

WANDERING MONTEL THEOREMS FOR HILBERT SPACE VALUED HOLOMORPHIC FUNCTIONS

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(Communicated by Stephan Ramon Garcia)

ABSTRACT. We prove that if $\{u^k\}$ is a sequence of holomorphic functions that takes values in an infinite dimensional Hilbert space \mathcal{H} , there are unitaries $\{U^k\}$ on \mathcal{H} so that $U^k u^k$ has a subsequence that converges locally uniformly. We also prove a non-commutative version of this result.

1. INTRODUCTION

1.1. Commutative theory. Let Ω be an open set in \mathbb{C}^d and assume that $\{u^k\}$ is a sequence in $\text{Hol}(\Omega)$, the algebra of holomorphic functions on Ω equipped with the topology of uniform convergence on compact subsets. The classical Montel Theorem asserts that if $\{u^k\}$ is locally uniformly bounded on Ω , then there exists a subsequence $\{u^{k_i}\}$ that converges in $\text{Hol}(\Omega)$.

It is well known that if \mathcal{X} is an infinite dimensional Banach space, then Montel's Theorem breaks down for $\text{Hol}_{\mathcal{X}}(\Omega)$, the space of \mathcal{X} -valued holomorphic functions; see e.g. [5, 14]. For example, if $\mathcal{X} = \ell^2$ and $\{f^k\}$ is a locally uniformly bounded sequence of holomorphic functions on Ω , then the sequence

$$\begin{pmatrix} f^1(\lambda) \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ f^2(\lambda) \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ f^3(\lambda) \\ \vdots \end{pmatrix}, \dots$$

is a locally uniformly bounded sequence that will have a convergent subsequence only if there exists a subsequence $\{f^{k_i}\}$ that converges uniformly to 0 on Ω .

Observe that the problem in the example given above is that while for all $\lambda \in \Omega$, u^k converges weakly to 0, it needn't be the case that $u^k(\lambda)$ converges in norm for any $\lambda \in \Omega$. However, just as in the case of the classical proof of Montel's theorem that uses the Arzela-Ascoli Theorem, if one assumes that $\{u^k\}$ is well behaved pointwise on a large enough set, then one *can* conclude uniform convergence in norm on compact sets. For example, consider the following theorem by Arendt and Nikolski [5, Cor. 2.3].

Received by the editors September 8, 2017, and, in revised form, January 5, 2018.

2010 *Mathematics Subject Classification.* Primary 32A19, 47L25.

The first author was partially supported by National Science Foundation Grant DMS 1665260.

The second author was partially supported by National Science Foundation Grant DMS 1565243.

Theorem 1.1. *Let Ω be an open connected set in \mathbb{C} , and let u^k be a sequence in $\text{Hol}_X(\Omega)$ that is locally bounded. Assume that*

$$\Omega_0 := \{z \in \Omega : \{u^k(z) : k \in \mathbb{N}\} \text{ is relatively compact in } X\}$$

has an accumulation point in Ω . Then there exists a subsequence which converges to a holomorphic function uniformly on compact subsets of Ω .

Theorem 1.1 deals with the difficulty by making strong additional assumptions about the pointwise behavior of $\{u^k\}$, assumptions that may not hold in desirable applications. The central idea of this paper, for Hilbert space valued functions, is instead to use a sequence of unitaries to push (most of) the range of the functions into a finite dimensional space. Here is our first main result.

Theorem 1.2. *If Ω is an open set in \mathbb{C}^d , \mathcal{H} is a Hilbert space, and $\{u^k\}$ is a locally uniformly bounded sequence in $\text{Hol}_{\mathcal{H}}(\Omega)$, then there exists a sequence $\{U^k\}$ of unitary operators on \mathcal{H} such that $\{U^k u^k\}$ has a subsequence that converges in $\text{Hol}_{\mathcal{H}}(\Omega)$.*

We prove Theorem 1.2 in Section 2. In Sections 3 and 4 we consider versions for non-commutative functions. These functions have been extensively studied recently; see e.g. [4, 6, 8–13, 15, 17]. Before stating our results, we must spend a little time explaining some definitions.

1.2. Non-commutative theory. In commutative analysis, one studies holomorphic functions defined on domains in \mathbb{C}^d . In non-commutative analysis one studies holomorphic functions defined on domains in \mathbb{M}^d , the d -dimensional nc universe. For each n we let \mathbb{M}_n^d denote the set of d -tuples of $n \times n$ matrices. We then let

$$\mathbb{M}^d = \bigcup_{n=1}^{\infty} \mathbb{M}_n^d.$$

When E is a subset of \mathbb{M}^d , for each n , we adopt the notation

$$E_n = E \cap \mathbb{M}_n^d.$$

In non-commutative analysis one studies *graded* functions, i.e., functions f defined on subsets E of \mathbb{M}^d , that satisfy

$$(1.3) \quad \forall_n \quad \forall_{\lambda \in E_n} \quad f(\lambda) \in \mathbb{M}_n.$$

\mathbb{M}^d carries a topology, the so-called *finite topology*,¹ wherein a set Ω is deemed to be open precisely when

$$\forall_n \quad \Omega_n \text{ is open in } \mathbb{M}_n^d.$$

With this definition, note that a graded function $f : E \rightarrow \mathbb{M}^1$ is finitely continuous if and only if $f|_{E_n}$ is continuous for each n and also that a set $K \subseteq \mathbb{M}^d$ is finitely compact if and only if there exists n such that $E_m = \emptyset$ when $m > n$ and E_m is compact when $m \leq n$.

If Ω is finitely open in \mathbb{M}^d , then for each n , Ω_n can be identified with an open set in \mathbb{C}^{dn^2} in an obvious way. If, in addition, f is a graded function on Ω , then we say that f is *holomorphic on Ω* if for each n , $f|_{\Omega_n}$ is a holomorphic mapping of Ω_n into \mathbb{M}_n . We let $\text{Hol}(\Omega)$ denote the collection of graded holomorphic functions.

¹Subsequently, we shall consider other topologies as well.

It is also possible to consider \mathcal{H} -valued holomorphic functions in the non-commutative setting. One particularly concrete way to do this is to realize in the scalar case just considered that (1.3) is equivalent to asserting that

$$\forall_n \quad \forall_{\lambda \in E_n} \quad f(\lambda) \in \mathcal{B}(\mathbb{C}^n, \mathbb{C}^n).$$

We therefore replace the former definition (that f be graded) with the requirement that f be a *graded \mathcal{H} -valued function*, i.e., that

$$\forall_n \quad \forall_{\lambda \in E_n} \quad f(\lambda) \in \mathcal{B}(\mathbb{C}^n, \mathbb{C}^n \otimes \mathcal{H}).$$

Just as before, we declare a graded \mathcal{H} -valued function defined on a finitely open set Ω in \mathbb{M}^d to be holomorphic if for each n , $f|_{\Omega_n}$ is a holomorphic mapping of Ω_n into $\mathcal{B}(\mathbb{C}^n, \mathbb{C}^n \otimes \mathcal{H})$. We let $\text{Hol}_{\mathcal{H}}(\Omega)$ denote the collection of graded \mathcal{H} -valued functions and view $\text{Hol}_{\mathcal{H}}(\Omega)$ as a complete metric space endowed with the topology of uniform convergence on finitely compact subsets of Ω .

A special class of graded functions arise by formalizing certain algebraic properties of free polynomials. If $E \subseteq \mathbb{M}^d$ we say that E is an *nc-set* if E is closed with respect to direct sums. We define the class of *nc-functions* as follows.

