

THE MEAN FIELD ANALYSIS OF THE KURAMOTO MODEL ON GRAPHS I. THE MEAN FIELD EQUATION AND TRANSITION POINT FORMULAS

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ABSTRACT. In his classical work on synchronization, Kuramoto derived the formula for the critical value of the coupling strength corresponding to the transition to synchrony in large ensembles of all-to-all coupled phase oscillators with randomly distributed intrinsic frequencies. We extend this result to a large class of coupled systems on convergent families of deterministic and random graphs. Specifically, we identify the critical values of the coupling strength (transition points), between which the incoherent state is linearly stable and is unstable otherwise. We show that the transition points depend on the largest positive or/and smallest negative eigenvalue(s) of the kernel operator defined by the graph limit. This reveals the precise mechanism, by which the network topology controls transition to synchrony in the Kuramoto model on graphs. To illustrate the analysis with concrete examples, we derive the transition point formula for the coupled systems on Erdős-Rényi, small-world, and k -nearest-neighbor families of graphs. As a result of independent interest, we provide a rigorous justification for the mean field limit for the Kuramoto model on graphs. The latter is used in the derivation of the transition point formulas.

In the second part of this work [8], we study the bifurcation corresponding to the onset of synchronization in the Kuramoto model on convergent graph sequences.

1. Introduction. Synchronization of coupled oscillators is a classical problem of nonlinear science with diverse applications in science and engineering [3, 46]. Physical and technological applications of synchronization include power, sensor, and communication networks [14], mobile agents [43], electrical circuits [1], coupled lasers [25], and Josephson junctions [52], to name a few. In biological and social sciences, synchronization is studied in the context of flocking, opinion dynamics,

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and voting [20, 38]. Synchronization plays a prominent role in physiology and in neurophysiology, in particular. It is important in the information processing in the brain [44] and in the mechanisms of several severe neurodegenerative diseases such as epilepsy [51] and Parkinsons Disease [29]. This list can be continued.

Identifying common principles underlying synchronization in such diverse models is a challenging task. In the seventies Kuramoto found an elegant approach to this problem. Motivated by problems in statistical physics and biology, he reduced a system of weakly coupled limit cycle oscillators to the system of equations for the phase variables only¹. The resultant equation is called the Kuramoto model (KM) [27]. Kuramoto's method applies directly to a broad class of models in natural science. Moreover, it provides a paradigm for studying synchronization. The analysis of the KM revealed one of the most striking results of the theory of synchronization. For a system of coupled oscillators with randomly distributed intrinsic frequencies, Kuramoto identified the critical value of the coupling strength, at which the gradual buildup of coherence begins. He introduced the order parameter, which describes the degree of coherence in a coupled system. Using the order parameter, Kuramoto predicted the bifurcation marking the onset of synchronization.

Kuramoto's analysis, while not mathematically rigorous, is based on the correct intuition for the transition to synchronization. His discovery initiated a line of fine research (see [48, 49, 47, 6] and references therein). It was shown in [48, 49] that the onset of synchronization corresponds to the loss of stability of the incoherent state, a steady state solution of the mean field equation. The latter is a nonlinear hyperbolic partial differential equation for the probability density function describing the distribution of phases on the unit circle at a given time. The bifurcation analysis of the mean field equation is complicated by the presence of the continuous spectrum of the linearized problem on the imaginary axis. To overcome this problem, in [6] the first author developed an analytical method, which uses the theory of generalized functions and rigged Hilbert spaces [18] (see [12, 17] for other approaches).

The Kuramoto's original analysis of the onset of synchronization and subsequent work [48, 49, 6] deal with all-to-all coupled systems. Real world applications feature complex and often random connectivity patterns [42]. The goal of our work is to extend the mathematical theory of synchronization to spatially structured networks. We are especially interested in identifying the contribution of the network connectivity to the the bifurcation underlying the transition to synchrony. To this end, we adopt the approach developed by the second author in [35, 36]. Specifically, we consider the KM on convergent families of deterministic and random weighted graphs. Our framework covers many random graphs widely used in applications, including Erdős-Rényi and small-world graphs. Furthermore, with minor modifications our approach applies to a large class of sparse graphs including power-law graphs (cf. [23]). For the KM on convergent graph sequences, we derive and rigorously justify the mean field equation and study stability and bifurcations of the incoherent state. The latter is a special solution of the mean field equation, whose loss of stability marks the onset of synchronization. For presentation purposes, we split our results in two sets. In the present paper, we deal with the derivation of the mean field equation and linear stability analysis of the incoherent state. In particular, we derive explicit formulas for the critical values of the coupling strength, at

¹For related reductions predating Kuramoto's work, see [33, 34, 3] and the discussion before Theorem 9.2 in [21].

which the incoherent state loses stability. These transition point formulas relate the onset of synchronization to the structure of the network through the extreme eigenvalues of the kernel operator defined by the graph limit. In the second part of our work [8], we deal with a more technical bifurcation analysis of the incoherent state. Here, we are also able to identify the contribution of the network connectivity to the bifurcation structure of the KM model. Together, the results of the present paper and those of [8] provide a complete theory of synchronization for the KM on convergent graph sequences.

The original KM with all-to-all coupling and random intrinsic frequencies has the following form:

$$\dot{\theta}_i = \omega_i + \frac{K}{n} \sum_{j=1}^n \sin(\theta_j - \theta_i). \quad (1)$$

Here, $\theta_i : \mathbb{R} \rightarrow \mathbb{S} := \mathbb{R}/2\pi\mathbb{Z}$, $i \in [n] := \{1, 2, \dots, n\}$ is the phase of the oscillator i , whose intrinsic frequency ω_i is drawn from the probability distribution with density $g(\omega)$, n is the number of oscillators, and K is the strength of coupling. The sum on the right-hand side of (1) describes the interactions of the oscillators in the network. The goal is to describe the distribution of $\theta_i(t)$, $i \in [n]$, for large times and $n \gg 1$.

Since the intrinsic frequencies are random, for small values of the coupling strength $K > 0$, the dynamics of different oscillators in the network are practically uncorrelated. For increasing values of $K > 0$, however, the dynamics of the oscillators becomes more and more synchronized. To describe the degree of synchronization, Kuramoto used the complex *order parameter*:

$$r(t)e^{i\psi(t)} := n^{-1} \sum_{j=1}^n e^{i\theta_j(t)}. \quad (2)$$

Here, $0 \leq r(t) \leq 1$ and $\psi(t)$ stand for the modulus and the argument of the order parameter defined by the right-hand side of (2). Note that if all phases are independent uniform random variables and $n \gg 1$ then with probability 1, $r = o(1)$ by the Strong Law of Large Numbers. If, on the other hand, all phase variables are equal then $r = 1$. Thus, one can interpret the value of r as the measure of coherence in the system dynamics. Numerical experiments with the KM (with normally distributed frequencies ω_i 's) reveal the phase transition at a certain critical value of the coupling strength $K_c > 0$. Specifically, numerics suggest that for $t \gg 1$ (cf. [47])

$$r(t) = \begin{cases} O(n^{-1/2}), & 0 < K < K_c, \\ r_\infty(K) + O(n^{-1/2}), & K > K_c. \end{cases}$$

Assuming that g is a smooth even function that is decreasing on $\omega \in \mathbb{R}^+$, Kuramoto derived the formula for the critical value

$$K_c = \frac{2}{\pi g(0)}. \quad (3)$$

Furthermore, he formally showed that in the partially synchronized regime ($K > K_c$), the steady-state value for the order parameter is given by

$$r_\infty(K) = \sqrt{\frac{-16}{\pi K_c^4 g''(0)}} \sqrt{K - K_c} + O(K - K_c). \quad (4)$$

Recently, Chiba and Nishikawa [10] and Chiba [6] confirmed Kuramoto's heuristic analysis with the rigorous derivation of (3) and analyzed the bifurcation at K_c .

In this paper, we initiate a mathematical investigation of the transition to coherence in the KM on graphs. To this end, we consider the following model:

$$\dot{\theta}_i = \omega_i + \frac{K}{n} \sum_{j=1}^n W_{nij} \sin(\theta_j - \theta_i), \quad (5)$$

$W_n = (W_{nij})$ is an $n \times n$ symmetric matrix of weights. Note that in the classical KM (1), every oscillator is coupled to every other oscillator in the network, i.e., the graph describing the interactions between the oscillators is the complete graph on n nodes. In the modified model (5), we supply the edges of the complete graph with the weights (W_{nij}) . Using this framework, we can study the KM on a variety of deterministic and random (weighted) graphs. For instance, let W_{nij} , $1 \leq i < j \leq n$ be independent Bernoulli random variables

$$\mathbb{P}(W_{nij} = 1) = p,$$

for some $p \in (0, 1)$. Complete the definition of W_n by setting $W_{nji} = W_{nij}$ and $W_{nii} = 0$, $i \in [n]$. With this choice of W_n , (5) yields the KM on Erdős-Rényi random graph.

The family of Erdős-Rényi graphs parameterized by n is one example of a convergent family of random graphs [30]. The limiting behavior of such families is determined by a symmetric measurable function on the unit square $W(x, y)$, called a graphon. In the case of the Erdős-Rényi graphs, the limiting graphon is the constant function $W \equiv p$. In this paper, we study the KM on convergent families of deterministic and random graphs. In each case the asymptotic properties of graphs are known through the limiting graphon W . The precise relation between the graphon W and the weight matrix W_n will be explained below.

