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H^∞-calculus for semigroup generators on BMO

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A B S T R A C T

We prove that the negative generator L of a semigroup of positive contractions on L^∞ has bounded $H^\infty(S_\eta)$-calculus on the associated Poisson semigroup-BMO space for any angle $\eta > \pi/2$, provided L satisfies Bakry-Émery's $\Gamma^2 \geq 0$ criterion. Our arguments only rely on the properties of the underlying semigroup and work well in the noncommutative setting. A key ingredient of our argument is a type of quasi monotone properties for the subordinated semigroup $T_{t,\alpha} = e^{-tL^\alpha}, 0 < \alpha < 1$, that is proved in the first part of this article.

Introduction

Let $\Delta = -\partial_x^2$ be the negative Laplacian operator on $\mathbb{R}^n$. The associated Poisson semigroup of operators $P_t = e^{-t\sqrt{\Delta}}, t \geq 0$ has many nice properties that make it a very useful tool in the classical analysis. In particular, the Poisson semigroup has a quasi

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monotone property that there exist constants $c_{r,j}$ such that, for any nonnegative function $f \in L^1(\mathbb{R}^n, \frac{1}{1+|x|^2}dx)$,

$$|t^j \partial_t^j P_t f| \leq c_{r,j} P_{rt} f, \tag{1}$$

for any $0 < r < 1, j = 0, 1, 2, \ldots$. As a first result of this article, we show that the quasi monotone property (1) extends to all subordinated semigroups $T_{t,\alpha} = e^{-tL^\alpha}$ for all $0 < \alpha < 1$ if L generates a semigroup of positive preserving operators on a Banach lattice X. The case of $0 < \alpha \leq \frac{1}{2}$ is easy and is previously known because of a precise subordination formula (see e.g. [28,24]). This type of quasi-monotonicity has been a useful tool in proving certain functional inequalities (see [16,28,24,23]).

Functional calculus is a theory of studying functions of operators. The so-called H^∞-calculus is a generalization of the Riesz-Dunford analytic functional calculus and defines $\Phi(L)$ via a Cauchy-type integral for an (unbounded) sectorial operator L and a function Φ that is bounded and holomorphic in a sector S_η of the complex plane. L is said to have the bounded H^∞-calculus property if the so-defined $\Phi(L)$ extend to bounded operators on X and $\|\Phi(L)\| \leq c\|\Phi\|_\infty$ for all such Φ's. The theory of bounded H^∞-calculus has developed rapidly in the last thirty years with many applications and interactions with harmonic analysis, Banach space theory, and the theory of evolution equations, starting with A. McIntosh's seminal work in 1986 (see [1], [7], [20], [27], [17], [26], [35]).

It is a major task in the study of the bounded H^∞-calculus theory to determine which operators have such a strong property. Cowling, Duong, and Hieber & Prüss ([8,12,19, 15,30]) prove that the infinitesimal generator of a semigroup of positive contractions on $L^p, 1 < p < \infty$ always has the bounded $H^\infty(S_\eta)$-calculus on L^p for any $\eta > \frac{\pi}{2}$. When the semigroup is symmetric, the angle can be reduced to $\eta > \omega_p = |\frac{\pi}{2} - \frac{\pi}{p}|$ by interpolation. It is not surprising that this result fails for L^∞ in general. One may want to seek a BMO-type space that could be an appropriate alternative for the $p = \infty$ case.

The main theorem of this article states that the negative generator L of a semigroup of positive contractions on L^∞ always has bounded $H^\infty(S_\eta)$-calculus on the space $\mathrm{BMO}(\sqrt{L})$ for any $\eta > \frac{\pi}{2}$, provided L satisfies Bakry-Émery's Γ^2 criterion. Junge and Mei attempted to prove this result (see Theorem 3.3 of [24]) under the same assumptions, but only managed to obtain a bounded $H^\infty(S_\eta)$ $(\eta > \frac{\pi}{2})$ calculus result for $\sqrt{L}$, instead of L. This is due to the fact that Lemma 3.2 and Theorem 3.3 of [24] are proved only for the operator M_a defined for the subordinated Poisson semigroup $P_t = e^{-t\sqrt{L}}$. The unknownness of the quasi-monotonicity for general subordinated semigroups e^{-tL^α} was a major obstacle that prevented Junge and Mei from reaching further. Please note that L is incorrectly written in place of $\sqrt{L}$ in the proof of Corollary 5.4 in [24]. Its corrected version is proved in this article as Corollary 3.

The classical BMO norm of a function $f \in L^1(\mathbb{R}^n, \frac{1}{1+|x|^2}dx)$ can be defined as

$$\|f\|_{BMO(\sqrt{\Delta})} = \sup_{t>0} \left\| e^{-t\sqrt{\Delta}} \left| f - e^{-t\sqrt{\Delta}} f \right|^2 \right\|_{L^\infty}^{\frac{1}{2}}. \tag{2}$$

BMO spaces associated with semigroup generators have been intensively studied recently (e.g. [14,13,11] and the subsequent works). When a cubic-BMO is available, one can often compare it with the semigroup BMO and they are equivalent in many cases. In this article, we consider the BMO($\sqrt{L}$)-(semi)norm studied in [24,28], which are defined similarly to (2), merely replacing Δ with the semigroup generator L. The corresponding space BMO($\sqrt{L}$) interpolates well with L^p-spaces when the semigroup is symmetric Markovian (see Lemma 11).

Under the assumptions of our main theorem, we also study semigroup-BMO spaces $BMO(L^\alpha), 0 < \alpha < 1$ and prove that they are all equivalent. We further prove that the imaginary power L^{is} is bounded on the associated semigroup-BMO space $BMO(L^\alpha)$ with a bound $\lesssim (1 + |s|)^{|\frac{3}{2}|} \exp(|\frac{|\pi s|}{2}|)$ (see (72), (75)). This complements Cowling's L^p-estimate (see [8, Corollary 1]) and fixes a mistake in [24] (see the Remark at the end of Section 3).

The related topics and estimates on semigroup generators have been studied with geometric/metric assumptions on the underlying measure space. This article is from a functional analysis point of view and tries to obtain a general result by abstract arguments. Cowling and Hieber/Prüss's method for their H^∞-calculus results on L^p is based on the transference techniques of Coifman and Weiss, which does not work for non-UMD Banach spaces, such as BMO. Our method is to consider the fractional power of the generator to take advantage of the quasi-monotone property (1). Our argument works well for the noncommutative case, that is for L that generates a semigroup of completely positive contractions on a semifinite von Neumann algebra.

We analyze a few examples to illustrate our results and demonstrate their applications to Fourier multipliers on non-classical L^p spaces at the end of the article. We use c for an absolute constant which may differ from line to line.

1. The complete monotonicity of a difference of exponential power functions

A nonnegative C^∞-function $f(t)$ on $(0, \infty)$ is *completely monotone* if

$$(-1)^k \partial_t^k f(t) \geq 0$$

for all t. Easy examples are $f(t) = e^{-\lambda t}$ for any $\lambda > 0$. It is well-known that completely monotonicity is preserved by addition, multiplication, and taking pointwise limits. So the Laplace transform of a positive Borel measure on $[0, \infty)$, which is an average of $e^{-\lambda t}$ in λ, is completely monotone. The Hausdorff-Bernstein-Widder Theorem says that the reverse is also true; namely that a function is completely monotone if and only if it is the

Laplace transform of a positive Borel measure on $[0, \infty)$. In particular, $g_s(t) = e^{-st^\alpha}$ is completely monotone and is the Laplace transform of a positive integrable C^∞ function $\phi_{s,\alpha}$ on $(0, \infty)$ for all $s > 0, 0 < \alpha < 1$.

$$e^{-st^\alpha} = \int_0^\infty e^{-\lambda t}\phi_{s,\alpha}(\lambda)d\lambda = \int_0^\infty e^{-s^{\frac{1}{\alpha}}\lambda t}\phi_{1,\alpha}(\lambda)d\lambda. \tag{3}$$

The function $\phi_{s,\alpha}$ is uniquely determined by the inverse Laplace transform

$$\phi_{s,\alpha}(\lambda) = s^{-\frac{1}{\alpha}}\phi_{1,\alpha}(s^{-\frac{1}{\alpha}}\lambda) = \mathcal{L}^{-1}(e^{-sz^\alpha})(\lambda) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{z\lambda}e^{-sz^\alpha}dz, \tag{4}$$

for $\sigma > 0, \lambda > 0$. The derivative $\partial_s\phi_{s,\alpha}$ is again an integrable function (see e.g. [36, page 263]), and

$$-t^\alpha e^{-st^\alpha} = \int_0^\infty e^{-\lambda t}\partial_s\phi_{s,\alpha}(\lambda)d\lambda. \tag{5}$$

The properties of $\phi_{s,\alpha}$ are important in the study of the fractional powers of semigroup generators.

The goal of this section is to prove a few pointwise inequalities for $\phi_{s,\alpha}$, which will be used in the next section. For that purpose, we first prove the complete monotonicity of several variants of e^{-st^α}.

For $k, n \in \mathbb{N}, 1 \le k \le n$, let $a_k^{(n)}$ be the real coefficients in the expansion

$$\frac{d^n}{dt^n}e^{-t^\alpha} = (-1)^n\sum_{k=1}^n a_k^{(n)}t^{-n+k\alpha}e^{-t^\alpha}.$$

It is easy to see that

$$\frac{d^n}{dt^n}e^{-ct^\alpha} = (-1)^n\sum_{k=1}^n c^k a_k^{(n)}t^{-n+k\alpha}e^{-ct^\alpha}.$$

Convention: We define $a_k^{(n)} = 0$ if $k > n$ or $k \le 0$.

The proof of the following lemma is simple and elementary. We leave it for the reader to verify.

Lemma 1. *The $a_k^{(n)}$'s satisfy the relation*

$$a_k^{(n+1)} = (n - k\alpha)a_k^{(n)} + \alpha a_{k-1}^{(n)} \tag{6}$$

for all $k \in \mathbb{Z}, n \in \mathbb{N}$.

Lemma 2. *Let $K_i, i = 1, 2$ be the first integer m such that $\frac{m}{m+i} \geq \alpha$. Then, for all $j \in \mathbb{Z}, n \in \mathbb{N}$, we have*

$$a_{k+j}^{(n)} - (j+1)a_{k+j+1}^{(n)} \geq 0 \quad \text{if } k \geq K_1 \tag{7}$$

$$(j+1)(a_{k+j+1}^{(n)} - (j+2)a_{k+j+2}^{(n)}) \leq a_{k+j}^{(n)} - (j+1)a_{k+j+1}^{(n)} \quad \text{if } k \geq K_2 \tag{8}$$

Proof. We only need to prove the case $j \geq 0$. Let D be the right derivative for discrete functions: $Df = f(j+1) - f(j)$. It is easy to see that the product rule holds $D(jf)(j) = jD_j f(j) + f(j+1)$. Fix $k \in \mathbb{Z}$. Let

$$f_n(j) = a_{k+j}^{(n)} j! \tag{9}$$

for $j \geq 0$, where we use the convention that $0! = 1$. By (6), we have

$$f_{n+1}(j) = (n - (k+j)\alpha)f_n(j) + \alpha j f_n(j-1),$$

for all $j \geq 1$ and $f_{n+1}(0) = (n - k\alpha)f_n(0) + \alpha a_{k-1}^{(n)}$. Taking the discrete derivative on both sides, we get

$$\begin{aligned} Df_{n+1}(j) &= (n - (k+j)\alpha)Df_n(j) - \alpha f_n(j+1) + \alpha j Df_n(j-1) + \alpha f_n(j) \\ &= (n - (k+j+1)\alpha)Df_n(j) + \alpha j Df_n(j-1), \end{aligned} \tag{10}$$

for $j \geq 1$ and $Df_{n+1}(0) = (n - (k+1)\alpha)Df_n(0) - \alpha a_{k-1}^{(n)}$. By induction, we get

$$D^i f_{n+1}(j) = (n - (k+j+i)\alpha)D^i f_n(j) + \alpha j D^i f_n(j-1), \tag{11}$$

for all $i \geq 1, j \geq 1$ and $D^i f_{n+1}(0) = (n - (k+i)\alpha)D^i f_n(0) + (-1)^i \alpha a_{k-1}^{(n)}$.

