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Optimization of on-condition thresholds for a system of degrading components with competing dependent failure processes

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ABSTRACT

An optimization model has been formulated and solved to determine on-condition failure thresholds and inspection intervals for multi-component systems with each component experiencing multiple failure processes due to simultaneous exposure to degradation and shock loads. In this new model, we consider on-condition maintenance optimization for systems of degrading components, which offers cost benefits over time-based preventive maintenance or replace-on-failure policies. For systems of degrading components, this can be a particularly difficult problem because of the dependent degradation and dependent failure times. In previous research, preventive maintenance and periodic inspection models have been considered; however, for systems whose costs due to failure are high, it is prudent to avoid the event of failure, i.e., the components or system should be repaired or replaced the before the failure happens. The determination of optimal on-condition thresholds for all components is effective to avoid failure and to minimize cost. Low on-condition thresholds can be inefficient because they waste component's life, and high on-condition thresholds are risky because the components are prone to costly failure. In this paper, we formulated and solved a new optimization model to determine optimal on-condition thresholds and inspection intervals. In our model, when the system is inspected, all components are inspected at that time. An inspection interval may be optimal for one component, but might be undesirable for another component, so the optimization requires a compromise. The on-condition maintenance optimization model is demonstrated on several examples.

1. Introduction

An effective maintenance policy maintains the system by achieving high safety and low cost, both of which are critical concerns in many modern industries [1]. Due to the inevitable deterioration of many components, systems may fail. To restore a failed system is often time-consuming and costly. Periodic and frequent inspection and repair/replacement can reduce the probability of deterioration and failure; however, it also incurs potentially expensive maintenance cost [2]. High quality operational performance and low maintenance cost can then become two conflicting objectives.

There has been much noteworthy research on reliability analysis for system subject to dependent failure processes, and accordingly, different maintenance policies have been considered [3–8]. For systems whose penalty cost due to downtime is high, detecting the component status and facilitating repair/replacement decision-making before system failure, leads to low risk of failure, and subsequently, lower maintenance cost. There have been previous studies on developing periodic inspection models for a degrading system with components sharing dependent degradation and dependent failure time [9–11]; however, those maintenance models are generally not combined with on-condition thresholds for components. In this paper, we formulate and solve an optimization model to determine on-condition thresholds and inspection intervals for multi-component systems with each component experiencing multiple failure processes.

We initially present a reliability model for systems in which failure processes for each component are dependent and failure times for all components are dependent [6]. Second, we introduce working principles for defining the on-condition thresholds and system status. A periodic inspection maintenance policy is selected so that the decision-making depends on the on-condition thresholds for all components. Finally, a maintenance cost rate model is developed and minimized for two different cases of replacement cost. In this model, system inspection interval and component on-condition thresholds are the decision

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variables. The new model offers cost benefits and performance improvement over time-based preventive maintenance or replace-on-failure policies.

The paper is organized as follows. Section 2 introduces and summarizes the relevant research previously done on reliability and maintenance policies for multi-component systems with multiple failure processes and provides the details of two failure processes. After presenting the system reliability model, Section 3 introduces on-condition thresholds and defines system status related to the on-condition thresholds. Section 4 describes the maintenance policy and cost rate optimization model based on system inspection interval and component on-condition thresholds. System examples are shown in Section 5 to illustrate the reliability and maintenance models.

The notation used in formulating the reliability and maintenance models is listed as follows:

$N(t)$	number of shock loads that have arrived by time t;
n	number of components in a series or parallel system;
λ	arrival rate of random shocks;
D_i	threshold for catastrophic/hard failure of ith component;
W_{ij}	size/magnitude of the jth shock load on the ith component;
$F_{Wi}(w)$	cumulative distribution function (cdf) of W_i;
H_i^1	critical wear degradation failure threshold of the ith component (a fixed parameter);
H_i^2	on-condition threshold of the ith component (a decision variable);
$X_i(t)$	wear volume of the ith component due to continuous degradation at t;
$X_{S_i}(t)$	total wear volume of the ith component at t due to both continual wear and instantaneous damage from shocks;
Y_{ij}	damage size contributing to soft failure of the ith component caused by the jth shock load;
$S_i(t)$	cumulative shock damage size of the ith component at t;
$\alpha_i(t)$, β_i	shape and scale parameter for gamma degradation process for component i;
$G_i(x_i,t)$	cumulative distribution function (cdf) of $X_i(t)$;
$F_{X_i}(x_i,t)$	cdf of $X_{S_i}(t)$;
$f_{Y_i}(y)$	probability density function (pdf) of Y_i;
$f_{Y_i}^{<k>}(y)$	pdf of the sum of k independent and identically distributed (*i.i.d.*) Y_i variables
$f_T(t)$, $F_T(t)$	pdf and cdf of the failure time, T;
$F_T^{H^1}(t)$	cdf of the failure time T for the whole system considering critical failure threshold;
$F_T^{H^2}(t)$	cdf of the time when an on-condition threshold is reached;
$C(t)$	cumulative maintenance cost by time t;
$E[TC]$	expected value of the total maintenance cost of the renewal cycle, TC;
τ	periodic inspection interval;
$CR(\tau)$	average long-run maintenance cost rate of the maintenance policy;
$E[K]$	expected renewal cycle length, K of the maintenance policy;
$E[N_I]$	expected number of inspections N_I;
$E[\rho]$	expected system downtime (the expected time from a system failure to the next inspection when the failure is detected);
C_R	replacement cost per unit;
C_{Rf}	fixed replacement cost per unit;
C_{Rl}	variable replacement cost per component cost per unit;
C_I	cost associated with each inspection;
C_ρ	penalty cost rate during downtime per unit of time;

