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Global Holomorphic Functions in Several Non-Commuting Variables II

Jim Agler and John McCarthy

Abstract. We give a new proof that bounded non-commutative functions on polynomial polyhedra can be represented by a realization formula, a generalization of the transfer function realization formula for bounded analytic functions on the unit disk.

1 Introduction

Let $\mathbb{M}_n$ denote the $n \times n$ matrices with complex entries, and let $\mathbb{M}^d = \bigcup_{n=1}^{\infty} \mathbb{M}_n^d$ be the set of all d-tuples of matrices of the same size. A *non-commutative function* (nc-function) on a set $E \subseteq \mathbb{M}^d$ is a function $\phi: E \to \mathbb{M}^1$ that satisfies

- ϕ is graded, which means that if $x \in E \cap \mathbb{M}_n^d$, then $\phi(x) \in \mathbb{M}_n$;
- ϕ is intertwining preserving, which means that if $x, y \in E$ and S is a linear operator satisfying $Sx = yS$, then $S\phi(x) = \phi(y)S$.

The points x and y are d-tuples, so we write $x = (x^1, \dots, x^d)$ and $y = (y^1, \dots, y^d)$. By $Sx = yS$, we mean that $Sx^r = y^r S$ for each $1 \leq r \leq d$. See [9] for a general reference to nc-functions.

The principal result of [2] was a realization formula for nc-functions that are bounded on polynomial polyhedra; the object of this note is to give a simpler proof of this formula, (see Theorem 1.2).

Let δ be an $I \times J$ matrix whose entries are non-commutative polynomials in d variables. If $x \in \mathbb{M}_n^d$, then $\delta(x)$ can be naturally thought of as an element of $\mathcal{B}(\mathbb{C}^J \otimes \mathbb{C}^n, \mathbb{C}^I \otimes \mathbb{C}^n)$, where $\mathcal{B}$ denotes the bounded linear operators, and all norms we use are operator norms on the appropriate spaces. We define

$$B_\delta := \{x \in \mathbb{M}^d : \|\delta(x)\| < 1\}. \tag{1.1}$$

Any set of the form (1.1) is called a *polynomial polyhedron*. Let $H^\infty(B_\delta)$ denote the nc-functions on B_δ that are bounded, and let $H_1^\infty(B_\delta)$ denote the closed unit ball, those nc-functions that are bounded by 1 for every $x \in B_\delta$.

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Definition 1.1 A *free realization* for ϕ consists of an auxiliary Hilbert space $\mathcal{M}$ and an isometry

$$\begin{array}{cc} & \begin{array}{cc}\mathbb{C} & \mathcal{M}\otimes\mathbb{C}^I\end{array} \\ \begin{array}{c}\mathbb{C}\\ \mathcal{M}\otimes\mathbb{C}^J\end{array} & \begin{pmatrix} A & B \\ C & D \end{pmatrix}\end{array}$$

such that for all $x \in B_\delta$, we have

$$\phi(x) = \begin{matrix}A\\ \otimes\\ 1\end{matrix} + \begin{matrix}B\\ \otimes\\ 1\end{matrix}\ \begin{matrix}1\\ \otimes\\ \delta(x)\end{matrix}\left[1 - \begin{matrix}D\\ \otimes\\ 1\end{matrix}\ \begin{matrix}1\\ \otimes\\ \delta(x)\end{matrix}\right]^{-1}\begin{matrix}C\\ \otimes\\ 1\end{matrix}. \tag{1.2}$$

The 1s need to be interpreted appropriately. If $x \in \mathbb{M}_n^d$, then (1.2) means

$$\phi(x) = \begin{matrix}A\\ \otimes\\ \mathrm{id}_{\mathbb{C}^n}\end{matrix} + \begin{matrix}B\\ \otimes\\ \mathrm{id}_{\mathbb{C}^n}\end{matrix}\ \begin{matrix}\mathrm{id}_{\mathcal{M}}\\ \otimes\\ \delta(x)\end{matrix}\left[\begin{matrix}\mathrm{id}_{\mathcal{M}}\\ \otimes\\ \mathrm{id}_{\mathbb{C}^I}\\ \otimes\\ \mathrm{id}_{\mathbb{C}^n}\end{matrix} - \begin{matrix}D\\ \otimes\\ \mathrm{id}_{\mathbb{C}^n}\end{matrix}\ \begin{matrix}\mathrm{id}_{\mathcal{M}}\\ \otimes\\ \delta(x)\end{matrix}\right]^{-1}\begin{matrix}C\\ \otimes\\ \mathrm{id}_{\mathbb{C}^n}\end{matrix}.$$

We adopt the convention of [11] and write tensors vertically to enhance legibility. The bottom-most entry corresponds to the space on which x originally acts; the top corresponds to the intrinsic part of the model on $\mathcal{M}$.