In studies of coupled systems on graphs, one of the main questions is the relation between the structure of the graph and network dynamics. For the problem at hand, this translates into the question of how the structure of the graph affects the transition to synchrony in the KM. For the KM on convergent families of graphs, in this paper, we derive the formulas for the critical values

$$K_c^+ = \frac{2}{\pi g(0) \zeta_{\max}(\mathcal{W})} \quad \text{and} \quad K_c^- = \frac{2}{\pi g(0) \zeta_{\min}(\mathcal{W})}, \quad (6)$$

where $\zeta_{\max}(\mathcal{W})$ ($\zeta_{\min}(\mathcal{W})$) is the largest positive (smallest negative) eigenvalue of the self-adjoint kernel operator $\mathcal{W} : L^2(I) \rightarrow L^2(I)$, $I := [0, 1]$, defined by

$$\mathcal{W}[f] = \int_I W(\cdot, y) f(y) dy, \quad f \in L^2(I). \quad (7)$$

If all eigenvalues of \mathcal{W} are positive (negative) then $K_c^- := -\infty$ ($K_c^+ := \infty$). The main result of this work shows that the incoherent state is linearly (neutrally) stable for $K \in [K_c^-, K_c^+]$ and is unstable otherwise. The transition point formulas in (6) reveal the effect of the network topology on the synchronization properties of the KM through the extreme eigenvalues $\zeta_{\max}(\mathcal{W})$ and $\zeta_{\min}(\mathcal{W})$. For the classical KM ($W \equiv 1$) $\zeta_{\max}(\mathcal{W}) = 1$ and there are no negative eigenvalues. Thus, we recover (3) from (6).

We derive (6) from the linear stability analysis of the mean field limit of (5). The latter is a partial differential equation for the probability density function corresponding to the distribution of the phase variables on the unit circle (see (16), (17)). For the classical KM, the mean field limit was derived by Strogatz and Mirollo in [48]. We derive the mean field limit for the KM on weighted graphs

(5) and show that its solutions approximate probability distribution of the phase variables on finite time intervals for $n \gg 1$. Here, we rely on the theory of Neunzert developed for the Vlasov equation [39, 40] (see also [4, 13, 19]), which was also used by Lancellotti in his treatment of the mean field limit for the classical KM [28]. In this paper, we assume that W is a Lipschitz continuous function on I^2 , so that we can use the classical results for the Vlasov equation in [39, 40]. Our analysis can be extended to the models with non-Lipschitz graphons W (cf. [22]). We do not pursue this extension in the present work, to keep the analysis simple.

With the mean field limit in hand, we proceed to study transition to coherence in (5). As for the classical KM, the density of the uniform distribution is a steady state solution of the mean field limit. The linear stability analysis in Section 3 shows that the density of the uniform distribution is neutrally stable for $K_c^- \leq K \leq K_c^+$ and is unstable otherwise. Thus, the critical values K_c^\pm given in (6) mark the loss of stability of the incoherent state. The bifurcations at K_c^\pm and the formula for the order parameter corresponding to (4) will be analyzed elsewhere using the techniques from [6, 7].

Sections 4 and 5 deal with applications. In the former section we collect approximation results, which facilitate application of our results to a wider class of models. Further, in Section 5, we discuss the KM for several representative network topologies : Erdős-Rényi, small-world, k -nearest-neighbor graphs, and the weighted ring model. We conclude with a brief discussion of our results in Section 6.

In addition to the work by Chiba [10, 5, 6] already mentioned above, the mean field limit and stability of the incoherent state in the classical KM (1) have been more recently treated by several authors [28, 12, 17]. In addition, the mean field limits for the KM forced by noise on lattices (albeit with singular weights) and on certain random graphs have been studied in [32] and [11] respectively. The goal of the present paper is twofold. First, we suggest a flexible framework for studying the mean field limit for the KM on convergent families of weighted deterministic and random graphs. Our framework covers many graphs common in applications including Erdős-Rényi and small-world graphs. Furthermore, with appropriate rescaling of the discrete model (5) our method applies without changes to certain sparse graphs, like sparse Erdős-Rényi and sparse stochastic block graphs (see Example 2.4 in [23]). Second, we show that linear stability of the incoherent state in the KM on graphs is fully determined by the spectral properties of the limiting graphon of the underlying graph sequence. This provides the link between the network structure and synchronization for the KM on graphs (5). This is the main message of this paper. In the follow-up work [8], we address several issues pertinent to nonlinear stability including asymptotic stability of the incoherent state, bifurcations, and the center manifold reduction.

2. The mean field limit. Throughout this paper, we will use a discretization of $I = [0, 1]$:

$$X_n = \{\xi_{n1}, \xi_{n2}, \dots, \xi_{nn}\}, \quad \xi_{ni} \in I, \quad i \in [n], \quad (8)$$

which satisfies the following property

$$\lim_{n \rightarrow \infty} n^{-1} \sum_{i=1}^n f(\xi_{ni}) = \int_I f(x) dx, \quad \forall f \in C(I). \quad (9)$$

Example 2.1. The following two examples of X_n will be used in constructions of various graphs throughout this paper.

1. The family of sets (8) with $\xi_{ni} = i/n$, $i \in [n]$ satisfies (9).
2. Let ξ_1, ξ_2, \dots be independent identically distributed (IID) random variables (RVs) with ξ_1 having the uniform distribution on I . Then with $X_n = \{\xi_1, \xi_2, \dots, \xi_n\}$ (9) holds almost surely (a.s.), by the Strong Law of Large Numbers.

Let W be a symmetric Lipschitz continuous function on I^2 :

$$|W(x_1, y_1) - W(x_2, y_2)| \leq L_W \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \quad \forall (x_{1,2}, y_{1,2}) \in I^2. \quad (10)$$

The weighted graph $\Gamma_n = G(W, X_n)$ on n nodes is defined as follows. The node and the edge sets of Γ_n are $V(\Gamma_n) = [n]$ and

$$E(\Gamma_n) = \{\{i, j\} : W(\xi_{ni}, \xi_{nj}) \neq 0, i, j \in [n]\}, \quad (11)$$

respectively. Each edge $\{i, j\} \in E(\Gamma_n)$ is supplied with the weight $W_{nij} := W(\xi_{ni}, \xi_{nj})$.

On Γ_n , we consider the KM of phase oscillators

$$\dot{\theta}_{ni} = \omega_i + K n^{-1} \sum_{j=1}^n W_{nij} \sin(\theta_{nj} - \theta_{ni}), \quad i \in [n]. \quad (12)$$

The phase variable $\theta_{ni} : \mathbb{R} \rightarrow \mathbb{S} := \mathbb{R}/2\pi\mathbb{Z}$ corresponds to the oscillator at node $i \in [n]$. Throughout this paper, we identify $\theta \in \mathbb{S}$ with its value in the fundamental domain, i.e., $\theta \in [0, 2\pi)$. Further, we equip \mathbb{S} with the distance

$$d_{\mathbb{S}}(\theta, \theta') = \min\{|\theta - \theta'|, 2\pi - |\theta - \theta'|\}. \quad (13)$$

The oscillators at the adjacent nodes interact through the coupling term on the right hand side of (12). The intrinsic frequencies $\omega_1, \omega_2, \dots$ are IID RVs. Assume that ω_1 has absolutely continuous probability distribution with a continuous density $g(\omega)$. The initial condition

$$\theta_{ni}(0) = \theta_i^0, \quad i \in [n], \quad (14)$$

are sampled independently from the conditional probability distributions with densities $\hat{\rho}_{\theta|\omega}^0(\theta, \omega_i, \xi_{ni})$, $i \in [n]$. Here, $\hat{\rho}_{\theta|\omega}^0(\theta, \omega, \xi)$ is a nonnegative continuous function on $G := \mathbb{S} \times \mathbb{R} \times I$ that is uniformly continuous in ξ . In addition, we assume

$$\int_{\mathbb{S}} \hat{\rho}_{\theta|\omega}^0(\phi, \omega, \xi) d\phi = 1 \quad \forall (\omega, \xi) \in \mathbb{R} \times I. \quad (15)$$

We want to show that the dynamics of (12) subject to the initial condition (14) can be described in terms of the probability density function $\hat{\rho}(t, \theta, \omega, x)$ satisfying the following Vlasov equation

$$\frac{\partial}{\partial t} \hat{\rho}(t, \theta, \omega, x) + \frac{\partial}{\partial \theta} \{\hat{\rho}(t, \theta, \omega, x) V(t, \theta, \omega, x)\} = 0, \quad (16)$$

where

$$V(t, \theta, \omega, x) = \omega + K \int_I \int_{\mathbb{R}} \int_{\mathbb{S}} W(x, y) \sin(\phi - \theta) \hat{\rho}(t, \phi, \lambda, y) d\phi d\lambda dy \quad (17)$$

and the initial condition

$$\hat{\rho}(0, \theta, \omega, x) = \hat{\rho}_{\theta|\omega}^0(\theta, \omega, x) g(\omega), \quad (18)$$

where g is a given probability density function describing the distribution of the intrinsic frequencies ω_i , $i \in [n]$, in the KM (5). By (15), $\hat{\rho}(0, \theta, \omega, x)$ is a probability

density on $(G, \mathcal{B}(G))$:

$$\begin{aligned} \int_{\mathbb{S}} \int_{\mathbb{R}} \int_I \hat{\rho}(0, \theta, \omega, x) dx d\omega d\theta &= \int_{\mathbb{R}} \int_I \left\{ \int_{\mathbb{S}} \hat{\rho}_{\theta|\omega}^0(\theta, \omega, x) d\theta \right\} g(\omega) dx d\omega \\ &= \int_{\mathbb{R}} g(\omega) d\omega = 1. \end{aligned} \quad (19)$$

Here, $\mathcal{B}(G)$ stands for the Borel σ -algebra of G .

Below, we show that the solutions of the IVPs for (12) and (16), generate two families of Borel probability measures parametrized by $t > 0$. To this end, we introduce the following empirical measure

$$\mu_t^n(A) = n^{-1} \sum_{i=1}^n \delta_{P_{ni}(t)}(A) \quad A \in \mathcal{B}(G), \quad (20)$$

where $P_{ni}(t) = (\theta_{ni}(t), \omega_i, \xi_{ni}) \in G$.