Let $k = K_1$ in (9). Note that the condition $Df_n(j) \leq 0$ trivially holds for $n \leq K_1 + j$ because $a_i^{(j)} = 0$ for $i > j$. In particular, $Df_n(j) \leq 0$ for all $j \geq 0, n = K_1$. We apply induction on n. Assume $Df_n(j) \leq 0$ holds for all $j \geq 0$. The equality (10) implies that $Df_{n+1}(j) \leq 0$ for all $j \geq 0$ satisfying $n \geq (K_1 + j + 1)\alpha$, which holds if $n+1 \geq K_1 + j + 1$ since $\frac{n}{n+1} \geq \alpha$. On the other hand, if $n + 1 \leq K_1 + j$ we have $Df_{n+1}(j) \leq 0$ trivially. So $Df_{n+1}(j) \leq 0$ for all $j \geq 0$. Therefore, $Df_n(j) \leq 0$ and equivalently (7) holds for all $n \in \mathbb{N}, j \geq 0$.

The argument for (8) is similar. Let $k = K_2$ in (9). Note that $D^2 f_n(j) \geq 0$ is equivalent to (8) for $j \geq 0$, which trivially holds for $n \leq K_2 + j$ since $K_2 \geq K_1$ and $a_{K_2+j}^{(n)} - (j+1)a_{K_2+j+1}^{(n)} \geq 0$. In particular, (8) holds for $n = K_2, j \geq 0$. Assume that (8) holds for $n = m, j \geq 0$. We consider the case $n = m + 1$. If $n = m + 1 \leq K_2 + j$, (8) holds trivially. Otherwise, $m + 1 \geq K_2 + j + 1$ and by applying (11) we see that $D^2 f_{n+1} \geq 0$. By induction, (8) holds for all $n \in \mathbb{N}, j \geq 0$. □

Remark. The argument of the previous lemma shows that $(-1)^i D^i f_n(j) \geq 0$ for all $n \in \mathbb{N}, j \geq 0$ if we choose k so that $\frac{k}{k+i} \leq \alpha$.

For a fixed $K \geq K_1$, let

$$F_n(x) = x^{-K} \sum_{j=1}^{n} a_j^{(n)} x^j = \sum_{j=-\infty}^{\infty} a_{K+j}^{(n)} x^j, \tag{12}$$

and for a fixed $K \geq K_2$, let

$$G_n(x) = x^{-K} \sum_{j=1}^{n+1} (a_{j-1}^{(n)} - (j-K) a_j^{(n)}) x^{j-1} = \sum_{j=-\infty}^{\infty} (a_{K+j-1}^{(n)} - j a_{K+j}^{(n)}) x^{j-1}. \tag{13}$$

Lemma 3. *Let $f(x) = F_n(x)$, or $G_n(x)$ for the given suitable K. We have $(f(x)e^{-x})' \leq 0$ and $f(x + rx) \leq e^{rx} f(x)$ for all $r, x > 0$.*

Proof. It is easy to see that $f(x) - f'(x) \geq 0$ for $x > 0$ by Lemma 2. So $(f(x)e^{-x})' = (f' - f)e^{-x} \leq 0$ and hence $f(x + rx) \leq e^{rx} f(x)$ for $r > 0$. □

We now come to the main result of this section.

Theorem 1. *Let $0 < \alpha$, $c < 1$, and $s \geq 0$ be fixed. Then*

(i) $e^{-cst^\alpha} - c^{K_1} e^{-st^\alpha}$ *is completely monotone in t.*
(ii) $K_1 e^{-st^\alpha} + st^\alpha e^{-st^\alpha}$ *is completely monotone in t.*
(iii) $\frac{1}{c^{K_2}(1-c)} e^{-cst^\alpha} - st^\alpha e^{-st^\alpha}$ *is completely monotone in t.*
(iv) $(\max\{\frac{jK_1}{c^{K_1}}, \frac{j}{c^{K_2}(1-c)}\})^j e^{-cst^\alpha} \pm s^j t^{j\alpha} e^{-st^\alpha}$ *are completely monotone in t for any $j \in \mathbb{N}$.*

Proof. By dilation, we may assume $s = 1$. We prove (i) first. Let $x = t^\alpha$ and F_n be as in 12,

$$\frac{d^n}{dt^n} e^{-t^\alpha} = (-1)^n t^{-n} \sum_{k=1}^{n} a_k^{(n)} x^k e^{-x} = (-1)^n t^{-n+K\alpha} e^{-x} F_n(x)$$

and

$$\frac{d^n}{dt^n} e^{-ct^\alpha} = (-1)^n t^{-n} \sum_{k=1}^{n} c^k a_k^{(n)} x^k e^{-x} e^{-rx} = (-1)^n t^{-n+K\alpha} c^K e^{-cx} F_n(cx). \tag{14}$$

Applying Lemma 2 and Lemma 3 to F_n gives us

$$\frac{\frac{d^n}{dt^n} e^{-ct^\alpha}}{\frac{d^n}{dt^n} e^{-t^\alpha}} \geq c^K,$$

for any $K \geq K_1$. This implies (i) since e^{-t^α} is completely monotone for any $0 < \alpha \leq 1$.

We now prove (ii). Let $g(s,t) = e^{-st^\alpha} s^{-K_1}$. Then $-\partial_s g(s,t)$, is the limit of the family of functions

$$\frac{1}{s^{K_1+1}(c-1)}(e^{-st^\alpha} - c^{-K_1} e^{-cst^\alpha})$$

as $c \to 1$, which are completely monotone in t by (i). So

$$K_1 e^{-st^\alpha} + st^\alpha e^{-st^\alpha} = -s^{K_1+1}\partial_s g(s,t)$$

is completely monotone in t.

For (iii), we denote by $f^{(n)}(t) = \partial_t^n f(t)$ and, for $K \geq K_2 \geq K_1$, write

$$\begin{aligned}
(t^\alpha e^{-t^\alpha})^{(n)} + K(e^{-t^\alpha})^{(n)} &= -\frac{1}{\alpha}[t(e^{-t^\alpha})']^{(n)} + K(e^{-t^\alpha})^{(n)} \\
&= -\frac{1}{\alpha}[t(e^{-t^\alpha})^{(n+1)} + n(e^{-t^\alpha})^{(n)}] + K(e^{-t^\alpha})^{(n)} \\
&= \frac{(-1)^n t^{-n}}{\alpha}\left[\sum_{k=1}^{\infty}(a_k^{(n+1)} - (n-K\alpha)a_k^{(n)})t^{k\alpha}e^{-t^\alpha}\right] \\
&= (-1)^n t^{-n}\left[\sum_{k=1}^{\infty}(a_{k-1}^{(n)} - (k-K)a_k^{(n)})t^{k\alpha}e^{-t^\alpha}\right] \\
&= (-1)^n t^{-n+K\alpha}\left[\sum_{k=-\infty}^{\infty}(a_{K+k-1}^{(n)} - ka_{K+k}^{(n)})t^{k\alpha}e^{-t^\alpha}\right] \\
&= (-1)^n t^{-n+K\alpha} x e^{-x} G_n(x) \qquad (15)
\end{aligned}$$

with $x = t^\alpha$ and $G_n(x)$ defined as in 13, which depends on K. Lemma 3 says that $G_n(x)e^{-x}$ deceases in x if $K \geq K_2$ and note that $G_n(x)e^{-x} = -(F_n(x)e^{-x})' \geq 0$. We have

$$\begin{aligned}
xG_n(x)e^{-x} &\leq \frac{1}{(1-c)}\int_{cx}^{x} G_n(s)e^{-s}ds \\
&= \frac{1}{(1-c)}\int_{cx}^{x} -(F_n(s)e^{-s})'ds \\
&\leq \frac{1}{(1-c)}F_n(cx)e^{-cx},
\end{aligned}$$

for $0 < c < 1$. Combining this inequality with (14) and (15) we get

$$\frac{(-1)^n \frac{d^n}{dt^n}(t^\alpha e^{-t^\alpha} + K_2 e^{-t^\alpha})}{(-1)^n \frac{d^n}{dt^n} e^{-ct^\alpha}} \leq \frac{1}{c^{K_2}(1-c)}.$$

This proves (iii) since e^{-ct^α} and e^{-t^α} are completely monotone.

For (iv), let $f(t) = \max\{\frac{K_1}{c^{K_1}}, \frac{1}{c^{K_2}(1-c)}\}e^{-cst^\alpha}, g(t) = st^\alpha e^{-st^\alpha}$. By (i), (ii) and (iii) we have that both $f+g, f-g$ are completely monotone in t. Recall that complete monotonicity is preserved by multiplication. Note that

$$f^{j+1} + g^{j+1} = \frac{1}{2}[(f^j - g^j)(f-g) + (f^j + g^j)(f+g)]$$

$$f^{j+1} - g^{j+1} = \frac{1}{2}[(f^j - g^j)(f+g) + (f^j + g^j)(f-g)].$$

We get, by induction, that $(\max\{\frac{K_1}{c^{K_1}}, \frac{1}{c^{K_2}(1-c)\alpha}\})^j e^{-jcst^\alpha} - s^j t^{j\alpha} e^{-jst^\alpha}$ is completely monotone for any $s > 0$, which implies (iv). □

We will apply Theorem 1 to pointwise estimates of $\phi_{s,\alpha}(\lambda)$. Let us first list a few basic properties of $\phi_{s,\alpha}$.

Lemma 4. *For any $s > 0, 0 < \alpha, \beta < 1$, we have*

$$\phi_{s,\frac{1}{2}}(\lambda) = \frac{1}{2\sqrt{\pi}} s e^{-\frac{s^2}{4\lambda}} \lambda^{-\frac{3}{2}}. \tag{16}$$

$$\phi_{1,\alpha\beta}(\lambda) = \int_0^\infty \phi_{s,\alpha}(\lambda)\phi_{1,\beta}(s)ds. \tag{17}$$

$$\phi_{s,\alpha}(\lambda) = s^{-\frac{1}{\alpha}}\phi_{1,\alpha}(s^{-\frac{1}{\alpha}}\lambda), \tag{18}$$

$$-\alpha s\partial_s\phi_{s,\alpha}(\lambda) = \phi_{s,\alpha}(\lambda) + \lambda\partial_\lambda\phi_{s,\alpha}(\lambda). \tag{19}$$

Proof. (16) is well-known (see e.g. [36], page 268). (17), (18) can be easily seen from (3) and (4). (18) implies (19). □

Corollary 1. *For all $\lambda, s > 0, 0 < c < 1, j \in \mathbb{N}$, we have*

$$c^{K_1}\phi_{s,\alpha}(\lambda) \le \phi_{cs,\alpha}(\lambda) \tag{20}$$

$$0 \le \partial_\lambda(\lambda^{1+\alpha K_1}\phi_{s,\alpha}(\lambda)), \tag{21}$$

$$|s^j\partial_s^j\phi_{s,\alpha}(\lambda)| \le \left(\max\left\{\frac{jK_1}{c^{K_1}}, \frac{j}{c^{K_2}(1-c)\alpha}\right\}\right)^j \phi_{cs,\alpha}, \tag{22}$$

$$|s\partial_s\phi_{s,\alpha}(\lambda)| \le \left(\frac{10}{1-\alpha}\right)\phi_{\alpha s,\alpha}(\lambda), \tag{23}$$

$$|s^j\partial_s^j\phi_{s,\alpha}(\lambda)| \le \left(\frac{10j}{1-\alpha}\right)^j \phi_{\alpha s,\alpha}(\lambda). \tag{24}$$

Proof. These are direct consequences of Theorem 1, the identity (3), and the Hausdorff-Bernstein-Widder Theorem because $K_i \leq \frac{i}{1-\alpha}$, except that (21) requires a little more calculation. To prove (21), note that (5) and Theorem 1 (ii) imply that

$$\partial_s \frac{\phi_{s,\alpha}(\lambda)}{s^{K_1}} = -s^{-K_1-1}(K_1\phi_{s,\alpha}(\lambda) - s\partial_s\phi_{s,\alpha}(\lambda)) \leq 0.$$

Since $\phi_{s,\alpha}(\lambda) = s^{-\frac{1}{\alpha}}\phi_{1,\alpha}(s^{-\frac{1}{\alpha}}\lambda)$, we get

$$-\left(\frac{1}{\alpha}+K_1\right)s^{-\frac{1}{\alpha}-K_1-1}\phi_{1,\alpha}(s^{-\frac{1}{\alpha}}\lambda) - \frac{1}{\alpha}s^{-\frac{1}{\alpha}-1}\lambda s^{-\frac{1}{\alpha}-K_1}(\partial_\lambda\phi_{1,\alpha})(s^{-\frac{1}{\alpha}}\lambda) \leq 0.$$