Definition 1.4. Let \mathcal{H} be a Hilbert space, let E be an nc-set, and assume that f is a function defined on E . We say that f is an *\mathcal{H} -valued nc-function on E* if the following conditions hold.

(i) f is *\mathcal{H} -graded*, i.e.,

$$\forall_n \quad \forall_{\lambda \in E \cap \mathbb{M}_n} \quad f(\lambda) \in \mathcal{B}(\mathbb{C}^n, \mathbb{C}^n \otimes \mathcal{H}).$$

(ii) f *preserves direct sums*, i.e.,

$$\forall_{\lambda, \mu \in E} \quad \lambda \oplus \mu \in E \implies f(\lambda \oplus \mu) = f(\lambda) \oplus f(\mu).$$

In this formula, if $\lambda \in E_m$ and $\mu \in E_m$, we identify $\mathbb{C}^m \oplus \mathbb{C}^n$ and \mathbb{C}^{m+n} and identify $(\mathbb{C}^m \otimes \mathcal{H}) \oplus (\mathbb{C}^n \otimes \mathcal{H})$ and $\mathbb{C}^{m+n} \otimes \mathcal{H}$.

(iii) f *preserves similarity*, i.e.,

$$f(S\lambda S^{-1}) = (S \otimes \text{id}_{\mathcal{H}})f(\lambda)S^{-1}$$

whenever $n \geq 1$, $S \in \mathbb{M}_n$ is invertible, and both λ and $S\lambda S^{-1}$ are in E_n .

When $f : E \rightarrow \mathbb{M}^1 \otimes \mathcal{H}$ is an nc-function and E is a finitely open nc-set condition (iii) above becomes very strong and yields the following proposition, which lies at the heart of nc analysis (see [10] or [13, Thm. 7.2]). We say a function f is bounded on E if $\sup_{\lambda \in E} \|f(\lambda)\| < \infty$.

Proposition 1.5. *Let Ω be a finitely open nc-set. If f is a bounded nc-function defined on Ω , then f is holomorphic on Ω .*

Proposition 1.5 suggests the following terminology. We say that a set $\Omega \subseteq \mathbb{M}^d$ is an *nc-domain* if Ω is a finitely open nc-set and we say that a topology τ on \mathbb{M}^d is an *nc-topology* if τ has a basis consisting of nc-domains. We then define special classes of functions in non-commuting variables as follows.

Definition 1.6. Let $\Omega \subseteq \mathbb{M}_n^d$, τ be an nc-topology, and assume that $f : \Omega \rightarrow \mathbb{M}^1 \otimes \mathcal{H}$ is an \mathcal{H} -valued function. We say that f is *τ -holomorphic* if f is a τ -locally bounded nc function on Ω .² We let $\text{Hol}_{\mathcal{H}}^{\tau}(\Omega)$ denote the collection of τ -holomorphic \mathcal{H} -valued functions defined on Ω .

²That is, f is an nc-function on Ω in the sense of Definition 1.4 and for each $\lambda \in \Omega$, there exists $B \subseteq \Omega$ such that $\lambda \in B \in \tau$ and $f|_B$ is bounded.

Evidently, Proposition 1.5 guarantees that if τ is an nc-topology and f is a τ -holomorphic function in the sense of Definition 1.6, then f is holomorphic, i.e.,

$$\text{Hol}_{\mathcal{H}}^{\tau}(\Omega) \subseteq \text{Hol}_{\mathcal{H}}^{\text{nc}}(\Omega) \subseteq \text{Hol}_{\mathcal{H}}(\Omega),$$

where $\text{Hol}_{\mathcal{H}}^{\text{nc}}(\Omega)$ denotes the set of functions in $\text{Hol}_{\mathcal{H}}(\Omega)$ that are nc.

We can now state our second main result, the non-commutative version of Theorem 1.2.

Theorem 1.7. *Assume that τ is an nc-topology, $\Omega \in \tau$, \mathcal{H} is a Hilbert space, and $\{u^k\}$ is a τ -locally uniformly bounded sequence in $\text{Hol}_{\mathcal{H}}^{\tau}(\Omega)$. There exist $u \in \text{Hol}_{\mathcal{H}}^{\tau}(\Omega)$, a sequence $\{U^k\}$ of unitary operators on \mathcal{H} , and an increasing sequence of indices $\{k_l\}$ such that $(\text{id}_n \otimes U^{k_l}) u^{k_l} \rightarrow u$ in $\text{Hol}(\Omega)$.*

As an application of Theorem 1.7, in Section 5 we prove that the cones

$$\mathcal{P} = \{u(\mu)^* u(\lambda) : u \in \text{Hol}_{\mathcal{H}}(\Omega) \text{ for some Hilbert space } \mathcal{H}\}$$

and

$$\mathcal{C} = \left\{ \begin{array}{c} \text{id}_{\mathbb{C}^J} \\ \otimes \\ u(\mu)^* \end{array} \left(\text{id} - \begin{array}{c} \delta(\mu)^* \delta(\lambda) \\ \otimes \\ \text{id}_{\mathcal{H}} \end{array} \right) \begin{array}{c} \text{id}_{\mathbb{C}^J} \\ \otimes \\ u(\lambda) \end{array} : u \in \text{Hol}_{\mathcal{H}}(B_{\delta}) \text{ and } u \text{ is nc} \right\}$$

are closed. In this last formula, δ is a J -by- J matrix of free polynomials, and $B_{\delta} = \{x : \|\delta(x)\| < 1\}$ is a non-commutative polynomial polyhedron. (We adopt the convention of [16] and write the tensors vertically for legibility.)

Proving that the cones are closed is the key step in proving realization formulas for free holomorphic functions; see [1, 2, 7].

In Section 6 we show that the assumptions in Proposition 3.2 below can be weakened to just requiring convergence on a set of uniqueness, which yields a graded version of the Arendt-Nikolski theorem.

2. A MONTEL THEOREM FOR HILBERT SPACE VALUED HOLOMORPHIC FUNCTIONS

In this section we prove Theorem 1.2 from the introduction.

2.1. Notation and definitions. If Ω is an open set in \mathbb{C}^d and \mathcal{H} is a Hilbert space, we let $\text{Hol}_{\mathcal{H}}(\Omega)$ denote the space of holomorphic \mathcal{H} -valued functions on Ω . If $u \in \text{Hol}_{\mathcal{H}}(\Omega)$ and $E \subseteq \Omega$, we let

$$\|u\|_E = \sup_{\lambda \in E} \|u(\lambda)\|_{\mathcal{H}}.$$

If $\|u\|_{\Omega} < \infty$, then we say that u is *bounded on Ω* . If $\{u^k\}$ is a sequence in $\text{Hol}_{\mathcal{H}}(\Omega)$, we say that $\{u^k\}$ is *uniformly bounded on Ω* if

$$\sup_k \|u^k\|_{\Omega} < \infty,$$

and we say that $\{u^k\}$ is *locally uniformly bounded on Ω* if for each $\lambda \in \Omega$ there exists a neighborhood B of λ such that $\{u^k\}$ is uniformly bounded on B . Recall that if such a neighborhood exists, then a Cauchy estimate implies that $\{u^k\}$ is *equicontinuous at λ* ; i.e., for each $\varepsilon > 0$ there exists a ball B_0 such that $\lambda \in B_0 \subseteq B$ and

