



Finite Blaschke products and the construction of rational Γ -inner functions [☆]



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ABSTRACT

Let

$$\Gamma \stackrel{\text{def}}{=} \{(z + w, zw) : |z| \leq 1, |w| \leq 1\} \subset \mathbb{C}^2.$$

A Γ -inner function is a holomorphic map h from the unit disc \mathbb{D} to Γ whose boundary values at almost all points of the unit circle \mathbb{T} belong to the distinguished boundary $b\Gamma$ of Γ . A rational Γ -inner function h induces a continuous map $h|_{\mathbb{T}}$ from \mathbb{T} to $b\Gamma$. The latter set is topologically a Möbius band and so has fundamental group \mathbb{Z} . The degree of h is defined to be the topological degree of $h|_{\mathbb{T}}$. In a previous paper the authors showed that if $h = (s, p)$ is a rational Γ -inner function of degree n then $s^2 - 4p$ has exactly n zeros in the closed unit disc \mathbb{D}^- , counted with an appropriate notion of multiplicity. In this paper, with the aid of a solution of an interpolation problem for finite Blaschke products, we explicitly construct the rational Γ -inner functions of degree n with the n zeros of $s^2 - 4p$ prescribed.

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1. Introduction

The *symmetrized bidisc* is the set

$$\Gamma \stackrel{\text{def}}{=} \{(z + w, zw) : |z| \leq 1, |w| \leq 1\} \subset \mathbb{C}^2.$$

Γ has attracted considerable interest in recent years because of its rich function theory [16,1,32], complex geometry [21,24,28,34,35,38], some associated operator theory [6,5,14,15,37,36,41] and its connection with

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the difficult problem of μ -synthesis [12,2,43]. The distinguished boundary of Γ , that is, the Šilov boundary of the algebra of continuous functions on Γ that are holomorphic in the interior of Γ , will be denoted by $b\Gamma$. Concretely, $b\Gamma$ is the symmetrization of the 2-torus [7, Theorem 2.4]:

$$b\Gamma = \{(z + w, zw) : |z| = |w| = 1\}.$$

A Γ -inner function is a holomorphic map h from the unit disc \mathbb{D} to Γ whose boundary values at almost all points of the unit circle \mathbb{T} (with respect to Lebesgue measure) belong to $b\Gamma$. The Γ -inner functions constitute a natural analog of the *inner functions* of A. Beurling [13], which play a central role in the function theory of the unit disc. For example, it was known to Nevanlinna and Pick that an n -point interpolation problem for functions in the Schur class is solvable if and only if it is solvable by a rational inner function of degree at most n . Likewise, every n -point interpolation problem for functions in the class $\text{Hol}(\mathbb{D}, \Gamma)$ of holomorphic maps from \mathbb{D} to Γ , if solvable, has a rational Γ -inner solution (for example, [22, Theorem 4.2]). Here, the *degree* of a rational Γ -inner function h is defined to be the topological degree of the restriction of h mapping \mathbb{T} continuously to $b\Gamma$. Since $b\Gamma$ is homeomorphic to a Möbius band, its fundamental group is \mathbb{Z} , and so the degree of h is an integer; it will be denoted by $\deg(h)$.

We shall address the analog for rational Γ -inner functions of a problem about rational inner functions solved by W. Blaschke [17]. The Argument Principle tells us that a rational inner function φ of degree n has exactly n zeros in \mathbb{D} , counted with multiplicity, from which fact one deduces that φ is a finite Blaschke product

$$\varphi(\lambda) = c \prod_{j=1}^n \frac{\lambda - \alpha_j}{1 - \bar{\alpha}_j \lambda}$$

where $|c| = 1$ and $\alpha_1, \dots, \alpha_n$ are the zeros of φ . In similar fashion, we should like to write down, as explicitly as possible, the general rational Γ -inner function of degree n . It was shown in [3] that if $h = (s, p)$ is a rational Γ -inner function of degree n then $s^2 - 4p$ has exactly n zeros in the closed unit disc \mathbb{D}^- , counted with an appropriate notion of multiplicity. The n zeros of $s^2 - 4p$ can be regarded as analogs of the α_j for present purposes.

The variety

$$\begin{aligned} \mathcal{R} &\stackrel{\text{def}}{=} \{(2z, z^2) : z \in \mathbb{C}\} \\ &= \{(s, p) \in \mathbb{C}^2 : s^2 = 4p\} \end{aligned} \tag{1.1}$$

plays a special role in the function theory of Γ : it is called the *royal variety*. For a rational Γ -inner function $h = (s, p)$, the zeros of $s^2 - 4p$ in \mathbb{D}^- are the points λ such that $h(\lambda) \in \mathcal{R}$; we shall call them the *royal nodes* of h . If $\sigma \in \mathbb{D}^-$ is a royal node of h then $h(\sigma) = (-2\eta, \eta^2)$ for some $\eta \in \mathbb{D}^-$; we call η the *royal value* of h corresponding to the royal node σ .

Let us formalize the problem of describing the general rational Γ -inner function in terms of its royal nodes and values.

Problem 1.1. Given distinct points $\sigma_1, \dots, \sigma_n$ in \mathbb{D}^- and values η_1, \dots, η_n in \mathbb{D}^- find if possible a rational Γ -inner function h of degree n such that

$$h(\sigma_j) = (-2\eta_j, \eta_j^2) \quad \text{for } j = 1, \dots, n.$$

The results of this paper show that there is a close connection between Problem 1.1 and an n -point interpolation problem for finite Blaschke products of degree n in which there are interpolation nodes in

both \mathbb{D} and \mathbb{T} and in which tangential information is specified at interpolation nodes in \mathbb{T} . To formulate this problem we introduce some terminology.

Definition 1.2. Let $n \geq 1$ and $0 \leq k \leq n$. By *Blaschke interpolation data* we mean a triple (σ, η, ρ) where

- (1) $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ is an n -tuple of distinct points such that $\sigma_j \in \mathbb{T}$ for $j = 1, \dots, k$ and $\sigma_j \in \mathbb{D}$ for $j = k+1, \dots, n$;
- (2) $\eta = (\eta_1, \eta_2, \dots, \eta_n)$ where $\eta_j \in \mathbb{T}$ for $j = 1, \dots, k$ and $\eta_j \in \mathbb{D}$ for $j = k+1, \dots, n$;
- (3) $\rho = (\rho_1, \rho_2, \dots, \rho_k)$ where $\rho_j > 0$ for $j = 1, \dots, k$.

For such data the *Blaschke interpolation problem* with data (σ, η, ρ) is the following:

Problem 1.3. Find if possible a rational inner function φ on \mathbb{D} (that is, a finite Blaschke product) of degree n with the properties

$$\varphi(\sigma_j) = \eta_j \quad \text{for } j = 1, \dots, n \quad (1.2)$$

and

$$A\varphi(\sigma_j) = \rho_j \quad \text{for } j = 1, \dots, k, \quad (1.3)$$

where $A\varphi(e^{i\theta})$ denotes the rate of change of the argument of $\varphi(e^{i\theta})$ with respect to θ .

Problem 1.3 has been much studied, for example [39,40,11,29,26,42,27]. Without the tangential conditions (1.3), or some other constraint (for example, a degree constraint), the problem would arguably be ill-posed: solvability would depend only on the interpolation conditions at nodes in \mathbb{D} , and the conditions at $\sigma_1, \dots, \sigma_k$ would be irrelevant. With the conditions (1.3), however, the problem has an elegant solution. There is a simple criterion for the existence of a solution of **Problem 1.3** in terms of an associated “Pick matrix”, and better still, there is an explicit parametrization of all solutions φ by a linear fractional expression in terms of a parameter $\zeta \in \mathbb{T}$. There are polynomials a, b, c and d of degree at most n such that the general solution of **Problem 1.3** is

$$\varphi = \frac{a\zeta + b}{c\zeta + d} \quad (1.4)$$

where the parameter ζ ranges over a cofinite subset of \mathbb{T} (see **Theorem 3.3** below). The polynomials a, b, c and d are unique subject to a certain normalization.

Analogously, **Problem 1.1** needs to be modified by the addition of tangential conditions at interpolation nodes in \mathbb{T} in order to be well posed. We are led to the following refinement of **Problem 1.1**.

Problem 1.4. Given Blaschke interpolation data (σ, η, ρ) with n interpolation nodes of which k lie in \mathbb{T} , find if possible a rational Γ -inner function $h = (s, p)$ of degree n such that

$$h(\sigma_j) = (-2\eta_j, \eta_j^2) \quad \text{for } j = 1, \dots, n$$

and

$$Ap(\sigma_j) = 2\rho_j \quad \text{for } j = 1, \dots, k.$$

We shall call this the *royal Γ -interpolation problem* with data (σ, η, ρ) .

The connection between [Problems 1.4 and 1.3](#) can be described with the aid of a certain 1-parameter family of rational functions Φ_ω on Γ , where $\omega \in \mathbb{T}$. These functions play a central role in the function theory of Γ (for example, [\[6,7\]](#)). They are defined by

$$\Phi_\omega(s, p) = \frac{2\omega p - s}{2 - \omega s}. \quad (1.5)$$

Φ_ω is holomorphic on Γ , except for a singularity at $(2\bar{\omega}, \bar{\omega}^2)$, and maps Γ into \mathbb{D}^- . They constitute a universal set of Carathéodory extremal functions for the interior of Γ [\[7, Corollary 3.4\]](#).

A consequence of [Theorems 4.4 and 4.9](#) is:

Theorem 1.5. *For Blaschke interpolation data (σ, η, ρ) the following two statements are equivalent.*

- (1) [Problem 1.4](#) with data (σ, η, ρ) is solvable by a rational Γ -inner function h such that $h(\mathbb{D}) \not\subset \mathcal{R}$;
- (2) [Problem 1.3](#) with data (σ, η, ρ) is solvable and there exist $s_0, p_0 \in \mathbb{C}$ such that

$$\begin{aligned} |s_0| &< 2, \quad |p_0| = 1, \\ s_0 &= \bar{s}_0 p_0, \\ s_0 a - 2b + 2p_0 c - s_0 d &= 0, \end{aligned}$$

where a, b, c and d are the polynomials in the normalized parametrization [\(1.4\)](#) of the solutions of [Problem 1.3](#).

The Γ -inner functions whose range is contained in \mathcal{R} , those of the form $(2f, f^2)$ for some inner f , behave differently from others.

[Theorem 4.9](#) gives a formula for a solution h of [Problem 1.4](#) in terms of s_0, p_0, a, b, c and d . Since the polynomials a, b, c and d are computed in [Theorem 3.9](#) and [Remark 3.11](#), we obtain an explicit solution of [Problem 1.4](#). The algorithm is presented in [Section 5](#).

The connection between the solution sets of the royal Γ -interpolation problem and the Blaschke interpolation problem can be made explicit with the aid of the functions Φ_ω .

Theorem 1.6. *Let (σ, η, ρ) be Blaschke interpolation data. Suppose that h is a solution of [Problem 1.4](#) with these data and that $h(\mathbb{D}) \not\subset \mathcal{R}$. For all $\omega \in \mathbb{T} \setminus \{-\bar{\eta}_1, -\bar{\eta}_2, \dots, -\bar{\eta}_k\}$, the function $\Phi_\omega \circ h$ is a solution of [Problem 1.3](#) with the same data. Conversely, for every solution φ of the Blaschke interpolation problem with data (σ, η, ρ) , there exists $\omega \in \mathbb{T}$ such that $\varphi = \Phi_\omega \circ h$.*

This theorem is a corollary to [Theorem 4.4](#).

In an earlier paper [\[3\]](#) the authors gave another construction of the general rational Γ -inner function $h = (s, p)$ of degree n , starting from different data, to wit, the royal nodes of h and the zeros of s . One step in the construction in [\[3\]](#) is to perform a Fejér–Riesz factorization of a non-negative trigonometric polynomial, whereas, in contrast, the construction in this paper can be carried out entirely in rational arithmetic.

2. Background material

In this section we establish some notation and terminology and present some elementary facts about the set Γ discussed in the introduction.

The following results afford useful criteria for membership of Γ and $b\Gamma$ [\[7\]](#).

Proposition 2.1. Let $(s, p) \in \mathbb{C}^2$. The point (s, p) lies in Γ if and only if

$$|s| \leq 2 \text{ and } |s - \bar{s}p| \leq 1 - |p|^2.$$

The point (s, p) lies in $b\Gamma$ if and only if

$$|s| \leq 2, \quad |p| = 1, \text{ and } s - \bar{s}p = 0.$$

The interior of Γ , the *open symmetrized bidisc*

$$\mathcal{G} \stackrel{\text{def}}{=} \{(z + w, zw) : |z| < 1, |w| < 1\} \quad (2.1)$$

will also arise.

Proposition 2.1 implies that if $h = (s, p) \in \text{Hol}(\mathbb{D}, \mathbb{C}^2)$ then h is Γ -inner if and only if p is inner, $|s|$ is bounded by 2 on \mathbb{D} and $s(\tau) - \overline{s(\tau)}p(\tau) = 0$ for almost all $\tau \in \mathbb{T}$ with respect to Lebesgue measure (by Fatou's theorem, s and p have non-tangential limits a.e. on \mathbb{T}). This paper focuses on the case that h is rational (that is, s and p are rational), in which case $s = \bar{s}p$ on the whole of \mathbb{T} .

Let us clarify the notion of the degree of a rational Γ -inner function h .

Definition 2.2. The degree $\deg(h)$ of a rational Γ -inner function h is defined to be $h_*(1)$, where $h_* : \mathbb{Z} = \pi_1(\mathbb{T}) \rightarrow \pi_1(b\Gamma)$ is the homomorphism of fundamental groups induced by h when it is regarded as a continuous map from \mathbb{T} to $b\Gamma$.

According to [3, Proposition 3.3], for any rational Γ -inner function $h = (s, p)$, $\deg(h)$ is equal to the degree $\deg(p)$ (in the usual sense) of the finite Blaschke product p .

We denote by \mathcal{S} the *Schur class*, which comprises all holomorphic maps from \mathbb{D} to \mathbb{D}^- .

Definition 2.3. For any differentiable function $f : \mathbb{T} \rightarrow \mathbb{C} \setminus \{0\}$ the *phasar derivative* of f at $z = e^{i\theta} \in \mathbb{T}$ is the derivative with respect to θ of the argument of $f(e^{i\theta})$ at z ; we denote it by $Af(z)$.