That is

$$(1+K_1\alpha)\phi_{1,\alpha}(s^{-\frac{1}{\alpha}}\lambda) + \lambda s^{-\frac{1}{\alpha}}(\partial_\lambda\phi_{1,\alpha})(s^{-\frac{1}{\alpha}}\lambda) \geq 0.$$

Therefore

$$(1+K_1\alpha)\phi_{s,\alpha}(\lambda) + \lambda\partial_\lambda\phi_{s,\alpha}(\lambda) \geq 0,$$

since $\partial_\lambda\phi_{s,\alpha}(\lambda) = s^{-\frac{2}{\alpha}}\partial_\lambda\phi_{s,\alpha}(s^{-\frac{1}{\alpha}}\lambda)$. This is (21). □

Lemma 5. *For any $s > 0, 0 < \beta < \alpha < 1$, we have that*

$$\int_0^\infty \left|\ln\left(s^{-\frac{1}{\alpha}}u\right)\right|\phi_{s,\alpha}(u)du < \frac{c}{\beta}. \tag{25}$$

$$\int_0^\infty\int_0^\infty \left|\ln\left(\frac{u}{v}\right)\right|\phi_{s,\alpha}(u)\phi_{s,\alpha}(v)dudv < \frac{c}{\beta^2}. \tag{26}$$

Proof. Since $\phi_{s,\alpha}(u) = s^{-\frac{1}{\alpha}}\phi_{1,\alpha}(s^{-\frac{1}{\alpha}}u)$, the left hand side of (25) is independent of s. We only need to prove the case $s = 1$. For $\alpha = \frac{1}{2}$, we can verify directly from (16) that (25) holds. Denote by $u(\alpha)$ the left hand side of (25). We then get $u(\frac{1}{2}) < \infty$. Using (17), we get $u(\frac{1}{2^n}) < \infty$. Now, for $\alpha > \frac{1}{2^n}$, we use (17) again and get

$$\begin{aligned}
\phi_{1,\frac{1}{2^n}}(\lambda) &= \int_0^\infty \phi_{s,\alpha}(\lambda)\phi_{1,\frac{1}{\alpha 2^n}}(s)ds \\
&\geq \int_0^1 \phi_{s,\alpha}(\lambda)\phi_{1,\frac{1}{\alpha 2^n}}(s)ds \\
(by\ (20)) &\geq \phi_{1,\alpha}(\lambda)\int_0^1 s^{K_1(\alpha)}\phi_{1,\frac{1}{\alpha 2^n}}(s)ds
\end{aligned}$$

$$\geq c_\alpha \phi_{1,\alpha}(\lambda).$$

We conclude that $u(\alpha) < \infty$ for all $0 < \alpha < 1$. Since $\phi_{1,\alpha}(\lambda)$ is continuous as a function in α and this continuity is uniform for $\lambda \in [\delta, N]$ for any $0 < \delta < N < \infty$, one can easily see that $u(\alpha)$ is continuous in α for $\alpha \in (0,1)$. We conclude that $u(\alpha)$ is bounded on $[\frac{1}{2^n}, \frac{1}{2}]$ for any $n \in \mathbb{N}$. Note that (17) also implies that

$$\begin{aligned}
&\int_0^\infty \phi_{1,\alpha\beta}(\lambda)|\ln\lambda| d\lambda \\
&= \int_0^\infty \int_0^\infty \phi_{s,\alpha}(\lambda)|\ln\lambda| d\lambda \phi_{1,\beta}(s) ds \\
&= \int_0^\infty \int_0^\infty \phi_{1,\alpha}(v)|\ln(s^{\frac{1}{\alpha}} v)| dv \phi_{1,\beta}(s) ds \\
&\geq \pm \int_0^\infty \int_0^\infty \phi_{1,\alpha}(v)(\frac{1}{\alpha}|\ln s| - |\ln v|) dv \phi_{1,\beta}(s) ds && (27) \\
&\Bigg(\leq \int_0^\infty \int_0^\infty \phi_{1,\alpha}(v)(\frac{1}{\alpha}|\ln s| + |\ln v|) dv \phi_{1,\beta}(s) ds \Bigg) && (28)
\end{aligned}$$

Our change in the order of integration is justified because all the terms are positive. Note $\int_0^\infty \phi_{t,\alpha}(s)ds = 1$ for any t, α. (27) and (28) imply that

$$|u(\alpha) - \frac{1}{\alpha} u(\beta)| \leq u(\alpha\beta) \leq u(\alpha) + \frac{1}{\alpha} u(\beta) \tag{29}$$

We then obtain (25). (26) follows from (25). $\square$

Remark *(Bell Polynomials).* We define the complete Bell polynomial $B_n(x_1, \ldots, x_n)$ by its generating function

$$\exp\left(\sum_{j=1}^\infty x_j \frac{u^j}{j!}\right) = \sum_{n=0}^\infty B_n(x_1, \ldots, x_n) \frac{u^n}{n!}$$

From this, we get the formula

$$B_n(x_1, \ldots, x_n) = \frac{d^n}{du^n} \exp\left(\sum_{j=1}^\infty x_j \frac{u^j}{j!}\right)\Big|_{u=0}$$

Now, for $s > 0$, let

$$x_j = -s\frac{d^j}{dt^j}t^\alpha = -s(\alpha)_j t^{\alpha-j}, \tag{30}$$

where $(\alpha)_j$ denotes the falling factorial. Then

$$\sum_{j=1}^{\infty} x_j \frac{u^j}{j!} = -st^\alpha \sum_{j=1}^{\infty} \frac{(\alpha)_j}{j!}\left(\frac{u}{t}\right)^j = st^\alpha - st^\alpha\left(1+\frac{u}{t}\right)^\alpha = st^\alpha - s(t+u)^\alpha$$

Applying Theorem 1 part (i), we see that for all $n \in \mathbb{N}$, $c \in (0,1)$, and $t > 0$ it holds that

$$\frac{\left.\frac{d^n}{du^n}e^{-sc(t+u)^\alpha}\right|_{u=0}}{\left.\frac{d^n}{du^n}e^{-s(t+u)^\alpha}\right|_{u=0}} \geq c^{K_1},$$

where K_1 is as in Lemma 2. We can rewrite this inequality as

$$e^{(1-c)st^\alpha}\frac{\left.\frac{d^n}{du^n}e^{sct^\alpha-sc(t+u)^\alpha}\right|_{u=0}}{\left.\frac{d^n}{du^n}e^{st^\alpha-s(t+u)^\alpha}\right|_{u=0}} \geq c^{K_1}.$$

We conclude that if we define x_j by (30), then

$$e^{(1-c)st^\alpha}\frac{B_n(cx_1,\ldots,cx_n)}{B_n(x_1,\ldots,x_n)} \geq c^{K_1} \tag{31}$$

for all $n \in \mathbb{N}$, $c \in (0,1)$, and $t > 0$. All of these calculations are easily reversible, and we conclude that (31) is actually equivalent to part (i) of Theorem 1.

2. Positive semigroups and BMO

Let (M, σ, μ) be a sigma-finite measure space. Let $L^1(M)$ be the space of all complex valued integrable functions and $L^\infty(M)$ be the space of all complex valued measurable and essentially bounded functions on M. Denote by f^* the pointwise complex conjugate of a function f on M and by $\langle f, g\rangle$ the duality bracket $\int fg^*$.

Definition 1. A map T from $L^\infty(M)$ to $L^\infty(M)$ is called *positive* if $Tf \geq 0$ for $f \geq 0$. If T is positive on $L^\infty(M)$, then $T \otimes id$ is positive on matrix valued function spaces $L^\infty(M) \otimes M_n$ for all $n \in \mathbb{N}$, i.e. T is *completely positive*.

A positive map T commutes with complex conjugation, i.e. $T(f^*) = T(f)^*$. For two positive maps S, T, we will write $S \geq T$ if $S - T$ is positive.

We will need the following Kadison-Schwarz inequality for completely positive maps T,

$$|T(f)|^2 \le \|T(1)\|_{L^\infty} T(|f|^2), \qquad f \in L^\infty(M). \tag{32}$$

2.1. Positive semigroups

We will consider a semigroup $(T_t)_{t\ge 0}$ of positive, weak*-continuous contractions on L^∞ with the weak* continuity at $t = 0+$. That is a family of positive, weak*-continuous contractions $T_t, t \ge 0$ on L^∞ such that $T_sT_t = T_{s+t}$, $T_0 = id$ and $\langle T_t(f), g\rangle \to \langle f, g\rangle$ as $t \to 0+$ for any $f \in L^\infty, g \in L^1$.

Such a semigroup (T_y) always admits an infinitesimal negative generator $L = \lim_{y\to 0} \frac{id-T_y}{y}$ which has a weak*-dense domain $D(L) \subset L^\infty$. We will write $T_y = e^{-yL}$. These definitions and facts extend to the noncommutative setting. Namely, given a semifinite von Neumann algebra $\mathcal{M}$ and a normal semifinite faithful trace τ, we let $L^\infty(\mathcal{M}) = \mathcal{M}$ and $L^1(\mathcal{M})$ be the completion of $\{f \in \mathcal{M} : \|f\|_{L^1} = \tau|f| < \infty\}$. Here $|g| = (g^*g)^{\frac{1}{2}}$ and g^* denotes the adjoint operators of g and we set $\langle f, g\rangle = \tau(fg^*)$. We say a map T on $\mathcal{M}$ is completely positive if $(T \otimes id)(f) \ge 0$ for any $f \ge 0, f \in \mathcal{M} \otimes M_n$. We say f_λ weak* converges to f if $\lim_\lambda \langle f_\lambda, g\rangle = \langle f, g\rangle$ for all $g \in L^1(\mathcal{M})$ (see [25] for details).

The so-called subordinated semigroups $T_{y,\alpha} = e^{-yL^\alpha}, 0 < \alpha < 1$ are defined as

$$T_{t,\alpha}f = \int_0^\infty T_u f \phi_{t,\alpha}(u)du = \int_0^\infty T_{t^{\frac{1}{\alpha}}u} f \phi_{1,\alpha}(u)du, \tag{33}$$

with $\phi_{t,\alpha}$ given in Section 1. The generator L^α is given by

$$L^\alpha(f) = \Gamma(-\alpha)^{-1} \int_0^\infty (T_t - id)(f) t^{-1-\alpha} dt, \tag{34}$$

for $f \in D(L)$. There are other (equivalent) formulations for L^α. The formula (34) is due to Balakrishnan (see [5] and [36, page 260]). For $T_t = e^{-tz}id$ with $Re(z) \ge 0$, $L^\alpha = z^\alpha$ with a chosen principal value so that $Re(z^\alpha) \ge 0$.

$(T_{y,\alpha})$ is again a semigroup of positive weak*-continuous contractions. The semigroup has an analytic extension and has the well-known norm estimate that

$$\sup_{y>0} \|y^k \partial_y^k T_{y,\alpha}\| < c_k. \tag{35}$$

What we wish is a pointwise estimate.

Note that (33) implies

$$\frac{T_{y,\frac{1}{2}}}{y}(f) \le \frac{T_{t,\frac{1}{2}}}{t}(f) \qquad \text{and} \qquad |y^k \partial_y T_{y,\frac{1}{2}} f| \le c_{k,t} T_{t,\frac{1}{2}} f, \tag{36}$$

for any $0 \leq t \leq y$ and $f \geq 0$ because of the positivity of T_u and the precise formulation of $\phi_{y,\frac{1}{2}}$.

Corollary 1 and the identity (33) actually imply the following corollary.

Corollary 2. *For all $f \geq 0, s > 0, 0 < c, \alpha < 1$, and $j \in \mathbb{N}$, we have*

$$c^{K_1} T_{s,\alpha} f \leq T_{cs,\alpha} f \tag{37}$$

$$|s^j \partial_s^j T_{s,\alpha}(f)| \leq (\frac{10j}{1-\alpha})^j T_{\alpha s,\alpha}(f). \tag{38}$$

Remark. Corollary 2 says that, when L generanates a positive semigroup, we have the point-wise estimate that

$$|s^j \partial_s^j u(x,s)| \leq c(\frac{10j}{1-\alpha})^j |u(x,s)|$$

for the canonical solution $u(x,s) = e^{-sL^\alpha} f(x)$ of the PDE

$$(L^\alpha + \partial_s) u(x,s) = 0; u(x,0) = f(x)$$

with $f(x) \geq 0$. When $\alpha = 1$, a similar estimate to Corollary 2 may hold for some special semigroups. For example, the heat semigroups generated by the Laplacian operator on $\mathbb{R}^n$ has a similar estimate with $c > 1$. But one can not hope this in general since (38) is already stronger than the analyticity of semigroup $T_{s,\alpha}$ on L^∞, which fails for $\alpha = 1$ in general.