$$\forall_{\mu \in B_0} \forall_k \|u^k(\mu) - u^k(\lambda)\| < \varepsilon.$$

We equip $\text{Hol}_{\mathcal{H}}(\Omega)$ with the usual topology of *uniform convergence on compacta*. Thus, a sequence $\{u^k\}$ in $\text{Hol}_{\mathcal{H}}(\Omega)$ is *convergent* precisely when there is a function $u \in \text{Hol}_{\mathcal{H}}(\Omega)$ such that

$$\lim_{k \rightarrow \infty} \|u^k - u\|_E = 0$$

for every compact $E \subseteq \Omega$. We say that a sequence $\{u^k\}$ in $\text{Hol}_{\mathcal{H}}(\Omega)$ is a *Cauchy sequence* if for each compact $E \subseteq \Omega$, $\{u^k\}$ is *uniformly Cauchy on E* ; i.e., for each $\varepsilon > 0$, there exists N such that

$$k, l \geq N \implies \|u^k - u^l\|_E < \varepsilon.$$

It is well known that $\text{Hol}_{\mathcal{H}}(\Omega)$ is *complete*; i.e., every Cauchy sequence in $\text{Hol}_{\mathcal{H}}(\Omega)$ converges. The following result is proved in [5, Thm. 2.1]; we include a proof that easily generalizes to Proposition 3.2.

Proposition 2.1. *Assume that Ω is an open set in \mathbb{C}^d , $\{\lambda_i\}$ is a dense sequence in Ω , and \mathcal{H} is a Hilbert space. If $\{u^k\}$ is a sequence in $\text{Hol}_{\mathcal{H}}(\Omega)$ that is locally uniformly bounded on Ω and for each fixed i , $\{u^k(\lambda_i)\}$ is a convergent sequence in \mathcal{H} , then $\{u^k\}$ is a convergent sequence in $\text{Hol}_{\mathcal{H}}(\Omega)$.*

Proof. Fix a compact set $E \subseteq \Omega$ and $\varepsilon > 0$. Note that as $\{u^k\}$ is assumed to be locally uniformly bounded on Ω , $\{u^k\}$ is equicontinuous at each point of Ω . Hence, as E is compact, we may construct a finite collection $\{B_r : r = 1, \dots, m\}$ of open balls in \mathbb{C}^d such that

$$(2.2) \quad E \subseteq \bigcup_{r=1}^m B_r \subseteq \Omega$$

and

$$(2.3) \quad \forall_r \quad \forall_{\mu_1, \mu_2 \in B_r} \quad \forall_k \quad \|u^k(\mu_1) - u^k(\mu_2)\| < \varepsilon/3.$$

As $\{\lambda_i\}$ is assumed dense in Ω ,

$$(2.4) \quad \forall_r \quad \exists_{i_r} \quad \lambda_{i_r} \in B_r.$$

Consequently, as for each fixed i we assume that $\{u^k(\lambda_i)\}$ is a convergent (and hence Cauchy) sequence in \mathcal{H} , there exists N such that

$$(2.5) \quad \forall_r \quad k, j \geq N \implies \|u^k(\lambda_{i_r}) - u^j(\lambda_{i_r})\| < \varepsilon/3.$$

Now, fix $\lambda \in E$. Use (2.2) to choose r such that $\lambda \in B_r$. Use (2.4) to choose i_r such that $\lambda_{i_r} \in B_r$. As λ and λ_{i_r} are both in B_r , we see from (2.3) that

$$\forall_k \quad \|u^k(\lambda) - u^k(\lambda_{i_r})\| < \varepsilon/3.$$

Hence, using (2.5), we have that if $k, j \geq N$, then

$$\begin{aligned} & \|u^k(\lambda) - u^j(\lambda)\| \\ & \leq \|u^k(\lambda) - u^k(\lambda_{i_r})\| + \|u^k(\lambda_{i_r}) - u^j(\lambda_{i_r})\| + \|u^j(\lambda_{i_r}) - u^j(\lambda)\| \\ & < \varepsilon. \end{aligned}$$

□

Since the concluding estimate in the previous paragraph holds for an arbitrary point $\lambda \in E$, $\{u^k\}$ is uniformly Cauchy on E . Since E is an arbitrary compact subset of Ω , $\{u^k\}$ is a Cauchy sequence in $\text{Hol}_{\mathcal{H}}(\Omega)$. Therefore, $\{u^k\}$ converges in $\text{Hol}_{\mathcal{H}}(\Omega)$.

2.2. The proof of Theorem 1.2. Theorem 1.2 follows quickly from Proposition 2.1 and the following lemma.

Lemma 2.6 (Wandering Isometry Lemma). *Assume that Ω is an open set in \mathbb{C}^d , $\{\lambda_i\}$ is a sequence in Ω , and \mathcal{H} is a Hilbert space. If $\{u^k\}$ is a sequence in $\text{Hol}_{\mathcal{H}}(\Omega)$ that is locally uniformly bounded on Ω , then there exists a subsequence $\{u^{k_l}\}$ and a sequence $\{V^l\}$ of unitary operators on \mathcal{H} such that for each fixed i , $\{V^l u^{k_l}(\lambda_i)\}$ is a convergent sequence in \mathcal{H} .*

Proof. If \mathcal{H} is finite dimensional, one can let each unitary be the identity, and the result is the regular Montel theorem. So we shall assume that \mathcal{H} is infinite dimensional. Let $\{e_i\}$ be an orthonormal basis for \mathcal{H} . For each fixed k let

$$\mathcal{H}^k = \text{span}\{e_1, e_2, \dots, e_k\},$$

$$\mathcal{M}_i^k = \text{span}\{u^k(\lambda_1), u^k(\lambda_2), \dots, u^k(\lambda_i)\}, \quad i = 1, \dots, k.$$

For each k choose a unitary $U^k \in \mathcal{B}(\mathcal{H})$ satisfying

$$U^k \mathcal{M}_i^k \subseteq \mathcal{H}^i, \quad i = 1, \dots, k.$$

Observe that with this construction, for each fixed i ,

$$\{U^k u^k(\lambda_i)\}_{k=i}^{\infty}$$

is a bounded sequence in \mathcal{H}^i , a finite dimensional Hilbert space. Therefore, there exist $v_i \in \mathcal{H}$ and an increasing sequence of indices $\{k_l\}$ such that

$$U^{k_l} u^{k_l}(\lambda_i) \rightarrow v_i \quad \text{in } \mathcal{H} \quad \text{as } l \rightarrow \infty.$$

Applying this fact successively with $i = 1, i = 2$, and so on, at each stage taking a subsequence of the previously selected subsequence, leads to a sequence $\{v_i\}$ in \mathcal{H} and an increasing sequence of indices $\{k_l\}$ such that

$$U^{k_l} u^{k_l}(\lambda_i) \rightarrow v_i \quad \text{in } \mathcal{H} \quad \text{as } l \rightarrow \infty$$

for all i . The lemma then follows if we let $V^l = U^{k_l}$. □

Proof of Theorem 1.2. Assume that Ω is an open set in \mathbb{C}^d , \mathcal{H} is a Hilbert space, and $\{u^k\}$ is a locally uniformly bounded sequence in $\text{Hol}_{\mathcal{H}}(\Omega)$. The theorem follows from the classical Montel theorem (with $U^k = \text{id}_{\mathcal{H}}$ for all k) if $\dim \mathcal{H} < \infty$. Therefore, we may assume that $\dim \mathcal{H} = \infty$.