2. Component and system reliability based on degradation analysis

Significant and meaningful prior research has been done on reliability and maintenance policies for systems with degradation, shocks and independent or dependent failure processes. In this new system model, we extend previously developed models and research to develop a new maintenance optimization model to determine optimal component on-condition thresholds and system inspection interval.

2.1. Research of reliability and maintenance for degrading systems

There is related literature and research work already dedicated to reliability analysis for systems subject to multiple failure processes. Song et al. [6] studied the reliability of multi-component systems with each component experiencing multiple failure processes. Chatwattanasiri et al. [12] then proposed a reliability model for a system of components with multiple competing and dependent failure processes when the future conditions are uncertain. Jiang et al. [13] further studied reliability of systems subjected to multiple competing dependent failure processes with changing dependent failure thresholds.

There have been other studies for systems experiencing degradation processes and external random shocks. Wang and Pham [14] developed a model considering the dependent relationship between random shocks and degradation processes by a time-scaled covariate factor. Rafiee et al. [15] studied reliability for systems subject to dependent competing failure processes with a changing degradation rate according to particular random shock patterns. Jiang et al. [16] developed reliability model for systems experiencing stochastic degradation processes and a random shock process, with shock effects falling into distinct zones.

Different maintenance policies for degrading systems with a single component or multiple components have also been extensively studied in the literature [17]. Bian and Gebraeel [18] proposed a stochastic model for the degradation processes of components and estimated residual lifetime distribution of each component. Levitin and Lisnianski [19] studied a preventive maintenance optimization problem for multi-state systems, which have a range of performance levels. Tsai [20] proposed a preventive maintenance model for systems with deteriorating components. A simple preventive maintenance task is to restore the degraded component to some level of the original condition and a preventive replacement task is to replace the aged component with a new one or to restore it to an as-new state. Li and Pham [21] developed a generalized condition-based maintenance model subject to multiple competing failure processes including two degradation processes and random shocks, in which the preventive maintenance thresholds for degradation processes and inspection sequences are the decision variables. Grall et al. [22] focused on the analytical modeling of a condition-based inspection/replacement policy for a stochastically and continuously deteriorating single-unit system, in which both the replacement threshold and the inspection schedule are considered as decision variables for this maintenance problem. Tian and Liao [23] investigated condition-based maintenance policies of multi-component systems based on a proportional hazards model, where economic dependency exists among different components subject to condition monitoring. Perez et al. [24] proposed a method for scheduling the maintenance in a wind farm with multiple turbines each having multiple components.

Jardine et al. [25] has performed important research considering diagnostics of mechanical systems implementing condition-based maintenance with an emphasis on models, algorithms and technologies for data processing and maintenance decision-making. Optimizing condition-based maintenance for equipment subject to vibration has been studied by Jardine et al. [26]. Zhu et al. [27] considered a maintenance model for systems with degradation which are continuously monitored, and units are immediately repaired when failure happened. This process was repeated until a predetermined time was reached for preventive maintenance to be performed. Wang and Pham [28] studied a multiple objective maintenance optimization problem for systems subject to dependent competing risks of degradation wear and random shocks. The number of preventive maintenance actions until replacement and the initial preventive maintenance interval were determined by simultaneously maximizing the asymptotic system availability, and minimizing the system cost rate using the fast elitist Non-dominated Sorting Genetic Algorithm (NSGA).

Ko and Byon [29] used asymptotic theory to analytically solve the large scale maintenance optimization problem when the maintenance set up cost is higher than repair cost. Abdul-Malak and Kharoufeh [30] developed a Markov decision process model to find the optimal replacement strategy for a system of multiple components in a shared environment. Wang et al. [31] considered a multi-phase inspection schedule for a system with degradation processes divided into more than two stages. Interaction between failure rates of units are considered for a two-unit system which is subjected to external shocks by Sung et al. [32].