2.2. Γ^2 criterion

P. A Meyer's gradient form Γ (also called "Carré du Champ") associated with T_t is defined as,

$$2\Gamma_L(f,g) = -L(f^* g) + (L(f^*)g) + f^*(L(g)), \tag{39}$$

for f, g with $f^*, g, f^*g \in D(L)$. It is easy to verify that for $L = -\triangle = -\frac{\partial^2}{\partial^2 x}$, $\Gamma_L(f,g) = \nabla f^* \cdot \nabla g$.

Convention. We will write $\Gamma(f)$ for $\Gamma_L(f,f)$.

It is well known that the completely positivity of the operators T_t implies that $\Gamma(f,g)$ is a completely positive bilinear form. We then have the Cauchy-Schwartz inequality

$$\Gamma\left(\int_0^\infty a_s d\mu(s), \int_0^\infty a_s d\mu(s)\right) \leq \int_0^\infty d|\mu|(s) \int_0^\infty \Gamma(a_s, a_s) d|\mu|(s) \tag{40}$$

Bakry-Émery's Γ^2 criterion plays an important role in this article. We use an equivalent definition.

Definition 2. A semigroup of positive operator $(T_t)_t$ satisfies the $\Gamma^2 \geq 0$ criterion if $\Phi(s) = T_{s-u}|T_u f|^2, s > u$ is (midpoint) convex in u, i.e.

$$T_t|T_u f|^2 - |T_t T_u f|^2 \leq T_u(T_t|f|^2 - |T_t f|^2) \tag{41}$$

for all $t, u > 0$ and $f \in L^\infty$.

For L equal to the Laplace-Beltrami operator on a complete manifold, the $\Gamma^2 \geq 0$ criterion holds if the manifold has nonnegative Ricci curvature everywhere. The "Γ^2" criterion is satisfied by a large class of semigroups including the heat, Ornstein-Uhlenbeck, Laguerre, and Jacobi semigroups (see [2,4]), and also by the semigroups of completely positive contractions on group von Neumann algebras. We refer the reader to [3] and references therein for the so-called curvature-dimension criterion which is more general than the "Γ^2" criterion.

D. Bakry usually assumes that there exists a *-algebra $\mathcal{A}$ which is weak* dense in $L^\infty(M)$ such that $T_s(\mathcal{A}) \subset \mathcal{A} \subset D(L)$. This is not needed in this article because we will only use the form $T_{t,\alpha}\Gamma_{L^\beta}(T_{s,\alpha}f, T_{s,\alpha}g), 0 < \alpha < 1, \alpha \leq \beta \leq 1$ which is well defined as

$$-L^\beta T_{t,\alpha}[(T_{s,\alpha}f^*)(T_{s,\alpha}g)] + T_{t,\alpha}[(T_{s,\alpha}f^*)(L^\beta T_{s,\alpha}g)] + T_{t,\alpha}[(L^\beta T_{s,\alpha}f^*)(T_{s,\alpha}g)] \tag{42}$$

for all $f, g \in L^\infty$ since $T_{s,\alpha}(L^\infty) \subset D(L) \subset D(L^\alpha)$ because of (33).

We will need the following Lemma due to P.A. Meyer. We add a short proof for the convenience of the reader.

Lemma 6. *For any $f \in L^\infty$ such that $T_s f, T_s f^*, T_s|f|^2 \in D(L)$ for all $s > 0$, we have*

$$T_s|f|^2 - |T_s f|^2 = 2\int_0^s T_{s-t}\Gamma(T_t f)dt.$$

In particular, for $0 < \alpha < 1$,

$$T_{s,\alpha}|f|^2 - |T_{s,\alpha}f|^2 = 2\int_0^s T_{s-t,\alpha}\Gamma_{L^\alpha}(T_{t,\alpha}f)dt \tag{43}$$

for any $f \in L^\infty$.

Proof. For s fixed, let

$$F_t = T_{s-t}(|T_t f|^2).$$

Then

$$\begin{aligned}\frac{\partial F_t}{\partial t} &= \frac{\partial T_{s-t}}{\partial t}(|T_t f|^2) + T_{s-t}[(\frac{\partial T_t}{\partial t} f^*) f] + T_{s-t}[f^*(\frac{\partial T_t}{\partial t} f)] \\ &= -2T_{s-t}\Gamma(T_t f). \end{aligned} \tag{44}$$

Therefore

$$T_s|f|^2 - |T_s f|^2 = -F_s + F_0 = 2\int_0^s T_{s-t}\Gamma(T_t f)dt.$$

Since $T_{s,\alpha}(L^\infty) \subset D(L^\alpha)$ we get (43) for all $f \in L^\infty$. □

Remark. Equation (44) shows that the $\Gamma^2 \geq 0$ criterion implies that

$$T_s\Gamma(T_{v+t}f) \leq T_{v+s}(\Gamma(T_t f)) \tag{45}$$

for all $v, s, t > 0$ and $f \in L^\infty$ such that $T_s f, T_s f^*, T_s|f|^2 \in D(L)$ for all $s > 0$.

The following lemma says that the $\Gamma^2 \geq 0$ criterion passes to fractional powers, which could be known to some experts. We add a proof as we do not find a reference.

Lemma 7. *If $T_t = e^{-tL}$ satisfies the $\Gamma^2 \geq 0$ criterion (41), then $T_{t,\alpha} = e^{-tL^\alpha}$ satisfies (41) and (45) for all $f \in L^\infty$ and $0 < \alpha < 1$. Moreover,*

$$\Gamma_{L^\alpha}(s^j\partial_s^j T_{s,\alpha}f) \leq \left(\frac{10}{1-\alpha}\right)^j T_{s,\alpha}\Gamma_{L^\alpha}(f) \tag{46}$$

Proof. Applying (34), we have that, with $c_\alpha = -(\Gamma(-\alpha))^{-1} > 0$,

$$\begin{aligned}\Gamma_{L^\alpha}(f,f) &= c_\alpha\int_0^\infty (T_t|f|^2 - (T_t f^*)f - f^*(T_t f) + |f|^2)t^{-1-\alpha}dt \\ &= c_\alpha\int_0^\infty (T_t|f|^2 - |T_t f|^2 + |T_t f - f|^2)t^{-1-\alpha}dt,\end{aligned} \tag{47}$$

if $f, f^*, |f|^2 \in D(L)$. The integration converges because

$$\|T_t|f|^2 - |T_t f|^2\| \leq c\min\{t, 1\}, \tag{48}$$

for $f \in D(L)$. In fact, by the $\Gamma^2 \geq 0$ criterion (41), we see that

$$T_t|T_t f|^2 - |T_{2t}f|^2 \leq \frac{1}{2}(T_{2t}|f|^2 - |T_{2t}f|^2).$$

So

$$\begin{aligned}\|T_t|f|^2 - |T_t f|^2\|^{\frac{1}{2}} &\leq \|T_t|f - T_t f|^2 - |T_t(f - T_t f)|^2\|^{\frac{1}{2}} + \|T_t|T_t f|^2 - |T_{2t} f|^2\|^{\frac{1}{2}} \\ &\leq ct + 2^{-\frac{1}{2}}\|T_{2t}|f|^2 - |T_{2t} f|^2\|^{\frac{1}{2}}.\end{aligned}$$

Let $u(t) = t^{-\frac{1}{2}}\|T_t|f|^2 - |T_t f|^2\|^{\frac{1}{2}}$. We get

$$u(t) \leq ct^{\frac{1}{2}} + u(2t).$$

Since $u(t)$ is uniformly bounded on $[1, \infty)$, we get $u(t)$ is uniformly bounded on $[0, \infty)$ by iteration. This proves (48).

Applying the Cauchy-Schwartz inequality (40) and the $\Gamma^2 \geq 0$ criterion for T_t to (47), we get

$$\Gamma_{L^\alpha}(T_u f, T_u f) \leq T_u \Gamma_{L^\alpha}(f, f). \tag{49}$$

Applying the subordination formula that $T_{t,\alpha} = \int_0^\infty T_u \phi_{t,\alpha}(u) du$ and the Cauchy-Schwartz inequality (40), we obtain

$$\Gamma_{L^\alpha}(T_{t,\alpha} f, T_{t,\alpha} f) \leq T_{t,\alpha} \Gamma_{L^\alpha}(f, f). \tag{50}$$

One can easily adapt the proof to get

$$T_{u,\alpha}\Gamma_{L^\alpha}(T_{t,\alpha}T_{v,\alpha}g, T_{t,\alpha}T_{v,\alpha}g) \leq T_{u,\alpha}T_{t,\alpha}\Gamma_{L^\alpha}(T_{v,\alpha}g, T_{v,\alpha}g), \tag{51}$$

for all $g \in L^\infty$ since $T_{v,\alpha}g, T_{u,\alpha}|T_{v,\alpha}g|^2 \in D(L)$. Applying (43), we get (45) for $T_{t,\alpha}$.

Now, apply (40) to Γ_{L^α} and $a(s) = T_s f, d\mu(s) = s^j \partial_j \phi_{t,\alpha}(s) ds$; we get (46) from (33), (24), and (51). □

2.3. BMO spaces associated with semigroups of operators

BMO spaces associated with semigroup generators have been intensively studied recently (see [14]). In this article, we follow the ones studied in [24] and [28] because they are defined in a pure semigroup language. Set

$$\|f\|_{\mathrm{bmo}(L^\alpha)} = \sup_{0<t<\infty} \|T_{t,\alpha}|f|^2 - |T_{t,\alpha} f|^2\|_{L^\infty}^{\frac{1}{2}}, \tag{52}$$

$$\|f\|_{\mathrm{BMO}(L^\alpha)} = \sup_{0<t<\infty} \|T_{t,\alpha}|f - T_{t,\alpha} f|^2\|_{L^\infty}^{\frac{1}{2}}, \tag{53}$$

for $f \in L^\infty, 0 < \alpha \leq 1$.

We wish to define the space $\mathrm{BMO}(L^\alpha), 0 < \alpha \leq 1$ so that it is a dual space and L_0^∞ is weak* dense in it, to be consistent with the classical ones (where $L_0^\infty(M) = L^\infty(M)/kerL^\alpha$). In [24] and [28], this is done by using a SOT- topology in the corresponding Hilbert C* modulars. In this article, we prefer to use the following detour to avoid introducing the theory of Hilbert C* modulars. Define, for $g \in L^1$,

$$\|g\|_{H^1(L^\alpha)} = \sup\{|\langle f, g\rangle| : f \in L^\infty, \|f\|_{BMO(L^\alpha)}, \|f^*\|_{BMO(L^\alpha)} \leq 1\}. \tag{54}$$

Let $H^1(L^\alpha) = \{g \in L^1; \|g\|_{H^1} < \infty\}$. For a net $f_\lambda \in L_0^\infty(M)$, we say f_λ converges in the *weak* topology* if $\langle f_\lambda, g\rangle$ converges for any $g \in H^1(L^\alpha)$. Let BMO(L^α) be the abstract closure of $L_0^\infty(M)$ with respect to this weak* topology, that is the linear space of all weak* convergent nets $f_\lambda \in L_0^\infty(M)$. For a weak* convergent f_λ, let

$$\|\lim_\lambda f_\lambda\|_{BMO(L^\alpha)} = \sup_{\|g\|_{H^1} \leq 1} \lim_\lambda \langle f_\lambda, g\rangle.$$

It is easy to see that this coincides with (53) if $\lim_\lambda f_\lambda \in L^\infty$.

As an application of Corollary 2, we show that these BMO and bmo norms with different $0 < \alpha < 1$ are all equivalent if we assume the $\Gamma^2 \geq 0$ criterion.