Fix a dense sequence $\{\lambda_i\}$ in Ω . By Lemma 2.6, there exists a subsequence $\{u^{k_l}\}$ and a sequence $\{V^l\}$ of unitary operators on \mathcal{H} such that for each fixed i , $\{V^l u^{k_l}(\lambda_i)\}$ is a convergent sequence in \mathcal{H} . Furthermore, as $\{u^k\}$ is locally uniformly bounded, so also $\{V^l u^{k_l}\}$ is locally uniformly bounded. Therefore, Proposition 2.1 implies that $\{V^l u^{k_l}\}$ is a convergent sequence in $\text{Hol}_{\mathcal{H}}(\Omega)$. The theorem then follows by choosing $\{U^k\}$ to be any sequence of unitaries in $\mathcal{B}(\mathcal{H})$ such that $U^{k_l} = V^l$ for all l . □

3. HOLOMORPHIC FUNCTIONS IN NON-COMMUTING VARIABLES

If Ω is finitely open in \mathbb{M}^d , we may construct a *finitely compact-open exhaustion* of Ω , i.e., an increasing sequence of compact sets $\{K_i\}$ that satisfy

$$K_1 \subset \text{int}(K_2) \subset K_2 \subset \text{int}(K_3) \subset \dots$$

and with $\Omega = \bigcup_i K_i$. For a set $E \subseteq \Omega$ and $f \in \text{Hol}(\Omega)$ we let

$$\|f\|_E = \sup_{\lambda \in E} \|f(\lambda)\|$$

and then in the usual way define a metric d on $\text{Hol}(\Omega)$ with the formula

$$d(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f - g\|_{K_n}}{1 + \|f - g\|_{K_n}}.$$

It then follows that $f_k \rightarrow f$ in the metric space (Ω, d) if and only if for each finitely compact set K in Ω , $\{f_k\}$ converges uniformly to f on K , i.e.,

$$\lim_{k \rightarrow \infty} \|f - f_k\|_K = 0.$$

Furthermore, $\text{Hol}(\Omega)$ is a complete metric space when endowed with this topology of uniform convergence on finitely compact subsets of Ω .

It is a straightforward exercise to extend Montel's theorem to the space $\text{Hol}_{\mathcal{H}}(\Omega)$ when $\dim \mathcal{H}$ is finite.

Proposition 3.1. *If Ω is a finitely open set in \mathbb{M}^d , \mathcal{H} is a Hilbert space with $\dim \mathcal{H} < \infty$, and $\{u^k\}$ is a finitely locally uniformly bounded sequence in $\text{Hol}_{\mathcal{H}}(\Omega)$, then $\{u^k\}$ has a convergent subsequence.*

Also, with the setup we have just described, mere notational changes to the proof of Proposition 2.1 yield a proof of the following proposition.

Proposition 3.2. *Assume that Ω is a finitely open set in \mathbb{M}^d , $\{\lambda_i\}$ is a dense sequence in Ω with $\lambda_i \in \mathbb{M}_{n_i}^d$ for each i , and \mathcal{H} is a Hilbert space. If $\{u^k\}$ is a sequence in $\text{Hol}_{\mathcal{H}}(\Omega)$ that is finitely locally uniformly bounded on Ω and for each fixed i , $\{u^k(\lambda_i)\}$ is a convergent sequence in $\mathcal{B}(\mathbb{C}^{n_i}, \mathbb{C}^{n_i} \otimes \mathcal{H})$, then $\{u^k\}$ is a convergent sequence in $\text{Hol}_{\mathcal{H}}(\Omega)$.*

Just as was the case for Proposition 2.1 in [5], it is possible to relax the assumption in Proposition 3.2 that $\{\lambda_i\}$ be a dense sequence in Ω to the assumption that $\{\lambda_i\}$ merely be a set of uniqueness for $\text{Hol}_{\mathcal{H}}(\Omega)$ (see Proposition 6.2).

We now turn to an analog of Theorem 1.2 in the non-commutative setting.

Lemma 3.3 (Wandering Isometry Lemma (non-commutative case)). *Assume that Ω is a finitely open set in \mathbb{M}^d and $\{\lambda_i\}$ is a sequence in Ω (where, for each i , $\lambda_i \in \mathbb{M}_{n_i}^d$). If \mathcal{H} is an infinite dimensional Hilbert space and $\{u^k\}$ is a sequence in $\text{Hol}_{\mathcal{H}}(\Omega)$ with the property that $\{u^k(\lambda_i)\}$ is bounded for each i , then there exists a subsequence $\{u^{k_i}\}$ and a sequence $\{V^l\}$ of unitary operators on \mathcal{H} such that for each fixed i , $\{(\text{id}_{n_i} \otimes V^l) u^{k_i}(\lambda_i)\}$ is a convergent sequence in $\mathcal{B}(\mathbb{C}^{n_i}, \mathbb{C}^{n_i} \otimes \mathcal{H})$.*

Proof. Choose an increasing sequence $\{\mathcal{H}_i\}$ of subspaces of \mathcal{H} with the property that

$$\dim \mathcal{H}_1 = n_1^2 \quad \text{and} \quad \forall_{i \geq 1} \quad \dim(\mathcal{H}_{i+1} \ominus \mathcal{H}_i) = n_{i+1}^2,$$

and for each n , let $\{e_1, \dots, e_n\}$ denote the standard basis of \mathbb{C}^n .

Fix k . For each $i = 1, \dots, k$, as $u^k(\lambda_i) : \mathbb{C}^{n_i} \rightarrow \mathbb{C}^{n_i} \otimes \mathcal{H}$, there exist n_i^2 vectors $x_{r,s}^{k,i} \in \mathcal{H}$, $r, s = 1, \dots, n_i$, such that

$$(3.4) \quad u^k(\lambda_i)e_r = \sum_{s=1}^{n_i} e_s \otimes x_{r,s}^{k,i}, \quad r = 1, \dots, n_i.$$

For each $i = 1, \dots, k$, define \mathcal{M}_i^k by

$$\mathcal{M}_i^k = \text{span} \{x_{r,s}^{k,i} : r, s = 1, \dots, n_i\}$$

and then define a sequence of spaces $\{\mathcal{N}_i^k\}$ by setting $\mathcal{N}_1^k = \mathcal{M}_1^k$ and

$$\mathcal{N}_i^k = (\mathcal{M}_1^k + \mathcal{M}_2^k + \dots + \mathcal{M}_i^k) \ominus (\mathcal{M}_1^k + \mathcal{M}_2^k + \dots + \mathcal{M}_{i-1}^k),$$

for $i = 2, \dots, k$. As for each $i = 1, \dots, k$, $\dim \mathcal{M}_i^k \leq n_i^2$, so also for $i = 1, \dots, k$, $\dim \mathcal{N}_i^k \leq n_i^2$. Consequently, as the spaces $\{\mathcal{N}_i^k\}$ are also pairwise orthogonal, it follows that there exists a unitary $U^k \in \mathcal{B}(\mathcal{H})$ such that

$$U^k(\mathcal{N}_1^k) \subseteq \mathcal{H}_1 \quad \text{and} \quad U^k(\mathcal{N}_i^k) \subseteq \mathcal{H}_i \ominus \mathcal{H}_{i-1} \quad \text{for } i = 2, \dots, k.$$

With this construction it follows using (3.4) that

$$(3.5) \quad (\text{id}_{n_i} \otimes U^k)u^k(\lambda_i)(\mathbb{C}^{n_i}) \subseteq \mathbb{C}^{n_i} \otimes \mathcal{H}_i, \quad i = 1, \dots, k.$$