The following theorem was proved in [2]; another proof appears in [6].

Theorem 1.2 *The function ϕ is in $H_1^\infty(B_\delta)$ if and only if it has a free realization.*

It is a straightforward calculation that any function of the form (1.2) is in $H_1^\infty(B_\delta)$. We wish to prove the converse. We shall use two other results: Theorems 1.4 and 1.5.

If $E \subset \mathbb{M}^d$, we let E_n denote $E \cap \mathbb{M}_n^d$. If $\mathcal{K}$ and $\mathcal{L}$ are Hilbert spaces, a $\mathcal{B}(\mathcal{K},\mathcal{L})$-valued nc function on a set $E \subseteq \mathbb{M}^d$ is a function ϕ such that

- ϕ is $\mathcal{B}(\mathcal{K},\mathcal{L})$ graded, which means if $x \in E_n$, then $\phi(x) \in \mathcal{B}(\mathcal{K}\otimes\mathbb{C}^n, \mathcal{L}\otimes\mathbb{C}^n)$;
- ϕ is intertwining preserving, which means if $x, y \in E$ and S is a linear operator satisfying $Sx = yS$, then

$$\begin{matrix}\mathrm{id}_{\mathcal{L}}\\ \otimes\\ S\end{matrix}\ \phi(x) = \phi(y)\ \begin{matrix}\mathrm{id}_{\mathcal{K}}\\ \otimes\\ S\end{matrix}.$$

Definition 1.3 An nc-model for $\phi \in H_1^\infty(B_\delta)$ consists of an auxiliary Hilbert space $\mathcal{M}$ and a $\mathcal{B}(\mathbb{C}, \mathcal{M}\otimes\mathbb{C}^J)$-valued nc-function u on B_δ such that, for all pairs $x, y \in B_\delta$ that are on the same level, *i.e.*, both in $B_\delta \cap \mathbb{M}_n^d$ for some n,

$$1 - \phi(y)^*\phi(x) = u(y)^*\left[\begin{matrix}1\\ \otimes\\ 1-\delta(y)^*\delta(x)\end{matrix}\right]u(x). \tag{1.3}$$

Again, the 1s have to be interpreted appropriately. If $x, y \in B_\delta \cap \mathbb{M}_n^d$, then (1.3) means

$$\mathrm{id}_{\mathbb{C}^n} - \phi(y)^*\phi(x) = u(y)^*\left[\begin{matrix}\mathrm{id}_{\mathcal{M}}\\ \otimes\\ \mathrm{id}_{\mathbb{C}^J\otimes\mathbb{C}^n} - \delta(y)^*\delta(x)\end{matrix}\right]u(x).$$

Theorem 1.4 *A graded function on B_δ has an nc-model if and only if it has a free realization.*

Theorem 1.4 was proved in [2], but a simpler proof was given by Balasubramanian [5]. Let us note for future reference that the functions u in (1.3) are locally bounded, and therefore holomorphic [2, Theorem. 4.6].

The finite topology on $\mathbb{M}^d$ (also called the disjoint union topology) is the topology in which a set Ω is open if and only if for every n, Ω_n is open in the Euclidean topology on $\mathbb{M}_n^d$. If $\mathcal{H}$ is a Hilbert space, and Ω is finitely open, we shall let $\mathrm{Hol}^{\mathrm{nc}}_{\mathcal{H}}(\Omega)$ denote the $\mathcal{B}(\mathbb{C},\mathcal{H})$ graded nc-functions on Ω that are holomorphic on each Ω_n. (A function u is holomorphic in this context if for each n, each $x \in \Omega_n$, and each $h \in \mathbb{M}_n^d$, the limit $\lim_{t\to 0} 1/t(u(x+th)-u(x))$ exists.) A sequence of functions u^k on Ω is finitely locally uniformly bounded if for each point $\lambda \in \Omega$, there is a finitely open neighborhood of λ inside Ω on which the sequence is uniformly bounded.

The following wandering Montel theorem was proved in [1]. If u is in $\mathrm{Hol}^{\mathrm{nc}}_{\mathcal{H}}(\Omega)$ and V is a unitary operator on $\mathcal{H}$, define $V * u$ by $(V * u)|_{\Omega_n} = \overset{V}{\underset{\mathrm{id}_{\mathbb{C}^n}}{\otimes}} u|_{\Omega_n} \quad \forall_n$.