To compare measures generated by the discrete and continuous systems, following [40], we use the bounded Lipschitz distance:

$$d(\mu, \nu) = \sup_{f \in \mathcal{L}} \left| \int_G f d\mu - \int_G f d\nu \right|, \quad \mu, \nu \in \mathcal{M}, \quad (21)$$

where \mathcal{L} is the set of functions

$$\mathcal{L} = \{f : G \rightarrow [0, 1] : |f(P) - f(Q)| \leq d_G(P, Q), P, Q \in G\} \quad (22)$$

and \mathcal{M} stands for the space of Borel probability measures on G . Here,

$$d_G(P, P') = \sqrt{d_{\mathbb{S}}(\theta, \theta')^2 + (\omega - \omega')^2 + (x - x')^2},$$

for $P = (\theta, \omega, x)$ and $P' = (\theta', \omega', x')$. The bounded Lipschitz distance metrizes the convergence of Borel probability measures on G [15, Theorem 11.3.3].

We are now in a position to formulate the main result of this section.

Theorem 2.2. *Suppose W is a Lipschitz continuous function on I^2 . Then for any $T > 0$, there exists a unique weak solution² of the IVP (16), (17), and (18), $\hat{\rho}(t, \cdot)$, $t \in [0, T]$, which provides the density for Borel probability measure on G :*

$$\mu_t(A) = \int_A \hat{\rho}(t, P) dP, \quad A \in \mathcal{B}(G), \quad (23)$$

parametrized by $t \in [0, T]$. Furthermore,

$$d(\mu_t^n, \mu_t) \rightarrow 0 \quad (24)$$

uniformly for $t \in [0, T]$, provided $d(\mu_0^n, \mu_0) \rightarrow 0$ as $n \rightarrow \infty$.

Proof. We rewrite (12) as follows

$$\begin{aligned} \dot{\theta}_{ni} &= \lambda_i + Kn^{-1} \sum_{j=1}^n W(x_{ni}, x_{nj}) \sin(\theta_{nj} - \theta_{ni}), \\ \dot{\lambda}_{ni} &= 0, \\ \dot{x}_{ni} &= 0, \quad i \in [n], \end{aligned} \quad (25)$$

subject to the initial condition

$$(\theta_{ni}(0), \lambda_{ni}(0), x_{ni}(0)) = (\theta_i^0, \omega_i, \xi_{ni}), \quad i \in [n]. \quad (26)$$

²See [39, Remark 1] for the definition of the weak solution.

As before, we consider the empirical measure corresponding to the solutions of (25), (26)

$$\mu_t^n(A) = n^{-1} \sum_{i=1}^n \delta_{P_{ni}(t)}(A), \quad A \in \mathcal{B}(G), \quad (27)$$

where $P_{ni}(t) = (\theta_{ni}(t), \lambda_{ni}(t), x_{ni}(t)) \in G$.

We need to show that μ_t^n and μ_t are close for large n . This follows from the Neunzert's theory [40]. Specifically, below we show

$$d(\mu_t^n, \mu_t) \leq C d(\mu_0^n, \mu_0), \quad t \in [0, T], \quad (28)$$

for some $C > 0$ independent from n .

Below, we prove (28). Theorem 2.2 will then follow.

Let $C(0, T; \mathcal{M})$ denote the space of weakly continuous \mathcal{M} -valued functions on $[0, T]$. Specifically, $\mu_\cdot \in C(0, T; \mathcal{M})$ means that

$$t \mapsto \int_G f(P) d\mu_t(P) \quad (29)$$

is a continuous function of $t \in [0, T]$ for every bounded continuous function $f \in C_b(G)$.

For a given $\nu_\cdot \in C(0, T; \mathcal{M})$, consider the following equation of characteristics:

$$\frac{dP}{dt} = \tilde{V}[\nu_\cdot](t, P), \quad P(s) = P^0 \in G. \quad (30)$$

where $P = (\theta, \omega, x)$ and

$$\tilde{V}[\nu_\cdot](t, P) = \begin{pmatrix} \omega + K \int_G W(x, y) \sin(v - \theta) d\nu_t(v, \omega, y) \\ 0 \\ 0 \end{pmatrix}. \quad (31)$$

Under our assumptions on W , (30) has a unique global solution, which depends continuously on initial data. Thus, (31) generates the flow $T_{t,s} : G \rightarrow G$ ($T_{s,s} = \text{id}$, $T_{s,t} = T_{t,s}^{-1}$):

$$P(t) = T_{t,s}[\nu_\cdot]P^0.$$

Following [39], we consider the fixed point equation:

$$\nu_t = \nu_0 \circ T_{0,t}[\nu_\cdot], \quad t \in [0, T], \quad (32)$$

which is interpreted as

$$\nu_t(A) = \nu_0(T_{0,t}[\nu_\cdot](A)) \quad \forall A \in \mathcal{B}(G).$$

It is shown in [39] that under the conditions (I) and (II) given below, for any $\nu_0 \in \mathcal{M}$ there is a unique solution of the fixed point equation (32) $\nu_\cdot \in C(0, T; \mathcal{M})$. Moreover, for any two initial conditions $\nu_0^{1,2} \in \mathcal{M}$, we have

$$\sup_{t \in [0, T]} d(\nu_t^1, \nu_t^2) \leq \exp\{CT\} d(\nu_0^1, \nu_0^2) \quad (33)$$

for some $C > 0$. By construction of $T_{t,s}$ and (27), the empirical measure μ_t^n satisfies the fixed point equation (32). By [39, Theorem 1], ν_t , the solution of the (32), is an absolutely continuous measure with density $\hat{\rho}(t, \cdot)$ for every $t \in [0, T]$, provided ν_0 is absolutely continuous with density $\hat{\rho}(0, \cdot)$ (cf. (19)). Furthermore, $\hat{\rho}(t, P)$ is a weak solution of the IVP for (16), (17), and (18). Therefore, since both the empirical measure μ_t^n and its continuous counterpart μ_\cdot (cf. (27) and (23)) satisfy the fixed point equation (32), we can use (33) to obtain (28). It remains to verify the following

two conditions on the vector field $\tilde{V}[\nu.]$, which guarantee the solvability of (32) and continuous dependence on initial data estimate (33) (cf. [40]):

- (I): $\tilde{V}[\mu.](t, P)$ is continuous in t and is globally Lipschitz continuous in P with Lipschitz constant³ L_1 , which depends on W .
- (II): The mapping $\tilde{V} : \mu. \mapsto \tilde{V}[\mu.]$ is Lipschitz continuous in the following sense:

$$\left| \tilde{V}[\mu.](t, P) - \tilde{V}[\nu.](t, P) \right| \leq L_2 d(\mu_t, \nu_t),$$

for some $L_2 > 0$ and for all $\mu., \nu. \in C(\mathbb{R}, \mathcal{M})$ and $(t, P) \in [0, T] \times G$.⁴

For the Lipschitz continuous function W , it is straightforward to verify conditions (I) and (II). In particular, (I) follows from the weak continuity of μ_t (cf. (29)) and Lipschitz continuity of W and $\sin x$. The second condition is verified following the treatment of the mechanical system presented in [40] (see also [28]). We include the details of the verification of (II) for completeness.

Let $P = (\theta, \omega, x) \in G$ be arbitrary but fixed and define

$$f(\phi, \lambda, y; P) = \frac{W(x, y) \sin(\phi - \theta) + \|W\|_{L^\infty(I^2)}}{2(\|W\|_{L^\infty(I^2)} + L_W)}, \quad (34)$$

where L_W is the Lipschitz constant of $W(x, y)$ (cf. (10)). Then $f \in \mathcal{L}$ (cf. (21)). Further,

$$\begin{aligned} & \left| \tilde{V}[\nu.](t, P) - \tilde{V}[\mu.](t, P) \right| \\ &= \left| K \int_G W(x, y) \sin(\phi - \theta) (d\nu_t(\phi, \lambda, y) - d\mu_t(\phi, \lambda, y)) \right| \\ &= 2K(\|W\|_{L^\infty(I^2)} + L_W) \left| \int_G f(\phi, \lambda, y) (d\nu_t(\phi, \lambda, y) - d\mu_t(\phi, \lambda, y)) \right| \\ &\leq L_2 d(\nu_t, \mu_t), \quad L_2 := 2K(\|W\|_{L^\infty(I^2)} + L_W), \end{aligned}$$

which verifies the condition (II). \square

Corollary 2.3. *For the empirical measure μ_t^n (20) and absolutely continuous measure μ_t (23) defined on the solutions of the IVPs (12), (14), (15) and (16), (17), (18) respectively, we have*

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} d(\mu_t^n, \mu_t) = 0 \quad \text{a.s..}$$

Proof. In view of Theorem 2.2, we need to show

$$\lim_{n \rightarrow \infty} d(\mu_0^n, \mu_0) = 0 \quad \text{a.s..}$$

By [15, Theorem 11.3.3], it is sufficient to show

$$\lim_{n \rightarrow \infty} \int_G f (d\mu_0^n - d\mu_0) = 0 \quad \forall f \in BL(G) \quad \text{a.s.}, \quad (35)$$

³A straightforward estimate shows that $L_1 = \sqrt{2} (L_W + \|W\|_{L^\infty(I^2)}) + 1$, where L_W is the Lipschitz constant in (10).