Now observe that as (3.5) holds for each k , for each fixed i ,

$$(\text{id}_{n_i} \otimes U^k)u^k(\lambda_i)(\mathbb{C}^{n_i}) \subseteq \mathbb{C}^{n_i} \otimes \mathcal{H}_i, \quad k = i, i+1, \dots;$$

i.e.,

$$\{(\text{id}_{n_i} \otimes U^k)u^k(\lambda_i)\}_{k=i}^\infty$$

is a bounded sequence in $\mathcal{B}(\mathbb{C}^{n_i}, \mathbb{C}^{n_i} \otimes \mathcal{H}_i)$, a finite dimensional Hilbert space. Therefore, for each fixed i , there exist $L \in \mathcal{H}$ and an increasing sequence of indices $\{k_l\}$ such that

$$U^{k_l}u^{k_l}(\lambda_i) \rightarrow L \quad \text{in } \mathcal{B}(\mathbb{C}^{n_i}, \mathbb{C}^{n_i} \otimes \mathcal{H}_i) \quad \text{as } l \rightarrow \infty.$$

Applying this fact successively with $i = 1$, $i = 2$, and so on, at each stage taking a subsequence of the previously selected subsequence leads to a sequence $\{L_i\}$ with $L_i \in \mathcal{B}(\mathbb{C}^{n_i}, \mathbb{C}^{n_i} \otimes \mathcal{H}_i)$ for each i and an increasing sequence of indices $\{k_l\}$ such that

$$\forall_i \quad U^{k_l}u^{k_l}(\lambda_i) \rightarrow L_i \quad \text{in } \mathcal{B}(\mathbb{C}^{n_i}, \mathbb{C}^{n_i} \otimes \mathcal{H}_i) \quad \text{as } l \rightarrow \infty.$$

The lemma then follows if we let $V^l = U^{k_l}$. □

Lemma 3.3 suggests the following notation. Let Ω be a finitely open set in \mathbb{M}^d and let \mathcal{H} be a Hilbert space. If U is a unitary acting on \mathcal{H} and $f \in \text{Hol}_{\mathcal{H}}(\Omega)$, then we may define $U * f \in \text{Hol}_{\mathcal{H}}(\Omega)$ by the formula

$$\forall_n \quad (U * f)|\Omega_n = (\text{id}_n \otimes U)f|\Omega_n.$$

With this notation we may formulate a non-commutative analog of Theorem 1.2 in the non-commutative setting.

Theorem 3.6. *If Ω is a finitely open set in \mathbb{M}^d , \mathcal{H} is a Hilbert space, and $\{u^k\}$ is a finitely locally uniformly bounded sequence in $\text{Hol}_{\mathcal{H}}(\Omega)$, then there exists a sequence $\{U^k\}$ of unitary operators on \mathcal{H} such that $\{U^k * u^k\}$ has a convergent subsequence.*

Proof. Assume that Ω is an open set in \mathbb{M}^d , \mathcal{H} is a Hilbert space, and $\{u^k\}$ is a finitely locally uniformly bounded sequence in $\text{Hol}_{\mathcal{H}}(\Omega)$. If $\dim \mathcal{H} < \infty$, then the theorem follows from Proposition 3.1 if we choose $U^k = \text{id}_{\mathcal{H}}$ for all k . Therefore, we assume that $\dim \mathcal{H} = \infty$.

Fix a dense sequence $\{\lambda_i\}$ in Ω . By Lemma 3.3, there exists a subsequence $\{u^{k_l}\}$ and a sequence $\{V^l\}$ of unitary operators on \mathcal{H} such that for each fixed i , $\{V^l u^{k_l}(\lambda_i)\}$ is a convergent sequence in $\mathcal{B}(\mathbb{C}^{n_i}, \mathbb{C}^{n_i} \otimes \mathcal{H})$. Furthermore, as $\{u^k\}$ is locally uniformly bounded, so also $\{V^l u^{k_l}\}$ is locally uniformly bounded. Therefore, Proposition 2.1 implies that $\{V^l u^{k_l}\}$ is a convergent sequence in $\text{Hol}_{\mathcal{H}}(\Omega)$. The theorem then follows by choosing $\{U^k\}$ to be any sequence of unitaries in $\mathcal{B}(\mathcal{H})$ such that $U^{k_l} = V^l$ for all l . \square

4. LOCALLY BOUNDED NC FUNCTIONS

Properties of τ -holomorphic functions can be very sensitive to the choice of nc-topology τ . For example, if τ is the *fat topology* studied in [3], then τ -holomorphic functions satisfy a version of the Implicit Function Theorem. On the other hand, if τ is the *free topology*, studied in [2], then τ -holomorphic functions satisfy the Oka-Weil Approximation Theorem. Remarkably, neither of these theorems holds for the other topology.

Also notice that if f is τ -holomorphic in the sense of Definition 1.6, then necessarily Ω , the domain of f , is an open set in the τ topology: for each $\lambda \in \Omega$ there exists $B_\lambda \subseteq \Omega$ such that $\lambda \in B_\lambda \in \tau$; hence, $\Omega = \bigcup_\lambda B_\lambda \in \tau$.

There are no such subtleties between the nc-topologies when it comes to understanding the implications of local boundedness.

Definition 4.1. Assume that τ is an nc-topology and $\Omega \in \tau$. If $\{u^k\}$ is a sequence in $\text{Hol}_{\mathcal{H}}^{\tau}(\Omega)$, we say that $\{u^k\}$ is τ -locally uniformly bounded on Ω if for each $\lambda \in \Omega$, there exists a τ -open $B \subseteq \Omega$ such that $\lambda \in B$ and

$$\sup_k \|u^k\|_B < \infty.$$

Lemma 4.2. Assume that τ is an nc-topology and $\Omega \in \tau$. Let $u \in \text{Hol}(\Omega)$ and let $\{u^k\}$ be a sequence in $\text{Hol}_{\mathcal{H}}^{\tau}(\Omega)$. If $\{u^k\}$ is τ -locally uniformly bounded on Ω and $u^k \rightarrow u$ in $\text{Hol}_{\mathcal{H}}(\Omega)$, then $u \in \text{Hol}_{\mathcal{H}}^{\tau}(\Omega)$.

Proof. Under the assumptions of the lemma, we need to prove the following two assertions:

$$(4.3) \quad u \text{ is an nc-function on } \Omega.$$

$$(4.4) \quad u \text{ is } \tau\text{-locally bounded on } \Omega.$$

To prove (4.3), note first that as $u \in \text{Hol}_{\mathcal{H}}(\Omega)$, condition (i) in Definition 1.4 holds. To verify condition (ii), assume that $\lambda, \mu, \lambda \oplus \mu \in \Omega$. Then, as $u^k \rightarrow u$ in $\text{Hol}_{\mathcal{H}}(\Omega)$ and $u^k \in \text{Hol}_{\mathcal{H}}^{\tau}$ for all k ,

$$\begin{aligned} u(\lambda \oplus \mu) &= \lim_{k \rightarrow \infty} u^k(\lambda \oplus \mu) \\ &= \lim_{k \rightarrow \infty} (u^k(\lambda) \oplus u^k(\mu)) \\ &= \lim_{k \rightarrow \infty} u^k(\lambda) \oplus \lim_{k \rightarrow \infty} u^k(\mu) \\ &= u(\lambda) \oplus u(\mu). \end{aligned}$$

Finally, note that if $n \geq 1$, $S \in \mathbb{M}_n$ is invertible, and both λ and $S\lambda S^{-1}$ are in Ω_n , then

$$\begin{aligned} u(S\lambda S^{-1}) &= \lim_{k \rightarrow \infty} u^k(S\lambda S^{-1}) \\ &= \lim_{k \rightarrow \infty} (S \otimes \text{id}_{\mathcal{H}}) u^k(\lambda) S^{-1} \\ &= (S \otimes \text{id}_{\mathcal{H}}) u(\lambda) S^{-1}, \end{aligned}$$

which proves condition (iii).