2.2. Review of gamma process models

In this paper, it is considered that each component degrades so that irreversible damage gradually occurs, and the degradation model is monotonically increasing. In this case, it is appropriate to use the gamma process to model the degradation path. A thorough review of the gamma process model and its applications can be found in Van Noortwijk [33]. For our applications, the gamma process with a shape parameter and a scale parameter β is a continuous time stochastic process with the following properties:

- It starts from 0 at time 0, i.e., $X(0) = 0$
- $X(t)$ has independent increment
- for $t > 0$ and $s > 0$, $X(t) - X(s) \sim gamma(\alpha(t-s), \beta)$.

In fact, the probability density function of degradation process for each component $X_i(t) - X_i(s)$ is given by:

$$g(x;\alpha_i(t-s),\beta_i) = \frac{\beta_i^{\alpha_i(t-s)}x^{\alpha_i(t-s)-1}\exp(-\beta_i x)}{\Gamma(\alpha_i(t-s))} \tag{1}$$

where $\alpha_i(t)$ and β_i are the shape parameter and scale parameter for component i.

Caballé et al. [34] proposed a condition-based maintenance strategy for a system that its degradation process follows a nonhomogenous Poisson process and its growth is modeled by gamma process. Yousefi and Coit [35] used gamma process to model the component degradation process, where each component subject to mutually dependent competing failure processes.

2.3. Component reliability with competing dependent failure processes

In this paper, we consider systems where each component can fail due to two competing dependent failure processes that share the same shock process; a soft failure process and a hard failure process [1,2], as depicted in Fig. 1. Each component in the system degrades with time, and when a shock arrives, if damage is greater than a hard failure threshold, catastrophic failure occurs. For components that survive the shocks, if total degradation which includes both pure degradation and additional incremental degradation caused by shock damage is greater than a defined soft failure threshold level, then soft failure occurs. The two failure processes are competing and dependent.

Specific assumptions used for the reliability and maintenance modeling in this paper are as follows [1,2]:

1. Soft failure occurs for the ith component when the total degradation of that component exceeds its critical threshold level H_i^1. Component degradation is accumulated by both continuous degradation over time and cumulative incremental damage due to random shocks.
2. When the shock size exceeds the hard failure threshold of any component i (D_i), hard failure occurs of that component.
3. Random shocks arrive as a Poisson process.
4. The model is for systems that are packaged and sealed together, making it impossible or impractical to repair or replace individual components within the system, e.g., MEMS.
5. For the maintenance policy, the system is inspected at periodic intervals and no continuous monitoring is performed. Replacements are assumed to be instantaneous and perfect.
6. At any inspection time, if the degradation of any component i is lower than its own on-condition threshold H_i^2, component i is in the safety level; hence, the system is within the high safety level area if all the components are in their own safety level areas. It should be noted that each component has its own unique on-condition threshold which can be distinctly different from the other components.
7. Upon an inspection, if the degradation of any component i is between its own failure threshold H_i^1 and its on-condition threshold H_i^2, it has not failed but it can be anticipated to fail, and for a series system, failure of any component causes system failure. Therefore, it is advantageous to replace the system to avoid downtime when any (or possibly more than one) component i exceeds H_i^2 for an increasing degradation path.
8. If the system fails, that is, the total degradation of any component i in a series system is higher than H_i^1 before the specified inspection interval, it is not immediately detected and not replaced until the next inspection. There is penalty cost per time associated with the failure of system during downtime, e.g., cost associated with loss of production, opportunity costs, etc.

We develop an optimization model to determine on-condition failure thresholds and inspection intervals for complex multi-component systems with each component experiencing multiple failure processes due to simultaneous exposure to degradation and shock loads. Two failure processes for each component are dependent, and failure times for all components are also dependent. Component hard failures occur when a shock load exceeds thresholds. Fig. 1(b) shows that component i may fail when damage from a shock exceeds D_i. W_{ij} is the shock size and it is an *i.i.d.* random variable with some defined distribution which is assumed in this paper as a normal distribution, $W_{ij} \sim N(\mu_{Wi}, \sigma_{Wi}^2)$, with parameters μ_{w_i} and σ_{w_i} such that the probability of having negative W_{ij} is insignificant. This is not a restriction for our model and depending on μ_{w_i} and σ_{w_i}, considering a truncated normal distribution is also an effective way to avoid having negative W_{ij}. The probability density function of truncated normal distribution is shown in Eq. (2)

$$f_{W_i}(w) = \begin{cases} 0, & \text{for } w < 0 \\ \dfrac{\frac{1}{\sqrt{2\pi\sigma_{W_i}^2}}e^{-\frac{(w-\mu_{W_i})^2}{2\sigma_{W_i}^2}}}{1-F_{W_i}(0)}, & \text{for } w \geq 0 \end{cases} \tag{2}$$

