# Recoverability for Holevo's just-as-good fidelity

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***Abstract*—Holevo's just-as-good fidelity is a similarity measure for quantum states that has found several applications. One of its critical properties is that it obeys a data processing inequality: the measure does not decrease under the action of a quantum channel on the underlying states. In this paper, I prove a refinement of this data processing inequality that includes an additional term related to recoverability. That is, if the increase in the measure is small after the action of a partial trace, then one of the states can be nearly recovered by the Petz recovery channel, while the other state is perfectly recovered by the same channel. The refinement is given in terms of the trace distance of one of the states to its recovered version and also depends on the minimum eigenvalue of the other state. As such, the refinement is universal, in the sense that the recovery channel depends only on one of the states, and it is explicit, given by the Petz recovery channel.**

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## I. Introduction

In Holevo's seminal 1972 work on the quasiequivalence of locally normal states [1], he established the following inequalities for quantum states $\rho$ and $\sigma$:

$$1-\sqrt{F_H(\rho,\sigma)} \leq \frac{1}{2}\left\|\rho-\sigma\right\|_1 \leq \sqrt{1-F_H(\rho,\sigma)}, \tag{1}$$

where $\left\|\rho-\sigma\right\|_1$ denotes the well known trace distance and the function $F_H$ is Holevo's "just-as-good fidelity," defined as

$$F_H(\rho,\sigma) \equiv \left[\operatorname{Tr}\{\sqrt{\rho}\sqrt{\sigma}\}\right]^2. \tag{2}$$

After writing it down, he then remarked that "it is evident that $F_H$ is just as good a measure of proximity of the states $\rho$ and $\sigma$ as $\left\|\rho-\sigma\right\|_1$." And so it is that the measure $F_H$ is known as Holevo's just-as-good fidelity.

Some years after this, Uhlmann defined the quantum fidelity as $F(\rho,\sigma) \equiv \left\|\sqrt{\rho}\sqrt{\sigma}\right\|_1^2$ [2]. It is evident that the following relation holds

$$F_H(\rho,\sigma) \leq F(\rho,\sigma), \tag{3}$$

due to the variational characterization of the trace norm of a square operator $X$ as

$$\left\|X\right\|_1 = \max_U \left|\operatorname{Tr}\{XU\}\right|, \tag{4}$$

where the optimization is with respect to a unitary operator $U$. Many years after this, at the dawn of quantum computing, with more growing interest in quantum information theory, Fuchs and van de Graaf presented the following widely employed inequalities [3]:

$$1-\sqrt{F(\rho,\sigma)} \leq \frac{1}{2}\left\|\rho-\sigma\right\|_1 \leq \sqrt{1-F(\rho,\sigma)}, \tag{5}$$

which bear a striking similarity to (1). Indeed the lower bound on $\frac{1}{2}\left\|\rho-\sigma\right\|_1$ in (5) is an immediate consequence of (3) and the lower bound in (1). The upper bound on $\frac{1}{2}\left\|\rho-\sigma\right\|_1$ in (5) can be proven by first showing that it is achieved for pure states, employing Uhlmann's "transition probability" characterization of $F(\rho,\sigma)$ [2], and then invoking monotonicity of trace distance with respect to partial trace. The latter inequalities in (5) have been more widely employed in quantum information theory than those in (1) due to Uhlmann's "transition probability" characterization of $F(\rho,\sigma)$ and its many implications.

Nevertheless, Holevo's just-as-good fidelity is clearly a useful measure of similarity for quantum states in light of (1), and it has found several applications in quantum information theory. For example, it serves as an upper bound on the probability of error in discriminating $\rho$ from $\sigma$ in a hypothesis testing experiment [4], [5], which in some sense is just a rewriting of the lower bound in (1) (see also [6, Lemma 3.2] in this context). In turn, this way of thinking has led to particular decoders for quantum polar codes [7], [8].

The function $F_H$ has also been rediscovered a number of times. For example, it is a particular case of Petz's quasi-entropies [9], [10]. It was studied under the name "quantum affinity" in [11] and shown to be equal to the fidelity of the canonical purifications of quantum states in [12].

One of the most important properties of $F_H$ is that it obeys the following data processing inequality:

$$F_H(\mathcal{N}(\rho),\mathcal{N}(\sigma)) \geq F_H(\rho,\sigma), \tag{6}$$

where $\mathcal{N}$ is a quantum channel (a completely positive and trace preserving map). This inequality is a consequence of data processing for Petz's more general quasi-entropies [9], [10]. This property is one reason that $F_H$ has an interpretation as a similarity measure: the states $\rho$ and $\sigma$ generally become more similar under the action of a quantum channel.