Thus, if $f(e^{i\theta}) = R(\theta)e^{ig(\theta)}$ is differentiable, where $g(\theta) \in \mathbb{R}$ and $R(\theta) > 0$, then g is differentiable on $[0, 2\pi)$ and the phasar derivative of f at $z = e^{i\theta} \in \mathbb{T}$ is equal to

$$Af(e^{i\theta}) = \frac{d}{d\theta} \arg f(e^{i\theta}) = g'(\theta). \quad (2.2)$$

The above is not standard notation, but we shall find it useful in the sequel. We summarize some elementary properties of phasar derivatives.

Proposition 2.4.

(1) For differentiable functions $\psi, \varphi : \mathbb{T} \rightarrow \mathbb{C} \setminus \{0\}$ and for any $c \in \mathbb{C} \setminus \{0\}$,

$$A(\psi\varphi) = A\psi + A\varphi \quad \text{and} \quad A(c\psi) = A\psi. \quad (2.3)$$

(2) For any rational inner function φ and for all $z \in \mathbb{T}$,

$$A\varphi(z) = z \frac{\varphi'(z)}{\varphi(z)}. \quad (2.4)$$

(3) If $\alpha \in \mathbb{D}$ and

$$B_\alpha(z) = \frac{z - \alpha}{1 - \bar{\alpha}z}$$

then

$$AB_\alpha(z) = \frac{1 - |\alpha|^2}{|z - \alpha|^2} > 0 \quad \text{for } z \in \mathbb{T}.$$

(4) For any rational inner function p ,

$$Ap(z) > 0 \quad \text{for all } z \in \mathbb{T}.$$

Recall that a point $\lambda \in \mathbb{D}^-$ is a royal node of a Γ -inner function h if and only if $h(\lambda)$ is in the royal variety $\mathcal{R} = \{(2z, z^2) : z \in \mathbb{C}\}$.

In the next proposition we shall use the notation $\Phi(z, s, p)$ as a synonym for $\Phi_z(s, p)$. Thus, for any function v on \mathbb{D} ,

$$\Phi \circ (v, h) = \frac{2vp - s}{2 - vs}.$$

Proposition 2.5. Let $h = (s, p)$ be a rational Γ -inner function and let σ be a royal node of h on \mathbb{T} . Then

- (i) there exists $\eta \in \mathbb{T}$ such that $p(\sigma) = \eta^2$ and $s(\sigma) = -2\eta$;
- (ii) σ is a zero of $s^2 - 4p$ of multiplicity at least 2;
- (iii) for any finite Blaschke product v ,

$$\Phi \circ (v, h)(\sigma) = \eta$$

independent of v ;

- (iv) for any finite Blaschke product v such that $v(\sigma) \neq -\bar{\eta}$,

$$A\Phi \circ (v, h)(\sigma) = \frac{1}{2}Ap(\sigma).$$

Proof. (i) By [1, Lemma 7.10], the royal nodes of $h = (s, p)$ on \mathbb{T} are precisely the points $\sigma \in \mathbb{T}$ such that $|s(\sigma)| = 2$. Thus there exists $\eta \in \mathbb{T}$ such that $s(\sigma) = -2\eta$ and, since $4p(\sigma) = s(\sigma)^2$, we have $p(\sigma) = \eta^2$.

(ii) By Proposition 2.1, on \mathbb{T} we have

$$\bar{p}(4p - s^2) = 4 - (\bar{p}s)s = 4 - \bar{s}s = 4 - |s|^2 \geq 0.$$

Since $|s(\sigma)| = 2$, the function $f(\theta) = 4 - |s(e^{i\theta})|^2$ has a local minimum at ξ where $\sigma = e^{i\xi}$. Therefore

$$\begin{aligned} 0 &= \frac{d}{d\theta} (4 - |s(e^{i\theta})|^2)_{|\xi} \\ &= \frac{d}{d\theta} (\bar{p}(4p - s^2)(e^{i\theta}))_{|\xi} \\ &= p(e^{i\xi})ie^{i\theta}(4p'(e^{i\xi}) - 2ss'(e^{i\xi})). \end{aligned} \tag{2.5}$$

Hence

$$(4p - s^2)'(\sigma) = 0,$$

and so σ is a zero of $s^2 - 4p$ of multiplicity at least 2.

(iii) If $vs(\sigma) \neq 2$, then

$$\begin{aligned}\Phi \circ (v, h)(\sigma) &= \frac{2vp - s}{2 - vs}(\sigma) \\ &= \frac{2v\frac{1}{4}s^2 - s}{2 - vs}(\sigma) \\ &= \frac{s(v\frac{1}{2}s - 1)}{2(1 - \frac{1}{2}vs)}(\sigma) \\ &= -\frac{1}{2}s(\sigma) = \eta.\end{aligned}\tag{2.6}$$

Thus $\Phi \circ (v, h)(\sigma) = \eta$ independent of v , as long as $vs(\sigma) \neq 2$, that is, $v(\sigma) \neq -\bar{\eta}$.

For any finite Blaschke product v such that $v(\sigma) = -\bar{\eta}$, by [Proposition 2.4](#), we have

$$Ap(\sigma) = \sigma \frac{p'(\sigma)}{p(\sigma)} = \sigma \bar{\eta}^2 p'(\sigma)$$

and

$$Av(\sigma) = \sigma \frac{v'(\sigma)}{v(\sigma)} = -\sigma \frac{v'(\sigma)}{\bar{\eta}}.$$

Since v and p are inner functions

$$Av(\sigma) \neq -\frac{1}{2}Ap(\sigma),$$

which is equivalent to

$$v'(\sigma) \neq \frac{1}{2}\bar{\eta}^3 p'(\sigma).$$

Note that $v(\sigma) = -\bar{\eta}$ implies that

$$2v(\sigma)p(\sigma) - s(\sigma) = 0 = 2 - v(\sigma)s(\sigma),$$

and so

$$\begin{aligned}\Phi \circ (v, h)(\sigma) &= \frac{(2vp - s)'(\sigma)}{(2 - vs)'(\sigma)} \\ &= \frac{2v'p + 2vp' - s'}{-v's - vs'}(\sigma) \\ &= \frac{2v'\eta^2 + 2(-\bar{\eta})p' + \bar{\eta}p'}{-v'(-2\eta) - (-\bar{\eta})(-\bar{\eta}p')}(\sigma) \\ &= \frac{2\eta^2 v' - \bar{\eta}p'}{2\eta v' - \bar{\eta}^2 p'}(\sigma) \\ &= \eta \frac{2\eta v' - \bar{\eta}^2 p'}{2\eta v' - \bar{\eta}^2 p'}(\sigma) = \eta.\end{aligned}\tag{2.7}$$

Thus $\Phi \circ (v, h)(\sigma) = \eta$ independent of v .

(iv) For any finite Blaschke product v such that $v(\sigma) \neq -\bar{\eta}$, by [Proposition 2.4](#), we have

$$\begin{aligned}
 A\Phi \circ (v, h)(\sigma) &= A(2vp - s)(\sigma) - A(2 - vs)(\sigma) \\
 &= \sigma \frac{2v'p + 2vp' - s'}{2vp - s}(\sigma) - \sigma \frac{-v's - vs'}{2 - vs}(\sigma) \\
 &= \sigma \frac{2v'\eta^2 + 2vp' + \bar{\eta}p'}{2v\eta^2 + 2\eta}(\sigma) + \sigma \frac{v'(-2\eta) + v(-\bar{\eta}p')}{2 + 2v\eta}(\sigma) \\
 &= \frac{\sigma}{2 + 2v\eta} (2v'\eta + 2\bar{\eta}vp' + \bar{\eta}^2p' - 2v'\eta - v\bar{\eta}p')(\sigma) \\
 &= \frac{\sigma \bar{\eta}^2 p'(\sigma)(1 + v\eta)}{2(1 + v\eta)} \\
 &= \frac{1}{2} \frac{\sigma p'(\sigma)}{p(\sigma)} \\
 &= \frac{1}{2} Ap(\sigma). \quad \square
 \end{aligned} \tag{2.8}$$

3. The Blaschke interpolation problem and rational Γ -inner functions

The Blaschke interpolation problem, [Problem 1.3](#), is an algebraic variant of the classical Pick interpolation problem. One seeks a Blaschke product of a given degree n satisfying n interpolation conditions, rather than merely a Schur-class function, and one admits interpolation nodes in both the open unit disc and the unit circle. As with the classical Nevanlinna–Pick problem, there is a criterion for the solvability of such a problem in terms of the positivity of a ‘Pick matrix’ formed from the interpolation data; however, to obtain a concise formulation, one has to impose additional interpolation conditions, on phasar derivatives at the interpolation nodes on the circle, and the bounds on these phasar derivatives appear on the diagonal entries of the Pick matrix. This modified Pick matrix appears in the work of several authors [\[10,4,40,25\]](#), but for simplicity we shall continue to speak of the Pick matrix. To be precise, the *Pick matrix* associated with Blaschke interpolation data (σ, η, ρ) as in [Definition 1.2](#) is defined to be the $n \times n$ matrix $M = [m_{ij}]_{i,j=1}^n$ with entries

$$m_{ij} = \begin{cases} \rho_i & \text{if } i = j \leq k \\ \frac{1 - \bar{\eta}_i \eta_j}{1 - \bar{\sigma}_i \sigma_j} & \text{otherwise.} \end{cases}$$

Remark 3.1. Of course, it can happen for n -point Blaschke interpolation data (σ, η, ρ) that there exists a Blaschke product φ of degree *strictly* less than n satisfying the conditions [\(1.2\)](#) to [\(1.3\)](#), but in the present context we are concerned with solutions of degree exactly n .

In the case that $n = k$, that is, where all the interpolation nodes lie on the unit circle there is an elegant solvability criterion due to D. Sarason [\[40\]](#). His result implies that, when $n = k$, [Problem 1.3](#) is solvable if and only if the corresponding Pick matrix M is minimally positive, that is, when $M \geq 0$ and there is no positive diagonal $n \times n$ matrix D , other than $D = 0$, such that $M \geq D$. Actually, Sarason considers interpolation by functions in the Schur class, not just Blaschke products, and so there is a subtlety concerning the existence of phasar derivatives at boundary points (related to the Julia–Carathéodory theorem), but since we are only concerned with rational functions, no such difficulty will arise here.

The following result is well known – see [\[10, Sections 21.1 and 21.4\]](#) or [\[4,40,25\]](#).

Proposition 3.2. *If Problem 1.3 is solvable then the corresponding Pick matrix M is positive definite and the solution of the problem is not unique.*

Several authors have developed deep and far-reaching machines to characterize solvability of interpolation problems for classes related to Problem 1.3, and to parametrize their sets of solutions [10,11,19,30,20,25]; there is a brief history in [10, Notes for Part V, p. 500]. A paper which addresses the combined interior and boundary problem specifically for finite Blaschke products is [26]. However, we have not found the precise statement that we need, and so, for the convenience of the reader, we give a self-contained treatment.

Our strategy for the construction of the general solution of Problem 1.3 is to adjoin an additional boundary interpolation condition; this augmented problem will have a unique solution, and in this way we obtain all solutions of Problem 1.3 in terms of a unimodular parameter.

The following is a refinement of the Sarason Interpolation Theorem [40], in that we consider interpolation nodes both on the circle and in the open disc. The result is contained in [18, Theorem 2.5]. See also [20, Theorem 5.2] for a solution to the analogous interpolation problem for the upper half plane.

Theorem 3.3. *Let M be the Pick matrix associated with Blaschke interpolation data (σ, η, ρ) .*

(1) *There exists a function φ in the Schur class such that*

$$\varphi(\sigma_j) = \eta_j \quad \text{for } j = 1, \dots, n, \quad (3.1)$$

and the phasar derivative $A\varphi(\sigma_j)$ exists and satisfies

$$A\varphi(\sigma_j) \leq \rho_j \quad \text{for } j = 1, \dots, k \quad (3.2)$$

if and only if $M \geq 0$;

(2) *if M is positive and of rank $r < n$ then there is a unique function φ in the Schur class satisfying conditions (3.1) and (3.2), and this function is a Blaschke product of degree r ;*

(3) *the unique function φ in statement (2) satisfies*

$$A\varphi(\sigma_j) = \rho_j \quad \text{for } j = 1, \dots, k \quad (3.3)$$

if and only if M is minimally positive.

Consider a point $\tau \in \mathbb{T}$ distinct from $\sigma_1, \dots, \sigma_k$. For each $\zeta \in \mathbb{T}$ we seek a solution φ to Problem 1.3 that satisfies the additional interpolation condition $\varphi(\tau) = \zeta$ and $A\varphi(\tau) = \rho_{\zeta, \tau}$, where $\rho_{\zeta, \tau} > 0$ is chosen to make the Pick matrix $B_{\zeta, \tau}$ of the augmented interpolation problem singular. We record the following simple lemma without proof.

Lemma 3.4. *If C is an $n \times n$ positive definite matrix, u is an $n \times 1$ column, $\rho = \langle C^{-1}u, u \rangle$ and the $(n+1) \times (n+1)$ matrix B is defined by*

$$B = \begin{bmatrix} C & u \\ u^* & \rho \end{bmatrix},$$

then B is positive semi-definite, $\text{rank}(B) = n$ and

$$B \begin{bmatrix} -C^{-1}u \\ 1 \end{bmatrix} = 0.$$

The Pick matrix $B_{\zeta,\tau}$ of the augmented problem is the $(n+1) \times (n+1)$ matrix,

$$B_{\zeta,\tau} = \begin{bmatrix} M & u_{\zeta,\tau} \\ u_{\zeta,\tau}^* & \rho_{\zeta,\tau} \end{bmatrix} \quad (3.4)$$

where M is the Pick matrix associated with [Problem 1.3](#), $u_{\zeta,\tau}$ is the $n \times 1$ column matrix defined by

$$u_{\zeta,\tau} = \begin{bmatrix} \frac{1-\bar{\eta}_1\zeta}{1-\bar{\sigma}_1\tau} \\ \vdots \\ \frac{1-\bar{\eta}_n\zeta}{1-\bar{\sigma}_n\tau} \end{bmatrix} \quad (3.5)$$

and

$$\rho_{\zeta,\tau} = \langle M^{-1}u_{\zeta,\tau}, u_{\zeta,\tau} \rangle.$$

Thus the *augmented problem* that we are considering is the Blaschke interpolation problem with data $(\tilde{\sigma}, \tilde{\eta}, \tilde{\rho})$ where

$$\tilde{\sigma} = (\sigma, \tau), \quad \tilde{\eta} = (\eta, \zeta), \quad \tilde{\rho} = (\rho, \rho_{\zeta,\tau}).$$

Proposition 3.5. *Let ψ be a Blaschke product of degree N . Let $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ be an n -tuple of distinct points in \mathbb{D}^- , let $\eta_j = \psi(\sigma_j)$ for $j = 1, \dots, n$ and let $\rho_j = A\psi(\sigma_j)$ for j such that $|\sigma_j| = 1$. The Pick matrix for the data (σ, η, ρ) has rank at most N .*

Proof. In the case that the σ_j all lie in \mathbb{D} the assertion is well known – see [\[4\]](#). It follows easily from the fact that in this case the Pick matrix M is given by

$$M = \left[\left\langle (1 - T_\psi T_\psi^*)k_{\lambda_j}, k_{\lambda_i} \right\rangle \right]_{i,j=1}^n,$$

where k_λ denotes the Szegő kernel and T_ψ is the analytic Toeplitz operator on the Hardy space H^2 with symbol ψ .