⁴With these assumptions the estimate (33) holds with $C = L_1 + L_2$ (see [40]).

where $BL(G)$ stands for the space of bounded real-valued Lipschitz functions on G with the supremum norm. Since $BL(G)$ is a separable space, let $\{f_m\}_{m=1}^\infty$ denote a dense set in $BL(G)$. Using (14) and (20), we have

$$\int_G f_m d\mu_0^n = n^{-1} \sum_{i=1}^n f_m(\theta_i^0, \omega_i, \xi_{ni}) =: n^{-1} \sum_{i=1}^n Y_{m,ni}. \quad (36)$$

RVs $Y_{m,ni}$, $i \in [n]$, are independent and uniformly bounded. Further,

$$\begin{aligned} \mathbb{E} \left(n^{-1} \sum_{i=1}^n Y_{m,ni} \right) &= n^{-1} \sum_{i=1}^n \int_{\mathbb{S}} \int_{\mathbb{R}} f_m(\phi, \lambda, \xi_{ni}) \hat{\rho}_{\theta|\omega}^0(\phi, \lambda, \xi_{ni}) g(\lambda) d\lambda d\phi \\ &=: n^{-1} \sum_{i=1}^n F_m(\xi_{ni}), \end{aligned} \quad (37)$$

Because f_m is Lipschitz and $\hat{\rho}_{\theta|\omega}^0$ is uniformly continuous in ξ , the function

$$F_m(\xi) = \int_{\mathbb{S}} \int_{\mathbb{R}} f_m(\phi, \lambda, \xi) \hat{\rho}_{\theta|\omega}^0(\phi, \lambda, \xi) g(\lambda) d\lambda d\phi$$

is continuous on I .

By (9), we have

$$\lim_{n \rightarrow \infty} \mathbb{E} \left(n^{-1} \sum_{i=1}^n Y_{m,ni} \right) = \int_I F_m(\xi) d\xi = \int_G f_m d\mu_0. \quad (38)$$

By the Strong Law of Large Numbers⁵, from (36), (37), and (38) we have

$$\lim_{n \rightarrow \infty} \int_G f_m (d\mu_0^n - d\mu_0) = \lim_{n \rightarrow \infty} n^{-1} \sum_{i=1}^n (Y_{m,ni} - \mathbb{E} Y_{m,ni}) = 0 \quad \text{a.s.} \quad (39)$$

Therefore,

$$\mathbb{P} \left\{ \lim_{n \rightarrow \infty} \int_G f_m (d\mu_0^n - d\mu_0) = 0 \quad \forall m \in \mathbb{N} \right\} = 1.$$

Using density of $\{f_m\}_{m=1}^\infty$ in $BL(G)$, we have

$$\mathbb{P} \left\{ \lim_{n \rightarrow \infty} \int_G f (d\mu_0^n - d\mu_0) = 0 \quad \forall f \in BL(G) \right\} = 1.$$

□

3. Linear stability. In the previous section, we established that the Vlasov equation (16), approximates discrete system (12) for $n \gg 1$. Next, we will use (16) to characterize the transition to synchrony for increasing K . To this end, in this section, we derive the linearized equation about the incoherent state, the steady state solution of the mean field equation. In the next section, we will study how the spectrum of the linearized equation changes with K .

In this section, we assume that probability density g is a continuous and even function monotonically decreasing on \mathbb{R}^+ .

⁵ It is easy to adjust the proof of Theorem 6.1 [2] so that it applies to the triangular array Y_{ni} , $i \in [n]$, $n \in \mathbb{N}$ (see [37, Lemma 3.1]).

3.1. Linearization. First, setting $\hat{\rho}(t, \theta, \omega, x) = \rho(t, \theta, \omega, x)g(\omega)$, from (16) we derive the equation for ρ :

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial \theta}\{V_\rho\rho\} = 0, \quad (40)$$

where

$$V_\rho(t, \theta, \omega, x) = \omega + K \int_I \int_{\mathbb{S}} \int_{\mathbb{R}} W(x, y) \sin(\phi - \theta) \rho(t, \phi, \lambda, y) g(\lambda) d\lambda d\phi dy.$$

By integrating (16) over \mathbb{S} , one can see that

$$\frac{\partial}{\partial t} \int_{\mathbb{S}} \hat{\rho}(t, \theta, \omega, x) d\theta = 0,$$

and, thus,

$$\int_{\mathbb{S}} \hat{\rho}(t, \theta, \omega, x) d\theta = \int_{\mathbb{S}} \hat{\rho}(0, \theta, \omega, x) d\theta = \int_{\mathbb{S}} \hat{\rho}_{\theta|\omega}^0(0, \theta, \omega, x) g(\omega) d\theta = g(\omega). \quad (41)$$

Thus,

$$\int_{\mathbb{S}} \rho(t, \theta, \omega, x) d\theta = 1 \quad \forall (t, \omega, x) \in \mathbb{R}^+ \times \mathbb{R} \times I. \quad (42)$$

In addition,

$$\int_{\mathbb{R}} g(\omega) d\omega = 1,$$

because g is a probability density function. The density of the uniform distribution on \mathbb{S} , $\rho_u = (2\pi)^{-1}$, is a steady-state solution of the mean field equation (40). It corresponds to the completely mixed state. We are interested in stability of this solution. In the remainder of this section, we derive the linearized equation around ρ_u .

Let

$$\rho = \rho_u + z(t, \theta, \omega, x). \quad (43)$$

By (42),

$$\int_{\mathbb{S}} z(t, \theta, \omega, x) d\theta = 0 \quad \forall (t, \omega, x) \in \mathbb{R}^+ \times \mathbb{R} \times I. \quad (44)$$

By plugging in (43) into (40), we obtain

$$\frac{\partial}{\partial t} z(t, \theta, \omega, x) + \frac{\partial}{\partial \theta} \left\{ V_{\rho_u+z}(t, \theta, \omega, x) \left(\frac{1}{2\pi} + z \right) \right\} = 0. \quad (45)$$

The expression in the curly brackets has the following form:

$$\begin{aligned} & V_{\rho_u+z}(t, \theta, \omega, x) \left(\frac{1}{2\pi} + z \right) \\ &= \left(\omega + K \int_I \int_{\mathbb{S}} \int_{\mathbb{R}} W(x, y) \sin(\phi - \theta) \times \right. \\ &\quad \times \left((2\pi)^{-1} + z(t, \phi, \lambda, y) \right) g(\lambda) d\lambda d\phi dy \left. \right) \left(\frac{1}{2\pi} + z \right) = \frac{\omega}{2\pi} + \omega z \\ &\quad + \frac{K}{2\pi} \int_I \int_{\mathbb{S}} \int_{\mathbb{R}} W(x, y) \sin(\phi - \theta) z(t, \phi, \lambda, y) g(\lambda) d\lambda d\phi dy + O(z^2). \end{aligned}$$

Thus,

$$\frac{\partial}{\partial t} z + \frac{\partial}{\partial \theta} \left\{ \omega z + \frac{K}{2\pi} \mathcal{G}[z] \right\} + O(z^2) = 0,$$

where the linear operator \mathcal{G} is defined by

$$\mathcal{G}[z] := \int_I \int_{\mathbb{S}} \int_{\mathbb{R}} W(x, y) \sin(\phi - \theta) z(t, \phi, \lambda, y) g(\lambda) d\lambda d\phi dy \quad (46)$$

We arrive at the linearized equation:

$$\frac{\partial}{\partial t} Z + \frac{\partial}{\partial \theta} \left\{ \omega Z + \frac{K}{2\pi} \mathcal{G}[Z] \right\} = 0. \quad (47)$$

3.2. Fourier transform. We expand Z into Fourier series

$$Z(t, \theta, \omega, x) = \sum_{k=1}^{\infty} \hat{Z}_k(t, \omega, x) e^{-ik\theta} + \overline{\left(\sum_{k=1}^{\infty} \hat{Z}_k(t, \omega, x) e^{-ik\theta} \right)}, \quad (48)$$

where \hat{Z}_k stands for the Fourier transform of Z

$$\hat{Z}_k = \frac{1}{2\pi} \int_{\mathbb{S}} Z(\theta, \cdot) e^{ik\theta} d\theta.$$

In (48), we are using the fact that Z is real and $\hat{Z}_0 = 0$ (cf. (44)).

The linear stability of ρ_u is, thus, determined by the time-asymptotic behavior of \hat{Z}_k , $k \geq 1$. To derive the differential equations for \hat{Z}_k , $k \geq 1$, we apply the Fourier transform to (47):

$$\frac{\partial}{\partial t} \hat{Z}_k + \left(\frac{\partial}{\partial \theta} \left\{ \omega Z + \frac{K}{2\pi} \mathcal{G}[Z] \right\} \right)_k = 0. \quad (49)$$

Using the definition of the Fourier transform and integration by parts, we have

$$\left(\frac{\partial}{\partial \theta} \left\{ \omega Z + \frac{K}{2\pi} \mathcal{G}[Z] \right\} \right)_k = \frac{1}{2\pi} \int_{\mathbb{S}} \frac{\partial}{\partial \theta} (\dots) e^{ik\theta} d\theta = -ik\omega \hat{Z}_k - ik \frac{K}{2\pi} (\widehat{\mathcal{G}[Z]})_k. \quad (50)$$

It remains to compute $(\widehat{\mathcal{G}[Z]})_k$, $k \geq 1$. To this, end we rewrite

$$\begin{aligned} \mathcal{G}[Z] &= \frac{1}{2i} \int_I \int_{\mathbb{S}} \int_{\mathbb{R}} W(x, y) \left(e^{i(\phi-\theta)} - e^{-i(\phi-\theta)} \right) Z(t, \phi, \lambda, y) g(\lambda) d\lambda d\phi dy \\ &= \frac{\pi}{i} \int_I \int_{\mathbb{R}} W(x, y) \left(e^{-i\theta} \hat{Z}_1 - e^{i\theta} \hat{Z}_{-1} \right) g(\lambda) d\lambda dy. \end{aligned} \quad (51)$$

Using (51), we compute

$$\begin{aligned} (\widehat{\mathcal{G}[Z]})_k &= \frac{1}{2\pi} \frac{\pi}{i} \int_{\mathbb{S}} \int_I \int_{\mathbb{R}} W(x, y) \left(e^{i(k-1)\theta} \hat{Z}_1 - e^{i(k+1)\theta} \hat{Z}_{-1} \right) g(\lambda) d\lambda dy d\theta \\ &= \begin{cases} \frac{\pi}{i} \mathcal{P}[\hat{Z}_1], & k = 1, \\ 0, & k > 1, \end{cases} \end{aligned} \quad (52)$$

where

$$\mathcal{P}[Z] := \int_I \int_{\mathbb{R}} W(x, y) Z(t, \lambda, y) g(\lambda) d\lambda dy. \quad (53)$$

The combination of (49), (50), and (52) yields the system of equations for \hat{Z}_k , $k \geq 1$:

$$\frac{\partial}{\partial t} \hat{Z}_1 = i\omega \hat{Z}_1 + \frac{K}{2} \mathcal{P}[\hat{Z}_1], \quad (54)$$

$$\frac{\partial}{\partial t} \hat{Z}_k = ik\omega \hat{Z}_k, \quad k > 1. \quad (55)$$

3.3. Spectral analysis. In this section, we study (54), which decides the linear stability of the uniform density ρ_u as a steady state solution of the mean field equation. We rewrite (54) as

$$\frac{\partial}{\partial t} \hat{Z}_1(t, \omega, x) = T[\hat{Z}_1](t, \omega, x). \quad (56)$$

where

$$T[Z] = i\omega Z + \frac{K}{2} \mathcal{P}[Z]. \quad (57)$$

Equations (53) and (57) define linear operators \mathcal{P} and T on the weighted Lebesgue space $L^2(X, g d\omega dx)$ with $X := \mathbb{R} \times I$.