The main contribution of this paper is the following refinement of the data processing inequality in (6), in the case that $\rho$ is a bipartite density operator, $\sigma$ is a positive definite bipartite operator, and the channel is a partial trace over the $B$ system:

$$\begin{aligned}\sqrt{F_H}(\rho_A,\sigma_A) \geq \sqrt{F_H}(\rho_{AB},\sigma_{AB}) \\ + \frac{\pi^2}{432}\frac{\lambda_{\min}(\sigma_{AB})}{\operatorname{Tr}\{\sigma_A\}}\left\|\mathcal{R}^{\sigma}_{A\to AB}(\rho_A)-\rho_{AB}\right\|_1^3,\end{aligned} \tag{7}$$

where $\lambda_{\min}(\sigma_{AB})$ is the minimum eigenvalue of $\sigma_{AB}$ and

$$\mathcal{R}^{\sigma}_{A\to AB}(\cdot) \equiv \sigma_{AB}^{1/2}\left[\sigma_A^{-1/2}(\cdot)_A\sigma_A^{-1/2}\otimes I_B\right]\sigma_{AB}^{1/2} \tag{8}$$

is a quantum channel known as the Petz recovery channel [13], [14]. The interpretation of this inequality is the same

as that given in previous work on this topic of refining data processing inequalities (see, e.g., [15]–[17]). If the difference $\sqrt{F_H}(\rho_A, \sigma_A) - \sqrt{F_H}(\rho_{AB}, \sigma_{AB})$ is small, then one can approximately recover the state $\rho_{AB}$ from its marginal $\rho_A$. A generalization of the result in (7) to arbitrary quantum channels is available in the appendix of arXiv:1801.02800.

The technique that I use for proving the above data processing refinement closely follows the elegant approach recently put forward by Carlen and Vershynina in [18]. This technique appears to be different from every other approach, given in recent years since [15], that has established refinements of data processing inequalities. It builds on Petz's approach from [9], [10] for proving data processing for the quantum relative entropy, as well as ideas in [19]. Here, I use this same technique and establish a general lemma regarding remainder terms for data processing with Petz's quasi-entropies, and then I specialize it to obtain the inequality in (7).

An interesting aspect of (7) is that the recovery channel is explicit, given in the Petz form, and universal, having no dependence on the state $\rho_{AB}$ while depending only on $\sigma_{AB}$.

In the rest of the paper, I begin by giving background and establish some notation. After that, I prove a general lemma that refines data processing for Petz's quasi-entropies. Then I specialize it to arrive at the inequality in (7).

## II. Background and Notation

I begin by reviewing some background and establish notation. Basic concepts of quantum information theory can be found in [6], [20], [21]. Let $f$ be an operator convex function defined on $[0, \infty)$. Examples include $f(x) = x \ln x$, $f(x) = -x^\alpha$ for $\alpha \in (0, 1)$, $f(x) = x^\alpha$ for $\alpha \in (1, 2]$. According to [22, Section 8], such a function has the following integral representation:

$$f(x) = f(0) + ax + bx^2 + \int_0^\infty d\mu(t) \ \left( \frac{x}{1+t} - 1 + \frac{t}{x+t} \right), \quad (9)$$

where $a \in \mathbb{R}$, $b \geq 0$, and $\mu$ is a non-negative measure on $(0, \infty)$ satisfying $\int_0^\infty d\mu(t) / (1+t)^2 < \infty$.

Define the maximally entangled vector as

$$|\Gamma\rangle_{S\tilde{S}} \equiv \sum_{i=0}^{|S|-1} |i\rangle_S |i\rangle_{\tilde{S}}, \quad (10)$$

for orthonormal bases $\{|i\rangle_S\}_i$ and $\{|i\rangle_{\tilde{S}}\}_i$, and for a positive semi-definite operator $\sigma$, define its canonical purification by

$$|\varphi^\sigma\rangle_{S\tilde{S}} \equiv \left( \sigma_S^{1/2} \otimes I_{\tilde{S}} \right) |\Gamma\rangle_{S\tilde{S}}. \quad (11)$$

Then, following Petz [9], [10], [23], [24], as well as what was discussed later in [25], [26], we define the $f$-quasi-relative entropy $Q_f(\rho\|\sigma)$ of a density operator $\rho$ and a positive definite operator $\sigma$ as

$$Q_f(\rho\|\sigma) \equiv \langle \varphi^\sigma |_{S\tilde{S}} \, f\left( \sigma_S^{-1} \otimes \rho_{\tilde{S}}^T \right) |\varphi^\sigma\rangle_{S\tilde{S}}. \quad (12)$$

For example, when $f(x) = x \ln x$, then $Q_f$ reduces to the quantum relative entropy from [27].

Now consider the bipartite case and define

$$|\Gamma\rangle_{A\hat{A}B\hat{B}} \equiv |\Gamma\rangle_{A\hat{A}} \otimes |\Gamma\rangle_{B\hat{B}}. \quad (13)$$

We can also write this as $|\Gamma\rangle_{AB\hat{A}\hat{B}}$ with it being understood that there is a permutation of systems. Then, by the above, we have for a density operator $\rho_{AB}$ and a positive definite operator $\sigma_{AB}$ that

$$Q_f(\rho_{AB}\|\sigma_{AB}) = \langle \varphi^{\sigma_{AB}} |_{AB\hat{A}\hat{B}} \, f\left( \sigma_{AB}^{-1} \otimes \rho_{\hat{A}\hat{B}}^T \right) |\varphi^{\sigma_{AB}}\rangle_{AB\hat{A}\hat{B}}. \quad (14)$$

