!}\prod_{i=1}^{n} k_i\, t_{2k_i-1}, \quad K \equiv \sum_i k_i, \tag{3.11}$$

yielding (expansion (3.13) was found in [13]),

$$f_0 = \sum_{k=1}^{\infty} t_{2k-1}\sigma^k(2k-1), \tag{3.12}$$

$$f_0 = 1 + \sum_{n=1}^{\infty} \sum_{k_1,\ldots,k_n=2}^{\infty} 2\frac{(-1)^n}{n!}\frac{(2K-n)!}{(2K-2n+2)!}\prod_{i=1}^{n} k_i\, t_{2k_i-1}, \quad K \equiv \sum_i k_i. \tag{3.13}$$

With σ being fixed, the remaining part of the partition function describes some auxiliary non-interacting bosons

$$\frac{\pi^2 \ell}{6\beta} f_1 = \log \sum_{n_1,n_2,\ldots} e^{-\sum_{r=1}^{\infty} n_r M_r} = -\sum_{r=1}^{\infty} \log\left(1 - e^{-M_r}\right). \tag{3.14}$$

In the thermodynamic limit $\ell \to \infty$ summation over r can be substituted by integration (Thomas-Fermi approximation), yielding (1.8).

4 Discussion

In this paper we have conjectured leading form of the spectrum of qKdV charges in $1/c$ expansion and verified it using recently obtained thermal averages for the first seven qKdV charges [14]. Using the conjectural form of the eigenvalues we have rewritten generalized partition function of 2d CFTs at large central charge in terms of non-interacting auxiliary bosons. The result of our calculation is the explicit form of the extensive part of free energy, exact up to $1/c$ corrections (1.8). We postpone discussing physical implications of our fundings until future work [19].

Acknowledgments

AD is supported by the National Science Foundation under Grant No. PHY-1720374. AD is grateful to KITP for hospitality, where this work was initiated. The research at KITP was supported in part by the National Science Foundation under Grant No. NSF PHY-1748958.

A Alternative representation of the partition function

The answer (1.8) was derived assuming $\beta \neq 0$ and $\mu_1 = \beta$ enters the expression for free energy differently from all other chemical potentials. In this section we obtain the answer for free energy F in another "coordinate patch," assuming some other chemical potential μ_{2r-1} for a given r is non-zero, $\mu_{2r-1} \neq 0$, while the rest of chemical potentials, including $\mu_1 = \beta$, could be zero.

Let us introduce $c'^{r-1}\mu_{2r-1} = \lambda \neq 0$ and the following set of independent variables

$$\tau_{2k-1} = \frac{\mu_{2k-1}}{\mu_{2r-1}} c'^{k-r} \left(\frac{\pi^2}{6\lambda^2 r^2}\right)^{\frac{k-r}{2r-1}}, \quad \tau_{2r-1} \equiv 1, \tag{A.1}$$

and functions $f_i(\tau)$, $\sigma(\tau)$,

$$F = c'\ell\lambda \left(\frac{\pi^2}{6\lambda^2 r^2}\right)^{\frac{r}{2r-1}} (f_0 + f_1/c' + O(1/c'^2)), \quad \Delta = c'\ell^2 \left(\frac{\pi^2}{6\lambda^2 r^2}\right)^{\frac{1}{2r-1}} \sigma. \tag{A.2}$$

Using these notations the expression for f_0 is as follows

$$f_0 = 2r\sqrt{\sigma} - \sum_{k=1}^{\infty} \tau_{2k-1}\sigma^k = \sum_{k=1} (2k-1)\tau_{2k-1}\sigma^k, \tag{A.3}$$

where the last equality holds "on-shell,"

$$r\sigma^{1/2} = \sum_{k=1}^{\infty} \tau_{2k-1} k\, \sigma^k, \qquad \sigma = 1 - \frac{2}{r(2r-1)} \sum_{k\neq r} k\, \tau_{2k-1} + \dots \tag{A.4}$$

Finally, the expression for f_1,

$$f_1 = -\frac{12r}{\pi} \int_0^\infty d\kappa \log\left(1 - \exp\left\{-\frac{2\pi}{r}\kappa\gamma\right\}\right), \tag{A.5}$$

$$\gamma = \sum_{k=1} \tau_{2k-1} k(2k-1)\sigma^{k-1} {}_2F_1(1, 1-k, 3/2, -\kappa^2/\sigma). \tag{A.6}$$

B $1/c$ versus $1/c'$ expansion

In a recent work [15] we were discussing free energy in $1/c$ expansion

$$F = \frac{\pi^2 \ell}{6\beta} \left(c\tilde{f}_0(\tilde{t}) + \tilde{f}_1(\tilde{t}) + O(1/c)\right), \tag{B.1}$$

using variables

$$\tilde{t}_{2k-1} = \left(\frac{\pi^2 c}{6\beta^2}\right)^{k-1} \frac{\mu_{2k-1}}{\beta}. \tag{B.2}$$

In this paper we used on $1/c'$ expansion

$$F = \frac{\pi^2 \ell}{6\beta} \left(c' f_0(t) + f_1(t) + O(1/c')\right), \tag{B.3}$$

and the variables

$$t_{2k-1} = \left(\frac{\pi^2 c'}{6\beta^2}\right)^{k-1} \frac{\mu_{2k-1}}{\beta}. \tag{B.4}$$