Consider the case that $\sigma_1, \dots, \sigma_k \in \mathbb{T}$ and $\sigma_{k+1}, \dots, \sigma_n \in \mathbb{D}$. Let M be the Pick matrix for the data (σ, η, ρ) . Choose $r \in (0, 1)$ and let $\lambda_j = r\sigma_j$ for $j = 1, \dots, n$. By the foregoing observation, the matrix

$$M(r) \stackrel{\text{def}}{=} \left[\frac{1 - \overline{\psi(r\sigma_i)}\psi(r\sigma_j)}{1 - r^2\bar{\sigma}_i\sigma_j} \right]_{i,j=1}^n$$

has rank at most N . Let $r \rightarrow 1-$. It follows from L'Hôpital's rule that the j th diagonal entry, for $j = 1, \dots, k$, tends to $A\psi(\sigma_j)$. The remaining entries of $M(r)$ also tend to the corresponding entries of M , and so $M(r) \rightarrow M$. It follows that $\text{rank}(M) \leq N$. \square

Proposition 3.6. *If the Pick matrix M associated with [Problem 1.3](#) is positive definite then, for any $\tau \in \mathbb{T} \setminus \{\sigma_1, \dots, \sigma_k\}$ and $\zeta \in \mathbb{T}$ there is at most one solution φ of [Problem 1.3](#) for which $\varphi(\tau) = \zeta$.*

Proof. Let ψ be a solution of [Problem 1.3](#) such that $\psi(\tau) = \zeta$ and let $\rho_\tau = A\psi(\tau)$. Thus ψ is in the Schur class and satisfies

$$\begin{aligned}
\psi(\sigma_j) &= \eta_j \quad \text{for } j = 1, \dots, n, \\
A\psi(\sigma_j) &= \rho_j \quad \text{for } j = 1, \dots, k, \\
\psi(\tau) &= \zeta, \\
A\psi(\tau) &= \rho_\tau.
\end{aligned} \tag{3.6}$$

Since ψ is a Blaschke product of degree n , it follows from [Proposition 3.5](#), applied to the augmented problem with data $(\tilde{\sigma}, \tilde{\eta}, (\rho, \rho_\tau))$, that the corresponding Pick matrix

$$\tilde{M} = \begin{bmatrix} M & u_{\zeta, \tau} \\ u_{\zeta, \tau}^* & \rho_\tau \end{bmatrix}$$

has rank less than or equal to n and so it is singular. Thus

$$\rho_\tau = \langle M^{-1}u_{\zeta, \tau}, u_{\zeta, \tau} \rangle = \rho_{\zeta, \tau},$$

and so $A\psi(\tau)$ is the same for every solution of [Problem 1.3](#) such that $\psi(\tau) = \zeta$. By [Theorem 3.3](#), there is a *unique* function ψ in the Schur class satisfying the conditions (3.6), and hence there is at most one solution of [Problem 1.3](#) such that $\psi(\tau) = \zeta$. \square

We denote by e_j the j th standard basis vector in \mathbb{C}^n .

Proposition 3.7. *If the Pick matrix M associated with [Problem 1.3](#) is positive definite, if $\tau \in \mathbb{T} \setminus \{\sigma_1, \dots, \sigma_k\}$, $\zeta \in \mathbb{T}$ and*

$$\langle M^{-1}u_{\zeta, \tau}, e_j \rangle \neq 0 \tag{3.7}$$

for $j = 1, \dots, k$, then there exists a unique solution φ to [Problem 1.3](#) such that $\varphi(\tau) = \zeta$.

Proof. Observe that by [Lemma 3.4](#), if $B_{\zeta, \tau}$ is defined by equation (3.4), then $B_{\zeta, \tau} \geq 0$ and $\text{rank}(B_{\zeta, \tau}) = n$. Further, [Lemma 3.4](#) guarantees that $\ker(B_{\zeta, \tau})$ is spanned by the vector $\begin{bmatrix} -M^{-1}u_{\zeta, \tau} \\ 1 \end{bmatrix}$. The inequation (3.7)

implies, for $j = 1, \dots, k$, that $\begin{bmatrix} -M^{-1}u_{\zeta, \tau} \\ 1 \end{bmatrix} \not\perp \begin{bmatrix} e_j \\ 0 \end{bmatrix}$ and therefore, for every $\varepsilon > 0$, we have

$$\begin{aligned}
\left\langle \left(B_{\zeta, \tau} - \varepsilon \begin{bmatrix} e_j \\ 0 \end{bmatrix} \otimes \begin{bmatrix} e_j \\ 0 \end{bmatrix} \right) \begin{bmatrix} -M^{-1}u_{\zeta, \tau} \\ 1 \end{bmatrix}, \begin{bmatrix} -M^{-1}u_{\zeta, \tau} \\ 1 \end{bmatrix} \right\rangle = \\
- \varepsilon \left| \left\langle \begin{bmatrix} -M^{-1}u_{\zeta, \tau} \\ 1 \end{bmatrix}, \begin{bmatrix} e_j \\ 0 \end{bmatrix} \right\rangle \right|^2 < 0.
\end{aligned}$$

Thus

$$B_{\zeta, \tau} - \varepsilon \begin{bmatrix} e_j \\ 0 \end{bmatrix} \otimes \begin{bmatrix} e_j \\ 0 \end{bmatrix} \not\geq 0.$$

It follows that $B_{\zeta, \tau}$ is minimally positive and the proposition follows from [Theorem 3.3](#). \square

In the light of [Proposition 3.7](#) we define the exceptional set Z_τ for [Problem 1.3](#) to be

$$Z_\tau = \{\zeta \in \mathbb{T} : \text{for some } j, 1 \leq j \leq k, \langle M^{-1}u_{\zeta, \tau}, e_j \rangle = 0\}. \tag{3.8}$$

Define $n \times 1$ vectors x_λ and y_λ for $\lambda \in \mathbb{D}^- \setminus \{\sigma_1, \dots, \sigma_k\}$ by the formulas

$$x_\lambda = \begin{bmatrix} \frac{1}{1-\bar{\sigma}_1\lambda} \\ \vdots \\ \frac{1}{1-\bar{\sigma}_n\lambda} \end{bmatrix}, \quad y_\lambda = \begin{bmatrix} \frac{\bar{\eta}_1}{1-\bar{\sigma}_1\lambda} \\ \vdots \\ \frac{\bar{\eta}_n}{1-\bar{\sigma}_n\lambda} \end{bmatrix}, \quad (3.9)$$

so that

$$u_{\zeta, \tau} = x_\tau - \zeta y_\tau. \quad (3.10)$$

Proposition 3.8.

(i) For any $\tau \in \mathbb{T} \setminus \{\sigma_1, \dots, \sigma_k\}$ if

$$\langle x_\tau, M^{-1}e_j \rangle = 0 = \langle y_\tau, M^{-1}e_j \rangle \text{ for some } j, 1 \leq j \leq k,$$

then $Z_\tau = \mathbb{T}$.

(ii) There exist uncountably many $\tau \in \mathbb{T} \setminus \{\sigma_1, \dots, \sigma_k\}$ such that the equation

$$\langle x_\tau, M^{-1}e_j \rangle = 0 = \langle y_\tau, M^{-1}e_j \rangle$$

does not hold for any $j, 1 \leq j \leq k$. Moreover, for such τ , the set Z_τ consists of at most k points.

Proof. (i) Let

$$Z_\tau^j = \{\zeta \in \mathbb{T} : (x_\tau - \zeta y_\tau) \perp M^{-1}e_j\};$$

By the definition (3.8) and equation (3.10),

$$Z_\tau = Z_\tau^1 \cup \dots \cup Z_\tau^k.$$

Note that $Z_\tau^j = \mathbb{T}$ if and only if, for every $\zeta \in \mathbb{T}$,

$$\langle x_\tau - \zeta y_\tau, M^{-1}e_j \rangle = \langle x_\tau, M^{-1}e_j \rangle - \zeta \langle y_\tau, M^{-1}e_j \rangle = 0.$$

Hence, for $\tau \in \mathbb{T} \setminus \{\sigma_1, \dots, \sigma_k\}$, the set $Z_\tau^j = \mathbb{T}$ if and only if

$$\langle x_\tau, M^{-1}e_j \rangle = 0 = \langle y_\tau, M^{-1}e_j \rangle.$$

Otherwise, Z_τ^j consists of at most one point $\zeta_\tau^j \in \mathbb{T}$.

We shall call a point $\tau \in \mathbb{T} \setminus \{\sigma_1, \dots, \sigma_k\}$ *unsuitable* if there exists $j, 1 \leq j \leq k$, such that $M^{-1}e_j \perp \{x_\tau, y_\tau\}$.

(ii) For $j, 1 \leq j \leq k$, let

$$E_j = \{\tau \in \mathbb{T} \setminus \{\sigma_1, \dots, \sigma_k\} : \{x_\tau, y_\tau\} \perp M^{-1}e_j\}.$$

Suppose that every $\tau \in \mathbb{T} \setminus \{\sigma_1, \dots, \sigma_k\}$ is unsuitable. Then

$$\mathbb{T} \setminus \{\sigma_1, \dots, \sigma_k\} = E_1 \cup \dots \cup E_k.$$

Pick j_0 such that E_{j_0} is uncountable. Thus, for every $\tau \in E_{j_0}$,

$$\langle x_\tau, M^{-1}e_{j_0} \rangle = 0 = \langle y_\tau, M^{-1}e_{j_0} \rangle.$$

Let

$$M^{-1}e_{j_0} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

By equations (3.9),

$$\langle x_\tau, M^{-1}e_{j_0} \rangle = \sum_{i=1}^n \frac{c_i}{1 - \overline{\sigma_i}\tau} = 0$$

and

$$\langle y_\tau, M^{-1}e_{j_0} \rangle = \sum_{i=1}^n \frac{c_i \overline{\eta_i}}{1 - \overline{\sigma_i}\tau} = 0.$$

Since the functions $f_i(\lambda) = \frac{1}{1 - \overline{\sigma_i}\lambda}$, $i = 1, \dots, n$, restricted to the infinite bounded set $E_{j_0} \subset \mathbb{C}$ are linearly independent, $c_i = 0$ for all i and $M^{-1}e_{j_0} = 0$. This is impossible. Therefore there exists $\tau \in \mathbb{T} \setminus \{\sigma_1, \dots, \sigma_k\}$ such that the equalities

$$\langle x_\tau, M^{-1}e_j \rangle = 0 = \langle y_\tau, M^{-1}e_j \rangle$$

do not hold for any j , $1 \leq j \leq k$. Hence there exists $\tau \in \mathbb{T} \setminus \{\sigma_1, \dots, \sigma_k\}$ such that the set Z_τ consists of at most k points. \square

Our final result concerning Problem 1.3 is that the particular solution guaranteed by Proposition 3.7 is uniquely determined by ζ and varies linear-fractionally in ζ . We suppose that Blaschke interpolation data (σ, η, ρ) are given, as in Definition 1.2.

Theorem 3.9. *Let the Pick matrix M for Problem 1.3 be positive definite, and let $\tau \in \mathbb{T} \setminus \{\sigma_1, \dots, \sigma_k\}$ be such that the set*

$$Z_\tau = \{\zeta \in \mathbb{T} : u_{\zeta, \tau} \perp M^{-1}e_j \text{ for some } j, 1 \leq j \leq k\}$$

contains at most k points, where $u_{\zeta, \tau}$ is defined by equation (3.5).

- (1) *If $\zeta \in \mathbb{T} \setminus Z_\tau$, then there is a unique solution φ_ζ of Problem 1.3 that satisfies $\varphi_\zeta(\tau) = \zeta$.*
- (2) *There exist unique polynomials a_τ , b_τ , c_τ , and d_τ of degree at most n such that*

$$\begin{bmatrix} a_\tau(\tau) & b_\tau(\tau) \\ c_\tau(\tau) & d_\tau(\tau) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (3.11)$$

and, for all $\zeta \in \mathbb{T}$, if φ is a solution of Problem 1.3 such that $\varphi(\tau) = \zeta$, then

$$\varphi(\lambda) = \frac{a_\tau(\lambda)\zeta + b_\tau(\lambda)}{c_\tau(\lambda)\zeta + d_\tau(\lambda)} \quad (3.12)$$

for all $\lambda \in \mathbb{D}$.

(3) If \tilde{a} , \tilde{b} , \tilde{c} , \tilde{d} are rational functions satisfying the equation

$$\begin{bmatrix} \tilde{a}(\tau) & \tilde{b}(\tau) \\ \tilde{c}(\tau) & \tilde{d}(\tau) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (3.13)$$

and such that for three distinct points ζ in $\mathbb{T} \setminus Z_\tau$, the equation

$$\frac{a_\tau(\lambda)\zeta + b_\tau(\lambda)}{c_\tau(\lambda)\zeta + d_\tau(\lambda)} = \frac{\tilde{a}(\lambda)\zeta + \tilde{b}(\lambda)}{\tilde{c}(\lambda)\zeta + \tilde{d}(\lambda)} \quad (3.14)$$

holds for all $\lambda \in \mathbb{D}$, then there exists a rational function X such that $\tilde{a} = Xa_\tau$, $\tilde{b} = Xb_\tau$, $\tilde{c} = Xc_\tau$ and $\tilde{d} = Xd_\tau$.

Proof. (1) By Proposition 3.8, there exists $\tau \in \mathbb{T} \setminus \{\sigma_1, \dots, \sigma_k\}$ such that the set Z_τ consists of at most k points. Proposition 3.7 asserts that if M is positive definite and $\zeta \in \mathbb{T} \setminus Z_\tau$ then there exists a solution φ to Problem 1.3 with $\varphi(\tau) = \zeta$. By Proposition 3.6, the solution (when it exists) is unique.