Theorem 1.5 *Let Ω be finitely open, $\mathcal{H}$ a Hilbert space, and $\{u^k\}$ a finitely locally uniformly bounded sequence in $\mathrm{Hol}^{\mathrm{nc}}_{\mathcal{H}}(\Omega)$. Then there exists a sequence $\{U^k\}$ of unitary operators on $\mathcal{H}$ such that $\{U^k * u^k\}$ has a subsequence that converges finitely locally uniformly to a function in $\mathrm{Hol}^{\mathrm{nc}}_{\mathcal{H}}(B_\delta)$.*

Let $\phi \in H_1^\infty(B_\delta)$. We shall prove Theorem 1.2 in the following steps.

- I For every $z \in B_\delta$, show that $\phi(z)$ is in $\mathrm{Alg}(z)$, the unital algebra generated by the elements of z.
- II Prove that for every finite set $F \subseteq B_\delta$, there is an nc-model for a function ψ that agrees with ϕ on F.
- III Show that these nc-models have a cluster point that gives an nc-model for ϕ.
- IV Use Theorem 1.4 to get a free realization for ϕ.

Remarks 1.6 Step I is noted in [2] as a corollary of Theorem 1.2; proving it independently allows us to streamline the proof of Theorem 1.2.

To prove Step II, we use one direction of [3, Theorem 1.3] that gives necessary and sufficient conditions to solve a finite interpolation problem on B_δ. The proof of necessity of this theorem used Theorem 1.2, but for Step II we only need the sufficiency of the condition, and the proof of this in [3] did not use Theorem 1.2.

All three known proofs of Theorem 1.2 start by proving a realization on finite sets, and then somehow taking a limit. In [2], this was done by considering partial nc-functions; in [6], it was done by using non-commutative kernels to get a compact set in which limit points must exist. In the current paper, we use the wandering Montel theorem.

2 Step I

Let $\{e_j\}_{j=1}^n$ be the standard basis for $\mathbb{C}^n$. For x in $\mathbb{M}_n$ or $\mathbb{M}_n^d$, let $x^{(k)}$ denote the direct sum of k copies of x. If $x \in \mathbb{M}_n^d$ and s is invertible in $\mathbb{M}_n$, then $s^{-1}xs$ denotes the d-tuple $(s^{-1}x^1s,\dots,s^{-1}x^ds)$.

Lemma 2.1 *Let $z \in \mathbb{M}_n^d$, with $\|z\| < 1$. Assume $w \notin \mathrm{Alg}(z)$. Then there is an invertible $s \in \mathbb{M}_{n^2}$ such that $\|s^{-1} z^{(n)} s\| < 1$ and $\|s^{-1} w^{(n)} s\| > 1$.*

Proof Let $\mathcal{A} = \mathrm{Alg}(z)$. Since $w \notin \mathcal{A}$, and $\mathcal{A}$ is finite dimensional and therefore closed, the Hahn–Banach theorem says that there is a matrix $K \in \mathbb{M}_n$ such that $\mathrm{tr}(aK) = 0$ for all $a \in \mathcal{A}$ and $\mathrm{tr}(wK) \neq 0$. Let $u \in \mathbb{C}^n \otimes \mathbb{C}^n$ be the direct sum of the columns of K, and $v = e_1 \oplus e_2 \oplus \cdots e_n$. Then for any $b \in \mathbb{M}_n$ we have

$$\mathrm{tr}(bK) = \langle b^{(n)} u, v \rangle.$$

Let $\mathcal{A} \otimes \mathrm{id}$ denote $\{a^{(n)} : a \in \mathcal{A}\}$. We have $\langle a^{(n)} u, v \rangle = 0$, for all $a \in \mathcal{A}$ and $\langle w^{(n)} u, v \rangle \neq 0$.

Let $\mathcal{N} = (\mathcal{A} \otimes \mathrm{id}) u$. This is an $\mathcal{A} \otimes \mathrm{id}$-invariant subspace, but it is not $w^{(n)}$ invariant (since $v \perp \mathcal{N}$, but v is not perpendicular to $w^{(n)} u$). So decomposing $\mathbb{C}^n \otimes \mathbb{C}^n$ as $\mathcal{N} \oplus \mathcal{N}^\perp$, every matrix in $\mathcal{A} \otimes \mathrm{id}$ has 0 in the $(2,1)$ entry, and $w^{(n)}$ does not.