Here we outline the relation between these two expansion schemes. Using

$$t_{2k-1} = \tilde{t}_{2k-1} \left(1 - \frac{1}{c}\right)^{k-1} \tag{B.5}$$

we readily find

$$\tilde{f}_0(t) = f_0(t), \tag{B.6}$$

and

$$\tilde{f}_1(t) = -f_0(t) - \sum_{k=1}^{\infty} (k-1) t_{2k-1} \frac{\partial f_0(t)}{\partial t_{2k-1}} + f_1(t). \tag{B.7}$$

Using the explicit form of f_0, (3.12), this can be simplified as

$$\tilde{f}_1(t) = -\sqrt{\sigma(t)} + f_1(t). \tag{B.8}$$

A comparison of f_1 from (1.8) with the equations (2.43), (2.52) of [15] confirms this result.

References

[1] V.V. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov, *Integrable structure of conformal field theory, quantum KdV theory and thermodynamic Bethe ansatz*, Commun. Math. Phys. **177** (1996) 381 [hep-th/9412229] [INSPIRE].

[2] V.V. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov, *Integrable structure of conformal field theory. 2. Q operator and DDV equation*, Commun. Math. Phys. **190** (1997) 247 [hep-th/9604044] [INSPIRE].

[3] V.V. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov, *Integrable structure of conformal field theory. 3. The Yang-Baxter relation*, Commun. Math. Phys. **200** (1999) 297 [hep-th/9805008] [INSPIRE].

[4] P. Calabrese and J. Cardy, *Quantum quenches in extended systems*, J. Stat. Mech. **0706** (2007) P06008 [arXiv:0704.1880] [INSPIRE].

[5] P. Calabrese, F.H.L. Essler and M. Fagotti, *Quantum quench in the transverse field Ising chain*, Phys. Rev. Lett. **106** (2011) 227203 [arXiv:1104.0154] [INSPIRE].

[6] J. Cardy, *Quantum quenches to a critical point in one dimension: some further results*, J. Stat. Mech. **1602** (2016) 023103 [arXiv:1507.07266] [INSPIRE].

[7] J. de Boer and D. Engelhardt, *Remarks on thermalization in 2D CFT*, Phys. Rev. **D 94** (2016) 126019 [arXiv:1604.05327] [INSPIRE].

JHEP05(2019)077

[8] A. Pérez, D. Tempo and R. Troncoso, *Boundary conditions for general relativity on AdS_3 and the KdV hierarchy*, *JHEP* **06** (2016) 103 [arXiv:1605.04490].

[9] S. He, F.-L. Lin and J.-j. Zhang, *Subsystem eigenstate thermalization hypothesis for entanglement entropy in CFT*, *JHEP* **08** (2017) 126 [arXiv:1703.08724] [INSPIRE].

[10] P. Basu, D. Das, S. Datta and S. Pal, *Thermality of eigenstates in conformal field theories*, *Phys. Rev.* **E 96** (2017) 022149 [arXiv:1705.03001] [INSPIRE].

[11] S. He, F.-L. Lin and J.-j. Zhang, *Dissimilarities of reduced density matrices and eigenstate thermalization hypothesis*, *JHEP* **12** (2017) 073 [arXiv:1708.05090] [INSPIRE].

[12] N. Lashkari, A. Dymarsky and H. Liu, *Universality of quantum information in chaotic CFTs*, *JHEP* **03** (2018) 070 [arXiv:1710.10458].

[13] A. Maloney, G.S. Ng, S.F. Ross and I. Tsiares, *Generalized Gibbs Ensemble and the Statistics of KdV Charges in* 2*D CFT*, *JHEP* **03** (2019) 075 [arXiv:1810.11054] [INSPIRE].

[14] A. Maloney, G.S. Ng, S.F. Ross and I. Tsiares, *Thermal Correlation Functions of KdV Charges in* 2*D CFT*, *JHEP* **02** (2019) 044 [arXiv:1810.11053] [INSPIRE].

[15] A. Dymarsky and K. Pavlenko, *Generalized Gibbs ensemble of* 2*d CFTs at large central charge in the thermodynamic limit*, *JHEP* **01** (2019) 098 [arXiv:1810.11025] [INSPIRE].

[16] L. Apolo, *Bounds on CFTs with $\mathcal{W}_3$ algebras and AdS_3 higher spin theories*, *Phys. Rev.* **D 96** (2017) 086003 [arXiv:1705.10402] [INSPIRE].

[17] J.L. Cardy, *Operator content of two-dimensional conformally invariant theories*, *Nucl. Phys.* **B 270** (1986) 186 [INSPIRE].

[18] P. Kraus and A. Maloney, *A Cardy formula for three-point coefficients or how the black hole got its spots*, *JHEP* **05** (2017) 160 [arXiv:1608.03284].

[19] A. Dymarsky and K. Pavlenko, *Generalized eigenstate thermalization in* 2*d CFTs*, arXiv:1903.03559 [INSPIRE].