Lemma 8. *Suppose L generates a weak* continuous semigroup of positive contractions, we have*

$$\|f\|_{BMO(L^\beta)} \leq \frac{c\alpha}{\beta} \|f\|_{BMO(L^\alpha)}, \tag{55}$$

$$\|f\|_{BMO(L^\beta)} \leq \frac{4}{1-\beta} \|f\|_{bmo(L^\beta)}, \tag{56}$$

for any $0 < \beta < \alpha \leq 1$. Assuming in addition that the semigroup $T_t = e^{-tL}$ satisfies the $\Gamma^2 \geq 0$ criterion (45), we have that

$$\|f\|_{BMO(L^\alpha)} \simeq \|f\|_{bmo(L^\alpha)} \simeq \|f\|_{bmo(L^\beta)}, \tag{57}$$

for all $0 < \beta, \alpha < 1$. In particular,

$$c(1-\alpha)^2 \|f\|_{BMO(L^\alpha)} \leq \|f\|_{BMO(\sqrt{L})} \leq c\|f\|_{BMO(L^\alpha)}, \tag{58}$$

for all $\frac{1}{2} < \alpha < 1$.

Proof. The argument for (55) is the same as that for the second inequality of [24, Theorem 2.6]. We sketch it here. By the Cauchy-Schwartz inequality,

$$\begin{aligned} T_{t,\beta}|f - T_{t,\beta} f|^2 &= \int_0^\infty \phi_{t,\frac{\beta}{\alpha}}(u) T_{u,\alpha} |\int_0^\infty \phi_{t,\frac{\beta}{\alpha}}(v)(f - T_{v,\alpha} f) dv|^2 du \\ &\leq \int_0^\infty \int_0^\infty \phi_{t,\frac{\beta}{\alpha}}(u) \phi_{t,\frac{\beta}{\alpha}}(v) T_{u,\alpha} |f - T_{v,\alpha} f|^2 du dv. \end{aligned}$$

It is easy to see that $\|T_{u,\alpha}|f - T_{v,\alpha} f|^2\| \leq (1 + |\ln \frac{u}{v}|)\|f\|^2_{BMO(L^\alpha)}$, so we get (55) from (26).

For the rest of this proof, we use Γ for Γ_{L^β}, T_t for $T_{t,\beta}$ and P_t for $T_{t,\frac{\beta}{2}}$ to simplify the notation. Since T_t has the quasi monotone property (37), we have

$$P_t = \int_0^\infty T_u \phi_{t,\frac{1}{2}}(u)du \geq \int_0^{t^2} \left(\frac{u}{t^2}\right)^{K_1} T_{t^2}\phi_{t,\frac{1}{2}}(u)du \geq \frac{1}{100K_1} T_{t^2}. \tag{59}$$

We now prove (56). Note

$$\begin{aligned} \|T_t|f - T_t f|^2\| &= \|T_t|f - T_t f|^2 - |T_t f - T_t T_t f|^2 + |T_t f - T_t T_t f|^2\| \\ &\leq \|f - T_t f\|^2_{bmo(L^\beta)} + \|T_t f - T_{2t} f\|^2. \end{aligned}$$

Let $\gamma = 2^{\frac{1}{K_1}}$ and $S = 2T_t - T_{\gamma t}$. Then S is a unital completely positive map because of (37). We have

$$\begin{aligned} |T_t f - T_{\gamma t} f|^2 + |Sf - T_t f|^2 &= -2|T_t f|^2 + |T_{\gamma t} f|^2 + |Sf|^2 \\ &\leq -2|T_t f|^2 + T_{\gamma t}|f|^2 + S|f|^2 \\ &\leq -2|T_t f|^2 + 2T_t|f|^2 \\ &\leq 2\|f\|^2_{bmo(L^\beta)}. \end{aligned}$$

We get by the triangle inequality that

$$\|T_t f - T_{2t} f\| \leq K_1 \sup_s \|T_s f - T_{\gamma s} f\| \leq \sqrt{2} K_1 \|f\|_{bmo(L^\beta)}.$$

Therefore,

$$\|f\|_{BMO(L^\beta)} \leq \sqrt{4 + 2K_1^2}\|f\|_{bmo(L^\beta)}.$$

To prove (57), we note that the $\Gamma^2 \geq 0$ assumption for L passes to L^α by Lemma 7. The inequality $\|f\|_{bmo} \leq (2 + \sqrt{2}\|f\|_{BMO})$ is proved in [24, Proposition 2.4] assuming the $\Gamma^2 \geq 0$ criterion. Together with (56), we get $\|f\|_{BMO(L^\alpha)} \simeq \|f\|_{bmo(L^\alpha)}$. We now show the second equivalence in (57). Note,

$$\begin{aligned} \int_0^t T_{t-s}\Gamma(T_s P_{\sqrt{t}} f)ds &= \int_0^t T_{t-s}\Gamma\left(\int_0^\infty \phi_{\sqrt{t},\frac{1}{2}}(v) T_v T_s f dv\right) ds \\ &\leq \int_0^\infty \phi_{\sqrt{t},\frac{1}{2}}(v) \int_0^t T_{t-s}\Gamma(T_v T_s f) ds dv \\ &\leq \int_0^\infty \phi_{\sqrt{t},\frac{1}{2}}(v) \int_0^t T_{t+v-\frac{t+v}{t}s}\Gamma(T_{\frac{t+v}{t}s} f) ds dv \end{aligned}$$

$$(u = \frac{t+v}{t}s) \le \int_0^\infty \phi_{\sqrt{t},\frac{1}{2}}(v)\frac{t}{t+v}\int_0^{t+v} T_{t+v-u}\Gamma(T_u f)dsdv$$
$$(43) = \int_0^\infty \phi_{\sqrt{t},\frac{1}{2}}(v)\frac{t}{t+v}(T_{t+v}|f|^2 - |T_{t+v}f|^2)dv$$
$$\le \int_0^\infty \phi_{\sqrt{t},\frac{1}{2}}(v)\frac{t}{t+v}\|f\|^2_{bmo(L^\beta)}dv < \frac{5}{6}\|f\|^2_{bmo(L^\beta)}.$$

We then have

$$\begin{aligned}
&(T_t|f|^2 - |T_t f|^2)^{\frac{1}{2}} \\
&\le (T_t|f - P_{\sqrt{t}}f|^2 - |T_t f - T_t P_{\sqrt{t}}f|^2)^{\frac{1}{2}} + (T_t|P_{\sqrt{t}}f|^2 - |T_t P_{\sqrt{t}}f|^2)^{\frac{1}{2}} \\
&\le 100K_1(P_{\sqrt{t}}|f - P_{\sqrt{t}}f|^2)^{\frac{1}{2}} + \sqrt{\frac{5}{6}}\|f\|_{bmo(L^\beta)} \\
&\le 100K_1\|f\|_{bmo(L^{\frac{\beta}{2}})} + \sqrt{\frac{5}{6}}\|f\|_{bmo(L^\beta)},
\end{aligned}$$

so

$$\|f\|_{bmo(L^\beta)} \le 1200K_1\|f\|_{bmo(L^{\frac{\beta}{2}})}.$$

Therefore,

$$\|f\|_{BMO(L^\beta)} \le 10000K_1^2\|f\|_{BMO(L^{\frac{\beta}{2}})}.$$

Applying (55), we have $\|f\|_{BMO(L^\alpha)} \simeq \|f\|_{BMO(L^\beta)}$ for all $0 < \beta, \alpha < 1$. □

Remark. The equivalence (57) fails for $\alpha = 1$ in general. See Section 4, Example 2.

3. Imaginary powers and H^∞-calculus

3.1. H^∞-calculus

Let us review some definitions and basic facts about H^∞-calculus. We refer the readers to [9,25,17] for details. For $0 < \theta < \pi$, let S_θ be the following open sector of the complex plane:

$$S_\theta = \{z \in \mathbb{C}, |\arg z| < \theta\}.$$

Recall that we say a closed operator A on a Banach space X is a *sectorial* operator of type $\omega < \pi$ if the spectrum of A is contained in $\overline{S}_\omega$, the closure of S_ω, and for any $\theta, \omega < \theta < \pi, z \notin S_\theta$, there exists c_θ such that

$$\|z(z-A)^{-1}\| \leq c_\theta.$$

We will assume that the domain of A is dense in X (or weak* dense in X when X is a dual space). We may also assume that A has dense range and is one to one by considering $A+\varepsilon$ (see [25, Lemma 3.2, 3.5]).

Let $H^\infty(S_\eta)$ be the space of all bounded analytic functions on S_η and $H_0^\infty(S_\eta)$ be the subspace of the functions $\Phi \in H^\infty(S_\eta)$ with an extra decay property that

$$|\Phi(z)| \leq \frac{c|z|^r}{(1+|z|)^{2r}},$$

for some $c, r > 0$. Then for any $\Phi \in H_0^\infty(S_\eta)$, and $\theta > \eta$,

$$\Phi(A) = \frac{1}{2\pi i} \int\limits_{\gamma_\theta} \Phi(z)(z-A)^{-1} dz \tag{60}$$

is a well defined bounded operator on $D(A)$ and its (weak*) extension is bounded on X. Here γ_θ is the boundary of S_θ oriented counterclockwise. For general $\Phi \in H^\infty(S_\eta)$, set

$$\Phi(A) = \psi(A)^{-1}(\Phi\psi)(A), \tag{61}$$

with $\psi(z) = \frac{z}{(1+z)^2}$. It turns out that the so defined $\Phi(A)$ is a closed (weak*) densely defined operator, which may not be bounded, and it coincides with $\Phi(A)$ defined as in (60) for $\Phi \in H_0^\infty(S_\eta)$. Moreover, these definitions are consistent with the definitions in the "older" functional calculus.

Definition 3. We say a (weak*) densely defined sectorial operator A of type ω has bounded $H^\infty(S_\eta)$-calculus, $\omega < \eta < \pi$, if the map $\Phi(A)$ extends to a bounded operator on X and there is a constant C such that

$$\|\Phi(A)\| \leq C\|\Phi\|_{H^\infty(S_\eta)} \tag{62}$$

for any bounded analytic function $\Phi \in H^\infty(S_\eta)$.

Remark. Suppose a densely defined sectorial A has bounded $H^\infty(S_\eta)$-calculus on Y and suppose Y is a weak* dense subspace of a dual Banach space X. Then the weak* extension of $\Phi(A)$ onto X, still denoted by $\Phi(A)$, is bounded and satisfies (62) with the same constant. So a weak* dense sectorial operator A has H^∞-calculus on X if and only if it has H^∞-calculus on the norm closure of $D(A)$.

The negative infinitesimal generator L of any uniformly bounded (weak*) strong continuous semigroup on a dual Banach space X is actually a (weak*) densely defined sectorial operator of type $\frac{\pi}{2}$ and L^α is of type $\frac{\alpha\pi}{2}$ on X. Cowling, Duong, and Hiebe &

Prüss ([8,12,19]) prove that the negative infinitesimal generator of a semigroup of positive contractions on $L^p, 1 < p < \infty$ always has the bounded $H^\infty(S_\eta)$-calculus for any $\eta > \frac{\pi}{2}$. One cannot hope to extend this to $p = \infty$. We will prove that the associated $\mathrm{BMO}(\sqrt{L})$ space is a good alternative, as desired.

Lemma 9. *Suppose A is a densely defined sectorial operator of type $\omega < \pi/2$ on a Banach space X. Assume $\int_0^\infty Ae^{-tA}a(t)dt$ is bounded on X with norm smaller than C for any function $a(t)$ with values in ± 1. Then A has a bound $H^\infty(S_\eta^0)$ calculus for any $\eta > \pi/2$.*

Proof. This is a consequence of [9, Example 4.8] by setting $a(t)$ to be the sign of $\langle Te^{-tT}u, v\rangle$ for any pair (u, v) in a dual pair (X, Y). $\square$

We are going to prove that the negative generator L of a semigroup of positive contractions satisfies the assumptions of Lemma 9. We follow an idea of E. Stein and consider scalar valued functions $a(t)$ such that

$$s\int_s^\infty \frac{|a(v-s)|^2}{v^2}dv \le c_a^2, \tag{63}$$

for all $s > 0$ and some constant c_a. Define M_a by

$$M_a(f) = \int_0^\infty a(t)\frac{\partial T_{t,\alpha}f}{\partial t}dt = \int_0^\infty a(t)L^\alpha T_{t,\alpha}f dt, \tag{64}$$

for $f \in L^\infty, 0 < \alpha < 1$. For now, we assume a is supported on a compact subset of $(0, \infty)$ so we do not worry about the convergence of the integration.