(2) Let $\zeta \in \mathbb{T}$ be such that there is a solution φ of Problem 1.3 satisfying $\varphi(\tau) = \zeta$. With the setup of the proof of Proposition 3.7, we have $B_{\zeta, \tau} \geq 0$ and

$$\text{ran}(B_{\zeta, \tau}) = \begin{bmatrix} -M^{-1}u_{\zeta, \tau} \\ 1 \end{bmatrix}^\perp. \quad (3.15)$$

For $\lambda \in \mathbb{D}$, we define a $(n+1) \times 1$ column matrix $v_{\zeta, \lambda}$, by

$$v_{\zeta, \lambda} = \begin{bmatrix} \frac{1 - \bar{\eta}_1 \varphi(\lambda)}{1 - \bar{\sigma}_1 \lambda} \\ \vdots \\ \frac{1 - \bar{\eta}_n \varphi(\lambda)}{1 - \bar{\sigma}_n \lambda} \\ \frac{1 - \bar{\zeta} \varphi(\lambda)}{1 - \bar{\tau} \lambda} \end{bmatrix},$$

and define a $(n+2) \times (n+2)$ matrix $C_{\zeta, \lambda}$ by

$$C_{\zeta, \lambda} = \begin{bmatrix} B_{\zeta, \tau} & v_{\zeta, \lambda} \\ v_{\zeta, \lambda}^* & \frac{1 - |\varphi(\lambda)|^2}{1 - |\lambda|^2} \end{bmatrix}.$$

As $C_{\zeta, \lambda}$ is the localization of the Pick matrix for φ to the points $\sigma_1, \dots, \sigma_n, \tau, \lambda$, it follows that $C_{\zeta, \lambda} \geq 0$. Hence, equation (3.15) implies that

$$\left\langle v_{\zeta, \lambda}, \begin{bmatrix} -M^{-1}u_{\zeta, \tau} \\ 1 \end{bmatrix} \right\rangle = 0. \quad (3.16)$$

Note that

$$u_{\zeta, \tau} = x_\tau - \zeta y_\tau \quad \text{and} \quad v_{\zeta, \lambda} = \begin{bmatrix} x_\lambda \\ \frac{1}{1 - \bar{\tau} \lambda} \end{bmatrix} - \varphi(\lambda) \begin{bmatrix} y_\lambda \\ \frac{\bar{\zeta}}{1 - \bar{\tau} \lambda} \end{bmatrix} \quad (3.17)$$

where $n \times 1$ vectors x_λ and y_λ are defined for $\lambda \in \mathbb{D}^- \setminus \{\sigma_1, \dots, \sigma_k\}$ by the formulas (3.9). Hence, by equations (3.16) and (3.17), we have

$$\begin{aligned}
0 &= \left\langle v_{\zeta, \lambda}, \begin{bmatrix} -M^{-1}u_{\zeta, \tau} \\ 1 \end{bmatrix} \right\rangle \\
&= \left\langle \begin{bmatrix} x_{\lambda} \\ \frac{1}{1-\bar{\tau}\lambda} \end{bmatrix} - \varphi(\lambda) \begin{bmatrix} y_{\lambda} \\ \frac{\bar{\zeta}}{1-\bar{\tau}\lambda} \end{bmatrix}, \begin{bmatrix} -M^{-1}(x_{\tau} - \zeta y_{\tau}) \\ 1 \end{bmatrix} \right\rangle \\
&= \langle (x_{\lambda} - \varphi(\lambda)y_{\lambda}), -M^{-1}(x_{\tau} - \zeta y_{\tau}) \rangle + \frac{1 - \bar{\zeta}\varphi(\lambda)}{1 - \bar{\tau}\lambda}.
\end{aligned} \tag{3.18}$$

Therefore

$$\langle x_{\lambda} - \varphi(\lambda)y_{\lambda}, \zeta M^{-1}y_{\tau} - M^{-1}x_{\tau} \rangle + \frac{1}{1 - \bar{\tau}\lambda} - \varphi(\lambda) \frac{\bar{\zeta}}{1 - \bar{\tau}\lambda} = 0. \tag{3.19}$$

Equation (3.19) for $\varphi(\lambda)$ yields, after simplification,

$$\varphi(\lambda) = \frac{A(\lambda)\zeta + B(\lambda)}{C(\lambda)\zeta + D(\lambda)}, \tag{3.20}$$

where

$$A(\lambda) = -\langle x_{\lambda}, M^{-1}x_{\tau} \rangle + \frac{1}{1 - \bar{\tau}\lambda}, \tag{3.21}$$

$$B(\lambda) = \langle x_{\lambda}, M^{-1}y_{\tau} \rangle, \tag{3.22}$$

$$C(\lambda) = -\langle y_{\lambda}, M^{-1}x_{\tau} \rangle, \tag{3.23}$$

and

$$D(\lambda) = \langle y_{\lambda}, M^{-1}y_{\tau} \rangle + \frac{1}{1 - \bar{\tau}\lambda}. \tag{3.24}$$

As the right hand sides of equations (3.21)–(3.24) depend only on the prescribed data of [Problem 1.3](#), equation (3.20) implies that φ is unique as claimed.

To define a_{τ} , b_{τ} , c_{τ} , d_{τ} with the desired properties, let

$$\pi(\lambda) = (1 - \bar{\tau}\lambda) \prod_{j=1}^n \frac{1 - \bar{\sigma}_j\lambda}{1 - \bar{\sigma}_j\tau},$$

and set

$$a_{\tau} = \pi A, \quad b_{\tau} = \pi B, \quad c_{\tau} = \pi C, \quad \text{and} \quad d_{\tau} = \pi D. \tag{3.25}$$

With these definitions, equation (3.11) follows immediately from equations (3.21)–(3.24) and equation (3.12) follows from equation (3.20).

(3) To prove the final assertion of [Theorem 3.9](#), assume that \tilde{a} , \tilde{b} , \tilde{c} , \tilde{d} are rational functions satisfying equation (3.13) and such that for 3 distinct points ζ in $\mathbb{T} \setminus Z_{\tau}$, equation (3.14) holds for all $\lambda \in \mathbb{D}$. Cross multiplication in equation (3.14) yields

$$\tilde{a}c_{\tau}\zeta^2 + (\tilde{a}d_{\tau} + \tilde{b}c_{\tau})\zeta + \tilde{b}d_{\tau} = a_{\tau}\tilde{c}\zeta^2 + (a_{\tau}\tilde{d} + b_{\tau}\tilde{c})\zeta + b_{\tau}\tilde{d}$$

for 3 distinct values of ζ . Hence,

$$\tilde{a}c_{\tau} = a_{\tau}\tilde{c}, \tag{3.26}$$

$$\tilde{a}d_{\tau} + \tilde{b}c_{\tau} = a_{\tau}\tilde{d} + b_{\tau}\tilde{c} \tag{3.27}$$

and

$$\tilde{b}d_\tau = b_\tau\tilde{d}. \quad (3.28)$$

Solving equation (3.26) for \tilde{c} and equation (3.28) for \tilde{b} and then substituting into equation (3.27), we deduce that

$$\frac{\tilde{a}}{a_\tau} = \frac{\tilde{d}}{d_\tau}.$$

Here, as equation (3.11) guarantees that a_τ and d_τ are not identically zero, \tilde{a}/a_τ and \tilde{d}/d_τ are well defined rational functions. Since equations (3.26) and (3.28) imply that

$$\tilde{c} = \frac{\tilde{a}}{a_\tau}c_\tau \text{ and } \tilde{b} = \frac{\tilde{d}}{d_\tau}b_\tau,$$

the final assertion of the theorem follows with $X = \tilde{a}/a_\tau$.

To see the uniqueness of polynomials $a_\tau, b_\tau, c_\tau, d_\tau$ assume that there is a second collection a_1, b_1, c_1, d_1 of polynomials of degree $\leq n$ such that equations (3.13) and (3.14) hold. By what was proved in the previous paragraph, it is not the case that both collections of polynomials are relatively prime. Otherwise, there is a third collection a_2, b_2, c_2, d_2 of polynomials of degree $\leq n-1$ such that equations (3.11) and (3.12) hold. This contradicts the fact that $\deg(\varphi) = n$ for all $\zeta \in \mathbb{T} \setminus Z_\tau$. \square

In view of Theorem 3.9 we can make precise what we mean by a parametrization of the solutions of a Blaschke interpolation problem.

Definition 3.10. Let (σ, η, ρ) be Blaschke interpolation data, with n distinct interpolation nodes of which k lie in \mathbb{T} . Suppose that Problem 1.3 is solvable. We say that

$$\varphi = \frac{a\zeta + b}{c\zeta + d}$$

is a *normalized linear fractional parametrization of the solutions of Problem 1.3* if

- (1) a, b, c, d are polynomials of degree at most n ;
- (2) for all but at most k values of $\zeta \in \mathbb{T}$, the function

$$\varphi(\lambda) = \frac{a(\lambda)\zeta + b(\lambda)}{c(\lambda)\zeta + d(\lambda)} \quad (3.29)$$

is a solution of Problem 1.3;

- (3) for some point $\tau \in \mathbb{T} \setminus \{\sigma_1, \dots, \sigma_k\}$,

$$\begin{bmatrix} a(\tau) & b(\tau) \\ c(\tau) & d(\tau) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};$$

- (4) every solution φ of Problem 1.3 has the form (3.29) for some $\zeta \in \mathbb{T}$.

Remark 3.11. Let (σ, η, ρ) be Blaschke interpolation data, with n distinct interpolation nodes of which k lie in \mathbb{T} . Suppose the Pick matrix M of this problem is positive definite. The above proof of Theorem 3.9 gives an *explicit* linear fractional parametrization of the solutions of Problem 1.3. As in Theorem 3.9 choose $\tau \in \mathbb{T} \setminus \{\sigma_1, \dots, \sigma_k\}$ such that the set Z_τ contains at most k points. A normalized linear fractional parametrization of the solution set of Problem 1.3 is

$$\varphi = \frac{a_\tau \zeta + b_\tau}{c_\tau \zeta + d_\tau},$$

where the polynomials a_τ , b_τ , c_τ and d_τ are defined by equations (3.25). Note that different choices of τ will yield different normalized parametrizations.

In the terminology of Definition 3.10, Theorem 3.9 tells us the following.

Corollary 3.12. *Let (σ, η, ρ) be Blaschke interpolation data, with n distinct interpolation nodes. Suppose the Pick matrix M of this problem is positive definite. There exists a normalized linear fractional parametrization*

$$\varphi = \frac{a\zeta + b}{c\zeta + d}$$

of the solutions of Problem 1.3. Moreover

- (1) at least one of the polynomials a , b , c , d has degree n ;
- (2) the polynomials a , b , c , d have no common zero in \mathbb{C} ;
- (3) $|c| \leq |d|$ on \mathbb{D}^- .

Proof. As in Theorem 3.9 choose $\tau \in \mathbb{T} \setminus \{\sigma_1, \dots, \sigma_k\}$ such that the set Z_τ contains at most k points. Let the polynomials $a = a_\tau$, $b = b_\tau$, $c = c_\tau$ and $d = d_\tau$ be defined by equations (3.25). Theorem 3.9 shows that (a, b, c, d) has the properties (1), (2) and (3) of Definition 3.10. Let φ be a solution of Problem 1.3 and let $\zeta = \varphi(\tau)$. By Theorem 3.9(2), φ is given by equation (3.29). Hence property (4) of Definition 3.10 holds.

Moreover (1) if all of a , b , c , d have degree strictly less than n then $\varphi = \frac{a\zeta+b}{c\zeta+d}$ is a rational function of degree strictly less than n , and so is not a solution of Problem 1.3.

(2) Suppose $\alpha \in \mathbb{C}$ is a common zero of the polynomials a , b , c , d . On canceling the common factor $\lambda - \alpha$ above and below in equations (3.29) and multiplying numerator and denominator by a suitable nonzero scalar we obtain a different normalized parametrization of solutions of Problem 1.3, with the same τ , contrary to the uniqueness statement in Theorem 3.9(2). Hence a , b , c and d have no common zero in \mathbb{C} .

(3) By the normalization property in Definition 3.10(3),

$$(ad - bc)(\lambda) \rightarrow 1 \quad \text{as } \lambda \rightarrow \tau.$$

Hence $ad - bc$ is a polynomial of degree at most $2n$ and is not identically zero. Therefore

$$Y \stackrel{\text{def}}{=} \{\lambda \in \mathbb{D} : (ad - bc)(\lambda) = 0\}$$

contains at most $2n$ points.

We claim that the real-valued function $f(\lambda) = |d(\lambda)| - |c(\lambda)|$ has no zeros in $\mathbb{D} \setminus Y$. For suppose that λ_0 is a zero of f . Then there exists $\zeta_0 \in \mathbb{T}$ such that $c(\lambda_0)\zeta_0 + d(\lambda_0) = 0$. Since $|\varphi| = \left| \frac{a\zeta+b}{c\zeta+d} \right| \leq 1$ on \mathbb{D} for almost all $\zeta \in \mathbb{T}$, it follows that also $a(\lambda_0)\zeta_0 + b(\lambda_0) = 0$, and therefore $(ad - bc)(\lambda_0) = 0$, that is, $\lambda_0 \in Y$.

Since $c(\tau) = 0$ and $d(\tau) = 1$, the continuous function f is strictly positive on a neighborhood of τ in \mathbb{D} . Suppose that $f(\lambda_1) < 0$ for some $\lambda_1 \in \mathbb{D}$. Then $f < 0$ on an open set, and hence there are infinitely many points in \mathbb{D} at which $f = 0$, a contradiction. Hence $f \geq 0$ on \mathbb{D} . \square

4. Prescribing the nodes and values

In this section we shall show how to construct rational Γ -inner functions with prescribed royal nodes and values. Our answer will be in terms of the solution to Problem 1.3 as described in Proposition 3.7 and Theorem 3.9. First we require a notion of multiplicity for royal nodes.

Definition 4.1. Let h be a rational Γ -inner function with royal polynomial R . If σ is a zero of R of order ℓ , we define the *multiplicity* $\#\sigma$ of σ (as a royal node of h) by

$$\#\sigma = \begin{cases} \ell & \text{if } \sigma \in \mathbb{D} \\ \frac{1}{2}\ell & \text{if } \sigma \in \mathbb{T}. \end{cases}$$

The *type* of h is the ordered pair (n, k) where n is the sum of the multiplicities of the royal nodes of h that lie in \mathbb{D}^- and k is the sum of the multiplicities of the royal nodes of h that lie in \mathbb{T} . We denote by $\mathcal{R}^{n,k}$ the collection of rational Γ -inner functions h of type (n, k) .