Lemma 3.1. *$T : L^2(X, g d\omega dx) \rightarrow L^2(X, g d\omega dx)$ is a closed operator. The residual spectrum of T is empty and the continuous spectrum $\sigma_c(T) = i \operatorname{supp}(g)$.*

Proof. Consider the multiplication operator $\mathcal{M}_{i\omega} : L^2(X, g d\omega dx) \rightarrow L^2(X, g d\omega dx)$ defined by

$$\mathcal{M}_{i\omega} z = i\omega z, \quad \omega \in \mathbb{R}. \quad (58)$$

It is well known that $\mathcal{M}_{i\omega}$ is closed and the (continuous) spectrum of $\mathcal{M}_{i\omega}$ lies on the imaginary axis $\sigma_c(\mathcal{M}_{i\omega}) = i \cdot \operatorname{supp}(g)$. Since $W(x, y)$ is square-integrable by the assumption, \mathcal{P} is a Hilbert-Schmidt operator on $L^2(X, g d\omega dx)$ and, therefore, is compact [54]. Then, the statement of the lemma follows from the perturbation theory for linear operators [24]. \square

Similarly, the spectrum of the operator $\mathcal{M}_{ij\omega}$ lies on the imaginary axis; $\sigma(\mathcal{M}_{ij\omega}) = ij \cdot \operatorname{supp}(g)$. Hence, the trivial solution $\hat{Z}_j \equiv 0$ of (55) for $j = 2, 3, \dots$ is neutrally stable.

We define a Fredholm integral operator \mathcal{W} on $L^2(I)$ by

$$\mathcal{W}[V](x) = \int_{\mathbb{R}} W(x, y) V(y) dy. \quad (59)$$

Since \mathcal{W} is compact, the set of eigenvalues $\sigma_p(\mathcal{W})$ is a bounded countable with the only accumulation point at the origin. Since \mathcal{W} is symmetric, all eigenvalues are real numbers.

Lemma 3.2. *The eigenvalues of T are given by*

$$\sigma_p(T) = \left\{ \lambda \in \mathbb{C} \setminus \sigma_c(T) : D(\lambda) = \frac{2}{\zeta K}, \zeta \in \sigma_p(\mathcal{W}) \setminus \{0\} \right\}, \quad (60)$$

where

$$D(\lambda) := \int_{\mathbb{R}} \frac{1}{\lambda - i\omega} g(\omega) d\omega. \quad (61)$$

Proof. Suppose $v \in L^2(X, g d\omega dx)$ is an eigenvector of T corresponding to the eigenvalue λ :

$$T[v] = \lambda v.$$

Then

$$v = 2^{-1} K(\lambda - i\omega)^{-1} \mathcal{P}[v]. \quad (62)$$

By multiplying both sides of (62) by $g(\omega)$ and integrating with respect to ω , we have

$$\int_{\mathbb{R}} v(\omega, x) g(\omega) d\omega = \frac{K}{2} \int_{\mathbb{R}} \frac{1}{\lambda - i\omega} g(\omega) d\omega \cdot \int_I \int_{\mathbb{R}} W(x, y) v(\omega, y) g(\omega) d\omega dy. \quad (63)$$

Rewrite (63) as

$$V = \frac{K}{2} D(\lambda) \mathcal{W}[V], \quad (64)$$

where

$$V := \int_{\mathbb{R}} v(\omega, \cdot) g(\omega) d\omega \in L^2(I).$$

Equations (64) and (59) reduce the eigenvalue problem for T to that for the Fredholm operator \mathcal{W} . Suppose $V \in L^2(I)$ is an eigenfunction of \mathcal{W} associated with the eigenvalue $\zeta \neq 0$. Then (64) implies $D(\lambda) = 2/(\zeta K)$.

If $0 \in \sigma_p(\mathcal{W})$ and V is a corresponding eigenvector, then Equation (62) yields

$$\begin{aligned} v &= \frac{K}{2} \frac{1}{\lambda - i\omega} \int_I \int_{\mathbb{R}} W(x, y) v(\omega, y) g(\omega) d\omega dy \\ &= \frac{K}{2} \frac{1}{\lambda - i\omega} \int_I W(x, y) V(y) dy = \frac{K}{2} \frac{1}{\lambda - i\omega} (\mathcal{W}[V])(x) = 0. \end{aligned}$$

Thus, $\zeta = 0$ is not an eigenvalue of T . \square

For $\zeta \neq 0$ denote

$$K(\zeta) = \frac{2}{\pi g(0)} \frac{1}{|\zeta|}. \quad (65)$$

Lemma 3.3. *For each $\zeta \in \sigma_p(\mathcal{W})$ and $K > K(\zeta)$ there is a unique eigenvalue of the operator T (cf. (57)), $\lambda = \lambda(\zeta, K)$ satisfying $D(\lambda) = 2/(\zeta K)$.*

For $\zeta \in \sigma_p(\mathcal{W}) \cap \mathbb{R}^+$, $\lambda(\zeta, K)$ is a positive increasing function of K satisfying

$$\lim_{K \rightarrow K(\zeta)+0} \lambda(\zeta, K) = 0+0, \quad \lim_{K \rightarrow \infty} \lambda(\zeta, K) = \infty. \quad (66)$$

For $\zeta \in \sigma_p(\mathcal{W}) \cap \mathbb{R}^-$, $\lambda(\zeta, K)$ is a negative decreasing function of K satisfying

$$\lim_{K \rightarrow K(\zeta)+0} \lambda(\zeta, K) = 0-0, \quad \lim_{K \rightarrow \infty} \lambda(\zeta, K) = -\infty. \quad (67)$$

Finally,

$$\sigma_p(T) = \{\lambda(\zeta, K) : \zeta \in \sigma_p(\mathcal{W}) \setminus \{0\}, K > K(\zeta)\} \subset \mathbb{R} \setminus \{0\}.$$

Proof. Since $\zeta \in \mathbb{R}$, setting $\lambda = x + iy$ for the equation $D(\lambda) = 2/(\zeta K)$ yields

$$\begin{cases} \int_{\mathbb{R}} \frac{x}{x^2 + (\omega - y)^2} g(\omega) d\omega = \frac{2}{\zeta K}, \\ \int_{\mathbb{R}} \frac{\omega - y}{x^2 + (\omega - y)^2} g(\omega) d\omega = 0. \end{cases} \quad (68)$$

With $\zeta = 1$ this system of equations was analyzed in [6] in the context of the classical Kuramoto model (1).

The second equation of (68) gives

$$0 = \int_{\mathbb{R}} \frac{\omega - y}{x^2 + (\omega - y)^2} g(\omega) d\omega = \int_0^\infty \frac{\omega}{x^2 + \omega^2} (g(y + \omega) - g(y - \omega)) d\omega. \quad (69)$$

Since g is even, $y = 0$ is a solution of (69). Furthermore, since g is unimodal, there are no other solutions of (69). Thus, λ is real.

With $y = 0$ the first equation of (68) yields

$$\int_{\mathbb{R}} \frac{x}{x^2 + \omega^2} g(\omega) d\omega = \frac{2}{\zeta K}. \quad (70)$$

For the uniqueness of x satisfying this equality, it is sufficient to show that the function

$$x \mapsto \int_{\mathbb{R}} \frac{x}{x^2 + \omega^2} g(\omega) d\omega$$

is monotonically decreasing in x except at the jump point $x = 0$. If this were not true, there would exist two eigenvalues for some interval $K_1 < K < K_2$, and two eigenvalues would collide and disappear at $K = K_1$ or $K = K_2$. This is impossible, because $D(\lambda)$ is holomorphic in λ .