We can obtain the probability that the ith component survives a shock [3]:

$$P_{Li} = P(W_{ij} < D_i) = F_{Wi}(D_i) = \Phi\left(\frac{D_i - \mu_{W_i}}{\sigma_{W_i}}\right) \quad \text{for } i = 1, 2, \ldots, n, \tag{3}$$

where $\Phi(\cdot)$ is the cdf of a standard normal random variable.

As shown in Fig. 1(a), total degradation of the ith component can be accumulated as $X_{Si}(t) = X_i(t) + S_i(t)$, and when $X_{Si}(t) > H_i^1$, soft failure occurs. Conditioning on the number of shocks and using a convolutional integral of $X_{Si}(t)$, we can obtain the probability that component i does not experience soft failure before time t as follow:

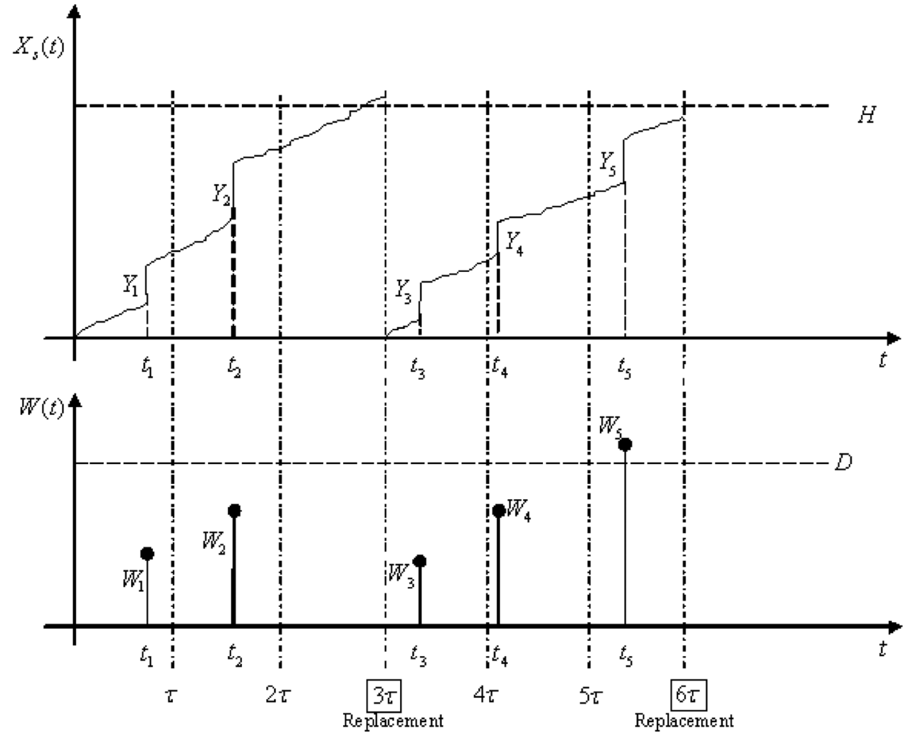

Fig. 1. Two dependent and competing failure processes for a component (a) soft failure process and (b) hard failure process [2].

$$P(X_{S_i}(t) < H_i^1) = F_{X_i}(H_i^1, t) = \sum_{m=0}^{\infty} P\left(X_i(t) + \sum_{j=1}^{m} Y_{ij} < H_i^1\right)\frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$

$$= \sum_{m=0}^{\infty}\left(\int_0^{H_i^1} G_i(H_i^1 - u, t) f_{Y_i}^{<m>}(u) du\right)\frac{\exp(-\lambda t)(\lambda t)^m}{m!} \tag{4}$$

$X_i(t)$ follows a gamma process, so $G_i(\bullet)$ is the cdf for a gamma distribution. It is convenient for Y_i to be gamma or normal distributed because the sum of m iid gamma random variables is also gamma, and the sum of m iid normal random variables is normal. In Song et al. [6], the assumption was made that Y_i was normally distributed, while in this paper, we assume Y_i is gamma distributed, but this is not a restriction.

2.4. Reliability analysis for multiple components system with MDCFP

We initially consider a series system, in which a component fails when either of the two dependent and competing failure modes occurs, and all components in the system behave similarly. Song at al. [6] developed a multi-component system reliability model when each component experiencing multiple failure processes. The reliability of this series system can be obtained, since the system fails when the first component fails. The concepts described in this paper can be extended to other system configurations, which is explained in more detail in Section 4.1.2.