To prove (4.4), fix $\lambda \in \Omega$. As $\{u^k\}$ is τ -locally uniformly bounded on Ω , Definition 4.1 implies that there exist $B \subseteq \Omega$ and a constant ρ such that $\lambda \in B \in \tau$ and

$$\sup_k \|u^k\|_B \leq \rho.$$

Fix $\mu \in B$. As we assume that $u^k \rightarrow u$ in $\text{Hol}_{\mathcal{H}}(\Omega)$, it follows that

$$\|u(\mu)\| = \lim_{k \rightarrow \infty} \|u^k(\mu)\| \leq \rho.$$

But then,

$$\|u\|_B \leq \rho.$$

As $B \in \tau$, this proves that u is τ -locally bounded on Ω . \square

Definition 4.1 and Lemma 4.2 allow one to easily deduce Theorem 1.7 as a corollary of Theorem 3.6.

Proof of Theorem 1.7. As we assume that $\{u^k\}$ is a τ -locally uniformly bounded sequence in $\text{Hol}_{\mathcal{H}}^{\tau}(\Omega)$, $\{u^k\}$ is finitely locally uniformly bounded in $\text{Hol}_{\mathcal{H}}(\Omega)$. Therefore, Theorem 3.6 implies that there exists a sequence $\{U^k\}$ of unitary operators on \mathcal{H} such that $\{U^k * u^k\}$ has a subsequence that converges in $\text{Hol}_{\mathcal{H}}(\Omega)$. Consequently, we may choose $u \in \text{Hol}_{\mathcal{H}}(\Omega)$ and an increasing sequence of indices $\{k_l\}$ such that $U^{k_l} * u^{k_l} \rightarrow u$ in $\text{Hol}(\Omega)$. The proof is completed by observing that Lemma 4.2 implies that $u \in \text{Hol}_{\mathcal{H}}^{\tau}(\Omega)$. \square

Let us emphasize that Theorem 1.7 asserts that $U^{k_l} * u^{k_l}$ converges to u , which is in $\text{Hol}_{\mathcal{H}}^{\tau}(\Omega)$, uniformly on sets that are compact in the finite topology; it does not say that it converges uniformly on compact sets in the τ topology.

Note that the proofs of Lemma 4.2 and Theorem 1.7 work identically if u^k are just assumed to be in $\text{Hol}_{\mathcal{H}}^{\text{nc}}(\Omega)$, so we get

Theorem 4.5. *Let Ω be a finitely open set in \mathbb{M}^d , let \mathcal{H} be a Hilbert space, and let $\{u^k\}$ be a finitely locally uniformly bounded sequence in $\text{Hol}_{\mathcal{H}}^{\text{nc}}(\Omega)$. Then there exists a sequence $\{U^k\}$ of unitary operators on \mathcal{H} such that $\{U^k * u^k\}$ has a subsequence that converges finitely locally uniformly to an element of $\text{Hol}_{\mathcal{H}}^{\text{nc}}(\Omega)$.*

5. SOME APPLICATIONS

A useful construct in the study of τ -holomorphic functions is the *duality construction*. If Ω is a finitely open set it is natural to consider the algebraic tensor product $\text{Hol}(\Omega)^* \otimes \text{Hol}(\Omega)$. This space can concretely be realized as the set of functions A defined on

$$\Omega \boxtimes \Omega = \bigcup_{n=1}^{\infty} (\Omega \cap \mathbb{M}_n^d) \times (\Omega \cap \mathbb{M}_n^d)$$

and such that there exist a finite dimensional Hilbert space \mathcal{H} and $u, v \in \text{Hol}_{\mathcal{H}}(\Omega)$ such that

$$A(\lambda, \mu) = v(\mu)^* u(\lambda), \quad (\lambda, \mu) \in \Omega \boxtimes \Omega.$$

As the functions in $\text{Hol}(\Omega)^* \otimes \text{Hol}(\Omega)$ are holomorphic in λ for each fixed μ and anti-holomorphic in μ for each fixed λ , we may complete $\text{Hol}(\Omega)^* \otimes \text{Hol}(\Omega)$ in the topology of uniform convergence on finitely compact subsets of $\Omega \boxtimes \Omega$ to obtain the space of *hereditary holomorphic functions on Ω* , $\text{Her}(\Omega)$. Inside $\text{Her}(\Omega)$, we may define a cone \mathcal{P} by

$$\mathcal{P} = \{u(\mu)^* u(\lambda) : u \in \text{Hol}_{\mathcal{H}}(\Omega) \text{ for some Hilbert space } \mathcal{H}\}.$$

Theorem 5.1. \mathcal{P} is closed in $\text{Her}(\Omega)$.

Proof. Assume that $\{v^k\}$ is a sequence with $v^k \in \text{Hol}_{\mathcal{H}_k}(\Omega)$ for each k and with $v^k(\mu)^* v^k(\lambda) \rightarrow A$ in $\text{Her}(\Omega)$. We may assume that \mathcal{H}_k is separable for each k . Fix a separable infinite dimensional Hilbert space \mathcal{H} and for each k choose an isometry $V^k : \mathcal{H}_k \rightarrow \mathcal{H}$. If for each k we let $u^k = V^k * v^k$, then $\{u^k\}$ is a sequence in $\text{Hol}_{\mathcal{H}}(\Omega)$ and $u^k(\mu)^* u^k(\lambda) \rightarrow A$ in $\text{Her}(\Omega)$.

Now, as $u^k(\mu)^* u^k(\lambda) \rightarrow A$ in $\text{Her}(\Omega)$ it follows that $\{u^k\}$ is a finitely locally uniformly bounded sequence in $\text{Hol}_{\mathcal{H}}(\Omega)$. Hence, by Theorem 3.6, there exists a sequence U^k of unitary operators on \mathcal{H} such that $\{U^k * u^k\}$ has a convergent subsequence; i.e., there exists $u \in \text{Hol}_{\mathcal{H}}(\Omega)$ and an increasing sequence of indices $\{k_l\}$ such that $U^{k_l} * u^{k_l} \rightarrow u$. But then, for each $(\lambda, \mu) \in \Omega \boxtimes \Omega$,

$$\begin{aligned} A(\lambda, \mu) &= \lim_{k \rightarrow \infty} u^k(\mu)^* u^k(\lambda) \\ &= \lim_{l \rightarrow \infty} u^{k_l}(\mu)^* u^{k_l}(\lambda) \\ &= \lim_{l \rightarrow \infty} (U^{k_l} * u^{k_l})(\mu)^* (U^{k_l} * u^{k_l})(\lambda) \\ &= u(\mu)^* u(\lambda), \end{aligned}$$

i.e., $A \in \mathcal{P}$. □

We also may use wandering Montel theorems to study sums of τ -holomorphic dyads. We let $\text{Her}^{\tau}(\Omega)$ denote the closure of

$$\{v(\mu)^* u(\lambda) : u, v \in \text{Hol}_{\mathcal{H}}^{\tau}(\Omega) \text{ for some finite dimensional Hilbert space } \mathcal{H}\}$$

inside $\text{Her}(\Omega)$ and define \mathcal{P}^{τ} in $\text{Her}^{\tau}(\Omega)$ by

$$\mathcal{P}^{\tau} = \{u(\mu)^* u(\lambda) : u \in \text{Hol}_{\mathcal{H}}^{\tau}(\Omega) \text{ for some Hilbert space } \mathcal{H}\}.$$

Theorem 5.2. Let τ be an nc-topology, and let $\Omega \in \tau$. Then \mathcal{P}^{τ} is closed in $\text{Her}^{\tau}(\Omega)$.