Lemma 10. *Assume that L generates a weak* continuous semigroup of positive contractions on L^∞ satisfying the $\Gamma^2 \ge 0$ criterion (45). We have*

$$\|M_a(f)\|_{bmo(L^\alpha)} \le \frac{cc_a}{(1-\alpha)^2}\|f\|_{bmo(L^\alpha)}, \tag{65}$$

$$\|M_a(f)\|_{BMO(L^\alpha)} \le \frac{cc_a}{(1-\alpha)^3}\|f\|_{BMO(L^\alpha)}, \tag{66}$$

$$\|M_a(f)\|_{BMO(L^\alpha)} \le \frac{cc_a}{(1-\alpha)^2}\|f\|_{L^\infty}, \tag{67}$$

for any $f \in L^\infty, 0 < \alpha < 1$.

Proof. We consider the case $\alpha \ge \frac{3}{4}$ only. The case $\alpha < \frac{3}{4}$ is easier and follows from this case by subordination. Recall that the $\Gamma^2 \ge 0$ assumption for L passes to L^α by Lemma 7, and $T_{t,\alpha}(L^\infty) \subset D(L^{2\alpha}), L^{2\alpha}T_{t,\alpha} = \partial_t^2 T_{t,\alpha}$. In this proof, we use Γ for Γ_{L^α} the gradient form associated with L^α, T_t for $T_{t,\alpha}$ and P_t for $T_{t,\frac{\alpha}{2}}$ to simplify the notation. Let $r = \frac{1}{1-\alpha} > 4$. We have that

$$\begin{aligned}
\int_0^t T_{rt-s}\Gamma(T_s f)ds &= \int_0^t T_{rt-s}\Gamma\left(\int_s^\infty L^\alpha T_v f dv\right) ds \\
&\leq \int_0^t T_{rt-s}\int_s^\infty \Gamma(L^\alpha T_v f)v^{\frac{3}{2}}dv \int_s^\infty v^{-\frac{3}{2}}dvds \\
&= \int_0^t T_{rt-s}\int_s^\infty \Gamma(L^\alpha T_v f)v^{\frac{3}{2}}dv 2s^{-\frac{1}{2}}ds \\
&= \int_0^\infty \int_0^{t\wedge v} 2s^{-\frac{1}{2}}T_{rt-s}ds\Gamma(L^\alpha T_v f)v^{\frac{3}{2}}dv
\end{aligned}$$

Let $S_v = \int_0^{t\wedge v} 2s^{-\frac{1}{2}}T_{rt-s}ds$. So by Lemma 6 and the $\Gamma^2 \geq 0$ criterion,

$$\begin{aligned}
\|f\|_{bmo}^2 = \sup_t &\left\| \int_0^{rt} T_{rt-s}\Gamma(T_s f)ds \right\| \\
&\leq \left\| \sup_t \int_0^{rt} T_{rt-\frac{s}{r}}\Gamma(T_{\frac{s}{r}} f)ds \right\| \\
&= \left\| \sup_t r\int_0^{t} T_{rt-s}\Gamma(T_s f)ds \right\| \\
&\leq \sup_t r\left\| \int_0^{\infty} S_v\Gamma(L^\alpha T_v)v^{\frac{3}{2}}dv \right\|.
\end{aligned}$$

So,

$$\begin{aligned}
\frac{1}{r}\|M_a f\|_{bmo}^2 &\leq \left\| \int_0^\infty S_v\Gamma(L^\alpha T_v M_a(f))v^{\frac{3}{2}}dv \right\| \\
&= \left\| \int_0^\infty S_v\Gamma(T_v \int_0^\infty a(u)L^{2\alpha}T_u f du)v^{\frac{3}{2}}dv \right\| \\
&= \left\| \int_0^\infty S_v\Gamma(\int_0^\infty a(u)L^{2\alpha}T_{u+v} f du)v^{\frac{3}{2}}dv \right\|
\end{aligned}$$

$$= \left\| \int_0^\infty v^{\frac{3}{2}} S_v \Gamma(\int_v^\infty a(u-v) \frac{1}{u} u L^{2\alpha} T_u f du) dv \right\|$$

$$\text{(Inequality (40))} \leq \left\| \int_0^\infty v^{\frac{3}{2}} S_v \Big(\int_v^\infty \frac{|a|^2}{u^2} du \int_v^\infty \Gamma\left(u L^{2\alpha} T_u f\right) du \Big) dv \right\|$$

$$\leq c_a^2 \left\| \int_0^\infty S_v \Big(\int_v^\infty \Gamma\left(u L^{2\alpha} T_u f\right) du \Big) v^{\frac{1}{2}} dv \right\|$$

$$= c_a^2 \left\| \int_0^\infty \int_0^{u\wedge t} v^{\frac{1}{2}} S_v dv \Gamma(u L^{2\alpha} T_u f) du \right\|.$$

Note $K_1 \leq r$ and $\sup_{r>4}(\frac{2}{1+\alpha}\frac{r}{r-1})^r \leq c$. By (37), we have, for $u \leq t$,

$$\int_0^{t\wedge u} v^{\frac{1}{2}} S_v dv \leq \int_0^{t\wedge u} v^{\frac{1}{2}} \int_0^{t\wedge v} s^{-\frac{1}{2}} T_{\frac{1+\alpha}{2}(rt-u)} \left(\frac{2}{1+\alpha}\cdot\frac{r}{r-1}\right)^r ds dv$$
$$\leq c T_{\frac{1+\alpha}{2}(rt-u)} t^2 \wedge u^2.$$

Applying (46), we get

$$\left\| \int_0^t \int_0^{u\wedge t} v^{\frac{1}{2}} S_v dv \Gamma(u L^{2\alpha} T_{\frac{u}{2}} T_{\frac{u}{2}} f) du \right\| \leq c r^2 \left\| \int_0^t \frac{T_{\frac{\alpha u}{2}}}{u^2} \int_0^{u\wedge t} v^{\frac{1}{2}} S_v dv \Gamma(T_{\frac{u}{2}} f) du \right\|$$
$$\leq c r^2 \left\| \int_0^t T_{\frac{(1+\alpha)rt}{2} - \frac{u}{2}} \Gamma(T_{\frac{u}{2}} f) du \right\|$$
$$\leq c r^2 \left\| \int_0^{\frac{t}{2}} T_{\frac{t}{2}-s} \Gamma(T_s f) ds \right\| \leq c r^2 \|f\|_{bmo}^2.$$

For $\alpha^{-n} t < u \leq \alpha^{-n-1} t, n \geq 0$, we use

$$\int_0^{t\wedge u} v^{\frac{1}{2}} S_v dv \leq \int_0^{t\wedge u} v^{\frac{1}{2}} \int_0^{t\wedge v} 2 s^{-\frac{1}{2}} T_{rt-t} \left(\frac{r}{r-1}\right)^r ds dv \leq c T_{rt-t} t^2 \wedge u^2.$$

Similar to (46), we get $\Gamma(u^2 L^{2\alpha} T_{\alpha^{-n}t} f) \leq c r^2 T_{2\alpha^{-n}t-u} \Gamma(f)$ because $\frac{r-1}{r-2} = \frac{1}{2-\alpha^{-1}} \leq \frac{\alpha^{-n}t}{2\alpha^{-n}t-u} \leq 1$. So

$$\Gamma(u L^{2\alpha} T_{\alpha^{-n}t} T_{u-\alpha^{-n}t} f)| \leq c \frac{r^2}{u^2} T_{2\alpha^{-n}t-u} \Gamma(T_{u-\alpha^{-n}t} f)$$

Therefore,

$$
\begin{aligned}
&\left\| \int_{\alpha^{-n}t}^{\alpha^{-n-1}t} \int_0^{u\wedge t} v^{\frac{1}{2}} S_v dv \Gamma(uL^{2\alpha} T_{\frac{u}{2}} T_{\frac{u}{2}} f) du \right\| \\
&\leq cr^2 \left\| \int_{\alpha^{-n}t}^{\alpha^{-n-1}t} \frac{t^2 \wedge u^2}{u^2} T_{2\alpha^{-n}t-u} \Gamma(T_{u-\alpha^{-n}t} f) du \right\| \\
&= cr^2 \alpha^{2n} \left\| \int_0^{\alpha^{-n-1}t(1-\alpha)} T_{\alpha^{-n}t-s} \Gamma(T_s f) ds \right\| \\
&\leq cr^2 \alpha^{2n} \|f\|_{bmo}^2.
\end{aligned}
$$

Summing up for $n \geq 0$, we get

$$
\left\| \int_t^{\infty} \int_0^{u\wedge t} v^{\frac{1}{2}} S_v dv \Gamma(uL^{2\alpha} T_{\frac{u}{2}} T_{\frac{u}{2}} f) du \right\| \leq cr^3 \|f\|_{bmo}^2.
$$

Combining the estimates above, we conclude that

$$
\|M_a(f)\|_{bmo(L^\alpha)} \leq cc_a r^2 \|f\|_{bmo(L^\alpha)}.
$$

Applying (57), we actually get

$$
\|M_a(f)\|_{BMO(L^\alpha)} \leq cc_a r^3 \|f\|_{BMO(L^\alpha)} \leq cc_a r^3 \|f\|_{L^\infty}.
$$

But we wish to get a better estimate. Note

$$
\begin{aligned}
(T_t - T_{2t}) M_a(f) &= \int_0^{\infty} a(s) \partial_s (T_{t+s} - T_{2t+s}) f ds \\
&= \int_t^{\infty} a(s-t) \partial_s (T_s - T_{t+s}) f ds \\
&\leq \left(\int_t^{\infty} \frac{|a(s-t)|^2}{s^2} ds \right)^{\frac{1}{2}} \left(\int_t^{\infty} s^2 \left| \int_0^t \partial_s^2 T_{v+s} f dv \right|^2 ds \right)^{\frac{1}{2}} \\
&\leq c_a \left(\int_t^{\infty} s^2 \int_0^t |\partial_s^2 T_{v+s} f|^2 dv ds \right)^{\frac{1}{2}}
\end{aligned}
$$

$$(by\ (38)) \leq \frac{25c_a}{(1-\alpha)^2}\left(\int_t^\infty s^{-2}\int_0^t T_{\alpha(v+s)}|f|^2 dvds\right)^{\frac{1}{2}}.$$

Therefore

$$\|(T_t - T_{2t})M_a(f)\|_{L^\infty} \leq \frac{25c_a}{(1-\alpha)^2}\|f\|_{L^\infty},$$

and hence

$$\|M_a(f)\|_{BMO(L^\alpha)} \leq \|M_a f\|_{bmo(L^\alpha)} + \sup_t \|(T_t - T_{2t})M_a(f)\|_{L^\infty} \leq cr^2 c_a\|f\|_{L^\infty}. \quad \square$$

Given $f \in L^\infty, g \in H^1(L^\alpha)$, let $\tilde{a}(t) = sign\langle L^\alpha T_{t,\alpha} f, g\rangle a(t)$. Then $\tilde{a}$ satisfies (63) if a does. We have from Lemma 10 that

$$\begin{aligned}\int_0^\infty |\langle a(t)L^\alpha T_{t,\alpha} f, g\rangle| dt &= \lim_{N,M\to\infty}\int_{\frac{1}{M}}^N |\langle a(t)L^\alpha T_{t,\alpha} f, g\rangle| dt \\ &= \lim_{N,M\to\infty}\left\langle \int_{\frac{1}{M}}^N \tilde{a}(t)L^\alpha T_{t,\alpha} f dt, g\right\rangle \\ &\leq cc_a\|M_{\tilde{a}} f\|_{BMO(L^\alpha)}\|g\|_{H^1} \\ &\leq \frac{cc_a}{(1-\alpha)^2}\|f\|_{L^\infty}\|g\|_{H^1}.\end{aligned}$$

This shows that $\lim_{N,M\to\infty}\int_{\frac{1}{M}}^N \langle a(t)L^\alpha T_{t,\alpha} f, g\rangle dt$ exists and $\int_{\frac{1}{M}}^N a(t)L^\alpha T_{t,\alpha} f dt$ weak* converges in BMO(L^α) as $N, M \to \infty$. So the integration in (64) weak* converges and M_a is well defined for all $f \in L^\infty$ and $a(t)$ satisfying (63). The weak* extension of M_a is then a bounded map from BMO(L^α) to BMO(L^α).