Now define the linear operator $V$ by

$$V_{A\hat{A}\to AB\hat{A}\hat{B}} \equiv \sigma_{AB}^{1/2} \left( \sigma_A^{-1/2} \otimes I_{\hat{A}} \right) |\Gamma\rangle_{B\hat{B}}. \quad (15)$$

This linear operator is an isometric extension of the Petz recovery channel, as discussed recently in [28]. One can readily verify that $V$ is an isometry and that

$$V_{A\hat{A}\to AB\hat{A}\hat{B}} \, |\varphi^{\sigma_A}\rangle_{A\hat{A}} = |\varphi^{\sigma_{AB}}\rangle_{AB\hat{A}\hat{B}}, \quad (16)$$

$$V^\dagger \left( \sigma_{AB}^{-1} \otimes \rho_{\hat{A}\hat{B}}^T \right) V = \sigma_A^{-1} \otimes \rho_{\hat{A}}^T. \quad (17)$$

For simple proofs of these properties, see, e.g., [25] or [28]. With all these notions in place, we can recall Petz's approach [9], [10], [23], [24] for establishing monotonicity of the $f$-quasi-relative entropy under partial trace:

$$\begin{aligned} Q_f(\rho_{AB}\|\sigma_{AB}) &= \langle \varphi^{\sigma_{AB}} |_{AB\hat{A}\hat{B}} \, f\left( \sigma_{AB}^{-1} \otimes \rho_{\hat{A}\hat{B}}^T \right) |\varphi^{\sigma_{AB}}\rangle_{AB\hat{A}\hat{B}} \\ &= \langle \varphi^{\sigma_A} |_{A\hat{A}} \, V^\dagger f\left( \sigma_{AB}^{-1} \otimes \rho_{\hat{A}\hat{B}}^T \right) V \, |\varphi^{\sigma_A}\rangle_{A\hat{A}} \\ &\geq \langle \varphi^{\sigma_A} |_{A\hat{A}} \, f\left( V^\dagger \left[ \sigma_{AB}^{-1} \otimes \rho_{\hat{A}\hat{B}}^T \right] V \right) |\varphi^{\sigma_A}\rangle_{A\hat{A}} \\ &= \langle \varphi^{\sigma_A} |_{A\hat{A}} \, f\left( \sigma_A^{-1} \otimes \rho_{\hat{A}}^T \right) |\varphi^{\sigma_A}\rangle_{A\hat{A}} = Q_f(\rho_A\|\sigma_A) \end{aligned} \quad (18)$$

where we made use of everything above and the operator Jensen inequality [29].

## III. General statement for quasi-entropies

I now modify the approach from [18] for lower bounds for relative entropy differences to use an arbitrary operator convex function $f$ instead. So we are considering the following $f$-quasi-relative entropy difference:

$$Q_f(\rho_{AB}\|\sigma_{AB}) - Q_f(\rho_A\|\sigma_A). \quad (19)$$

Recall the integral representation of $f$ from (9). Let

$$\Delta_{AB\hat{A}\hat{B}} \equiv \sigma_{AB}^{-1} \otimes \rho_{\hat{A}\hat{B}}^T, \qquad \Delta_{A\hat{A}} \equiv \sigma_A^{-1} \otimes \rho_{\hat{A}}^T, \quad (20)$$

$$V_{A\hat{A}\to AB\hat{A}\hat{B}} \equiv \sigma_{AB}^{1/2} \left( \sigma_A^{-1/2} \otimes I_{\hat{A}} \right) |\Gamma\rangle_{B\hat{B}} \quad (21)$$

and recall from (17) that $V^\dagger \Delta_{AB\hat{A}\hat{B}} V = \Delta_{A\hat{A}}$. This implies

$$\begin{aligned} \left\| \Delta_{A\hat{A}} \right\|_\infty &= \left\| V^\dagger \Delta_{AB\hat{A}\hat{B}} V \right\|_\infty = \left\| V V^\dagger \Delta_{AB\hat{A}\hat{B}} V V^\dagger \right\|_\infty \\ &\leq \left\| \Delta_{AB\hat{A}\hat{B}} \right\|_\infty \end{aligned} \quad (22)$$

with the last equality following from isometric invariance of the operator norm and the inequality from submultiplicativity of the operator norm and the fact that $VV^\dagger$ is a projection.

*Lemma 1:* Let $\mu$ be a measure. For an operator $X$, define

$$\nu(X) = \int_0^\infty \mathrm{d}\mu(t) \ t \left( \frac{1}{t} - \frac{1}{t+X} \right), \quad (23)$$

and for $T > 0$, define $\mu([0,T]) \equiv \int_0^T \mathrm{d}\mu(t)$. For $c > 0$, define $g(c,T) \equiv \int_T^\infty \mathrm{d}\mu(t)\ \frac{1}{1+t/c}$. Let $\rho_{AB}$ be a density operator and $\sigma_{AB}$ a positive definite operator. Then for all $T > 0$, the following inequality holds

$$\begin{aligned}\left\| \left[\sigma_{AB}^{1/2}\sigma_A^{-1/2}\nu(\Delta_{A\hat{A}})\sigma_A^{1/2} - \nu(\Delta_{AB\hat{A}\hat{B}})\sigma_{AB}^{1/2}\right] |\Gamma\rangle_{A\hat{A}B\hat{B}} \right\|_2 \\ \leq \mu([0,T])^{1/2}\left[Q_f(\rho_{AB}\|\sigma_{AB}) - Q_f(\rho_A\|\sigma_A)\right]^{1/2} \\ + 2g(\left\|\Delta_{AB\hat{A}\hat{B}}\right\|_\infty, T)\operatorname{Tr}\{\sigma_A\}. \quad (24)\end{aligned}$$