Let $s = \alpha I_{\mathcal{N}} + \beta I_{\mathcal{N}^\perp}$, with $\alpha \gg \beta > 0$. Then

$$s^{-1} \begin{bmatrix} A & B \\ C & D \end{bmatrix} s = \begin{bmatrix} A & \frac{\beta}{\alpha} B \\ \frac{\alpha}{\beta} C & D \end{bmatrix}.$$

If the ratio α/β is large enough, then for each of the d matrices z^r, the corresponding $s^{-1}(z^r \otimes \mathrm{id}) s$ will have strict contractions in the (1,1) and (2,2) slots, and each $(1,2)$ entry will be small enough so that the whole thing is a contraction.

For w, however, as the $(2,1)$ entry is non-zero, the norm of $s^{-1} w^{(n)} s$ can be made arbitrarily large. ∎

Lemma 2.2 *Let $z \in B_\delta \cap \mathbb{M}_n^d$, and $w \in \mathbb{M}_n$ not be in $\mathcal{A} := \mathrm{Alg}(z)$. Then there is an invertible $s \in \mathbb{M}_{n^2}$ such that $s^{-1} z^{(n)} s \in B_\delta$ and $\|s^{-1} w^{(n)} s\| > 1$.*

Proof As in the proof of Lemma 2.1, we can find an invariant subspace $\mathcal{N}$ for $\mathcal{A} \otimes \mathrm{id}$ that is not w-invariant. Decompose $\delta(z^{(n)})$ as a map from $(\mathcal{N} \otimes \mathbb{C}^J) \oplus (\mathcal{N}^\perp \otimes \mathbb{C}^J)$ into $(\mathcal{N} \otimes \mathbb{C}^I) \oplus (\mathcal{N}^\perp \otimes \mathbb{C}^I)$. With s as in Lemma 2.1, and $\alpha \gg \beta > 0$, and P the projection from $\mathbb{C}^n \otimes \mathbb{C}^n$ onto $\mathcal{N}$, we get

$$\delta(s^{-1} z^{(n)} s) = \begin{bmatrix} \begin{smallmatrix} P \\ \otimes \\ \mathrm{id} \end{smallmatrix} \delta(z^{(n)}) \begin{smallmatrix} P \\ \otimes \\ \mathrm{id} \end{smallmatrix} & \frac{\beta}{\alpha} \begin{smallmatrix} P \\ \otimes \\ \mathrm{id} \end{smallmatrix} \delta(z^{(n)}) \begin{smallmatrix} P^\perp \\ \otimes \\ \mathrm{id} \end{smallmatrix} \\ 0 & \begin{smallmatrix} P^\perp \\ \otimes \\ \mathrm{id} \end{smallmatrix} \delta(z^{(n)}) \begin{smallmatrix} P^\perp \\ \otimes \\ \mathrm{id} \end{smallmatrix} \end{bmatrix}. \tag{2.1}$$

The matrix is upper triangular because every entry of δ is a polynomial, and $\mathcal{N}$ is $\mathcal{A}$-invariant. For α/β large enough, every matrix of the form (2.1) with $z \in B_\delta$ is a contraction, so $s^{-1} z^{(n)} s \in B_\delta$. But $s^{-1} w^{(n)} s$ will contain a non-zero entry multiplied by $\frac{\alpha}{\beta}$, so we achieve the claim. ∎

Theorem 2.3 *If ϕ is in $H^\infty(B_\delta)$, then for all $z \in B_\delta$, we have $\phi(z) \in \mathrm{Alg}(z)$.*

Proof We can assume that $z \in B_\delta$ and that $\|\phi\| \leq 1$ on B_δ. Let $w = \phi(z)$. If $w \notin \mathrm{Alg}(z)$, then by Lemma 2.2, there is an s such that $s^{-1} z^{(n)} s \in B_\delta$ and $\|\phi(s^{-1} z^{(n)} s)\| = \|s^{-1} w^{(n)} s\| > 1$, a contradiction. ∎

Note that Theorem 2.3 does not hold for all nc-functions. In [4] it was shown that there is a class of nc functions, called fat functions, for which the implicit function theorem holds, but Theorem 2.3 fails.

3 Step II

Let $F = \{x_1, \dots, x_N\}$. Define $\lambda = x_1 \oplus \cdots \oplus x_N$, and define $w = \phi(x_1) \oplus \cdots \oplus \phi(x_N)$. As nc functions preserve direct sums (a consequence of being intertwining preserving) we need to find a function ψ in $H_1^\infty(B_\delta)$ that has an nc model, and satisfies $\psi(\lambda) = w$.