By [3, Theorem 3.8], if $h = (s, p)$ belongs to $\mathcal{R}^{n,k}$ then $\deg(h) = n$ and p is a Blaschke product of degree n .

The following example of rational Γ -inner functions from $\mathcal{R}^{n,k}$ for even $n \geq 2$ can be found in [1, Proposition 12.1].

Example 4.2. For all $\nu \geq 0$ and $0 < r < 1$, the function

$$h_\nu(\lambda) = \left(2(1-r) \frac{\lambda^{\nu+1}}{1+r\lambda^{2\nu+1}}, \frac{\lambda(\lambda^{2\nu+1}+r)}{1+r\lambda^{2\nu+1}} \right), \quad \lambda \in \mathbb{D}, \quad (4.1)$$

belongs to $\mathcal{R}^{2\nu+2, 2\nu+1}$. The royal nodes of h_ν that lie in \mathbb{T} , being the points at which $|s| = 2$, are the $(2\nu+1)$ th roots of -1 , that is,

$$\omega_j = e^{i\pi(2j+1)/(2\nu+1)}, \quad j = 0, \dots, 2\nu.$$

They are all of multiplicity 1. Note that there is a simple royal node at 0.

In this section we are concerned only with rational Γ -inner functions whose royal nodes all have multiplicity 1.

The following elementary calculation will be useful.

Lemma 4.3. Let $a, b, c, d, s_0, p_0 \in \mathbb{C}$ and suppose that $|p_0| = 1$, $s_0 = \overline{s_0}p_0$, $s_0c \neq 2d$ and $|s_0| < 2$. Let

$$s = 2 \frac{2p_0c - s_0d}{s_0c - 2d}. \quad (4.2)$$

Then

$$|s| \leq 2 \iff |c| \leq |d|. \quad (4.3)$$

Proof.

$$|s| \leq 2 \iff |2p_0c - s_0d|^2 \leq |s_0c - 2d|^2 \quad (4.4)$$

$$\iff 4|c|^2 - 2\operatorname{Re}(2p_0c\overline{s_0d}) + |s_0|^2|d|^2$$

$$\leq |s_0|^2|c|^2 - 2\operatorname{Re}(2s_0c\overline{d}) + 4|d|^2$$

$$\iff (4 - |s_0|^2)(|c|^2 - |d|^2) \leq 4\operatorname{Re}(s_0c\overline{d} - s_0c\overline{d})$$

$$\iff |c|^2 - |d|^2 \leq 0$$

$$\iff |c| \leq |d|. \quad \square \quad (4.5)$$

The next result provides a necessary condition for the existence of a rational Γ -inner function with prescribed royal interpolation data.

Theorem 4.4. *Let $h = (s, p)$ be a rational Γ -inner function of type (n, k) having distinct royal nodes $\sigma_1, \dots, \sigma_n$ and corresponding royal values η_1, \dots, η_n , where $\sigma_1, \dots, \sigma_k \in \mathbb{T}$. Let $\rho_j = \frac{1}{2}Ap(\sigma_j)$ for $j = 1, \dots, k$.*

- (1) *There exists a rational inner function φ that solves Problem 1.3, that is, such that $\deg(\varphi) = n$,*

$$\varphi(\sigma_j) = \eta_j \quad \text{for } j = 1, \dots, n \quad (4.6)$$

and

$$A\varphi(\sigma_j) = \rho_j \quad \text{for } j = 1, \dots, k. \quad (4.7)$$

Any such function φ is expressible in the form $\varphi = \Phi_\omega \circ h$ for some $\omega \in \mathbb{T}$.

- (2) *There exist polynomials a, b, c, d of degree at most n such that a normalized parametrization of the solutions of Problem 1.3 is*

$$\varphi = \frac{a\zeta + b}{c\zeta + d}, \quad \zeta \in \mathbb{T}.$$

- (3) *For any polynomials a, b, c, d as in (2), there exist $s_0, p_0 \in \mathbb{C}$ such that*

$$|p_0| = 1, \quad (4.8)$$

$$s_0 = \overline{s_0}p_0, \quad (4.9)$$

$$|s_0| < 2 \quad (4.10)$$

$$|c| \leq |d|, \quad (4.11)$$

$$s_0a - 2b + 2p_0c - s_0d = 0 \quad (4.12)$$

and

$$(2p_0c - s_0d)^2 \neq (-2p_0a + s_0b)(s_0c - 2d). \quad (4.13)$$

Moreover

$$s = 2 \frac{2p_0c - s_0d}{s_0c - 2d}, \quad (4.14)$$

$$p = \frac{-2p_0a + s_0b}{s_0c - 2d} \quad (4.15)$$

Proof. (1) For $\omega \in \mathbb{T}$ consider the rational function

$$\psi_\omega = \Phi_\omega \circ h = \frac{2\omega p - s}{2 - \omega s}. \quad (4.16)$$

Then, if $\omega \neq -\bar{\eta}_1, -\bar{\eta}_2, \dots, -\bar{\eta}_k$,

$$\psi_\omega(\sigma_j) = \frac{2\omega\eta_j^2 + 2\eta_j}{2 + \omega 2\eta_j} = \eta_j \quad \text{for } j = 1, 2, \dots, n. \quad (4.17)$$

We claim that, for $\omega \in \mathbb{T} \setminus \{-\bar{\eta}_1, \dots, \bar{\eta}_k\}$, the function $\varphi = \psi_\omega$ is a solution of [Problem 1.3](#). By [\[1, Proposition 3.2\]](#), for any $\omega \in \mathbb{T}$ and any point $(s(\lambda), p(\lambda)) \in \Gamma$,

$$|\Phi_\omega(s(\lambda), p(\lambda))| = 1 \quad \text{if and only if} \quad \omega(s(\lambda) - \bar{s}(\lambda)p(\lambda)) = 1 - |p(\lambda)|^2.$$

Thus it is easy to see that φ is inner. The equation [\(4.17\)](#) shows that ψ_ω takes the required values at $\sigma_1, \dots, \sigma_n$. By [Proposition 2.5\(iv\)](#),

$$A\psi_\omega(\sigma_j) = \frac{1}{2}Ap(\sigma_j) = \rho_j \quad \text{for } j = 1, 2, \dots, k. \quad (4.18)$$

It is also true that $\deg(\psi_\omega) = n$ for $\omega \neq -\bar{\eta}_1, \dots, -\bar{\eta}_k$. By [\[3, Proposition 2.2\]](#), for a rational Γ -inner function $h = (s, p)$ such that $\deg(p) = n$ and if D is the denominator when p is written in its lowest terms then s can also be written with denominator D . It follows that

$$\deg(\psi_\omega) = \deg(p) - \#\{\text{cancellations between } 2\omega p - s \text{ and } 2 - \omega s\}. \quad (4.19)$$

By [\[1, Theorem 7.12\]](#), such cancellations can occur only at the royal nodes $\sigma_j \in \mathbb{T}$, $j = 1, \dots, k$, and then only when $\omega = \frac{1}{2}\overline{s(\sigma_j)} = -\bar{\eta}_j$. Hence there are *no* cancellations in equation [\(4.19\)](#), and so $\deg(\psi_\omega) = n$. We have shown that, if $\omega \neq -\bar{\eta}_1, -\bar{\eta}_2, \dots, -\bar{\eta}_k$, then $\varphi = \psi_\omega$ is a solution of [Problem 1.3](#).

(2) Since [Problem 1.3](#) is solvable, its Pick matrix is positive definite and so [Theorem 3.9](#) tells us that there exist polynomials a, b, c, d of degree at most n which parametrize the solutions of [Problem 1.3](#). Let us choose a particular such 4-tuple of polynomials, as described in [Theorem 3.9](#). By [Proposition 3.8](#), there exists $\tau \in \mathbb{T} \setminus \{\sigma_1, \dots, \sigma_k\}$ such that the set Z_τ (defined in equation [\(3.8\)](#)) consists of at most k points. Fix such a $\tau \in \mathbb{T}$; then there exist unique polynomials $a_\tau, b_\tau, c_\tau, d_\tau$ of degree at most n such that

$$\begin{bmatrix} a_\tau(\tau) & b_\tau(\tau) \\ c_\tau(\tau) & d_\tau(\tau) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (4.20)$$

and, for all $\zeta \in \mathbb{T} \setminus Z_\tau$, the function

$$\varphi = \frac{a_\tau\zeta + b_\tau}{c_\tau\zeta + d_\tau} \quad (4.21)$$

is the unique solution of [Problem 1.3](#) that satisfies $\varphi(\tau) = \zeta$. Moreover, the general 4-tuple of polynomials that parametrizes the solutions of [Problem 1.3](#) is expressible in the form

$$(a, b, c, d) = (Xa_\tau, Xb_\tau, Xc_\tau, Xd_\tau) \quad (4.22)$$

for some rational function X .

Let $s_0 = s(\tau)$, $p_0 = p(\tau)$. Since h is Γ -inner, equations [\(4.8\)](#) and [\(4.9\)](#) hold by virtue of [Proposition 2.1](#). Since τ is chosen not to be a royal node of h , the inequation [\(4.10\)](#) also holds. Moreover $|s_0| < 2$, since, for any point (z_1, z_2) in the distinguished boundary $b\Gamma$ of Γ , we have $|z_1| = 2$ if and only if $z_1^2 = 4z_2$ – see [\[1, Proposition 3.2\(3\)\]](#). It remains to prove equations [\(4.12\)](#) and [\(4.11\)](#).

Lemma 4.5. *Let*

$$Z_\tau^\sim \stackrel{\text{def}}{=} \left\{ \frac{-2\bar{\eta}_1 p_0 - s_0}{2 + \bar{\eta}_1 s_0}, \frac{-2\bar{\eta}_2 p_0 - s_0}{2 + \bar{\eta}_2 s_0}, \dots, \frac{-2\bar{\eta}_k p_0 - s_0}{2 + \bar{\eta}_k s_0} \right\}.$$

If $\zeta \in \mathbb{T} \setminus Z_\tau^\sim$ then the function

$$\varphi = \frac{(4p - s_0s)\zeta + 2s_0p - 2p_0s}{(2s_0 - 2s)\zeta + 4p_0 - s_0s} \quad (4.23)$$

is a solution of [Problem 1.3](#) and satisfies $\varphi(\tau) = \zeta$.

Proof. Observe that, by equation (4.16), for any $\omega \in \mathbb{T}$,

$$\psi_\omega(\tau) = \frac{2\omega p_0 - s_0}{2 - \omega s_0},$$

which is well defined since $|s_0| < 2$. We have, for $\zeta \in \mathbb{T}$,

$$\begin{aligned} \psi_\omega(\tau) = \zeta &\Leftrightarrow \frac{2\omega p_0 - s_0}{2 - \omega s_0} = \zeta \\ &\Leftrightarrow \omega = \frac{2\zeta + s_0}{2p_0 + \zeta s_0}. \end{aligned}$$

Hence, as long as

$$\frac{2\zeta + s_0}{2p_0 + \zeta s_0} \neq -\bar{\eta}_1, \dots, -\bar{\eta}_k, \quad (4.24)$$

the function

$$\varphi = \psi_\omega = \psi_{\frac{2\zeta + s_0}{2p_0 + \zeta s_0}} \quad (4.25)$$

is a solution of [Problem 1.3](#) which satisfies in addition $\varphi(\tau) = \zeta$. Condition (4.24) can equally be written

$$\zeta \neq -\frac{2\bar{\eta}_j p_0 + s_0}{2 + \bar{\eta}_j s_0}, \quad \text{for } j = 1, 2, \dots, k$$

or equivalently $\zeta \notin Z_\tau^\sim$.

On computing φ from equations (4.25) and (4.16) we find that φ is indeed given by equation (4.23); this establishes the Lemma. \square

We conclude the proof of [Theorem 4.4](#) (2). For $\zeta \in \mathbb{T} \setminus (Z_\tau \cup Z_\tau^\sim)$ we have two expressions for the unique solution of [Problem 1.3](#) for which $\varphi(\tau) = \zeta$, to wit equations (4.21) (with the normalizing condition (4.20)) and (4.23). Note that

$$\begin{aligned} \begin{bmatrix} 4p(\tau) - s_0s(\tau) & 2s_0p(\tau) - 2p_0s(\tau) \\ 2s_0 - 2s(\tau) & 4p_0 - s_0s(\tau) \end{bmatrix} &= \begin{bmatrix} 4p_0 - s_0s_0 & 2s_0p_0 - 2p_0s_0 \\ 2s_0 - 2s_0 & 4p_0 - s_0s_0 \end{bmatrix} \\ &= (4p_0 - s_0^2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Since the set $Z_\tau \cup Z_\tau^\sim$ is finite, the linear fractional transformations in equations (4.21) and (4.23) are equal at infinitely many points, hence coincide. On taking account of the normalizing condition we obtain

$$\begin{bmatrix} a_\tau & b_\tau \\ c_\tau & d_\tau \end{bmatrix} = \frac{1}{4p_0 - s_0^2} \begin{bmatrix} 4p - s_0s & 2s_0p - 2p_0s \\ 2s_0 - 2s & 4p_0 - s_0s \end{bmatrix}.$$

Suppose that a , b , c , and d are polynomials that parametrize the solutions of [Problem 1.3](#), as in [Theorem 4.4](#) (2). By the observation (4.22), there exists a rational function X such that

$$Xa = 4p - s_0s, \quad (4.26)$$

$$Xb = 2s_0p - 2p_0s, \quad (4.27)$$

$$Xc = 2s_0 - 2s, \text{ and} \quad (4.28)$$

$$Xd = 4p_0 - s_0s. \quad (4.29)$$

Thus

$$\begin{aligned} X(s_0a - 2b + 2p_0c - s_0d) = \\ s_0(4p - s_0s) - 2(2s_0p - 2p_0s) + 2p_0(2s_0 - 2s) - s_0(4p_0 - s_0s) \end{aligned}$$

which is zero. Hence equation (4.12) holds.