**Proof.** The proof follows [18] quite closely at times but also features some departures. Since $V$ is an isometry satisfying $V^\dagger V = I_{A\hat{A}}$, it follows that $VV^\dagger$ is a projection, so that $VV^\dagger \leq I_{AB\hat{A}\hat{B}}$. Using the integral representation in (9), we arrive at the chain of inequalities in (25), where we made use of (17) and the fact that $VV^\dagger$ is a projection so that $VV^\dagger \leq I_{AB\hat{A}\hat{B}}$. Similarly, we find that

$$\begin{aligned}&Q_f(\rho_A\|\sigma_A) = \langle\varphi^{\sigma_A}|_{A\hat{A}}\left[f(\Delta_{A\hat{A}})\right]|\varphi^{\sigma_A}\rangle_{A\hat{A}} \\ &= f(0) + \langle\varphi^{\sigma_A}|_{A\hat{A}}\left[a\Delta_{A\hat{A}} + b\Delta_{A\hat{A}}^2\right]|\varphi^{\sigma_A}\rangle_{A\hat{A}} \\ &+ \int_0^\infty d\mu(t)\,\langle\varphi^{\sigma_A}|_{A\hat{A}}\left(\frac{\Delta_{A\hat{A}}}{1+t} - 1 + \frac{t}{\Delta_{A\hat{A}}+t}\right)|\varphi^{\sigma_A}\rangle_{A\hat{A}}. \quad (26)\end{aligned}$$

Thus, we find that

$$\begin{aligned}&Q_f(\rho_{AB}\|\sigma_{AB}) - Q_f(\rho_A\|\sigma_A) \geq \\ &\int_0^\infty d\mu(t)\ t\,\langle\varphi^{\sigma_A}|_{A\hat{A}}\left[V^\dagger\left(\Delta_{AB\hat{A}\hat{B}}+t\right)^{-1}V\right. \\ &\left. - \left(\Delta_{A\hat{A}}+t\right)^{-1}\right]|\varphi^{\sigma_A}\rangle_{A\hat{A}}. \quad (27)\end{aligned}$$

Now consider that for $t > 0$

$$\begin{aligned}&t\,\langle\varphi^{\sigma_A}|_{A\hat{A}}\left[V^\dagger(\Delta_{AB\hat{A}\hat{B}}+t)^{-1}V - \left(\Delta_{A\hat{A}}+t\right)^{-1}\right]|\varphi^{\sigma_A}\rangle_{A\hat{A}} \\ &= t\,\langle\varphi^{w^t}|_{AB\hat{A}\hat{B}}\left(\Delta_{AB\hat{A}\hat{B}}+t\right)|\varphi^{w^t}\rangle_{AB\hat{A}\hat{B}} \\ &\geq t^2\left\||\varphi^{w^t}\rangle_{AB\hat{A}\hat{B}}\right\|_2^2, \quad (28)\end{aligned}$$

where

$$\begin{aligned}|\varphi^{w^t}\rangle_{AB\hat{A}\hat{B}} \equiv V\left(\Delta_{A\hat{A}}+t\right)^{-1}|\varphi^{\sigma_A}\rangle_{A\hat{A}} \\ - \left(\Delta_{AB\hat{A}\hat{B}}+t\right)^{-1}|\varphi^{\sigma_{AB}}\rangle_{AB\hat{A}\hat{B}}. \quad (29)\end{aligned}$$

Consider that

$$\begin{aligned}|\varphi^{w^t}\rangle_{AB\hat{A}\hat{B}} &= \sigma_{AB}^{1/2}\sigma_A^{-1/2}\left(\Delta_{A\hat{A}}+t\right)^{-1}\sigma_A^{1/2}|\Gamma\rangle_{A\hat{A}}|\Gamma\rangle_{B\hat{B}} \\ &- \left(\Delta_{AB\hat{A}\hat{B}}+t\right)^{-1}\sigma_{AB}^{1/2}|\Gamma\rangle_{A\hat{A}}|\Gamma\rangle_{B\hat{B}} \quad (30) \\ &= \left[\sigma_{AB}^{1/2}\sigma_A^{-1/2}\left(\Delta_{A\hat{A}}+t\right)^{-1}\sigma_A^{1/2}\right. \\ &\left. - \left(\Delta_{AB\hat{A}\hat{B}}+t\right)^{-1}\sigma_{AB}^{1/2}\right]|\Gamma\rangle_{A\hat{A}}|\Gamma\rangle_{B\hat{B}}. \quad (31)\end{aligned}$$