Fig. 2 shows a series system with n components. The reliability of this series system at time t is the probability that each component survives each of the $N(t)$ shock loads ($W_{ij} < D_i$ for $j = 1, 2, \ldots$) and the total degradation of each component is less than the soft failure threshold level ($X_{Si}(t) < H_i^1$ for all i) .

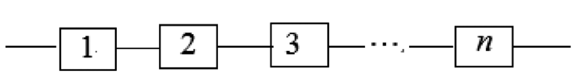

Fig. 2. Series system example.

In this model, shocks arriving at random time intervals are modeled as a Poisson process. When the system receives a shock (at rate λ), all components experience a shock. If we consider the component survival probabilities conditioned on the number of shocks, then the failure processes for all components become independent for a fixed number of shocks. The system reliability function can be derived for the general case for a series system as follows [6]:

$$R(t) = \sum_{m=0}^{\infty}\prod_{i=1}^{n}\left[P(W_i < D_i)^m P\left(X_i(t) + \sum_{j=1}^{m} Y_{ij} < H_i^1\right)\right]\frac{\exp(-\lambda t)(\lambda t)^m}{m!} \tag{5}$$

Using a convolution integral, the reliability model can be obtained as follow [6]:

$$R(t) = \sum_{m=0}^{\infty}\prod_{i=1}^{n}\left[P(W_i < D_i)^m \int_0^{H_i^1} G_i(H_i^1 - u, t) f_{Y_i}^{<m>}(u) du\right]\frac{\exp(-\lambda t)(\lambda t)^m}{m!} \tag{6}$$

3. Operational principle of the on-condition rule

For systems whose costs associated with failure are high, it is advantageous to repair or replace the components or system before a failure occurs. The concept of condition monitoring and on-condition thresholds for the components is used to evaluate and measure system status. This can be an effective way to improve opportunities to detect the component critical and degraded status and to avoid costly failures. The maintenance optimization is challenging because of the dependent degradation and dependent failure times among all components.

3.1. Definition of system status related to on-condition threshold

For some systems, the cost and consequence of failure are excessive compared to comparable preventive repair cost, replacement cost or other kinds of cost. Therefore, it is prudent to prevent failure from

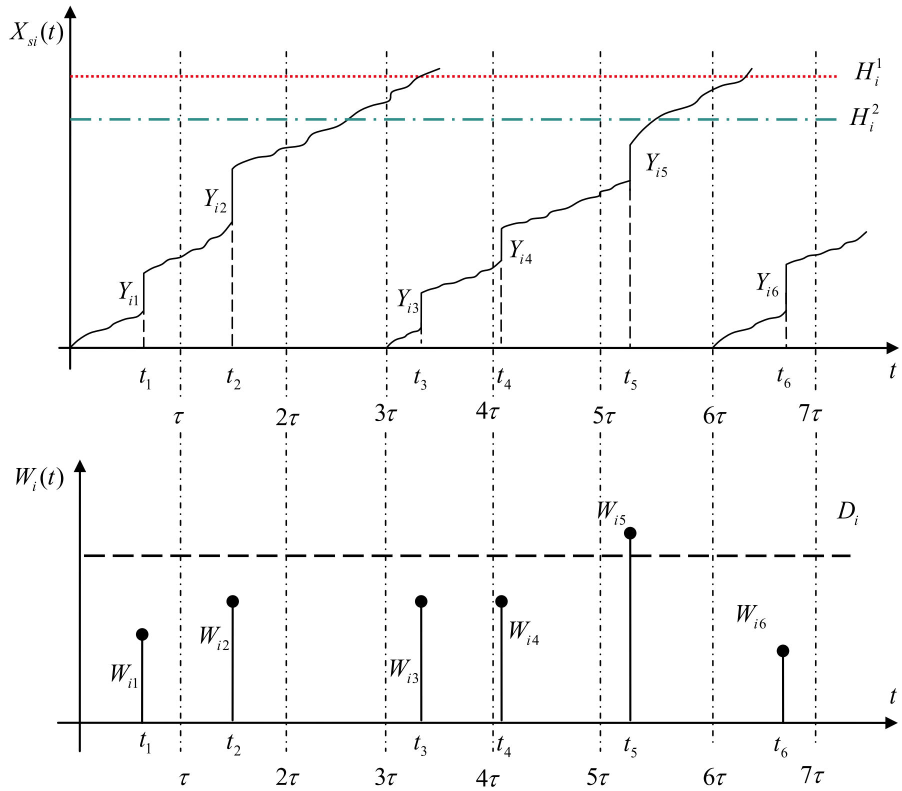

Fig. 3. Two thresholds divide system status into three regions.

occurring and replace the system after it has sufficiently aged, rather than allowing to fail and possibly cause more severe consequences. For the multi-component system considered in this research, the components are packaged and/or sealed together and it is reasonable or necessary to replace the whole system before the critical degradation thresholds are reached. On-condition rules provide the capability to measure system status and replace the system before failure to avoid system downtime. Based on the defined rules, the implementation of a lower degradation threshold can be useful to avoid failure by providing criteria to detect the degradation status of the components.