Theorem 2. *Suppose $a(t)$ satisfies (63). M_a extends to a bounded operator from BMO(L^α) to BMO(L^α) for $0 < \alpha < 1$. The estimates are as in Lemma 10.*

Theorem 3. *Suppose $T_t = e^{-tL}$ is a weak* continuous semigroup of positive contractions on L^∞ satisfying the $\Gamma^2 \geq 0$ criterion. Then L has a complete bounded $H^\infty(S_\eta)$ calculus on BMO($\sqrt{L}$) for any $\eta > \frac{\pi}{2}$.*

Proof. Given $\alpha \in (\frac{1}{2}, 1)$, let Y^α be the norm closure of $D(L)$ in BMO(L^α). It is easy to check that $T_{t,\alpha} = e^{-tL^\alpha}$ are contractions on Y^α. Then L^α is a densely defined sectorial operator of type $\frac{\pi}{2}$ in Y^α. Lemma 9 and Lemma 10 imply that L^α has a bounded $H^\infty(S_\eta)$ calculus on Y^α for any $\eta > \frac{\pi}{2}$. Note $\Phi(z) = \Psi(z^{\frac{1}{\alpha}}) \in S_\eta$ if $\Psi \in S_{\frac{\eta}{\alpha}}$ and $\Phi(L^\alpha) = \Psi(L)$. We conclude that L has a bounded $H^\infty(S_\eta)$ calculus on Y^α for any $\eta > \frac{\pi}{2\alpha}$. Given

$\theta > \frac{\pi}{2}$, choose $\frac{1}{2} < \alpha < 1$ so that $\alpha\theta > \frac{\pi}{2}$. Then L has a bounded $H^\infty(S_\theta)$ calculus on Y^α. Lemma 8 then implies that L has a bounded $H^\infty(S_\theta)$ calculus on $Y^{\frac{1}{2}} \simeq Y^\alpha$ and on $BMO(\sqrt{L})$ for any $\theta > \frac{\pi}{2}$, since $Y^{\frac{1}{2}}$ is weak * dense in BMO$(\sqrt{L})$ and $\Phi(L)$ is the weak* extension of its restriction on $Y^{\frac{1}{2}}$ by definition. The same argument applies to $id \otimes L$. We then obtain the completely bounded $H^\infty(S_\eta)$ calculus as well. □

3.2. Imaginary Power and Interpolation

Given $0 < \alpha < 1$, choose $\frac{\pi}{2} < \theta < \frac{\pi}{2\alpha}$. By (61), we have the identities

$$L^\alpha e^{-tL^\alpha} = \frac{1}{2\pi i}\psi(L)^{-1}\int_{\gamma_\theta} z^\alpha \psi(z) e^{-tz^\alpha}(z-L)^{-1}dz, \tag{68}$$

$$L^{i\alpha s} = \frac{1}{2\pi i}\psi(L)^{-1}\int_{\gamma_\theta} z^{i\alpha s}\psi(z)(z-L)^{-1}dz. \tag{69}$$

Note

$$z^{i\alpha s} = \Gamma(1-is)^{-1}\int_0^\infty t^{-is} z^\alpha e^{-tz^\alpha} dt. \tag{70}$$

Since these integrals converge absolutely, we can exchange the order of the integrations and get

$$L^{i\alpha s} = \Gamma(1-is)^{-1}\int_0^\infty t^{-is} L^\alpha e^{-tL^\alpha} dt. \tag{71}$$

The inequality (67) of Lemma 10 implies that

$$\|L^{i\alpha s} f\|_{BMO(L^\alpha)} \le \frac{c}{(1-\alpha)^2}\Gamma(1-is)^{-1}\|f\|_{L^\infty}.$$

Thus for $\alpha > \frac{1}{2}$,

$$\begin{aligned}\|L^{is} f\|_{BMO(L^\alpha)} &\le c\Gamma\left(1 - i\frac{s}{\alpha}\right)^{-1}\|f\|_{L^\infty} \\ &\le \frac{c}{(1-\alpha)^2(1+|s|)^{\frac{1}{2}}}\exp\left(\frac{\pi|s|}{2\alpha}\right)\|f\|_{L^\infty}.\end{aligned}$$

Choosing $\alpha = \frac{|s|}{|s|+1}$ for s large, we get

$$\|L^{is} f\|_{BMO(L^\alpha)} \le c(1+|s|)^{\frac{3}{2}}\exp\left(\frac{\pi|s|}{2}\right)\|f\|_{L^\infty}. \tag{72}$$

The same estimate holds with bmo(L^α)-norms putting on both sides of (72) because we can apply the same argument to (65) of Lemma 10 instead of (67). Applying the inequality (55) to (72), we get

$$\|L^{is}f\|_{BMO(\sqrt{L})} \leq c(1+|s|)^{\frac{3}{2}} \exp\left(\frac{\pi|s|}{2}\right) \|f\|_{L^\infty}. \tag{73}$$

Applying the inequalities (66), (58) instead of (67), (55), we will have similarly

$$\|L^{is}f\|_{BMO(\sqrt{L})} \leq c(1+|s|)^{\frac{9}{2}} \exp\left(\frac{\pi|s|}{2}\right) \|f\|_{BMO(\sqrt{L})}. \tag{74}$$

Definition 4. We say a weak* continuous semigroup of positive contractions is a *symmetric Markov* semigroup if $\langle T_t f, g\rangle = \langle f, T_t g\rangle$ for $f \in L^\infty, g \in L^1$ and it admits a standard Markov dilation in the sense of [24, page 717].

Remark. The Markov dilation assumption in the above definition holds automatically in many cases. In the commutative case (i.e. the underlying von Neumann algebra $\mathcal{M} = L^\infty(M)$), this is due to Rota (see [34, page 106, Theorem 9]). Therefore every weak* continuous semigroup of unital symmetric positive contractions is automatically a symmetric Markov semigroup. In [33] it is proven that this is the case for convolution semigroups on group von Neumann algebras. In [10,22] it is proven that this holds for the finite von Neumann algebras case. The case of a general semifinite von Neumann algebra is conjectured but there has not been a written proof.

Lemma 11. *([JM12]) Assume that $T_t = e^{-tA}$ (e.g. $A = L^\alpha$) is a symmetric Markov semigroup on a semifinite von Neumann algebra $\mathcal{M}$. Then, the following interpolation result holds*

$$[BMO(A), L_0^1(\mathcal{M})]_{\frac{1}{p}} = L_0^p(\mathcal{M})$$

for $1 < p < \infty$. Here $L_0^p(\mathcal{M}) = L^p(\mathcal{M})/ker A$.

Since $\|L^{is}\|_{L^2\to L^2} = 1$ if L generates a symmetric Markov semigroup, by interpolation, we get from (73) the following result.

Corollary 3. *Suppose $T_t = e^{-tL}$ is a symmetric Markov semigroup of operators on a semifinite von Neumann algebra $\mathcal{M}$ and satisfies the $\Gamma^2 \geq 0$ criterion. Then, L has the completely bounded $H^\infty(S_\eta)$-calculus on L^p for any $\eta > \omega_p = |\frac{\pi}{2} - \frac{\pi}{p}|, 1 < p < \infty$ and*

$$\|L^{is}\|_{L^p\to L^p} \leq c(1+|s|)^{|\frac{3}{2}-\frac{3}{p}|} \exp\left(|\frac{\pi s}{2} - \frac{\pi s}{p}|\right), \tag{75}$$

for all $1 < p < \infty$.

Remark. Let us point out that the left-hand side of the inequality on [24, line 4, page 728] misses a "$\frac{1}{2}$". It should be $\|L^{\frac{is}{2}}f\|$ instead of $\|L^{is}f\|$, because Theorem 3.3 of [24] is for the semigroup generated by $\sqrt{L}$. So the estimate of the constants $c_{s,p}$ given in [24, Corollary 5.4] is not correct. Also [21] contains a similar estimate to (75) without assuming the $\Gamma^2 \geq 0$ criterion. Their method is the transference principle and works for L^p only.

Junge, Le Merdy, and Xu ([25]) studied the H^∞-calculus in the noncommutative setting. In particular, they prove a $H^\infty(S_\eta)$-calculus property of $L : \lambda_g \mapsto |g|\lambda_g$ on $L^p(\hat{\mathbb{F}}_n)$ for all $1 < p < \infty, \eta > |\frac{\pi}{2} - \frac{\pi}{p}|$. Here $L^p(\hat{\mathbb{F}}_n)$ is the noncommutative L^p-space associated with the free group von Neumann algebra.

4. Examples

The "$\Gamma^2 \geq 0$" criterion is known to be satisfied by a large class of semigroups including the heat, Ornstein-Uhlenbeck, and Jacobi semigroups (see [2]). The results proved in this article apply to all of them. The main example in the noncommutative setting, is the semigroup of operators on a group von Neumann algebra, generated from a conditionally negative function on the underlying group (see Example 4). We will analyze a few of them in the following.

Example 1. Let $-L = \Delta$ be the Laplace-Beltrami operator on a complete Riemannian manifold with nonnegative Ricci curvatures. Then the associated heat semigroup $T_t = e^{-tL}$ is symmetric Markovian and satisfies the $\Gamma^2 \geq 0$ criterion. All the theorems of this article hold for L, and it has bounded $H^\infty(S_\eta)$ calculus on $BMO(\sqrt{L})$ for any $\eta > \frac{\pi}{2}$.

In the special case that $L = -\partial_x^2$ the Laplacian on Euclidean space $\mathbb{R}^n$, the BMO(L), bmo(L), and BMO($\sqrt{L}$) spaces are all equivalent to the classical BMO space of all functions $f \in L^1(\mathbb{R}^n, \frac{1}{1+|x|^2}dx)$ with a finite BMO norm,

$$\|f\|_{BMO(\mathbb{R}^n)} = \sup_{B \subset \mathbb{R}^n} (E_B|f - E_B f|^2)^{\frac{1}{2}} < \infty.$$

Here the supremum runs on all balls (or cubes) in $\mathbb{R}^n$ and $E_B = \frac{1}{|B|}\int_B f dx$ denotes the mean value operator. This can be verified by the integral representation of $T_t, T_{t,\frac{1}{2}}$, the convexity of $|\cdot|^2$ and the fact that $|E_B f - E_{kB} f| \lesssim \log k \|f\|_{BMO(\mathbb{R}^n)}$. By Lemma 8 we then get the equivalence between BMO($\mathbb{R}^n$) and BMO(L^α) for all $0 < \alpha \leq 1$.

Example 2. Let $L = \partial_x$ on $\mathbb{R}$. Then $T_t = e^{-tL}$ is the translation operator sending $f(\cdot)$ to $f(\cdot - t)$. It is a Markov semigroup and the $\Gamma^2 \geq 0$ criterion holds trivially. The BMO(L) space is equivalent to L_0^∞ and the bmo(L) (semi)norm vanishes. For any $0 < \alpha < 1$, BMO(L^α) is equivalent to the classical BMO($\mathbb{R}^n$) space. Indeed, by the subordination formula, we get the following integral representation for $T_{t,\frac{1}{2}} = e^{-t\sqrt{L}}$:

$$T_{t,\frac{1}{2}}f(x) = \frac{1}{2\sqrt{\pi}} \int_0^\infty f(x-s)te^{-\frac{t^2}{4s}}s^{-\frac{3}{2}}ds.$$

From this, it is easy to check that, for $I_{x,k} = [x - 2^k \frac{t^2}{4}, x - 2^{-k}\frac{t^2}{4}], k \in \mathbb{N}$,

$$c^{-1}E_{I_{x,1}}|f| \leq T_{t,\frac{1}{2}}|f|(x) \leq c\sum_k 2^{-\frac{k}{2}}E_{I_{x,k}}|f|.$$

After an elementary calculation and using the fact that

$$|E_B f - E_{kB} f| \lesssim \log k \|f\|_{BMO(\mathbb{R}^n)},$$

one can see that $\|\cdot\|_{BMO(\sqrt{L})} \simeq \|\cdot\|_{BMO(\mathbb{R}^n)}$, thus $\|\cdot\|_{BMO(L^\alpha)} \simeq \|\cdot\|_{BMO(\mathbb{R}^n)}$ for all $0 < \alpha < 1$ by Lemma 8.