So we set

$$\begin{aligned}w_{AB\hat{A}\hat{B}}^t \equiv \sigma_{AB}^{1/2}\sigma_A^{-1/2}\left(\Delta_{A\hat{A}}+t\right)^{-1}\sigma_A^{1/2} \\ - \left(\Delta_{AB\hat{A}\hat{B}}+t\right)^{-1}\sigma_{AB}^{1/2}, \quad (32)\end{aligned}$$

so that

$$|\varphi^{w^t}\rangle_{AB\hat{A}\hat{B}} = w_{AB\hat{A}\hat{B}}^t|\Gamma\rangle_{A\hat{A}B\hat{B}}. \quad (33)$$

Now invoking the definition in (23) we find that

$$\begin{aligned}&\sigma_{AB}^{1/2}\sigma_A^{-1/2}\nu(\Delta_{A\hat{A}})\sigma_A^{1/2} - \nu(\Delta_{AB\hat{A}\hat{B}})\sigma_{AB}^{1/2} \\ &= \int_0^\infty \mathrm{d}\mu(t)\ t\ \sigma_{AB}^{1/2}\sigma_A^{-1/2}\left(\frac{1}{t} - \frac{1}{t+\Delta_{A\hat{A}}}\right)\sigma_A^{1/2} \\ &\quad - \int_0^\infty \mathrm{d}\mu(t)\ t\left(\frac{1}{t} - \frac{1}{t+\Delta_{AB\hat{A}\hat{B}}}\right)\sigma_{AB}^{1/2} \quad (34) \\ &= \int_0^\infty \mathrm{d}\mu(t)\ t\ \left[\sigma_{AB}^{1/2}\sigma_A^{-1/2}\left(\frac{1}{t} - \frac{1}{t+\Delta_{A\hat{A}}}\right)\sigma_A^{1/2}\right. \\ &\quad \left. - \left(\frac{1}{t} - \frac{1}{t+\Delta_{AB\hat{A}\hat{B}}}\right)\sigma_{AB}^{1/2}\right] \quad (35) \\ &= \int_0^\infty \mathrm{d}\mu(t)\ t\ \left[-\sigma_{AB}^{1/2}\sigma_A^{-1/2}\frac{1}{t+\Delta_{A\hat{A}}}\sigma_A^{1/2}\right. \\ &\quad \left. + \frac{1}{t+\Delta_{AB\hat{A}\hat{B}}}\sigma_{AB}^{1/2}\right] \quad (36) \\ &= -\int_0^\infty \mathrm{d}\mu(t)\ t w_{AB\hat{A}\hat{B}}^t. \quad (37)\end{aligned}$$

Thus, for any $T > 0$, we have that

$$\begin{aligned}&\left\|\left[\sigma_{AB}^{1/2}\sigma_A^{-1/2}\nu(\Delta_{A\hat{A}})\sigma_A^{1/2} - \nu(\Delta_{AB\hat{A}\hat{B}})\sigma_{AB}^{1/2}\right]|\Gamma\rangle_{A\hat{A}B\hat{B}}\right\|_2 \\ &= \left\|\int_0^\infty \mathrm{d}\mu(t)\ t w_{AB\hat{A}\hat{B}}^t|\Gamma\rangle_{A\hat{A}B\hat{B}}\right\|_2 \quad (38) \\ &\leq \int_0^T \mathrm{d}\mu(t)\ t\left\|w_{AB\hat{A}\hat{B}}^t|\Gamma\rangle_{A\hat{A}B\hat{B}}\right\|_2 \\ &\quad + \left\|\int_T^\infty \mathrm{d}\mu(t)\ t\ w_{AB\hat{A}\hat{B}}^t|\Gamma\rangle_{A\hat{A}B\hat{B}}\right\|_2 \quad (39)\end{aligned}$$

Let us study the two terms separately. For the first term, from Cauchy–Schwarz

$$\left|\int_0^T \mathrm{d}\mu(t)f(t)g(t)\right|^2 \leq \left[\int_0^T \mathrm{d}\mu(t)f^2(t)\right]\left[\int_0^T \mathrm{d}\mu(t)g^2(t)\right],$$

we have that

$$\begin{aligned}&\left[\int_0^T \mathrm{d}\mu(t)\ t\left\|w_{AB\hat{A}\hat{B}}^t|\Gamma\rangle_{A\hat{A}B\hat{B}}\right\|_2\right]^2 \\ &\leq \mu([0,T])\int_0^T \mathrm{d}\mu(t)\ t^2\left\|w_{AB\hat{A}\hat{B}}^t|\Gamma\rangle_{A\hat{A}B\hat{B}}\right\|_2^2 \quad (40) \\ &\leq \mu([0,T])\int_0^\infty \mathrm{d}\mu(t)\ t^2\left\|w_{AB\hat{A}\hat{B}}^t|\Gamma\rangle_{A\hat{A}B\hat{B}}\right\|_2^2 \quad (41) \\ &\leq \mu([0,T])\left[Q_f(\rho_{AB}\|\sigma_{AB}) - Q_f(\rho_A\|\sigma_A)\right]. \quad (42)\end{aligned}$$