As depicted in Fig. 3, H_i^1 is defined as the soft failure threshold for component i and H_i^2 is the on-condition threshold for component i, with $H_i^2 \le H_i^1$. At each inspection time, we determine component condition for each component by inspection and compare it to the corresponding threshold. The action taken depends on a selection of condition-based operational status and the defined maintenance condition rules. We adopt rules related to this on-condition degradation threshold to define the component degradation state.

At each inspection interval, if no hard failure occurs, and at the same time, total degradation of the ith component is less than H_i^2, we then consider the component is in the safe region. The safe region is defined as the combination of soft failure process and hard failure process both below their respective thresholds and this status is defined as event A shown in Table 1. If no hard or soft failure occurs and total degradation is between H_i^2 and H_i^1 for any component i, this component has not failed; however, probabilistically it may fail within a short period of time. This status can be described by the combination of soft failure process area between H_i^2 and H_i^1, and hard failure process area below the hard failure threshold, which is defined as event B in Table 1. If there has been a hard failure or the total degradation of any component i is greater than H_i^1, the system has failed. The status can be defined as the union of the soft failure process area above the red dashed line, and hard failure process area above black dashed line, and this status is defined as event C.

Considering the safe region for example, conditioning on m shocks arriving to the system by time t with probability $\frac{\exp(-\lambda t)(\lambda t)^m}{m!}$, the probability of no hard failure is $P(W_i < D_i)^m$, and the probability that total degradation is less than H_i^2 is $\int_0^{H_i^2} G_i(H_i^2 - u, t) f_{Y_i}^{<m>}(u) du$. Combining both soft failure process and hard failure process, the probability for event A, i.e., the component i is in safe region, is:

$$P(A_i) = \sum_{m=0}^{\infty} P(W_i < D_i)^m \left(\int_0^{H_i^2} G_i(H_i^2 - u, t) f_{Y_i}^{<m>}(u) du \right) \frac{\exp(-\lambda t)(\lambda t)^m}{m!} \tag{7}$$

Similarly, for event B, component i is still working, but it is probabilistically more likely to fail within the next inspection interval. The probability of no hard failure considering m shocks is $P(W_i < D_i)^m$ and the probability that total degradation is between H_i^1 and H_i^2 is $\int_{H_i^2}^{H_i^1} G_i(H_i^1 - u, t) f_{Y_i}^{<m>}(u) du$. Combining both the soft failure process and hard failure process; we can obtain the probability for event B. For event C, either a soft failure or a hard failure occurs, with probability which equals to one minus the probability that neither of these two failure happens. The policy is summarized in Table 1.

Given this reliability model for systems with each component experiencing multiple failure processes due to simultaneous exposure to degradation and shock loads, we can then define a maintenance cost optimization objective function. The system is inspected periodically, and the condition of each component is observed and compared to a threshold. Upon an inspection, we replace the system with a new one when we observe that a hard failure has occurred or total degradation is greater than the on-condition threshold for any component i.

The expected number of inspections N_I, for a vector of on-condition thresholds $\boldsymbol{H}^2 = (H_1^2, H_2^2, \ldots, H_n^2)$ is given by,

$$E(N_I) = \sum_{k=1}^{\infty} k \left(F_T^{\mathbf{H}^2}(k\tau) - F_T^{\mathbf{H}^2}((k-1)\tau) \right) \tag{8}$$

$F_T^{\mathbf{H}^2}(t)$ is the probability that the degradation of at least one component is above its own on-condition threshold by time t. $F_T^{\mathbf{H}^2}(t)$can be calculated using Eq. (9)

Table 1
Component status defined with two soft failure thresholds and hard failure threshold.