By Theorem 3, L has $H^\infty(S_\eta)$-calculus on $BMO(\sqrt{L}) \simeq BMO(\mathbb{R}^n)$ for any $\eta > \frac{\pi}{2}$. It is easy to see that

$$L^{is} = P_+ e^{\frac{-s\pi}{2}}\Delta^{\frac{is}{2}} + P_- e^{\frac{s\pi}{2}}\Delta^{\frac{is}{2}}.$$

So L does not have $H^\infty(S_\theta)$-calculus on $BMO(L) \simeq L^\infty(\mathbb{R})/\mathbb{C}$ for any positive θ and

$$\|L^{is}\|_{BMO\to BMO} \simeq e^{\frac{\pi|s|}{2}}\|\Delta^{\frac{is}{2}}\|_{BMO\to BMO}$$

for $|s|$ large. This indicates that it is better to consider $BMO(\sqrt{L})$ instead of $BMO(L)$ for the purpose of this article.

Example 3. Let $-L = \frac{\partial_x^2}{2} - x\cdot\partial_x$ be the Ornstein-Uhlenbeck operator on $(\mathbb{R}^n, e^{-|x|^2}dx)$. Let $O_t f = O_{t,1} = e^{-tL}$. O_t is a symmetric Markov semigroup with respect to the Gaussian measure $d\mu = e^{-|x|^2}dx$ and satisfies the $\Gamma^2 \geq 0$ criterion. Theorem 3 says that $L = -\frac{\partial_x^2}{2} + x\cdot\partial_x$ has bounded $H^\infty(S_\eta)$-calculus on $BMO(\sqrt{L})$ for any $\eta > \frac{\pi}{2}$.

Mauceri and Meda (see [31]) introduced the following BMO space for the Ornstein-Uhlenbeck semigroup

$$\|f\|_{BMO(MM)} = \sup_{r_B \leq \min\{1, \frac{1}{|c_B|}\}} (E_B^\mu|f - E_B^\mu f|^2)^{\frac{1}{2}}, \tag{76}$$

with r_B, c_B the radius and the center of B, and $E_B^\mu = \frac{1}{\mu(B)}\int \cdot d\mu$ the mean value operator with respect to the Gaussian measure $d\mu$. Note, for the balls B satisfying $r_B \leq \min\{1, \frac{1}{|c_B|}\}$, we have the equivalence $E_B^\mu|f| \simeq E_B|f|$. One may replace E_B^μ by E_B, the mean value operator with respect to the Lebesgue measure dx in (76). The resulted BMO norms are equivalent to each other. From the integral presentation

$$O_t(f) = \frac{1}{(\pi - \pi e^{-2t})^{\frac{n}{2}}} \int_{\mathbb{R}^n} \exp\left(-\frac{|e^{-t}x - y|^2}{1 - e^{-2t}}\right) f(y)dy, \tag{77}$$

one easily see that, for $t \leq 4$ and $\sqrt{t}|x| \leq 1$,

$$O_t|f|(x) \geq \frac{1}{(\pi - \pi e^{-2t})^{\frac{n}{2}}} \int\limits_{B(x,\sqrt{t})} \exp\left(-\frac{2|x-y|^2}{1-e^{-2t}}\right) f(y)dy$$
$$\geq c_n E_{B(x,\sqrt{t})}|f|(x). \tag{78}$$

Note $E_{B(x,\sqrt{t})}|f| \leq c_n E_{B(x,\sqrt{s})}|f|$ for all $t < s < 2t$. We then have from (78) that, for $O_{t,\frac{1}{2}} = e^{-tL^{\frac{1}{2}}}, t \leq 1, tx \leq 1$,

$$O_{t,\frac{1}{2}}|f|(x) = \int\limits_0^\infty O_s|f|(x)\phi_{t,\frac{1}{2}}(s)ds \geq \frac{c}{\sqrt{t}} \int\limits_{t^2}^{4t^2} O_s|f|(x)ds \geq c_n E_{B(x,t)}|f|(x).$$

We then easily get

$$4O_{t,\alpha}|f - O_{t,\alpha}f|^2(x) \geq c_n E_{B(x,t^{\frac{1}{2\alpha}})}|f - E_{B(x,t^{\frac{1}{2\alpha}})}f(x)|^2(x), \tag{79}$$

by the convexity of $|\cdot|^2$, for $\alpha = \frac{1}{2}, 1$. Therefore,

$$\|\cdot\|_{BMO(MM)} \lesssim \|\cdot\|_{BMO(L)}, \|\cdot\|_{bmo(L)}, \|\cdot\|_{BMO(\sqrt{L})},$$

and by Lemma 8,

$$\|\cdot\|_{BMO(MM)} \lesssim \|\cdot\|_{BMO(L^\alpha)}$$

for all $0 < \alpha \leq 1$. By Theorem 3, the Ornstein-Uhlenbeck operator $L = -\frac{\partial_x^2}{2} + x \cdot \partial_x$ has bounded $H^\infty(S_\eta)$ calculus from $L^\infty(\mathbb{R}^n)$ to Mauceri-Meda's BMO(MM) for any $\eta > \frac{\pi}{2}$.

Let $f(y) = \frac{1}{\sqrt{4\pi s}} \exp(-\frac{|y|^2}{4s})$, with $s > 100$. We have

$$\begin{aligned}
&(O_t|f|^2 - |O_t f|^2)(x) \\
&= \frac{1}{4\pi\sqrt{(s+2v)s}} \exp\left(-\frac{|e^{-t}x|^2}{2s+4v}\right) - \frac{1}{4\pi(s+v)} \exp\left(-\frac{|e^{-t}x|^2}{2s+2v}\right) \\
&= (\frac{1}{4\pi\sqrt{(s+2v)s}} - \frac{1}{4\pi(s+v)}) \exp\left(-\frac{|e^{-t}x|^2}{2s+4v}\right) \\
&\quad + \frac{1}{4\pi(s+v)} \left(\exp\left(-\frac{|e^{-t}x|^2}{2s+4v}\right) - \exp\left(-\frac{|e^{-t}x|^2}{2s+2v}\right)\right) \\
&\lesssim \frac{1}{s^3} + \frac{1}{s^2} \lesssim \frac{1}{s^2}.
\end{aligned}$$

On the other hand, for $v = \frac{1-e^{-2t}}{4}, v' = \frac{1-e^{-4t}}{4}$,

$$(O_t f - O_{2t} f)(x) = \frac{1}{\sqrt{4\pi(s+v)}} e^{-\frac{|e^{-t}x|^2}{4s+4v}} - \frac{1}{\sqrt{4\pi(s+v')}} e^{-\frac{|e^{-2t}x|^2}{4s+4v'}}$$

For $x^2 = e^{2t}(4s+4v), t = 10$, we get

$$\begin{aligned} |(O_t f - O_{2t} f)(x)| &\geq |\frac{1}{\sqrt{4\pi(s+v)}} e^{-1} - \frac{1}{\sqrt{4\pi(s+v')}} e^{-\frac{1}{100}}| \\ &\geq \frac{1}{2\sqrt{4\pi(s+v')}} \geq \frac{1}{10\sqrt{s}}. \end{aligned}$$

So,

$$\|f\|_{BMO(L)} \geq \sup_{t>0} \|O_t f - O_{2t} f\|_{L^\infty} \geq \frac{\sqrt{s}}{5} \|f\|_{bmo(L)}.$$

Therefore, the BMO(L) and bmo(L)-norms are not equivalent for the Ornstein-Uhlenbeck semigroup, by letting $s \to \infty$. This shows that one can not extend Lemma 8 to the case of $\alpha = 1$.

Example 4. Let (G, μ) be a locally compact unimodular group with its Haar measure. Let $\lambda_g, g \in G$ be the translation-operator on $L^2(G)$ defined as

$$\lambda_g(f)(h) = f(g^{-1}h).$$

The so-called group von Neumann algebra $L^\infty(\hat{G})$ is the weak* closure in $B(L_2(G))$ of the operators $f = \int_G \hat{f}(g)\lambda_g d\mu(g)$ with $\hat{f} \in C_c(G)$. The canonical trace τ on $L^\infty(\hat{G})$ is defined as $\tau f = \hat{f}(e)$. If G is abelian, then $L^\infty(\hat{G})$ is the canonical L^∞ space of functions on the dual group $\hat{G}$. In particular, if $G = \mathbb{Z}$, the integer group, then $\lambda_k = e^{ikt}, k \in \mathbb{Z}$ and $L^p(\hat{\mathbb{Z}}) = L^p(\mathbb{T})$, the function space on the unit circle. Please refer to [32] for details on noncommutative L^p spaces.

Let φ be a scalar valued function on G. We say φ is *conditionally negative* if $\varphi(g^{-1}) = \varphi(g)^*$ and

$$\sum_{g,h} \overline{a_g} a_h \varphi(g^{-1}h) \leq 0 \tag{80}$$

for any finite collection of coefficients $a_g \in \mathbb{C}$ with $\sum_g a_g = 0$. Schöenberg's theorem says that

$$T_t : \lambda_g = e^{-t\varphi(g)}\lambda_g$$

extends to Markov semigroups of operators on the group von Neumann algebra $L^\infty(\hat{G})$ if and only if φ is a conditionally negative function with $\varphi(e) = 0$. The negative generator of the semigroup is the unbounded map

$$L : \lambda_g \mapsto \varphi(g)\lambda_g$$

which is weak* densely defined on $L^\infty(\hat{G})$.

Let $K_\varphi(g,h) = \frac{1}{2}(\varphi(g) + \varphi(h) - \varphi(g^{-1}h))$, the Gromov form associated with φ. Then one can directly verify from (80) that K_φ is a positive definite function on $G \times G$. Thus K_φ^2 is a positive definite function too. This is equivalent to the $\Gamma^2 \geq 0$ criterion for T_t, and therefore Theorem 3 applies to all such $(T_t)_t$'s. If in addition, φ is real valued, then (T_t) is a symmetric Markov semigroup. We then obtain the following corollary.

Corollary 4. *Let G be a locally compact unimodular group. Suppose φ is a conditionally negative function on G with $\varphi(e) = 0$. Let L be the weak* densely defined linear map on $L^\infty(\hat{G})$ such that $L(\lambda_g) = \varphi(g)\lambda_g$. Then,*

(i) For any $\eta > \frac{\pi}{2}$ and any bounded analytic Φ on S_η, the map $\Phi(L) : \lambda_g \mapsto \Phi(\varphi(g))\lambda_g$ extends to a completely bounded operator on BMO($\sqrt{L}$) and $\|\Phi(L)\| \leq C_\eta \|\Phi\|_\infty$.

(ii) Suppose in addition that φ is real valued. If Φ is a bounded analytic function on S_η with $\eta > |\frac{\pi}{2} - \frac{\pi}{p}|$, then the map $\Phi(L)$ extends to a completely bounded operator on $L^p(\hat{G})$ for $1 < p < \infty$.

Remark. Corollary 4 (i) was proved in [29] for $L : \lambda_g \mapsto \sqrt{\varphi(g)}\lambda_g$ with φ a symmetric conditionally negative function on G.

Example 5. Let $G = \mathbb{F}_\infty$ be the nonabelian free group with a countably infinite number of generators. Let $|g|$ be the reduced word length of $g \in G$. Then $\varphi : g \to |g|$ is a conditionally negative function (see [18]) and $L : \lambda_g \mapsto |g|\lambda_g$ generates a symmetric Markov semigroup on the free group von Neumann algebra. Fix $\theta \in (\frac{\pi}{2}, \pi)$, let $\Phi(z) = (\ln(z+2))^{-1}$ for $z \in S_\theta$. Then $\Phi \in H^\infty(S_\theta)$. Corollary 4 then implies that the Fourier multiplier

$$\lambda_g \mapsto \frac{1}{\ln(|g|+2)}\lambda_g$$

extends to a bounded operator on BMO($\sqrt{L}$). By the interpolation result Lemma 11, we conclude that this multiplier is bounded on $L^p(\hat{\mathbb{F}}_\infty)$ with constant $\lesssim \frac{p^2}{p-1}$.

This produces a slowly decreasing multiplier which is bounded on $L^p(\hat{\mathbb{F}}_\infty)$ for all $1 < p < \infty$. Note that Bożejko and Fendler disproved the uniform L^p boundedness of the ℓ_1-length projections P_N, that map λ_g to $\chi_{\{g:|g|<N\}}\lambda_g$, for all $p > 3$ (see [6]). So the classical method of producing slowing decreasing L^p-multipliers through P_N fails on free groups.

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