Moving to the second term, from the reasoning in the proof of [18, Theorem 1.7], we find that for any positive operator $X$

$$\begin{aligned}t\left(\frac{1}{t} - \frac{1}{t+X}\right) &\leq t\left(\frac{1}{t} - \frac{1}{t+\|X\|_\infty}\right)I \quad (43) \\ &= \frac{1}{1+t/\|X\|_\infty}I \quad (44)\end{aligned}$$

$$
\begin{aligned}
Q_f(\rho_{AB}\|\sigma_{AB}) &= \langle\varphi^{\sigma_A}|_{A\hat{A}} V^\dagger \left[f\left(\Delta_{AB\hat{A}\hat{B}}\right)\right] V |\varphi^{\sigma_A}\rangle_{A\hat{A}} \\
&= \langle\varphi^{\sigma_A}|_{A\hat{A}} V^\dagger \left[f(0) + a\Delta_{AB\hat{A}\hat{B}} + b\Delta^2_{AB\hat{A}\hat{B}} + \int_0^\infty d\mu(t)\ \left(\frac{\Delta_{AB\hat{A}\hat{B}}}{1+t} - 1 + \frac{t}{\Delta_{AB\hat{A}\hat{B}}+t}\right)\right] V |\varphi^{\sigma_A}\rangle_{A\hat{A}} \\
&= f(0) + \langle\varphi^{\sigma_A}|_{A\hat{A}} \left[aV^\dagger\Delta_{AB\hat{A}\hat{B}}V + bV^\dagger\Delta^2_{AB\hat{A}\hat{B}}V\right] |\varphi^{\sigma_A}\rangle_{A\hat{A}} \\
&\quad + \langle\varphi^{\sigma_A}|_{A\hat{A}} \left[\int_0^\infty d\mu(t)\ \left(\frac{V^\dagger\Delta_{AB\hat{A}\hat{B}}V}{1+t} - 1 + V^\dagger\frac{t}{\Delta_{AB\hat{A}\hat{B}}+t}V\right)\right] |\varphi^{\sigma_A}\rangle_{A\hat{A}} \\
&\geq f(0) + \langle\varphi^{\sigma_A}|_{A\hat{A}} \left[aV^\dagger\Delta_{AB\hat{A}\hat{B}}V + bV^\dagger\Delta_{AB\hat{A}\hat{B}}VV^\dagger\Delta_{AB\hat{A}\hat{B}}V\right] |\varphi^{\sigma_A}\rangle_{A\hat{A}} \\
&\quad + \langle\varphi^{\sigma_A}|_{A\hat{A}} \left[\int_0^\infty d\mu(t)\ \left(\frac{V^\dagger\Delta_{AB\hat{A}\hat{B}}V}{1+t} - 1 + V^\dagger\frac{t}{\Delta_{AB\hat{A}\hat{B}}+t}V\right)\right] |\varphi^{\sigma_A}\rangle_{A\hat{A}} \\
&= f(0) + \langle\varphi^{\sigma_A}|_{A\hat{A}} \left[a\Delta_{A\hat{A}} + b\Delta^2_{A\hat{A}} + \int_0^\infty d\mu(t)\ \left(\frac{\Delta_{A\hat{A}}}{1+t} - 1 + V^\dagger\left(\frac{t}{\Delta_{AB\hat{A}\hat{B}}+t}\right)V\right)\right] |\varphi^{\sigma_A}\rangle_{A\hat{A}}, \quad (25)
\end{aligned}
$$

so that $\int_T^\infty \mathrm{d}\mu(t)\ t\left(\frac{1}{t} - \frac{1}{t+X}\right) \leq g(\|X\|, T)\ I$. This leads to the development in (45), and after putting everything together, we get (24). ■

## IV. Application to Holevo's just-as-good fidelity

I now specialize the above analysis to the case of the operator convex function $-x^\alpha$ for $\alpha \in (0,1)$, and I abbreviate the corresponding quasi-entropy as $Q_\alpha$. For this case, from [22, Section 8], we have that $d\mu(t) = \frac{\sin(\alpha\pi)}{\pi} t^{\alpha-1}\,\mathrm{d}t$. Plugging into the quantities in Lemma 1, we find that

$$
\mu([0,T]) = \frac{\sin(\alpha\pi)}{\pi}\int_0^T t^{\alpha-1}\,\mathrm{d}t = \frac{\sin(\alpha\pi)}{\alpha\pi}T^\alpha. \quad (46)
$$

We also find that

$$
\begin{aligned}
&g(\left\|\Delta_{AB\hat{A}\hat{B}}\right\|_\infty, T) \\
&= \int_T^\infty \mathrm{d}\mu(t)\ \frac{1}{1 + t/\left\|\Delta_{AB\hat{A}\hat{B}}\right\|_\infty} \quad (47)\\
&= \frac{\sin(\alpha\pi)}{\pi}\int_T^\infty \mathrm{d}t\ t^{\alpha-1}\frac{1}{1 + t/\left\|\Delta_{AB\hat{A}\hat{B}}\right\|_\infty} \quad (48)\\
&\leq \frac{\sin(\alpha\pi)}{\pi}\int_T^\infty \mathrm{d}t\ t^{\alpha-1}\frac{1}{t/\left\|\Delta_{AB\hat{A}\hat{B}}\right\|_\infty} \quad (49)\\
&= \frac{\sin(\alpha\pi)\left\|\Delta_{AB\hat{A}\hat{B}}\right\|_\infty}{\pi T^{1-\alpha}(1-\alpha)}. \quad (50)
\end{aligned}
$$