Proof. Assume that $u^k(\mu)^* u^k(\lambda) \rightarrow A$ in $\text{Her}(\Omega)$, where, as in the proof of Theorem 5.1, we may assume that $u^k \in \text{Hol}_{\mathcal{H}}^{\tau}(\Omega)$ for each k . By Theorem 1.7, there exist $u \in \text{Hol}_{\mathcal{H}}^{\tau}(\Omega)$, a sequence U^k of unitary operators on \mathcal{H} , and an increasing sequence of indices $\{k_l\}$ such that $U^{k_l} * u^{k_l} \rightarrow u$. But then as in the proof of Theorem 5.1, $A(\lambda, \mu) = u(\mu)^* u(\lambda)$ for all $(\lambda, \mu) \in \Omega \boxtimes \Omega$, i.e., $A \in \mathcal{P}^{\tau}$. □

Finally, we shall prove that the model cone is closed; this is the key ingredient in the proof of the realization formula for free holomorphic functions [1, 2, 7]. Let

δ be a J -by- L matrix whose entries are free polynomials in d variables. We define B_δ to be the polynomial polyhedron

$$B_\delta := \{x \in \mathbb{M}^d : \|\delta(x)\| < 1\}.$$

The free topology is the nc-topology generated by the sets B_δ , as δ ranges over all matrices of polynomials. The model cone \mathcal{C} is the set of hereditary functions on B_δ of the form

$$(5.3) \quad \mathcal{C} := \left\{ \begin{array}{c} \text{id}_{\mathbb{C}^J} \\ \otimes \\ u(\mu)^* \end{array} \left(\text{id} - \begin{array}{c} \delta(\mu)^* \delta(\lambda) \\ \otimes \\ \text{id}_{\mathcal{H}} \end{array} \right) \begin{array}{c} \text{id}_{\mathbb{C}^J} \\ \otimes \\ u(\lambda) \end{array} : u \in \text{Hol}_{\mathcal{H}}(B_\delta) \text{ and } u \text{ is nc,} \right. \\ \left. \text{for some Hilbert space } \mathcal{H} \right\}.$$

We write the tensors vertically just to enhance readability.

Theorem 5.4. *The model cone \mathcal{C} , defined in (5.3), is closed in $\text{Her}(B_\delta)$.*

Proof. Suppose u^k is a sequence of nc functions in $\text{Hol}_{\mathcal{H}}(B_\delta)$ (we may assume the space \mathcal{H} is the same for each u^k , as in the proof of Theorem 5.1), so that

$$(5.5) \quad \begin{array}{c} \text{id}_{\mathbb{C}^J} \\ \otimes \\ u^k(\mu)^* \end{array} \left(\text{id} - \begin{array}{c} \delta(\mu)^* \delta(\lambda) \\ \otimes \\ \text{id}_{\mathcal{H}} \end{array} \right) \begin{array}{c} \text{id}_{\mathbb{C}^J} \\ \otimes \\ u^k(\lambda) \end{array}$$

converges in $\text{Her}(B_\delta)$ to $A(\lambda, \mu)$. On any finitely compact set, $\|\delta(x)\|$ will be bounded by a constant that is strictly less than one. Since (5.5) converges uniformly on finitely compact subsets of $B_\delta \boxtimes B_\delta$, this means that u^k is a finitely locally uniformly bounded sequence. Therefore by Theorem 3.6, there exist unitaries U^k such that $U^k * u^k$ has a convergent subsequence which converges to some nc function $u \in \text{Hol}_{\mathcal{H}}(B_\delta)$. Then

$$A(\lambda, \mu) = \begin{array}{c} \text{id}_{\mathbb{C}^J} \\ \otimes \\ u(\mu)^* \end{array} \left(\text{id} - \begin{array}{c} \delta(\mu)^* \delta(\lambda) \\ \otimes \\ \text{id}_{\mathcal{H}} \end{array} \right) \begin{array}{c} \text{id}_{\mathbb{C}^J} \\ \otimes \\ u(\lambda) \end{array},$$

as desired. □

6. SETS OF UNIQUENESS

In this section we shall show that the assumption in Propositions 2.1 and 3.2 that $\{\lambda_i\}$ is a dense sequence in Ω can be relaxed to the assumption that $\{\lambda_i\}$ is a set of uniqueness for $\text{Hol}(\Omega)$. We remark that it is an elementary fact that if \mathcal{H} is a Hilbert space, then $\{\lambda_i\}$ is a set of uniqueness for $\text{Hol}_{\mathcal{H}}(\Omega)$ if and only if $\{\lambda_i\}$ is a set of uniqueness for $\text{Hol}(\Omega)$.

The following proposition is essentially the same as the Arendt-Nikolski Theorem 1.1, so we shall omit the proof.

Proposition 6.1. *Assume that Ω is an open set in \mathbb{C}^d , $\{\lambda_i\}$ is a sequence in Ω that is a set of uniqueness for $\text{Hol}_{\mathcal{H}}(\Omega)$,³ and \mathcal{H} is a Hilbert space. If $\{u^k\}$ is a sequence in $\text{Hol}_{\mathcal{H}}(\Omega)$ that is locally uniformly bounded on Ω and for each fixed i , $\{u^k(\lambda_i)\}$ is a convergent sequence in \mathcal{H} , then $\{u^k\}$ converges in $\text{Hol}_{\mathcal{H}}(\Omega)$.*

Here is the graded version.

Proposition 6.2. *Assume that Ω is a finitely open set in \mathbb{M}^d , $\{\lambda_i\}$ is a sequence in Ω (with $\lambda_i \in \mathbb{M}_{n_i}^d$ for each i) that is a set of uniqueness for $\text{Hol}_{\mathcal{H}}(\Omega)$, and \mathcal{H} is a Hilbert space. If $\{u^k\}$ is a sequence in $\text{Hol}_{\mathcal{H}}(\Omega)$ that is finitely locally uniformly*

³That is, if $f \in \text{Hol}(\Omega)$ and $f(\lambda_i) = 0$ for all i , then $f(\lambda) = 0$ for all $\lambda \in \Omega$.

bounded on Ω and for each fixed i , $\{u^k(\lambda_i)\}$ is a convergent sequence in \mathcal{H} , then $\{u^k\}$ converges in $\text{Hol}_{\mathcal{H}}(\Omega)$.