Let $\mathcal{P}_d$ denote the nc polynomials in d variables, and define

$$I_\lambda = \{q \in \mathcal{P}_d : q(\lambda) = 0\}.$$

Let $V_\lambda = \{x \in \mathbb{M}^d : q(x) = 0 \text{ whenever } q \in I_\lambda\}$. We will need the following theorem from [3].

Theorem 3.1 *Let $\lambda \in B_\delta \cap \mathbb{M}_n^d$ and $w \in \mathbb{M}_n$. There exists a function ψ in the closed unit ball of $H^\infty(B_\delta)$ such that $\psi(\lambda) = w$ if*

(i) $w \in \mathrm{Alg}(\lambda)$, *so there exists $p \in \mathcal{P}_d$ such that $p(\lambda) = w$.*

(ii) $\sup\{\|p(x)\| : x \in V_\lambda \cap B_\delta\} \leq 1$.

Moreover, if the conditions are satisfied, ψ can be chosen to have a free realization.

Since $\phi(\lambda) = w$, by Theorem 2.3, there is a free polynomial p such that $p(\lambda) = w$; so condition (i) is satisfied. To see condition (ii), note that for all $x \in V_\lambda \cap B_\delta$, we have $p(x) = \phi(x)$. Indeed, by Theorem 2.3, there is a polynomial q so that $q(\lambda \oplus x) = \phi(\lambda \oplus x)$. Therefore $q(\lambda) = p(\lambda)$, so, since $x \in V_\lambda$, we also have $q(x) = p(x)$, and hence $p(x) = \phi(x)$. But ϕ is in the unit ball of $H_1^\infty(B_\delta)$, so $\|\phi(x)\| \leq 1$ for every x in B_δ.

So we can apply Theorem 3.1 to conclude that there is a function ψ in $H^\infty(B_\delta)$ that has a free realization, and that agrees with ϕ on the finite set F.

We note that the converse of Theorem 3.1 is also true. Given Theorem 2.3, the converse is almost immediate.

4 Steps III and IV

Let $\Lambda = \{x_j\}_{j=1}^\infty$ be a countable dense set in B_δ. For each k, let $F_k = \{x_1, \dots, x_k\}$. By Step II, there is a function $\psi^k \in H_1^\infty(B_\delta)$ that has a free realization and agrees with ϕ on F_k. By Theorem 1.4, there exists a Hilbert space $\mathcal{M}^k$ and a $\mathcal{B}(\mathbb{C}, \mathcal{M}^k \otimes \mathbb{C}^J)$ valued nc function u^k on B_δ so that, for all n, for all $x, y \in B_\delta \cap \mathbb{M}_n^d$, we have

$$1 - \psi^k(y)^* \psi^k(x) = u^k(y)^* \left[\begin{array}{c} 1 \\ \otimes \\ 1 - \delta(y)^* \delta(x) \end{array} \right] u^k(x). \tag{4.1}$$

Embed each $\mathcal{M}^k$ in a common Hilbert space $\mathcal{H}$. Since the left-hand side of (4.1) is bounded, it follows that u^k are locally bounded, so we can apply Theorem 1.5 to find a sequence of unitaries U^k such that, after passing to a subsequence, $U^k * u^k$ converges to a function ν in $\mathrm{Hol}^{\mathrm{nc}}_{\mathcal{H}}(\Omega)$. We have therefore that

$$1 - \phi(y)^* \phi(x) = \nu(y)^* \left[\begin{array}{c} 1 \\ \otimes \\ 1-\delta(y)^*\delta(x) \end{array} \right] \nu(x) \tag{4.2}$$

holds for all pairs (x, y) that are both in $\Lambda \cap \mathbb{M}^d_n$ for any n. So by continuity, we get that (4.2) is an nc model for ϕ on all B_δ, completing Step III.

Finally, Step IV follows by applying Theorem 1.4.

5 Closing Remarks

One can modify the argument to get a realization formula for $\mathcal{B}(\mathcal{K}, \mathcal{L})$-valued bounded nc functions on B_δ, or to prove Leech theorems (also called Toeplitz-corona theorems [8, 10]. For finite-dimensional $\mathcal{K}$ and $\mathcal{L}$, this was done in [2]; for infinite-dimensional $\mathcal{K}$ and $\mathcal{L}$, the formula was proved in [6] using results from [7].

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U.C. San Diego, La Jolla, California, USA
e-mail: jagler@ucsd.edu

Washington University, St. Louis, Missouri, USA
e-mail: mccarthy@wustl.edu