Furthermore, we have that

$$
\begin{aligned}
\nu(X) &= \int_0^\infty \mathrm{d}\mu(t)\ t\left(\frac{1}{t} - \frac{1}{t+X}\right) \quad (51)\\
&= \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty \mathrm{d}t\ t^{\alpha-1} t\left(\frac{1}{t} - \frac{1}{t+X}\right) = X^\alpha. \quad (52)
\end{aligned}
$$

Substituting into (24), we find that

$$
\begin{aligned}
&\left\|\left[\sigma_{AB}^{1/2}\sigma_A^{-1/2}\Delta^\alpha_{A\hat{A}}\sigma_A^{1/2} - \Delta^\alpha_{AB\hat{A}\hat{B}}\sigma_{AB}^{1/2}\right]|\Gamma\rangle_{A\hat{A}B\hat{B}}\right\|_2 \leq \\
&\left[\frac{\sin(\alpha\pi)}{\alpha\pi}T^\alpha\right]^{1/2}\left[Q_\alpha(\rho_{AB}\|\sigma_{AB}) - Q_\alpha(\rho_A\|\sigma_A)\right]^{1/2} \\
&\qquad + \frac{2\sin(\alpha\pi)\left\|\Delta_{AB\hat{A}\hat{B}}\right\|_\infty}{\pi T^{1-\alpha}(1-\alpha)}\operatorname{Tr}\{\sigma_A\}. \quad (53)
\end{aligned}
$$

We can consider this for an arbitrary $\alpha \in (0,1)$, but the most interesting and physically relevant case seems to occur when $\alpha = 1/2$. So I now prove the claim in (7).

For $\alpha = 1/2$, the lower bound in (53) simplifies to

$$
\begin{aligned}
&\left\|\left[\sigma_{AB}^{1/2}\sigma_A^{-1/2}\rho_A^{1/2} - \rho_{AB}^{1/2}\right]|\Gamma\rangle_{A\hat{A}B\hat{B}}\right\|_2 \\
&\qquad = \left\|\sigma_{AB}^{1/2}\sigma_A^{-1/2}\rho_A^{1/2} - \rho_{AB}^{1/2}\right\|_2, \quad (54)
\end{aligned}
$$

while the upper bound in (53) becomes

$$
\begin{aligned}
&[2/\pi]^{1/2}\,T^{1/4}\left[Q_{1/2}(\rho_{AB}\|\sigma_{AB}) - Q_{1/2}(\rho_A\|\sigma_A)\right]^{1/2} \\
&\qquad + \frac{4\left\|\Delta_{AB\hat{A}\hat{B}}\right\|_\infty}{\pi T^{1/2}}\operatorname{Tr}\{\sigma_A\}. \quad (55)
\end{aligned}
$$

Now minimizing over $T > 0$ gives the choice

$$
T = \left[\frac{8\left\|\Delta_{AB\hat{A}\hat{B}}\right\|_\infty \operatorname{Tr}\{\sigma_A\}}{[2\pi]^{1/2}\left[Q_{1/2}(\rho_{AB}\|\sigma_{AB}) - Q_{1/2}(\rho_A\|\sigma_A)\right]^{1/2}}\right]^{4/3}, \quad (56)
$$

leading to the upper bound

$$
\begin{aligned}
&(3/[2^{2/3}])\Big[(8/\pi^2)[Q_{1/2}(\rho_{AB}\|\sigma_{AB}) \\
&\qquad - Q_{1/2}(\rho_A\|\sigma_A)]\left\|\Delta_{AB\hat{A}\hat{B}}\right\|_\infty \operatorname{Tr}\{\sigma_A\}\Big]^{1/3}. \quad (57)
\end{aligned}
$$

Thus, the final inequality is

$$
\begin{aligned}
&\frac{\pi^2}{54\left\|\Delta_{AB\hat{A}\hat{B}}\right\|_\infty \operatorname{Tr}\{\sigma_A\}}\left\|\sigma_{AB}^{1/2}\sigma_A^{-1/2}\rho_A^{1/2} - \rho_{AB}^{1/2}\right\|_2^3 \\
&\qquad \leq Q_{1/2}(\rho_{AB}\|\sigma_{AB}) - Q_{1/2}(\rho_A\|\sigma_A). \quad (58)
\end{aligned}
$$

Using definitions, this is then equivalent to

$$
\begin{aligned}
&\frac{\pi^2}{54\left\|\Delta_{AB\hat{A}\hat{B}}\right\|_\infty \operatorname{Tr}\{\sigma_A\}}\left\|\sigma_{AB}^{1/2}\sigma_A^{-1/2}\rho_A^{1/2} - \rho_{AB}^{1/2}\right\|_2^3 \\
&\qquad \leq \operatorname{Tr}\{\rho_A^{1/2}\sigma_A^{1/2}\} - \operatorname{Tr}\{\rho_{AB}^{1/2}\sigma_{AB}^{1/2}\}. \quad (59)
\end{aligned}
$$