Since g is nonnegative and $K > 0$, from (70) we have $\zeta x > 0$. Further, the left-hand side of (70) satisfies

$$\lim_{x \rightarrow \pm 0} \int_{\mathbb{R}} \frac{x}{x^2 + \omega^2} g(\omega) d\omega = \pm \pi g(0), \quad (71)$$

$$\lim_{x \rightarrow \pm \infty} \int_{\mathbb{R}} \frac{x}{x^2 + \omega^2} g(\omega) d\omega = 0. \quad (72)$$

The first identity follows from the Poisson's integral formula for the upper half-plane [45]. The combination of (70) and (71) implies that $x \rightarrow 0$ as $K \rightarrow K(\zeta) + 0$, while that of (70) and (72) yields $x \rightarrow \pm \infty$ as $K \rightarrow \infty$. \square

To formulate the main result of this section, we will need the following notation:

$$\begin{aligned} \xi_{\max}(\mathcal{W}) &= \begin{cases} \max\{\zeta : \zeta \in \sigma_p(\mathcal{W}) \cap \mathbb{R}^+\}, & \sigma_p(\mathcal{W}) \cap \mathbb{R}^+ \neq \emptyset, \\ 0 + 0, & \text{otherwise.} \end{cases} \\ \xi_{\min}(\mathcal{W}) &= \begin{cases} \min\{\zeta : \zeta \in \sigma_p(\mathcal{W}) \cap \mathbb{R}^-\}, & \sigma_p(\mathcal{W}) \cap \mathbb{R}^- \neq \emptyset, \\ 0 - 0, & \text{otherwise.} \end{cases} \end{aligned} \quad (73)$$

Further, let

$$K_c^+ = \frac{2}{\pi g(0) \xi_{\max}(\mathcal{W})} \quad \text{and} \quad K_c^- = \frac{2}{\pi g(0) \xi_{\min}(\mathcal{W})}. \quad (74)$$

Theorem 3.4. *The spectrum of T consists of the continuous spectrum on the imaginary axis and possibly one or more negative eigenvalues, if $K \in [K_c^-, K_c^+]$, and there is at least one positive eigenvalue of T , otherwise.*

Theorem 3.4 follows from Lemma 3.3. It shows that the incoherent state is linearly (neutrally) stable for $K \in [K_c^-, K_c^+]$ and is unstable otherwise. The critical values K_c^\pm mark the loss of stability the incoherent state. The detailed analysis of the bifurcations at K_c^\pm will be presented elsewhere.

Remark 3.5. For the classical Kuramoto model on the complete graph, $W \equiv 1$ and $\zeta_{\max}(\mathcal{W}) = 1$. Thus, we recover the well-known Kuramoto's transition formula (3) [27]. In the general case, the transition points depend on the graph structure through the extreme eigenvalues of the kernel operator \mathcal{W} .

Remark 3.6. For nonnegative graphons W , $\zeta_{\max}(\mathcal{W})$ coincides with the spectral radius of the limiting kernel operator \mathcal{W} (cf. [50]):

$$\varrho(\mathcal{W}) = \max\{|\zeta| : \zeta \in \sigma_p(\mathcal{W})\},$$

This can be seen from the variational characterization of the eigenvalues of a self-adjoint compact operator, which also implies

$$\varrho(\mathcal{W}) = \max_{\|f\|_{L^2(I)}=1} (\mathcal{W}[f], f). \quad (75)$$

4. Approximation. Equation (12) may be viewed as a base model. To apply our results to a wider class of deterministic and random networks, we will need approximation results, which are collected in this section.

4.1. Deterministic networks. Consider the Kuramoto model on the weighted graph $\tilde{\Gamma}_n = \langle [n], E(\tilde{\Gamma}_n), \tilde{W} \rangle$

$$\dot{\tilde{\theta}}_{ni} = \omega_i + Kn^{-1} \sum_{j=1}^n \tilde{W}_{nij} \sin(\tilde{\theta}_{nj} - \tilde{\theta}_{ni}), \quad i \in [n], \quad (76)$$

where $\tilde{W}_n = (\tilde{W}_{nij})$ is a symmetric matrix.

Denote the corresponding empirical measure by

$$\tilde{\mu}_t^n(A) = n^{-1} \sum_{i=1}^n \delta_{\tilde{P}_{ni}(t)}(A), \quad A \in \mathcal{B}(G), \quad (77)$$

where $\tilde{P}_{ni}(t) = (\tilde{\theta}_{ni}(t), \omega_i, \xi_{ni})$, $i \in [n]$.

First, we show that if W_n and \tilde{W}_n are close, so are the solutions of the IVPs for (12) and (76) with the same initial conditions. To measure the proximity of $W_n = (W_{nij}) \in \mathbb{R}^{n \times n}$ and $\tilde{W}_n = (\tilde{W}_{nij})$ and the corresponding solutions θ_n and $\tilde{\theta}_n$, we will use the following norms:

$$\|W_n\|_{2,n} = \sqrt{n^{-2} \sum_{i,j=1}^n W_{nij}^2}, \quad \|\theta_n\|_{1,n} = \sqrt{n^{-1} \sum_{i=1}^n \theta_{ni}^2}, \quad (78)$$

where $\theta_n = (\theta_{n1}, \theta_{n2}, \dots, \theta_{nn})$ and $W_n = (W_{nij})$.

The following lemma will be used to extend our results for the KM (12) to other networks.

Lemma 4.1. *Let $\theta_n(t)$ and $\tilde{\theta}_n(t)$ denote solutions of the IVP for (12) and (76) respectively. Suppose that the initial data for these problems coincide*

$$\theta_n(0) = \tilde{\theta}_n(0). \quad (79)$$

Then for any $T > 0$ there exists $C = C(T) > 0$ such that

$$\max_{t \in [0, T]} \left\| \theta_n(t) - \tilde{\theta}_n(t) \right\|_{1,n} \leq C \left\| W_n - \tilde{W}_n \right\|_{2,n}, \quad (80)$$

where the positive constant C is independent from n .

Corollary 4.2.

$$\sup_{t \in [0, T]} d(\mu_t^n, \tilde{\mu}_t^n) \leq C \left\| W_n - \tilde{W}_n \right\|_{2,n}. \quad (81)$$

Proof. (Lemma 4.1) Denote $\phi_{ni} = \theta_{ni} - \tilde{\theta}_{ni}$. By subtracting (76) from (12), multiplying the result by $n^{-1}\phi_{ni}$, and summing over $i \in [n]$, we obtain

$$\begin{aligned} (2K)^{-1} \frac{d}{dt} \|\phi_n\|_{1,n}^2 &= n^{-2} \sum_{i,j=1}^n (W_{nij} - \tilde{W}_{nij}) \sin(\theta_{nj} - \theta_{ni}) \phi_{ni} \\ &+ n^{-2} \sum_{i,j=1}^n \tilde{W}_{nij} \left(\sin(\theta_{nj} - \theta_{ni}) - \sin(\tilde{\theta}_{nj} - \tilde{\theta}_{ni}) \right) \phi_{in} \\ &=: I_1 + I_2. \end{aligned} \quad (82)$$

Using an obvious bound $|\sin(\theta_{nj} - \theta_{ni})| \leq 1$ and an elementary inequality $|ab| \leq 2^{-1}(a^2 + b^2)$, we obtain

$$|I_1| \leq 2^{-1} \left(\|W_n - \tilde{W}_n\|_{2,n}^2 + \|\phi_n\|_{1,n}^2 \right). \quad (83)$$

Further, from Lipschitz continuity of \sin and the definition of ϕ_{ni} , we have

$$\left| \sin(\theta_{ni} - \theta_{nj}) - \sin(\tilde{\theta}_{ni} - \tilde{\theta}_{nj}) \right| \leq |\phi_{ni} - \phi_{nj}| \leq |\phi_{ni}| + |\phi_{nj}|.$$

Therefore,

$$|I_2| \leq 2\|\tilde{W}\|_{L^\infty(I^2)}\|\phi_n\|_{1,n}^2. \quad (84)$$

The combination of (82), (83), and (84) yields

$$K^{-1} \frac{d}{dt} \|\phi_n(t)\|_{1,n}^2 \leq \left(4\|\tilde{W}\|_{L^\infty(I^2)} + 1 \right) \|\phi_n(t)\|_{1,n}^2 + \|W_n - \tilde{W}_n\|_{2,n}^2. \quad (85)$$

By the Gronwall's inequality [16, Appendix B.2.j],

$$\begin{aligned} \|\phi_n(t)\|_{1,n}^2 &\leq e^{KC_1 t} \left(\|\phi_n(0)\|_{1,n}^2 + \int_0^t K \|W_n - \tilde{W}_n\|_{2,n}^2 ds \right) \\ &\leq e^{KC_1 t} K t \|W_n - \tilde{W}_n\|_{2,n}^2, \end{aligned}$$

where we used $\phi_n(0) = 0$ and $C_1 := (4\|\tilde{W}\|_{L^\infty(I^2)} + 1)$. Thus,

$$\sup_{t \in [0, T]} \|\phi_n(t)\|_{1,n}^2 \leq e^{KC_1 T} K T \|W_n - \tilde{W}_n\|_{2,n}^2.$$

□

Proof. (Corollary 4.2) Let $f \in \mathcal{L}$ (cf. (22)) and consider

$$\begin{aligned} \left| \int_G f(d\mu_t^n - d\tilde{\mu}_t^n) \right| &= \left| n^{-1} \sum_{i=1}^n f(\theta_{ni}(t)) - f(\tilde{\theta}_{ni}(t)) \right| \\ &\leq n^{-1} \sum_{i=1}^n \left| \theta_{ni}(t) - \tilde{\theta}_{ni}(t) \right| \\ &\leq \|\theta_n(t) - \tilde{\theta}_n(t)\|_{1,n}. \end{aligned}$$

By Lemma 4.1,

$$\max_{t \in [0, T]} d(\mu_t^n, \tilde{\mu}_t^n) = \sup_{f \in \mathcal{L}} \left| \int_G f(d\mu_t^n - d\tilde{\mu}_t^n) \right| \leq C \|W_n - \tilde{W}_n\|_{2,n}.$$

□

4.2. Random networks. We now turn to the KM on random graphs. To this end, we use W-random graph $\bar{\Gamma}_n = G_r(X_n, W)$, which we define next. As before, X_n is a set of points (8), (9). $\bar{\Gamma}_n$ is a graph on n nodes, i.e., $V(\bar{\Gamma}_n) = [n]$. The edge set is defined as follows:

$$\mathbb{P}(\{i, j\} \in E(\Gamma_n)) = W(\xi_{ni}, \xi_{nj}). \quad (86)$$

The decision for each pair $\{i, j\}$ is made independently from the decisions on other pairs.