Let us find connections between s , p and the polynomials a , b , c , d . Solving equations (4.28) and (4.29) for s and X we find that

$$X = \frac{2s_0^2 - 8p_0}{s_0c - 2d} \quad (4.30)$$

and

$$s = 2 \frac{2p_0c - s_0d}{s_0c - 2d}. \quad (4.31)$$

Eliminating s from equations (4.26) and (4.27) we deduce that

$$(8p_0 - 2s_0^2)p = X(2p_0a - s_0b), \quad (4.32)$$

which implies via equation (4.30) that

$$p = \frac{-2p_0a + s_0b}{s_0c - 2d}. \quad (4.33)$$

Since $h = (s, p)$ is a rational Γ -inner function, $|s| \leq 2$ on \mathbb{D} and, by Lemma 4.3, equation (4.11) holds. Since, by assumption, $h(\mathbb{D}) \not\subset \mathcal{R}$, we have $s^2 \neq 4p$ on \mathbb{D}^- and inequation (4.13) holds. The proof of Theorem 4.4 is complete. \square

Remark 4.6. The above proof shows that, if $\omega \neq -\bar{\eta}_1, \dots, -\bar{\eta}_k$, then $\Phi_\omega \circ h$ is a solution of the corresponding Blaschke interpolation problem. What if $\omega = -\bar{\eta}_j$ for some $j \in \{1, \dots, k\}$? Then the rational function $\Phi_\omega \circ h$ has a removable singularity at σ_j . After cancellation, it is still true (by Proposition 2.5) that $\Phi_\omega \circ h(\sigma_j) = \eta_j$, but we cannot assert that $A(\Phi_\omega \circ h)(\sigma_j) = 2\rho_j$. In any case $\Phi_\omega \circ h$ has degree $n - 1$, and so is not a solution of Problem 1.3.

There is a converse to Theorem 4.4. To prove it we need the following purely algebraic observation, which is proved by a routine calculation.

Proposition 4.7. *Let a , b , c , d be polynomials in the indeterminate λ and suppose that $s_0, p_0 \in \mathbb{C}$ satisfy $s_0^2 \neq 4p_0$, $s_0c \neq 2d$ and*

$$s_0a - 2b + 2p_0c - s_0d = 0.$$

Let rational functions s , p be defined by

$$s = 2 \frac{2p_0c - s_0d}{s_0c - 2d}, \quad p = \frac{-2p_0a + s_0b}{s_0c - 2d} \quad (4.34)$$

and let

$$\zeta(\omega) = \frac{2\omega p_0 - s_0}{2 - \omega s_0}. \quad (4.35)$$

Then, as rational functions in (ω, λ) ,

$$\frac{2\omega p(\lambda) - s(\lambda)}{2 - \omega s(\lambda)} = \frac{a(\lambda)\zeta(\omega) + b(\lambda)}{c(\lambda)\zeta(\omega) + d(\lambda)}.$$

This algebraic relation has implications for rational maps from \mathbb{D} to Γ .

Proposition 4.8. *Let a, b, c, d be polynomials having no common zero in \mathbb{D}^- and satisfying $|c| \leq |d|$ on \mathbb{D} . Suppose that $s_0, p_0 \in \mathbb{C}$ satisfy $s_0c \neq 2d$ and*

$$s_0a - 2b + 2p_0c - s_0d = 0.$$

Suppose in addition that $|p_0| = 1$, $|s_0| < 2$ and $s_0 = \bar{s}_0p_0$. Let rational functions s, p be defined by equations (4.34) and let

$$\psi_\zeta(\lambda) = \frac{a(\lambda)\zeta + b(\lambda)}{c(\lambda)\zeta + d(\lambda)}. \quad (4.36)$$

(i) *If, for all but finitely many values of $\lambda \in \mathbb{D}$,*

$$|\psi_\zeta(\lambda)| \leq 1 \quad (4.37)$$

for all but finitely many $\zeta \in \mathbb{T}$, then $s_0c - 2d$ has no zeros in \mathbb{D}^- and (s, p) is a holomorphic map from \mathbb{D} to Γ .

(ii) *If, for all but finitely many $\zeta \in \mathbb{T}$, the function ψ_ζ is inner, then $h = (s, p)$ is a rational Γ -inner function.*

Proof. (i) Notice first that the hypotheses on s_0 and p_0 imply that $\zeta(\cdot)$ (defined by equation (4.35)) is an automorphism of \mathbb{D} and so defines a bijective self-map of \mathbb{T} .

By hypothesis there is a finite subset E of \mathbb{D} such that, for all $\lambda \in \mathbb{D} \setminus E$, there is a finite subset F_λ of \mathbb{T} such that the inequality (4.37) holds for all $\zeta \in \mathbb{T} \setminus F_\lambda$.

We claim that the denominator $s_0c - 2d$ of s and p in equations (4.34) has no zeros in \mathbb{D}^- . For suppose that α is such a zero. Since $|c| \leq |d|$ on \mathbb{D}^- and $|s_0| < 2$,

$$\begin{aligned} 0 = |s_0c - 2d| &\geq 2|d| - |s_0c| \\ &\geq (2 - |s_0|)|d| \end{aligned}$$

at α , and hence $d(\alpha) = 0$, and consequently $c(\alpha) = 0$.

Choose a sequence α_j in $\mathbb{D} \setminus E$ such that $\alpha_j \rightarrow \alpha$. For each j , for $\zeta \in \mathbb{T} \setminus F(\lambda_j)$ we have $|\psi_\zeta| \leq 1$ on $\mathbb{D} \setminus E$. Hence, for all but countably many $\zeta \in \mathbb{T}$ (that is, for $\zeta \in \mathbb{T} \setminus \cup_j F(\lambda_j)$)

$$\left| \frac{a(\alpha_j)\zeta + b(\alpha_j)}{c(\alpha_j)\zeta + d(\alpha_j)} \right| \leq 1.$$

Since $c(\alpha_j)\zeta + d(\alpha_j) \rightarrow 0$ uniformly almost everywhere for $\zeta \in \mathbb{T}$ as $j \rightarrow \infty$, the same holds for $a(\alpha_j)\zeta + b(\alpha_j)$. Hence $a(\alpha_j) \rightarrow 0$ and $b(\alpha_j) \rightarrow 0$. Thus $a(\alpha) = b(\alpha) = 0$. Hence a, b, c, d all vanish at α , contrary to hypothesis. It follows that $s_0c - 2d$ has no zeros in \mathbb{D}^- .

Thus s and p are rational functions having no poles in \mathbb{D}^- .

Consider $\lambda \in \mathbb{D} \setminus E$. By [Proposition 4.7](#),

$$\Phi_\omega(s(\lambda), p(\lambda)) = \frac{a(\lambda)\zeta(\omega) + b(\lambda)}{c(\lambda)\zeta(\omega) + d(\lambda)} \quad (4.38)$$

whenever both sides are defined, that is, for all $\omega \in \mathbb{T} \setminus \Omega_\lambda$ where

$$\Omega_\lambda = \{\omega \in \mathbb{T} : \omega s(\lambda) = 2 \quad \text{or} \quad c(\lambda)\zeta(\omega) = -d(\lambda)\}.$$

Ω_λ contains at most two points. On combining the relations [\(4.36\)](#), [\(4.37\)](#) and [\(4.38\)](#) we deduce that

$$|\Phi_\omega(s(\lambda), p(\lambda))| \leq 1$$

for all $\omega \in \mathbb{T}$ such that $\omega \notin \Omega_\lambda \cup \zeta^{-1}(F_\lambda)$, hence for all but finitely many $\omega \in \mathbb{T}$. By [Lemma 4.3](#), $|s(\lambda)| \leq 2$. It follows from [\[1, Proposition 3.2\(2\)\]](#) that $(s(\lambda), p(\lambda)) \in \Gamma$. Since this is true for all but finitely many $\lambda \in \mathbb{D}$ and s, p are rational functions without poles in \mathbb{D}^- , (s, p) maps the whole of \mathbb{D}^- into Γ .

(ii) Suppose that, for some finite subset F of \mathbb{T} , the function ψ_ζ is inner for all $\zeta \in \mathbb{T} \setminus F$. By Part (i), (s, p) maps \mathbb{D} into Γ and therefore extends to a continuous map of \mathbb{D}^- into Γ . Consider $\lambda \in \mathbb{T}$. By [Proposition 4.7](#) and equation [\(4.36\)](#),

$$\Phi_\omega(s(\lambda), p(\lambda)) = \psi_{\zeta(\omega)}(\lambda) \quad (4.39)$$

whenever both sides are defined, that is, for all $\omega \in \mathbb{T} \setminus \Omega_\lambda$ where

$$\Omega_\lambda = \{\omega \in \mathbb{T} : \omega s(\lambda) = 2 \quad \text{or} \quad c(\lambda)\zeta(\omega) = -d(\lambda)\}.$$

Ω_λ contains at most two points. For $\omega \in \mathbb{T} \setminus \zeta^{-1}(F)$ the function $\psi_{\zeta(\omega)}$ is inner. Hence, for $\omega \in \mathbb{T} \setminus (\zeta^{-1}(F) \cup \Omega_\lambda)$,

$$|\Phi_\omega(s(\lambda), p(\lambda))| = |\psi_{\zeta(\omega)}(\lambda)| = 1. \quad (4.40)$$

[\[1, Proposition 3.2\]](#) asserts that, for any $\omega \in \mathbb{T}$ and any point $(s_1, p_1) \in \Gamma$,

$$|\Phi_\omega(s_1, p_1)| = 1 \quad \text{if and only if} \quad \omega(s_1 - \bar{s}_1 p_1) = 1 - |p_1|^2.$$

Hence, if $|\Phi_\omega(s_1, p_1)| = 1$ for two distinct $\omega \in \mathbb{T}$, then $|p_1| = 1$ and $s_1 = \bar{s}_1 p_1$, which is to say that (s_1, p_1) is in the distinguished boundary $b\Gamma$ of Γ . Therefore, since equation [\(4.40\)](#) holds for many $\omega \in \mathbb{T}$, $(s(\lambda), p(\lambda)) \in b\Gamma$. Thus $h = (s, p)$ is a rational Γ -inner function. \square

The following result gives the promised explicit construction of a solution of the royal Γ -interpolation problem in terms of a normalized parametrization of solutions of the corresponding Blaschke interpolation problem.

Theorem 4.9. *Let (σ, η, ρ) be Blaschke interpolation data with n distinct interpolation nodes of which k lie in \mathbb{T} , as in [Definition 1.2](#). Suppose that [Problem 1.3](#) with these data is solvable and the solutions φ of [Problem 1.3](#) have normalized parametrization*

$$\varphi = \frac{a\zeta + b}{c\zeta + d}.$$

Suppose that there exist scalars s_0 and p_0 such that

$$|p_0| = 1, \quad s_0 = \overline{s_0}p_0, \quad |s_0| < 2 \quad (4.41)$$

and

$$s_0a - 2b + 2p_0c - s_0d = 0. \quad (4.42)$$

Then there exists a rational Γ -inner function $h = (s, p)$ such that

- (i) $h \in \mathcal{R}^{n,k}$,
- (ii) $h(\sigma_j) = (-2\eta_j, \eta_j^2)$ for $j = 1, 2, \dots, n$,
- (iii) $Ap(\sigma_j) = 2\rho_j$ for $j = 1, 2, \dots, k$,
- (iv) for all but finitely many $\omega \in \mathbb{T}$, the function $\Phi_\omega \circ h$ is a solution of [Problem 1.3](#).

An explicit function $h = (s, p)$ satisfying conditions (i)–(iv) is given by

$$s = 2 \frac{2p_0c - s_0d}{s_0c - 2d}, \quad (4.43)$$

$$p = \frac{-2p_0a + s_0b}{s_0c - 2d}. \quad (4.44)$$

Proof. By [Corollary 3.12](#) (3), $|c| \leq |d|$ on \mathbb{D}^- . Hence $\left| \frac{d(\lambda)}{c(\lambda)} \right| \geq 1$ for $\lambda \in \mathbb{D}^-$. By assumption $|s_0/2| < 1$, and therefore $s_0c \neq 2d$ on \mathbb{D}^- . By [Proposition 4.8](#), h is a rational Γ -inner function. Since a, b, c, d are polynomials of degree at most n , the rational function h has degree at most n . Recall that the degree of h coincides with the degree of p .

By [Definition 3.10](#) of a normalized linear fractional parametrization of the solutions of [Problem 1.3](#), for some point $\tau \in \mathbb{T} \setminus \{\sigma_1, \dots, \sigma_k\}$,

$$\begin{bmatrix} a(\tau) & b(\tau) \\ c(\tau) & d(\tau) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus it is easy to see that

$$s(\tau) = 2 \frac{2p_0c(\tau) - s_0d(\tau)}{s_0c(\tau) - 2d(\tau)} = s_0, \quad (4.45)$$

$$p(\tau) = \frac{-2p_0a(\tau) + s_0b(\tau)}{s_0c(\tau) - 2d(\tau)} = p_0. \quad (4.46)$$

By assumption, $|p_0| = 1$ and $|s_0| < 2$, and hence $s(\tau)^2 \neq 4p(\tau)$. Therefore $h(\mathbb{D}^-)$ is not in the royal variety \mathcal{R} .

Let us show that h satisfies the interpolation conditions

$$h(\sigma_j) = (-2\eta_j, \eta_j^2) \quad (4.47)$$

for $j = 1, \dots, n$, which is to say that σ_j is a royal node of h with corresponding royal value η_j . By hypothesis, there is a finite set $F \subset \mathbb{T}$ such that, for all $\zeta \in \mathbb{T} \setminus F$, the function

$$\varphi(\lambda) = \psi_{\zeta}(\lambda) \stackrel{\text{def}}{=} \frac{a(\lambda)\zeta + b(\lambda)}{c(\lambda)\zeta + d(\lambda)}$$

is a solution of [Problem 1.3](#), and so

$$\psi_{\zeta}(\sigma_j) = \eta_j \quad \text{for } j = 1, \dots, n \quad (4.48)$$

and

$$A\psi_{\zeta}(\sigma_j) = \rho_j \quad \text{for } j = 1, \dots, k \quad (4.49)$$

for all $\zeta \in \mathbb{T} \setminus F$. By [Proposition 4.7](#)

$$\psi_{\zeta(\omega)}(\lambda) = \frac{a(\lambda)\zeta(\omega) + b(\lambda)}{c(\lambda)\zeta(\omega) + d(\lambda)} = \frac{2\omega p(\lambda) - s(\lambda)}{2 - \omega s(\lambda)} = \Phi_{\omega} \circ h(\lambda) \quad (4.50)$$

as rational functions in (ω, λ) , where $\zeta(\omega) = \frac{2\omega p_0 - s_0}{2 - \omega s_0}$. Hence, for $\omega \in \mathbb{T} \setminus \zeta^{-1}(F)$, $\Phi_{\omega} \circ h$ is a solution of [Problem 1.3](#); this proves statement (iv).