Proof. The theorem will follow if we can show that $\{u^k|\Omega_n\}$ is a convergent sequence for each n . Accordingly, fix n and adopt the notation H_n for the holomorphic $\mathcal{B}(\mathbb{C}^n, \mathbb{C}^n \otimes \mathcal{H})$ -valued functions defined on Ω_n . Thus, $\{u^k|\Omega_n\}$ is a locally uniformly bounded sequence in H_n . Furthermore, if $\{\eta_j\}$ is an enumeration of $\{\lambda_i : i \geq 1\} \cap \Omega_n$, as $\{\lambda_i\}$ is a set of uniqueness for $\text{Hol}(\Omega)$, $\{\eta_i\}$ is a set of uniqueness for both $\text{Hol}(\Omega_n)$ and H_n . Finally, let $u^k(\eta_j) \rightarrow u_j$ as $k \rightarrow \infty$ for each j .

For fixed $\alpha \in \mathbb{C}^n$ and $\beta \in \mathbb{C}^n$, define $f_{\alpha,\beta}^k \in \text{Hol}(\Omega_n)$ by

$$(6.3) \quad f_{\alpha,\beta}^k(\lambda) = \langle u^k(\lambda)\alpha, \beta \rangle_{\mathbb{C}^n \otimes \mathcal{H}}, \quad \lambda \in \Omega.$$

Noting that

$$(6.4) \quad |f_{\alpha,\beta}^k(\lambda)| = |\langle u^k(\lambda)\alpha, \beta \rangle| \leq \|u^k(\lambda)\| \|\alpha\| \|\beta\|,$$

it follows that $\{f_{\alpha,\beta}^k\}$ is locally uniformly bounded on Ω_n . Therefore by Montel's theorem, $\{f_{\alpha,\beta}^k\}$ has compact closure in $\text{Hol}(\Omega_n)$.

We claim that $\{f_{\alpha,\beta}^k\}$ has a unique cluster point, for assume that $\{f_{\alpha,\beta}^{k_r}\}$ and $\{f_{\alpha,\beta}^{k_s}\}$ are subsequences of $\{f_{\alpha,\beta}^k\}$ with $\{f_{\alpha,\beta}^{k_r}\} \rightarrow f$ and $\{f_{\alpha,\beta}^{k_s}\} \rightarrow g$. Then, as we assume for each j , $u^k(\eta_j) \rightarrow u_j$ as $k \rightarrow \infty$,

$$\begin{aligned} f(\eta_i) &= \lim_{r \rightarrow \infty} f_{\alpha,\beta}^{k_r}(\eta_i) \\ &= \lim_{r \rightarrow \infty} \langle u^{k_r}(\eta_i)\alpha, \beta \rangle \\ &= \langle u_i\alpha, \beta \rangle \\ &= \lim_{s \rightarrow \infty} \langle u^{k_s}(\eta_i)\alpha, \beta \rangle \\ &= \lim_{s \rightarrow \infty} f_{\alpha,\beta}^{k_s}(\eta_i) \\ &= g(\eta_i). \end{aligned}$$

Hence, as $\{\eta_i\}$ is a set of uniqueness, $f = g$. Since $\{f_{\alpha,\beta}^k\}$ has a unique cluster point, we have shown that for each $\alpha \in \mathbb{C}^n$ and $\beta \in \mathbb{C}^n \otimes \mathcal{H}$, there exists $f_{\alpha,\beta} \in \text{Hol}(\Omega_n)$ such that

$$(6.5) \quad f_{\alpha,\beta}^k \rightarrow f_{\alpha,\beta} \text{ in } \text{Hol}(\Omega_n) \text{ as } k \rightarrow \infty.$$

Now fix $\lambda \in \Omega_n$ and define L_λ by

$$(6.6) \quad L_\lambda(\alpha, \beta) = f_{\alpha,\beta}(\lambda), \quad \alpha \in \mathbb{C}^n, \beta \in \mathbb{C}^n \otimes \mathcal{H}.$$

Observe that (6.3) and (6.5) imply that L_λ is a sesquilinear functional on $\mathbb{C}^n \times (\mathbb{C}^n \otimes \mathcal{H})$. Furthermore, (6.4) and (6.5) imply that L_λ is bounded. Therefore, by the Riesz Representation Theorem, there exists $u(\lambda) \in \mathcal{B}(\mathbb{C}^n, \mathbb{C}^n \otimes \mathcal{H})$ such that

$$\forall \alpha \in \mathbb{C}^n \quad \forall \beta \in \mathbb{C}^n \otimes \mathcal{H} \quad L_\lambda(\alpha, \beta) = \langle u(\lambda)\alpha, \beta \rangle$$

or, equivalently,

$$\forall \alpha \in \mathbb{C}^n \quad \forall \beta \in \mathbb{C}^n \otimes \mathcal{H} \quad \langle u(\lambda)\alpha, \beta \rangle = f_{\alpha,\beta}(\lambda).$$

The function u constructed in the previous paragraph has the following properties: it is holomorphic,

$$(6.7) \quad \forall \lambda \in \Omega_n \quad u^k(\lambda) \rightarrow u(\lambda) \quad \text{weakly in } \mathcal{B}(\mathbb{C}^n, \mathbb{C}^n \otimes \mathcal{H}) \quad \text{as } k \rightarrow \infty,$$

and

$$(6.8) \quad \forall_j \quad u^k(\eta_j) \rightarrow u(\eta_j) \quad \text{in norm in } \mathcal{B}(\mathbb{C}^n, \mathbb{C}^n \otimes \mathcal{H}) \quad \text{as } k \rightarrow \infty.$$

Claim 6.9. Let

$$u^k(\mu)^* u^k(\lambda) \rightarrow u(\mu)^* u(\lambda) \quad \text{in } \text{Her}(\Omega_n) \quad \text{as } k \rightarrow \infty.$$

To prove this claim, first note that as we are assuming $\{u^k\}$ is a locally uniformly bounded sequence in $\text{Hol}_{\mathcal{H}}(\Omega_n)$, $\{u^k(\mu)^* u^k(\lambda)\}$ is a locally uniformly bounded sequence in $\text{Her}(\Omega_n)$. Therefore, the claim follows from Montel's theorem if we can show that

$$(6.10) \quad A(\lambda, \mu) = u(\mu)^* u(\lambda)$$

whenever $\{k_r\}$ is a sequence of indices such that

$$(6.11) \quad u^{k_r}(\mu)^* u^{k_r}(\lambda) \rightarrow A(\lambda, \mu) \quad \text{in } \text{Her}(\Omega_n) \quad \text{as } r \rightarrow \infty.$$

But if (6.11) holds, then (6.8) implies that for each independently chosen i and j ,

$$A(\eta_j, \eta_i) = \lim_{r \rightarrow \infty} u^{k_r}(\eta_i)^* u^{k_r}(\eta_j) = u^{k_r}(\eta_i)^* u^{k_r}(\eta_j).$$

Since both sides of (6.10) are holomorphic in λ and anti-holomorphic in μ and $\{\eta_i\}$ is a set of uniqueness, it follows that (6.10) holds for all $\lambda, \mu \in \Omega$. This completes the proof of Claim 6.9.

Finally, fix $\lambda \in \Omega$. By (6.7), $\{u^k(\lambda)\}$ converges weakly in $\mathcal{B}(\mathbb{C}^n, \mathbb{C}^n \otimes \mathcal{H})$ to $u(\lambda)$, and by Claim 6.9, $u^k(\lambda)^* u^k(\lambda) \rightarrow u(\lambda)^* u(\lambda)$. Therefore, $u^k(\lambda) \rightarrow u(\lambda)$ is norm in $\mathcal{B}(\mathbb{C}^n, \mathbb{C}^n \otimes \mathcal{H})$. Since this holds for all $\lambda \in \Omega$, the proof of Proposition 6.2 may be completed by an application of Proposition 3.2. \square

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