The KM on the W-random graph $\bar{\Gamma}_n = G_r(X_n, W)$ has the following form:

$$\dot{\bar{\theta}}_{ni} = \omega_i + K n^{-1} \sum_{j=1}^n e_{nij} \sin(\bar{\theta}_{nj} - \bar{\theta}_{ni}), \quad i \in [n] := \{1, 2, \dots, n\} \quad (87)$$

where e_{nij} , $1 \leq i \leq j \leq n$ are independent Bernoulli RVs:

$$\mathbb{P}(e_{nij} = 1) = W(\xi_{ni}, \xi_{nj}),$$

and $e_{nij} = e_{nji}$.

Lemma 4.3. *Let $\theta_n(t)$ and $\bar{\theta}_n(t)$ denote solutions of the IVP for (12) and (87) respectively. Suppose that the initial data for these problems coincide*

$$\theta_n(0) = \bar{\theta}_n(0). \quad (88)$$

Then

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \|\theta_n(t) - \bar{\theta}_n(t)\|_{1,n} = 0 \quad a.s.. \quad (89)$$

Proof. The proof follows the lines of the proof of Lemma 4.1. As before, we set up the equation for $\phi_{ni} := \bar{\theta}_{ni} - \theta_{ni}$:

$$\begin{aligned} (2K)^{-1} \frac{d}{dt} \|\phi_n\|_{1,n}^2 &= n^{-2} \sum_{i,j=1}^n (e_{nij} - W_{nij}) \sin(\theta_{nj} - \theta_{ni}) \phi_{ni} \\ &+ n^{-2} \sum_{i,j=1}^n e_{nij} (\sin(\bar{\theta}_{nj} - \bar{\theta}_{ni}) - \sin(\theta_{nj} - \theta_{ni})) \phi_{in} \\ &=: I_1 + I_2. \end{aligned} \quad (90)$$

As in (84), we have bound

$$|I_2| \leq n^{-2} \sum_{i,j=1}^n (|\phi_{ni}| + |\phi_{nj}|) |\phi_{ni}| \leq 2 \|\phi_n\|_{1,n}^2, \quad (91)$$

where we used $0 \leq e_{nij} \leq 1$.

Next, we turn to the first term on the right hand side of (90). For this, we will need the following definitions:

$$\begin{aligned} Z_{ni}(t) &= n^{-1} \sum_{j=1}^n a_{nij}(t) \eta_{nij}, \\ a_{nij}(t) &= \sin(\theta_{nj}(t) - \theta_{ni}(t)), \\ \eta_{nij} &= e_{nij} - W_{nij}. \end{aligned}$$

and $Z_n = (Z_{n1}, Z_{n2}, \dots, Z_{nn})$. With these definitions in hand, we estimate I_1 as follows:

$$|I_1| = |n^{-1} \sum_{i=1}^n Z_{ni} \phi_{ni}| \leq 2^{-1} (\|Z_n\|_{1,n}^2 + \|\phi_n\|_{1,n}^2), \quad (92)$$

The combination of (90)-(92) yields,

$$\frac{d}{dt} \|\phi_n(t)\|_{1,n}^2 \leq 5K \|\phi_n(t)\|_{1,n}^2 + K \|Z_n(t)\|_{1,n}^2. \quad (93)$$

Using the Gronwall's inequality and (88), we have

$$\|\phi_n(t)\|_{1,n}^2 \leq K \exp\{5Kt\} \int_0^t \|Z_n(s)\|_{1,n}^2 ds$$

and

$$\sup_{t \in [0, T]} \|\phi_n(t)\|_{1,n}^2 \leq K \exp\{5KT\} \int_0^T \|Z_n(t)\|_{1,n}^2 dt. \quad (94)$$

Our next goal is to estimate $\int_0^T \|Z_n(t)\|_{1,n}^2 dt$. Below, we show that

$$\int_0^T \|Z_n(t)\|_{1,n}^2 dt \rightarrow 0 \quad \text{as } n \rightarrow \infty \text{ a.s.} \quad (95)$$

To this end, we will use the following observations. Note that η_{nik} and η_{nil} are independent for $k \neq l$ and

$$\mathbb{E} \eta_{nij} = \mathbb{E} e_{nij} - W_{nij} = 0, \quad (96)$$

where we used $\mathbb{P}(e_{nij} = 1) = W_{nij}$.

Further,

$$\begin{aligned} \mathbb{E} \eta_{nij}^2 &= \mathbb{E} (e_{nij} - W_{nij})^2 = \mathbb{E} e_{nij}^2 - W_{nij}^2 \\ &= W_{nij} - W_{nij}^2 \leq 1/4. \end{aligned} \quad (97)$$

and

$$\begin{aligned} \mathbb{E} (\eta_{nij}^4) &= \mathbb{E} (e_{nij} - W_{nij})^4 \\ &= \mathbb{E} (e_{nij}^4 - 4e_{nij}^3 W_{nij} + 6e_{nij}^2 W_{nij}^2 - 4e_{nij} W_{nij}^3 + W_{nij}^4) \\ &= W_{nij} - 4W_{nij}^2 + 6W_{nij}^3 - 3W_{nij}^4 \\ &= W_{nij} (1 - W_{nij})^4 + W_{nij}^4 (1 - W_{nij}) \leq 2^{-4}. \end{aligned} \quad (98)$$

Next,

$$\int_0^T Z_{ni}(t)^2 dt = n^{-2} \sum_{k,l=1}^n c_{nikl} \eta_{nik} \eta_{nil}, \quad (99)$$

where

$$c_{nikl} = \int_0^T a_{nik}(t) a_{nil}(t) dt \quad \text{and} \quad |c_{nikl}| \leq T. \quad (100)$$

Further,

$$\int_0^T \|Z_n(t)\|_{1,n}^2 dt = n^{-3} \sum_{i,k,l=1}^n c_{nikl} \eta_{nik} \eta_{nil} \quad (101)$$

and, finally,

$$\mathbb{E} \left(\int_0^T \|Z_n(t)\|_{1,n}^2 dt \right)^2 = n^{-6} \sum_{i,k,l,j,p,q=1}^n c_{nikl} c_{njpq} \mathbb{E} (\eta_{nik} \eta_{nil} \eta_{njp} \eta_{njq}). \quad (102)$$

We have six summation indices i, k, l, j, p, q ranging from 1 to n . Since $\mathbb{E} \eta_{mik} = 0$ for $i, k \in [n]$, and RVs η_{mik} and η_{njp} are independent whenever $\{i, k\} \neq \{j, p\}$, the nonzero terms on the right-hand side of (102) fall into two groups:

$$\begin{aligned} I : & \quad c_{nikk} c_{njpp} \eta_{nik}^2 \eta_{njp}^2 \quad (i \neq j) \quad \text{or} \quad c_{nikl}^2 \eta_{nik}^2 \eta_{nil}^2 \quad (k \neq l). \\ II : & \quad c_{nikk} c_{njpp} \eta_{nik}^2 \eta_{njp}^2 \quad (i \neq j) \quad \text{or} \quad c_{nikl}^2 \eta_{nik}^2 \eta_{nil}^2 \quad (k \neq l). \end{aligned}$$

There are n^2 terms of type I and $3n^3(n-1)$ terms of type II . Thus,

$$\mathbb{E} \left(\int_0^T \|Z_n(t)\|_{1,n}^2 dt \right)^2 \leq T^2 n^{-6} (n^2 + 3n^3(n-1)) = O(n^{-2}), \quad (103)$$

where we used (102), (97), (98), and the bound on $|c_{nikl}|$ in (100).

Next, for a given $\epsilon > 0$ we denote

$$A_n^\epsilon = \left\{ \left| \int_0^T \|Z_n(t)\|_{1,n}^2 dt \right| \geq \epsilon \right\}. \quad (104)$$

and use Markov's inequality and (103) to obtain

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n^\epsilon) \leq \epsilon^{-2} \sum_{n=1}^{\infty} \mathbb{E} \left(\int_0^T \|Z_n(t)\|_{1,n}^2 dt \right)^2 < \infty. \quad (105)$$

By the Borel-Cantelli Lemma, (104) and (105) imply (95). The latter combined with (94) concludes the proof of Lemma 4.3. \square

5. Examples.

5.1. Erdős-Rényi graphs. Let $p \in (0, 1)$, X_n be defined in (8), (9), and $W_p(x, y) \equiv p$. Then $\Gamma_{n,p} = G_r(X_n, W_p)$ is a family of Erdős-Rényi random graphs. To apply the transition point formula (74), we need to compute the largest eigenvalue of the self-adjoint compact operator $\mathcal{W}_p : L^2(I) \rightarrow L^2(I)$ defined by

$$\mathcal{W}_p[f](x) = \int_I W(x, y) f(y) dy = p \int_I f(y) dy, \quad f \in L^2(I). \quad (106)$$

Lemma 5.1. *The largest eigenvalue of \mathcal{W}_p is $\zeta_{\max}(\mathcal{W}_p) = p$.*

Proof. Suppose $\lambda \in \mathbb{R} \setminus \{0\}$ is an eigenvalue of \mathcal{W}_p and $v \in L^2(I)$ is the corresponding eigenvector. Then

$$p \int_I v(y) dy = \zeta v(x). \quad (107)$$

Since the right-hand side is not identically 0, $v = \text{constant} \neq 0$. By integrating both sides of (107), we find that $\zeta = p$. \square

Thus, for the KM on Erdős-Rényi graphs, we have

$$K_c^+ = \frac{2}{\pi g(0)p}, \quad K_c^- = -\infty.$$

5.2. Small-world network. Small-world (SW) graphs interpolate between regular nearest-neighbor graphs and completely random Erdős-Rényi graphs. They found widespread applications, because they combine features of regular symmetric graphs and random graphs, just as seen in many real-world networks [53].