For any $\lambda \in \mathbb{D}^-$ equation (4.50) holds whenever both denominators are nonzero, hence for all but at most two values of $\omega \in \mathbb{T}$. On combining equations (4.48) and (4.50) (with $\lambda = \sigma_j$) we infer that, for $j = 1, \dots, n$ and for all but finitely many $\omega \in \mathbb{T}$,

$$\frac{2\omega p(\sigma_j) - s(\sigma_j)}{2 - \omega s(\sigma_j)} = \psi_{\zeta(\omega)}(\sigma_j) = \eta_j.$$

Therefore, for almost all $\omega \in \mathbb{T}$,

$$2\omega p(\sigma_j) - s(\sigma_j) = \eta_j(2 - \omega s(\sigma_j)).$$

It follows that $s(\sigma_j) = -2\eta_j$ and $p(\sigma_j) = \eta_j^2$, $j = 1, 2, \dots, n$, and so the interpolation conditions (4.47) hold.

We have already observed that $\deg(h) \leq n$ and that $h(\mathbb{D})$ is not in \mathcal{R} . Thus [\[3, Theorem 3.8\]](#) tells us that, in this case, the number of royal nodes of h is equal to the degree of h . Therefore h has at most n royal nodes. Since the points σ_j , $j = 1, 2, \dots, n$ are royal nodes, they comprise *all* the royal nodes of h and $\deg(h) = n$. Precisely k of the σ_j lie in \mathbb{T} , and so h has exactly k royal nodes in \mathbb{T} . Thus $h \in \mathcal{R}^{n,k}$ and statement (i) holds.

Next we show that $Ap(\sigma_j) = 2\rho_j$. Fix $j \in \{1, \dots, k\}$. By [Proposition 2.5\(iv\)](#), for $\omega \in \mathbb{T}$, $\omega \neq -\bar{\eta}_j$ (and so $2 - \omega s(\sigma_j) = 2(1 + \omega\eta_j) \neq 0$),

$$A(\Phi_{\omega} \circ h)(\sigma_j) = \frac{1}{2}Ap(\sigma_j). \quad (4.51)$$

There is also a set Ω_j containing at most one $\omega_j \in \mathbb{T}$ such that $c(\sigma_j)\zeta(\omega) + d(\sigma_j) = 0$ for $\omega \in \Omega_j$. Hence if $\omega \in \mathbb{T} \setminus (\{-\bar{\eta}_j\} \cup \Omega_j)$, it follows from equation (4.50) that $\psi_{\zeta(\omega)} = \Phi_{\omega} \circ h$ in a neighborhood of σ_j , and consequently, for such ω ,

$$A\psi_{\zeta(\omega)}(\sigma_j) = A(\Phi_{\omega} \circ h)(\sigma_j). \quad (4.52)$$

Each of the equations (4.51), (4.52) and (4.49) hold for ω in a cofinite subset of \mathbb{T} . Hence, for ω in the intersection of these cofinite subsets,

$$Ap(\sigma_j) = 2A(\Phi_{\omega} \circ h)(\sigma_j) = 2A\psi_{\zeta(\omega)}(\sigma_j) = 2\rho_j$$

as required. \square

Remark 4.10. Under the assumptions of [Theorem 4.9](#), the condition [\(4.13\)](#):

$$(2p_0c - s_0d)^2 \neq (-2p_0a + s_0b)(s_0c - 2d) \quad (4.53)$$

is satisfied automatically and the rational Γ -inner function h is such that $h(\mathbb{D}^-)$ is not in the royal variety \mathcal{R} .

Remark 4.11. Every solution of a royal Γ -interpolation problem is obtainable by the method in the theorem. Let data (σ, η, ρ) be as in [Theorem 4.9](#). Suppose that [Problem 1.3](#) with these data is solvable and the solutions φ of [Problem 1.3](#) have normalized parametrization

$$\varphi = \frac{a\zeta + b}{c\zeta + d}.$$

By [Theorem 4.4](#), every rational Γ -inner function $h = (s, p) \in \mathcal{R}^{n,k}$ satisfying

- (i) $h(\sigma_j) = (-2\eta_j, \eta_j^2)$ for $j = 1, 2, \dots, n$,
- (ii) $Ap(\sigma_j) = 2\rho_j$ for $j = 1, 2, \dots, k$ is expressible by the equations

$$s = 2 \frac{2p_0c - s_0d}{s_0c - 2d}, \quad (4.54)$$

$$p = \frac{-2p_0a + s_0b}{s_0c - 2d} \quad (4.55)$$

for some choice of s_0, p_0 satisfying conditions [\(4.41\)](#) and [\(4.42\)](#).

Example 4.12. Consider 3 distinct points $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{T}$ and 3 distinct points $\eta_1, \eta_2, \eta_3 \in \mathbb{T}$ in the same cyclic order as $\sigma_1, \sigma_2, \sigma_3$. There is a Blaschke factor φ such that $\varphi(\sigma_j) = \eta_j$ for $j = 1, 2, 3$; let $\rho_j = A\varphi(\sigma_j)$ for $j = 1, 2, 3$. [Problem 1.3](#) with data (σ, η, ρ) is solvable and φ is a solution. Let $h = (-2\varphi, \varphi^2)$; then $h(\mathbb{D}) \subset \mathcal{R}$. Every point of \mathbb{D}^- is a royal node of h ; in particular, h has the 3 distinct royal nodes $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{T}$ with corresponding royal values $\eta_1, \eta_2, \eta_3 \in \mathbb{T}$, and

$$Ap(\sigma_j) = A\varphi^2(\sigma_j) = 2A\varphi(\sigma_j) = 2\rho_j, \quad j = 1, 2, 3.$$

At the same time $\deg(h) = 2$. The example shows that for the rational Γ -inner functions whose range is contained in \mathcal{R} , it can happen that $\deg(h)$ is strictly less than n .

5. The algorithm

In this section we summarize the steps in the solution of the royal Γ -interpolation problem in the form of a concrete algorithm.

We suppose given Blaschke interpolation data (σ, η, ρ) as in [Definition 1.2](#). Here there are n prescribed royal nodes σ_j , of which the first k lie in \mathbb{T} and the remaining $n - k$ are in \mathbb{D} . To construct a rational Γ -inner function or functions of degree n having royal nodes σ_j , royal values η_j and phasar derivatives $2\rho_j$ at σ_j we proceed as follows.

- (1) Form the Pick matrix $M = [m_{ij}]_{i,j=1}^n$ for the data (σ, η, ρ) , with entries

$$m_{ij} = \begin{cases} \rho_i & \text{if } i = j \leq k \\ \frac{1 - \overline{\eta_i}\eta_j}{1 - \overline{\sigma_i}\sigma_j} & \text{otherwise.} \end{cases}$$

If M is not positive definite then the interpolation problem is not solvable. Otherwise, introduce the notation

$$x_\lambda = \begin{bmatrix} \frac{1}{1-\bar{\sigma}_1\lambda} \\ \vdots \\ \frac{1}{1-\bar{\sigma}_n\lambda} \end{bmatrix}, \quad y_\lambda = \begin{bmatrix} \frac{\bar{\eta}_1}{1-\bar{\sigma}_1\lambda} \\ \vdots \\ \frac{\bar{\eta}_n}{1-\bar{\sigma}_n\lambda} \end{bmatrix}, \quad (5.1)$$

as in equations (3.9).

(2) Choose a point $\tau \in \mathbb{T} \setminus \{\sigma_1, \dots, \sigma_k\}$ such that the set of $\zeta \in \mathbb{T}$ for which

$$\langle M^{-1}x_\tau, e_j \rangle = \zeta \langle M^{-1}y_\tau, e_j \rangle \quad \text{for some } j \in \{1, \dots, n\}$$

(where e_j is the j th standard basis vector in \mathbb{C}^n) is finite.

(3) Find $s_0, p_0 \in \mathbb{C}$ such that $|s_0| < 2$, $|p_0| = 1$, $s_0 = \bar{s}_0 p_0$ and, for all $\lambda \in \mathbb{D}$,

$$s_0 (\langle x_\lambda, M^{-1}x_\tau \rangle + \langle y_\lambda, M^{-1}y_\tau \rangle) + 2 \langle x_\lambda, M^{-1}y_\tau \rangle + 2p_0 \langle y_\lambda, M^{-1}x_\tau \rangle = 0. \quad (5.2)$$

If there is no pair (s_0, p_0) satisfying these conditions, then the interpolation problem is not solvable; otherwise

(4) Let

$$g(\lambda) = \prod_{j=1}^n \frac{1 - \bar{\sigma}_j \lambda}{1 - \bar{\sigma}_j \tau}$$

and let polynomials a, b, c, d be given by

$$a(\lambda) = g(\lambda) (1 - (1 - \bar{\tau}\lambda) \langle x_\lambda, M^{-1}x_\tau \rangle),$$

$$b(\lambda) = g(\lambda) (1 - \bar{\tau}\lambda) \langle x_\lambda, M^{-1}y_\tau \rangle,$$

$$c(\lambda) = -g(\lambda) (1 - \bar{\tau}\lambda) \langle y_\lambda, M^{-1}x_\tau \rangle,$$

$$d(\lambda) = g(\lambda) (1 + (1 - \bar{\tau}\lambda) \langle y_\lambda, M^{-1}y_\tau \rangle).$$

Note that

$$\begin{bmatrix} a(\tau) & b(\tau) \\ c(\tau) & d(\tau) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(5) Let

$$s = 2 \frac{2p_0 c - s_0 d}{s_0 c - 2d},$$

$$p = \frac{-2p_0 a + s_0 b}{s_0 c - 2d}.$$

It is easy to see that

$$s(\tau) = s_0 \quad \text{and} \quad p(\tau) = p_0.$$

Then $h = (s, p)$ is a rational Γ -inner function of degree at most n such that $h(\sigma_j) = (-2\eta_j, \eta_j^2)$ for $j = 1, \dots, n$ and $Ap(\sigma_j) = 2\rho_j$ for $j = 1, \dots, k$. By assumption, $|p_0| = 1$ and $|s_0| < 2$, and hence $s(\tau)^2 \neq 4p(\tau)$. Therefore $h(\mathbb{D}^-)$ is not in the royal variety \mathcal{R} and the degree of h is exactly n .

The following comments relate the steps of the algorithm to results in the paper.

(1) If the royal Γ -interpolation problem is solvable, then the Blaschke interpolation problem with the same data is solvable, by [Theorem 4.4](#). By [Proposition 3.2](#), $M > 0$.

(2) This amounts to saying that Z_τ is finite, in the notation of equation (3.8). By [Proposition 3.8](#), there are uncountably many such $\tau \in \mathbb{T}$.

(3) The necessity of the existence of s_0, p_0 is given in [Theorem 4.4](#) equation (4.12), together with the equations (3.21) to (3.25) for a, b, c and d .

The conditions that $|s_0| < 2$, $|p_0| = 1$ and $s_0 = \bar{s}_0 p_0$ are equivalent to $(s_0, p_0) \in b\Gamma$ and $|s_0| < 2$. By a standard parametrization of $b\Gamma$ [[7, Theorem 2.4](#)], we can take $s_0 = 2t\omega$, $p_0 = \omega^2$ for some $t \in (-1, 1)$ and $\omega \in \mathbb{T}$. The condition (5.2) then becomes: for all $\lambda \in \mathbb{D}$,

$$\langle y_\lambda, M^{-1}x_\tau \rangle \omega^2 + t(\langle x_\lambda, M^{-1}x_\tau \rangle + \langle y_\lambda, M^{-1}y_\tau \rangle) \omega + \langle x_\lambda, M^{-1}y_\tau \rangle = 0. \quad (5.3)$$

After multiplication of both sides by $\prod_{j=1}^n (1 - \bar{\sigma}_j \lambda)$, the coefficients in this equation relating t and ω become polynomials in λ of degree at most n , and so the equation is in effect a system of $2n + 2$ real equations in two real variables. Consequently the system is over-determined. The existence of s_0, p_0 satisfying equations (5.2) is thus in principle a stringent condition for the solvability of a royal Γ -interpolation problem. Remarkably, in the two examples in the next section, the λ terms factor out entirely from equation (5.3), and one obtains a single real equation in t and ω , which has a 1-parameter family of solutions.

(4) The equations for a, b, c and d are equations (3.21) to (3.25).

(5) The equations for s and p are (4.45) and (4.46).

6. Two examples

Even the simplest case of [Problem 1.4](#), the royal Γ -interpolation problem with only one interpolation node, demands a surprising amount of calculation to solve. This problem is so simple that it can be readily solved without the foregoing theory, but it is instructive to see how the algorithm in [Section 5](#) works in this case.

Example 6.1. Consider the case $n = 1, k = 0$ of [Problem 1.4](#). There are prescribed a single royal node $\sigma_1 \in \mathbb{D}$ and a single royal value $\eta \in \mathbb{D}$, and we seek a Γ -inner function h of degree 1 such that $h(\sigma_1) = (-2\eta, \eta^2)$. By composition with an automorphism of \mathbb{D} we may reduce to the case that $\sigma_1 = 0$. There is clearly at least a 1-parameter family of solutions, if any, since if h is a solution then so is $h(\omega\lambda)$ for any $\omega \in \mathbb{T}$.

The recipe for h in [Section 4](#) proceeds as follows. Choose an arbitrary $\tau \in \mathbb{T}$. The normalized parametrization of the solution set of the associated Blaschke interpolation problem, according to equations (3.25), is given by

$$\begin{aligned} a_\tau(\lambda) &= \frac{\bar{\tau}\lambda - |\eta|^2}{1 - |\eta|^2}, \\ b_\tau(\lambda) &= \frac{\eta(1 - \bar{\tau}\lambda)}{1 - |\eta|^2}, \\ c_\tau(\lambda) &= -\frac{\bar{\eta}(1 - \bar{\tau}\lambda)}{1 - |\eta|^2}, \\ d_\tau(\lambda) &= \frac{1 - |\eta|^2 \bar{\tau}\lambda}{1 - |\eta|^2}. \end{aligned}$$

The next step is to determine whether there exist s_0, p_0 such that equations (4.8) to (4.13) hold. A little calculation shows that there is a 1-parameter family of such (s_0, p_0) , given by

$$s_0 = -\frac{4\omega \operatorname{Re}(\bar{\omega}\eta)}{1 + |\eta|^2}, \quad p_0 = \omega^2$$

for any $\omega \in \mathbb{T}$. Substitution of these values into equations (4.45) and (4.46) yields the degree 1 Γ -inner function

$$h(\lambda) = \left(-2\frac{\eta + \bar{\eta}\kappa\lambda}{1 + \bar{\eta}^2\kappa\lambda}, \frac{\kappa\lambda + \eta^2}{1 + \bar{\eta}^2\kappa\lambda} \right)$$

where

$$\kappa = \bar{\tau} \frac{\omega^2 - \eta^2}{1 - \bar{\eta}^2\omega^2}.$$

κ is a general point of \mathbb{T} , and so we do obtain a 1-parameter family of Γ -inner functions of degree 1 satisfying $h(0) = (-2\eta, \eta^2)$. An alternative expression for h is

$$h(\lambda) = (\beta + \bar{\beta}p(\lambda), p(\lambda))$$

where

$$\beta = -\frac{2\eta}{1 + |\eta|^2}, \quad p(\lambda) = \frac{\kappa\lambda + \eta^2}{1 + \bar{\eta}^2\kappa\lambda}.$$

Example 6.2. Next consider the case of a single interpolation node on the unit circle – say $\sigma = 1$. A point $\eta \in \mathbb{T}$ and a $\rho > 0$ are prescribed, and we seek a Γ -inner function $h = (s, p)$ of degree 1 such that $h(1) = (-2\eta, \eta^2)$ and $Ap(1) = 2\rho$.