$$
\begin{aligned}
&\left\| \int_T^\infty \mathrm{d}\mu(t)\ t\ w^t_{AB\hat{A}\hat{B}} \left|\Gamma\right\rangle_{AB\hat{A}\hat{B}} \right\|_2 \\
&\leq \left\| \int_T^\infty \mathrm{d}\mu(t)\ t \left[ \left( t^{-1} - \left( \Delta_{AB\hat{A}\hat{B}} + t \right)^{-1} \right) \sigma_{AB}^{1/2} - \sigma_{AB}^{1/2} \sigma_A^{-1/2} \left( t^{-1} - \left( \Delta_{A\hat{A}} + t \right)^{-1} \right) \sigma_A^{1/2} \right] \left|\Gamma\right\rangle_{AB\hat{A}\hat{B}} \right\|_2 \\
&\leq \left\| \int_T^\infty \mathrm{d}\mu(t)\ t \left( t^{-1} - \left( \Delta_{AB\hat{A}\hat{B}} + t \right)^{-1} \right) \sigma_{AB}^{1/2} \left|\Gamma\right\rangle_{AB\hat{A}\hat{B}} \right\|_2 + \left\| \int_T^\infty \mathrm{d}\mu(t)\ t\ \sigma_{AB}^{1/2} \sigma_A^{-1/2} \left( t^{-1} - \left( \Delta_{A\hat{A}} + t \right)^{-1} \right) \sigma_A^{1/2} \left|\Gamma\right\rangle_{AB\hat{A}\hat{B}} \right\|_2 \\
&= \left\| \int_T^\infty \mathrm{d}\mu(t)\ t \left( t^{-1} - \left( \Delta_{AB\hat{A}\hat{B}} + t \right)^{-1} \right) \sigma_{AB}^{1/2} \left|\Gamma\right\rangle_{AB\hat{A}\hat{B}} \right\|_2 + \left\| \int_T^\infty \mathrm{d}\mu(t)\ t \left( t^{-1} - \left( \Delta_{A\hat{A}} + t \right)^{-1} \right) \sigma_A^{1/2} \left|\Gamma\right\rangle_{A\hat{A}} \right\|_2 \\
&\leq g(\left\| \Delta_{AB\hat{A}\hat{B}} \right\|_\infty, T) \left\| \sigma_{AB}^{1/2} \left|\Gamma\right\rangle_{AB\hat{A}\hat{B}} \right\|_2 + g(\left\| \Delta_{A\hat{A}} \right\|_\infty, T) \left\| \sigma_A^{1/2} \left|\Gamma\right\rangle_{A\hat{A}} \right\|_2 \\
&= \left[ g(\left\| \Delta_{AB\hat{A}\hat{B}} \right\|_\infty, T) + g(\left\| \Delta_{A\hat{A}} \right\|_\infty, T) \right] \mathrm{Tr}\{\sigma_A\} \leq 2g(\left\| \Delta_{AB\hat{A}\hat{B}} \right\|_\infty, T) \mathrm{Tr}\{\sigma_A\}. \qquad (45)
\end{aligned}
$$

The estimate from [18, Lemma 2.2] then gives

$$
\frac{\pi^2/432}{\left\| \Delta_{AB\hat{A}\hat{B}} \right\|_\infty \mathrm{Tr}\{\sigma_A\}} \left\| \sigma_{AB}^{1/2} \sigma_A^{-1/2} \rho_A \sigma_A^{-1/2} \sigma_{AB}^{1/2} - \rho_{AB} \right\|_1^3 \leq \mathrm{Tr}\{\rho_A^{1/2} \sigma_A^{1/2}\} - \mathrm{Tr}\{\rho_{AB}^{1/2} \sigma_{AB}^{1/2}\}. \qquad (60)
$$

Observe that $\left\| \Delta_{AB\hat{A}\hat{B}} \right\|_\infty = \left\| \sigma_{AB}^{-1} \otimes \rho_{\hat{A}\hat{B}}^T \right\|_\infty \leq \frac{1}{\lambda_{\min}(\sigma_{AB})}$ because $\rho_{AB}$ is a density operator. So we then get

$$
\frac{\pi^2}{54} \frac{\lambda_{\min}(\sigma_{AB})}{\mathrm{Tr}\{\sigma_A\}} \left\| \sigma_{AB}^{1/2} \sigma_A^{-1/2} \rho_A^{1/2} - \rho_{AB}^{1/2} \right\|_2^3 \leq \mathrm{Tr}\{\rho_A^{1/2} \sigma_A^{1/2}\} - \mathrm{Tr}\{\rho_{AB}^{1/2} \sigma_{AB}^{1/2}\}, \qquad (61)
$$

$$
\frac{\pi^2}{432} \frac{\lambda_{\min}(\sigma_{AB})}{\mathrm{Tr}\{\sigma_A\}} \left\| \sigma_{AB}^{1/2} \sigma_A^{-1/2} \rho_A \sigma_A^{-1/2} \sigma_{AB}^{1/2} - \rho_{AB} \right\|_1^3 \leq \mathrm{Tr}\{\rho_A^{1/2} \sigma_A^{1/2}\} - \mathrm{Tr}\{\rho_{AB}^{1/2} \sigma_{AB}^{1/2}\}, \qquad (62)
$$

the latter of which being what was claimed in (7).

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