Following [36, 37], we construct SW graphs as W-random graphs [31]. To this end let X_n be a set of points from (8) satisfying (9), and define $W_{p,r} : I^2 \rightarrow I$ by

$$W_{p,r}(x, y) = \begin{cases} 1 - p, & d_{\mathbb{S}}(2\pi x, 2\pi y) \leq 2\pi r, \\ p, & \text{otherwise,} \end{cases} \quad (108)$$

where $p, r \in (0, 1/2)$ are two parameters.

Definition 5.2. [36] $\Gamma_n = G_r(X_n, W_{p,r})$ is called W-small-world graph.

The justification of the mean field limit in Section 2 relies on the assumption that the graphon W is Lipschitz continuous. A rigorous treatment of the mean field limit for the KM with non-Lipschitz graphon $W_{p,r}$ is beyond the scope of this paper. For the mathematical analysis of the non-Lipschitz case, we refer an interested reader to [22]. In this paper, we apply Theorem 2.2 to the KM with regularized graphon $W_{p,r}^\epsilon$.

Let $W_{p,r}^\epsilon$ be a Lipschitz continuous graphon such that

$$\|W_{p,r} - W_{p,r}^\epsilon\|_{L^2(I^2)} < \epsilon, \quad (109)$$

and consider $\Gamma_n^\epsilon = G_r(X_n, W_{p,r}^\epsilon)$. Denote μ_t^n and $\mu_t^{n,\epsilon}$ empirical measures generated by the KMs on Γ_n and Γ_n^ϵ (with the same initial data) respectively. By (81),

$\sup_{t \in [0, T]} d(\mu_t^n, \mu_t^{n, \epsilon}) = O(\epsilon)$ a.s., i.e., the KM on Γ_n^ϵ approximates the KM on Γ_n . Below, we derive the transition point formula for the KM on Γ_n^ϵ .

Lemma 4.3 and Theorem 2.2 justify (16), (17) as the continuum limit for the KM on the sequence of SW graphs (Γ_n^ϵ) . Thus, (74) yields the transition point formula for the KM on SW graphs. To use this formula, we need to compute the extreme eigenvalues of $\mathcal{W}_{p,r}^\epsilon : L^2(I) \rightarrow L^2(I)$ defined by

$$\mathcal{W}_{p,r}^\epsilon[f] = \int_I W_{p,r}^\epsilon(\cdot, y) f(y) dy, \quad f \in L^2(I).$$

Lemma 5.3. *The largest eigenvalue of $\mathcal{W}_{p,r}^\epsilon$ is*

$$\zeta_{\max}(\mathcal{W}_{p,r}^\epsilon) = 2r + p - 4pr + o_\epsilon(1). \quad (110)$$

Proof. We calculate the largest eigenvalue of $\mathcal{W}_{p,r}$. Using the definition (108), one can write

$$\mathcal{W}_{p,r}[f](x) = \int_{\mathbb{S}} K_{p,r}(x - y) f(y) dy,$$

where $K_{p,r}$ is a 1-periodic function on \mathbb{R} , whose restriction to the interval $[-1/2, 1/2)$ is defined as follows

$$K_{p,r}(x) = \begin{cases} 1 - p, & |x| \leq r, \\ p, & \text{otherwise,} \end{cases} \quad (111)$$

The eigenvalue problem for $\mathcal{W}_{p,r}$ can be rewritten as

$$K_{p,r} * v = \zeta v.$$

Thus, the eigenvalues of $\mathcal{W}_{p,r}$ are given by the Fourier coefficients of $K_{p,r}(x)$ as

$$\zeta_k = (\hat{K}_{p,r})_k := \int_{\mathbb{S}} K_{p,r}(x) e^{-2\pi i k x} dx,$$

for $k \in \mathbb{Z}$. The corresponding eigenvectors $v_k = e^{i2\pi k x}$, $k \in \mathbb{Z}$, form a complete orthonormal set in $L^2(I)$.

A straightforward calculation yields

$$\zeta_k = \begin{cases} 2r + p - 4rp, & k = 0, \\ (\pi k)^{-1} (1 - 2p) \sin(2\pi k r), & k \neq 0. \end{cases} \quad (112)$$

Further,

$$\zeta_{\max}(\mathcal{W}_{p,r}) = 2r + p - 4rp. \quad (113)$$

The estimate (110) follows from (113) and (109) via continuous dependence of the eigenvalues with respect to the parameter ϵ . \square

Thus, for the KM on Γ_n^ϵ , we have

$$K_c^+ = \frac{2}{\pi g(0)} \frac{1}{2r + p - 4pr} + O(\epsilon).$$

The smallest negative eigenvalue is given by $\zeta_{k(r)}$ for some $k(r) \neq 0$. While it is difficult to obtain an explicit expression $k(r)$, from (112) we can find a lower bound for it:

$$\xi_{\min}(\mathcal{W}) > \frac{-1 + 2p}{\pi}. \quad (114)$$

In particular, for sufficiently small ϵ , the incoherent state in the KM on Γ_n^ϵ is stable for $K \in \left(2(g(0)(-1 + 2p))^{-1}, K_c^+\right]$.

5.3. Coupled oscillators on a ring. A common in applications type of network connectivity can be described as follows. Consider n oscillators placed uniformly around a circle. They are labelled by integers from 1 to n in the counterclockwise direction. To each potential edge $\{i, j\} \in [n]^2$ we assign a weight

$$W_{nij} = G(\xi_{ni} - \xi_{nj}), \quad (115)$$

where G is a 1-periodic even measurable bounded function.

Example 5.4. Let $X_n = \{1/n, 2/n, \dots, 1\}$ and the restriction of the 1-periodic even function G_r on $[0, 1/2]$ is defined by

$$G_r(x) = \begin{cases} 1, & 0 \leq x \leq r, \\ 0, & x > r, \end{cases} \quad (116)$$

where $r \in (0, 1/2)$ is a parameter. With this choice of $G := G_r$, we obtain a k -nearest-neighbor model, in which each node is connected to $k = \lfloor rn \rfloor$ from each side.

Example 5.5. Another representative example was used by Kuramoto and Battogtokh [26]. Here, let $X_n = \{1/n, 2/n, \dots, 1\}$ and the restriction of the 1-periodic even function G to $[0, 1/2]$ is defined by $G(x) := e^{-\kappa x}$, where $\kappa > 0$ is a parameter. With this choice of G , we obtain the KM where the strength of interactions decreases exponentially with the distance between oscillators.

As in our treatment of the KM on SW graphs in §5.2, the integral operator $\mathcal{W} : L^2(I) \rightarrow L^2(I)$ can be written as a convolution

$$\mathcal{W}[f](x) = \int_I G(x-y)f(y)dy, \quad f \in L^2(I). \quad (117)$$

It is easy to verify that the eigenvalues of \mathcal{W} are given by the Fourier coefficients of $G(x)$.

For instance, for the k -nearest-neighbor network in Example 5.4, by setting $p = 0$ in (113), for the network at hand we obtain $\zeta_{\max}(\mathcal{W}) = r$ and, thus,

$$K_c^+ = \frac{1}{\pi g(0)r}.$$

Note that, in accord with our physical intuition, the transition point is inversely proportional to the coupling range.

For the network in Example 5.5, eigenvalues are given by

$$\zeta_k := \begin{cases} \frac{2\kappa}{\kappa^2 + 4\pi^2 k^2} (1 - e^{-\kappa/2}), & (k : \text{even}), \\ \frac{2\kappa}{\kappa^2 + 4\pi^2 k^2} (1 + e^{-\kappa/2}), & (k : \text{odd}), \end{cases} \quad k = 0, 1, 2, \dots \quad (118)$$

The largest positive eigenvalue is $\zeta_{\max}(\mathcal{W}) = \zeta_0$, and we obtain

$$K_c^+ = \frac{\kappa}{\pi g(0)} \frac{1}{1 - e^{-\kappa/2}}, \quad K_c^- = -\infty.$$

This recovers the classical result (3) as $\kappa \rightarrow 0$.

If the explicit expression for the largest positive eigenvalue of \mathcal{W} is not available, for a network with nonnegative graphon W , the transition point can be estimated using the variational characterization of the largest eigenvalue (75). Specifically, from (75), we have:

$$\zeta_{\max}(\mathcal{W}) \geq \int_{I^2} W dx dy = \|W\|_{L^1(I^2)},$$

and, thus,

$$K_c^+ \leq \frac{2}{\pi g(0) \|W\|_{L^1(I^2)}}.$$

6. Discussion. In this work, we derived and rigorously justified the mean field equation for the KM on convergent families of graphs. Our theory covers a large class of coupled systems. In particular, it clarifies the mathematical meaning of the mean field equation used in the analysis of chimera states (see, e.g., [41]). Moreover, we show how to write the mean field equation for the KM on many common in applications random graphs including Erdős-Rényi and small-world graphs, for which it has not been known before.

We used the mean field equation to study synchronization in the KM on large deterministic and random graphs. We derived the transition point formulas for the critical values of the coupling strength, at which the incoherent state loses stability. The transition point formulas show explicit dependence of the stability boundaries of the incoherent state on the spectral properties of the limiting graphon. This reveals the precise mechanism by which the network topology affects the stability of the incoherent state and the onset of synchronization. In the follow-up work, we will show that the linear stability analysis of this paper can be extended to show nonlinear stability of the incoherent state albeit with respect to the weak topology. There we will also present the bifurcation analysis for the critical values K_c^\pm . The analysis of the KM on small-world networks shows that, unlike in the original KM (1), on graphs the incoherent state may remain stable even for negative values of K , i.e., for repulsive coupling. In fact, the bifurcations at the left and right endpoints can be qualitatively different. In the small-world case, the center manifold at K_c^- is two-dimensional, whereas it is one-dimensional at K_c^+ . These first findings indicate that the bifurcation structure of the KM on graphs (12) is richer than that of its classical counterpart (1) and motivates further investigations of this interesting problem [9]. In the future, we also plan to extend our analysis to the KM on certain sparse graphs, including sparse power law networks considered in [23].

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