Choose $\tau \in \mathbb{T} \setminus \{1\}$. Again calculate the normalized parametrization of the solution set of the associated Blaschke interpolation problem according to equations (3.25):

$$\begin{aligned} a_\tau(\lambda) &= \frac{1 - \lambda}{1 - \tau} - \frac{1 - \bar{\tau}\lambda}{\rho|1 - \tau|^2}, \\ b_\tau(\lambda) &= \frac{\eta(1 - \bar{\tau}\lambda)}{\rho|1 - \tau|^2}, \\ c_\tau(\lambda) &= -\frac{\bar{\eta}(1 - \bar{\tau}\lambda)}{\rho|1 - \tau|^2}, \\ d_\tau(\lambda) &= \frac{1 - \bar{\tau}\lambda}{\rho|1 - \tau|^2} + \frac{1 - \lambda}{1 - \tau}. \end{aligned}$$

Equations (4.8) to (4.13) for (s_0, p_0) have solution

$$s_0 = -\eta - \omega^2\bar{\eta}, \quad p_0 = \omega^2$$

for any $\omega \in \mathbb{T} \setminus \{\eta\}$. Then equations (4.45) and (4.46) yield the degree 1 Γ -inner function $h = (s, p)$ where

$$\begin{aligned} s(\lambda) &= 2\frac{\rho(\eta + \omega^2\bar{\eta})(1 - \bar{\tau})(1 - \lambda) + (\eta - \omega^2\bar{\eta})(1 - \bar{\tau}\lambda)}{-2\rho(1 - \bar{\tau})(1 - \lambda) + (\omega^2\bar{\eta}^2 - 1)(1 - \bar{\tau}\lambda)}, \\ p(\lambda) &= \frac{-2\omega^2\rho(1 - \bar{\tau})(1 - \lambda) + (\omega^2 - \eta^2)(1 - \bar{\tau}\lambda)}{-2\rho(1 - \bar{\tau})(1 - \lambda) + (\omega^2\bar{\eta}^2 - 1)(1 - \bar{\tau}\lambda)}. \end{aligned} \quad (6.1)$$

One can check directly that $h = (s, p)$ is a Γ -inner function of degree 1 satisfying $h(1) = (-2\eta, \eta^2)$ and $Ap(1) = 2\rho$. It appears at first sight that we have constructed a 2-parameter family of functions with the

prescribed royal node, value and phasar derivative, since the parameters ω and τ range through \mathbb{T} (or at least, cofinite subsets thereof). However, by means of some entertaining algebra, one can express h in terms of a single unimodular parameter (the same thing happened, though more simply, in [Example 6.1](#)). Let

$$\kappa = \tau \frac{2\rho(1 - \bar{\tau})\omega - \bar{\tau}(\omega - \bar{\omega})}{2\rho(1 - \tau)\bar{\omega} - \tau(\bar{\omega} - \omega)}.$$

Clearly κ is unimodular. Now let

$$\alpha(\kappa) = \frac{2\rho - \bar{\kappa}}{1 + 2\rho}.$$

It transpires that $|\alpha(\kappa)| < 1$ and

$$\alpha(\kappa) = \frac{2\rho(1 - \bar{\tau}) - 1 + \bar{\omega}^2}{2\rho(1 - \bar{\tau}) - \bar{\tau}(1 - \bar{\omega}^2)}.$$

One may verify that the functions s and p in equations [\(6.1\)](#) can be written

$$\begin{aligned} s(\lambda) &= -\eta - \bar{\eta}p(\lambda), \\ p(\lambda) &= \eta^2 \kappa \frac{\lambda - \alpha(\kappa)}{1 - \overline{\alpha(\kappa)}\lambda}, \end{aligned} \tag{6.2}$$

with $\kappa \in \mathbb{T}$, evidently a 1-parameter family.

It is noteworthy that the function $h = (s, p)$ defined by equations [\(6.2\)](#) maps \mathbb{D} into the disc $\{(\eta + \bar{\eta}z, z) : z \in \mathbb{D}\}$, which is a subset of the topological boundary $\partial\Gamma$ of Γ . Inner functions h such that $h(\mathbb{D}) \subset \partial\Gamma$ were called *superficial* in [\[1\]](#) and discussed in [\[1, Proposition 8.3\]](#). The example shows that the solutions of a royal Γ -interpolation problem can be superficial.

7. Concluding remarks

In this section we relate the results of the paper to some classical results in the theory of invariant distances and thereby describe some of the original motivation for our work.

The algorithm which is developed in this paper provides constructions of n -extremal maps and m -geodesics in $\text{Hol}(\mathbb{D}, \mathcal{G})$ with prescribed royal nodes σ_j , royal values η_j and phasar derivatives at σ_j . The n -extremal maps simultaneously generalize both Blaschke products and complex geodesics and constitute a significant class.

Recall that for a domain G in \mathbb{C}^N the *Carathéodory distance* C_G on G is defined by

$$C_G(z_1, z_2) = \sup_{F \in \text{Hol}(G, \mathbb{D})} \rho(F(z_1), F(z_2)). \tag{7.1}$$

In equation [\(7.1\)](#) z_1 and z_2 are two points in G , ρ denotes the pseudohyperbolic distance on \mathbb{D} ,

$$\rho(z, w) = \left| \frac{z - w}{1 - \bar{w}z} \right|,$$

and, for any domain G and any set E , $\text{Hol}(G, E)$ denotes the space of holomorphic mappings from G to E . A dual notion is the *Kobayashi distance* of G , which is defined to be the largest pseudodistance K_G subordinate to the Lempert function ρ_G of G (e.g. [\[33, 23\]](#)). The *Lempert function* of G is given by

$$\rho_G(z_1, z_2) = \inf_{\substack{h \in \text{Hol}(\mathbb{D}, G) \\ h(\lambda_1) = z_1 \\ h(\lambda_2) = z_2}} \rho(\lambda_1, \lambda_2). \tag{7.2}$$

The *Kobayashi extremal problem* for a pair of points $z_1, z_2 \in G$ is to find the quantity $\rho_G(z_1, z_2)$ [31]. Any function $h \in \text{Hol}(\mathbb{D}, G)$ for which the infimum is attained is called a *Kobayashi extremal function* for the domain G and the points z_1, z_2 . In the special case when $G = \mathcal{G}$ it turns out that the 1-parameter family $\Phi_\omega \in \text{Hol}(\mathcal{G}, \mathbb{D})$, which we encountered in equation (1.5), is “universal” for the Carathéodory extremal problem [7, Corollary 3.4], the following sense.

Theorem 7.1. *If $z_1, z_2 \in \mathcal{G}$ then there exists $\omega \in \mathbb{T}$ such that*

$$C_G(z_1, z_2) = \rho(\Phi_\omega(z_1), \Phi_\omega(z_2)). \quad (7.3)$$

Another fact about the complex geometry of \mathcal{G} is that

$$\rho_{\mathcal{G}} = K_{\mathcal{G}} = C_{\mathcal{G}}.$$

This corresponds to the geometric property of \mathcal{G} that if h is an extremal function for the Kobayashi problem (7.2), then the range $\text{ran}(h)$ of h is a totally geodesic analytic disc in \mathcal{G} [7, Corollary 5.7].

The Kobayashi extremal problem can be viewed as an extremal 2-point interpolation problem. Specifically, by a *finite interpolation problem in $\text{Hol}(\mathbb{D}, G)$* , one means the following.

Problem 7.2. Given n distinct points $\lambda_1, \dots, \lambda_n$ in \mathbb{D} and n points z_1, \dots, z_n in an open or closed set $G \subset \mathbb{C}^N$, to determine whether there exists a function $h \in \text{Hol}(\mathbb{D}, G)$ such that $h(\lambda_j) = z_j$ for $j = 1, \dots, n$.

We say that *Problem 7.2 is solvable*, or that the data $\lambda_j \mapsto z_j$ *are solvable*, if there does exist an $h \in \text{Hol}(\mathbb{D}, G)$ that satisfies these interpolation conditions. We say that the problem is (or the data are) *extremal* when the problem is solvable but there do not exist an open neighborhood U of the closure of \mathbb{D} and a map $h \in \text{Hol}(U, G)$ such that the conditions

$$h(\lambda_j) = z_j \quad \text{for } j = 1, \dots, n, \quad (7.4)$$

hold.

A map $h \in \text{Hol}(\mathbb{D}, G)$ is said to be *n-extremal* if, for any choice of n distinct points $\lambda_1, \dots, \lambda_n \in \mathbb{D}$, the interpolation data $\lambda_j \in \mathbb{D} \mapsto h(\lambda_j) \in G$ are extremally solvable.

With this perspective, if h and λ_1, λ_2 minimize the right hand side of equation (7.2), then the 2-point interpolation problem $\lambda_j \mapsto z_j, j = 1, 2$ for $\text{Hol}(\mathbb{D}, G)$ is extremal and h is an extremal solution to it. Just as the Kobayashi extremal functions on \mathcal{G} are both rational and Γ -inner, more generally, the following result obtains (see [22] or [2, Theorem 3.1]).

Proposition 7.3. *If $\lambda_j \mapsto (s_j, p_j), j = 1, \dots, n$, is a solvable n -point interpolation problem for $\text{Hol}(\mathbb{D}, \mathcal{G})$ then it has a rational Γ -inner solution.*

The royal variety (or more precisely, $\mathcal{R} \cap \mathcal{G}$) is a complex geodesic of \mathcal{G} , with extremal function given by $h_{\mathcal{R}}(\lambda) = (2\lambda, \lambda^2)$. Furthermore, among the complex geodesics in \mathcal{G} , the royal variety is characterized by the property that

$$\alpha(\mathcal{R} \cap \mathcal{G}) = \mathcal{R} \cap \mathcal{G}$$

whenever α is a biholomorphic self map (automorphism) of \mathcal{G} [9, Lemma 4.3]. In addition, the automorphism group of \mathcal{G} acts transitively on $\mathcal{R} \cap \mathcal{G}$.

If F is a Carathéodory extremal function for some pair of points in \mathcal{G} then so is $m \circ F$ for any Möbius transformation m of the disc. The universal set described in [Theorem 7.1](#) above is normalized so as to satisfy $\Phi_\omega \circ h_{\mathcal{R}} = -\text{id}_{\mathbb{D}}$. As a result,

$$\Phi_\omega|_{\mathcal{R}} \text{ does not depend on } \omega. \quad (7.5)$$

A Kobayashi extremal function on any domain for which the Lempert function and the Carathéodory distance coincide has a holomorphic left inverse. L. Kosinski and W. Zwonek [\[32\]](#) introduced a generalization of this notion: a map $h : \mathbb{D} \rightarrow G$, for any domain G , is said to be an n -complex geodesic if there exists a holomorphic map $F : G \rightarrow \mathbb{D}$ such that $F \circ h$ is a Blaschke product of degree at most n . The following result shows that rational Γ -inner functions enjoy this property.

Proposition 7.4. *Let h be a rational Γ -inner function of degree n which is not superficial and let $h(\mathbb{D}) \not\subset \mathcal{R}$. Then*

- (1) *h is an $(n+1)$ -extremal holomorphic map in $\text{Hol}(\mathbb{D}, \Gamma)$ and is an n -complex geodesic of \mathcal{G} ;*
- (2) *if in addition h has at least one royal node $\sigma \in \mathbb{T}$ then h is an n -extremal holomorphic map in $\text{Hol}(\mathbb{D}, \Gamma)$ and is an $(n-1)$ -complex geodesic of \mathcal{G} .*

Proof. As in [Theorem 4.4](#), $\varphi = \Phi_\omega \circ h$ for some $\omega \in \mathbb{T}$ is a rational inner function $\varphi \in \text{Hol}(\mathbb{D}, \mathbb{D}^-)$ such that $\deg(\varphi) \leq n$. Thus h is an n -complex geodesic. By a version of Pick's result, the $(n+1)$ -extremal holomorphic self-maps of \mathbb{D} are precisely the Blaschke products of degree at most n . Thus φ is a $(n+1)$ -extremal in $\text{Hol}(\mathbb{D}, \mathbb{D}^-)$. Therefore, by [\[1, Proposition 2.2\]](#), h is $(n+1)$ -extremal in $\text{Hol}(\mathbb{D}, \Gamma)$.

(ii) If $h(\sigma) = (-2\eta, \eta^2)$ and $\omega = -\bar{\eta}$ then the rational function $\Phi_\omega \circ h$ has a removable singularity at σ . After cancellation $\Phi_\omega \circ h$ has degree $(n-1)$. As above h is an n -extremal holomorphic map in $\text{Hol}(\mathbb{D}, \Gamma)$ and is an $(n-1)$ -complex geodesic of \mathcal{G} . \square

Corollary 7.5. *All non-superficial functions h in $\mathcal{R}^{1,0} \cup \mathcal{R}^{2,1}$ are complex geodesics of \mathcal{G} .*

Proof. First we recall a result from [\[8\]](#) that an analytic function $h : \mathbb{D} \rightarrow \mathcal{G}$ is a complex geodesic of \mathcal{G} if and only if there is an $\omega \in \mathbb{T}$ such that $\Phi_\omega \circ h \in \text{Aut } \mathbb{D}$ and that every complex geodesic of \mathcal{G} is Γ -inner. By [Proposition 7.4](#), each non-superficial function from the set $\mathcal{R}^{1,0} \cup \mathcal{R}^{2,1}$ is a complex geodesic. \square

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