A	Component is in safe region	$P(A) = \sum_{m=0}^{\infty} P(W_i < D_i)^m (\int_0^{H_i^2} G_i(H_i^2 - u,\, t) f_{Y_i}^{<m>}(u) du) \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$
B	Component is working, but probabilistically fails soon	$P(B) = \sum_{m=0}^{\infty} P(W_i < D_i)^m (\int_0^{H_i^1} G_i(H_i^1 - u,\, t) f_{Y_i}^{<m>}(u) du - \int_0^{H_i^2} G_i(H_i^2 - u,\, t) f_{Y_i}^{<m>}(u) du) \times \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$
C	Component fails	$P(C) = \sum_{m=0}^{\infty} (1 - P(W_i < D_i)^m \int_0^{H_i^1} G(H_i^1 - u,\, t) f_Y^{<m>}(u) du) \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$

$$F_T^{\mathbf{H}^2}(t) = 1 - \sum_{m=0}^{\infty} \prod_{i=1}^{n} \left[P(W_i < D_i)^m \int_0^{H_i^2} G_i(H_i^2 - u,\, t) f_{Y_i}^{<m>}(u) du \right] \frac{\exp(-\lambda t)(\lambda t)^m}{m!} \tag{9}$$

From Fig. 4, we can observe that system downtime is the time duration between when a failure occurs and the next time an inspection is performed, and the failure is detected. Conditioning on the event that there is a failure at time t between the (k-1)th and kth inspection [(k-1)τ, $k\tau$] with probability $F_T^{\mathbf{H}^2}(k\tau) - F_T^{\mathbf{H}^2}((k-1)\tau)$, and defining the failure time as t, the system downtime is $k\tau$ - t. The expected value of system downtime or the expected time from a system failure to the next inspection when the failure is detected, can then be determined as $\int_{(k-1)\tau}^{k\tau} (k\tau - t) dF_T^{\mathbf{H}^1}(t)$. Summing over the probability that failure can occur in any inspection interval, we can obtain expected system downtime as follows:

$$E[\rho] = \sum_{k=1}^{\infty} E[\rho | N_I = k] P(N_I = k) = \sum_{k=1}^{\infty} \left((F_T^{\mathbf{H}^2}(k\tau) - F_T^{\mathbf{H}^2}((k-1)\tau)) \int_{(k-1)\tau}^{k\tau} (k\tau - t) dF_T^{\mathbf{H}^1}(t) \right) \tag{10}$$

where $F_T^{\mathbf{H}^1}(t) = 1 - \left(\sum_{m=0}^{\infty} \prod_{i=1}^{n} \left[P(W_i < D_i)^m \int_0^{H_i^1} G_i(H_i^1 - u,\, t) f_{Y_i}^{<m>}(u) du \right] \frac{\exp(-\lambda t)(\lambda t)^m}{m!} \right)$ (11)

The expected time between two replacements or expected cycle length is

$$E[K] = \sum_{k=1}^{\infty} E[K | N_I = k] P(N_I = k) = \sum_{k=1}^{\infty} k\tau (F_T^{\mathbf{H}^2}(k\tau) - F_T^{\mathbf{H}^2}((k-1)\tau)) \tag{12}$$

4. Condition-based maintenance modeling and optimization

Condition-based maintenance offers the promise of enhancing the effectiveness of maintenance programs. For some cases, the penalty cost due to downtime is relatively higher than the comparable corrective maintenance costs, so it is cost-effective to replace the whole system before the wear volumes of components reach their failure thresholds, However, there are other cases that replacing the system upon failure is more beneficial because you obtain maximum system life and downtime costs are small. In this paper, if the optimal on-condition threshold is the same as failure threshold, i.e., $H_i^2 = H_i^1$, we have the case that implementing preventive maintenance before failure is not necessary or even beneficial.

4.1. Description of the maintenance model

Effective on-condition degradation thresholds can achieve our goal of replacing the system before failure by providing the criteria to detect component degradation beyond the threshold. If the on-condition threshold is too low and far away from the nominal threshold level, then we have to replace the whole system more frequently, and it results in excessive cost. Alternatively, if the threshold is too high, then the system may fail before the next inspection leading to potentially expensive downtime cost. Therefore, on-condition degradation thresholds for all components and an inspection interval for the whole system are chosen to be decision variables in this maintenance optimization problem.

To evaluate the performance of the condition-based maintenance policy, we use an average long-run maintenance cost rate model as the objective function, in which the periodic inspection interval τ for the whole system and on-condition thresholds H_i^2 for all components are the decision variables. At time τ, and subsequent inspection intervals of time τ, the entire assembled system is inspected. If the system is still operating satisfactorily with no detected component degradation above the on-condition threshold, nothing is done. If degradation thresholds for all component are below the fixed critical degradation thresholds H_i^1 but some are above the on-condition threshold H_i^2, the whole system is replaced preventively. If there is a hard failure or at least one component's wear volume is above the critical degradation threshold H_i^1 prior to inspection, then the system is not replaced with a new one correctively until the next inspection. The average long-run maintenance cost per unit time can be evaluated by:

$$\lim_{t\to\infty} (C(t)/t) = \frac{\text{Expected maintenance cost between two replacements}}{\text{Expected time between two replacements}} = \frac{E[TC]}{E[K]} \tag{13}$$

where TC is the total maintenance cost of a renewal cycle, and K is the length of a cycle that takes a value of a multiple of τ [36]. The expected total maintenance cost is given as:

$$E[TC] = C_I E[N_I] + C_\rho E[\rho] + C_R \tag{14}$$

where C_I is the cost of each inspection. C_R is the replacement cost, C_ρ is the penalty cost incurred during down time, and τ is the time interval for periodic inspection. In this model C_R is a fixed value independent of the number of components with threshold exceeding H_i^2. However, an alternate formulation is presented in 4.1.1 with variable replacement cost. Based on Eqs. (8)–(10), the average long-run maintenance cost rate is given as

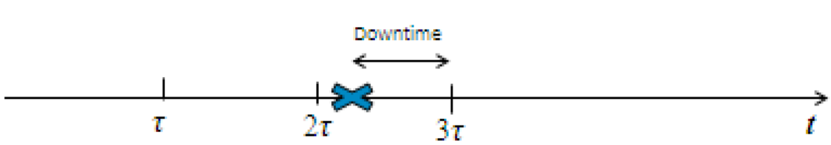

Fig. 4. System downtime under periodic inspection maintenance policy.

$$CR(\tau, \mathbf{H}^2) = \frac{C_I \sum_{k=1}^{\infty} k(F_T^{\mathbf{H}^2}(k\tau) - F_T^{\mathbf{H}^2}((k-1)\tau)) + C_\rho \sum_{k=1}^{\infty} \left((F_T^{\mathbf{H}^2}(k\tau) - F_T^{\mathbf{H}^2}((k-1)\tau)) \int_{(k-1)\tau}^{k\tau} (k\tau - t) dF_T^{\mathbf{H}^1}(t) \right) + C_R}{\sum_{k=1}^{\infty} k\tau(F_T^{\mathbf{H}^2}(k\tau) - F_T^{\mathbf{H}^2}((k-1)\tau))} \tag{15}$$

4.1.1. Maintenance model considering different replacement costs

Another model has also been developed for cases with different replacement costs, depending on the number of aged components close to the failure threshold, i.e., exceeding H_i^2. The applications in this paper are for systems that are packaged and sealed together, so whenever system replacement is required, the assembled system should be replaced with a new one. In the previous sections, it is assumed that at each inspection time when a replacement is required, the whole system is replaced with a new one and the replacement cost is fixed regardless of how many components have conditions above the on-condition thresholds or failed. However, other systems and applications behave differently. To consider the dependency of replacement cost on component's status, a new cost rate model has been derived, where the expected maintenance cost between two replacements can be calculated as follow:

$$E[TC] = C_I E[N_I] + C_\rho E[\rho] + \sum_{l=1}^{n} (C_{Rf} + lC_{Rc})P_l$$
$$P_l = P(l \text{ components above } H_i^2) \tag{16}$$

C_{Rf} includes fixed setup cost for replacement, and C_{Rc} is the replacement cost for each additional component with degradation level above its own on-condition threshold. It is assumed that the replacement cost is a function of the number of components affected, but not which specific components. When the whole system is replaced with a new one in the previous sections is equivalent to the case that all the components have degradation level greater than their on-condition threshold in this section or $C_R = C_{Rf} + nC_{Rc}$.

Case 1: If all components in the system are identical, the probability of having l aged beyond the on-condition threshold or failed components (l component with degradation level greater than its own on-condition threshold) can be calculated as follow.

$$P_l = \binom{n}{l} (1 - P(A))^l P(A)^{n-l} \tag{17a}$$

$P(A) = P(A_i)$ for all i, with $P(A_i)$ given by Eq. (7).

Case 2: If some or all of the components are different, then computation of P_l is more complex. Define $S(l)$ as a set of all n-dimension vectors $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, whose values sum to l with $x_i \in \{0, 1\}$, where x_i is 1 if component i has degradation level greater than its own on-condition threshold, and 0 otherwise. Therefore, P_l for Case 2 can be computed as:

$$P_l = \sum_{\mathbf{x} \in S(l)} \prod_{i=0}^{n} (1 - P(A_i))^{x_i} P(A_i)^{1-x_i}, \quad S(l) = \left\{ \mathbf{x};\ \sum_{i=1}^{n} x_i = l \right\} \tag{17b}$$

Based on Table 1, $P(A_i)$ for both cases is the probability that degradation level of component i is less than its own on-condition threshold, and there is no hard failure for component i by time t. $P(A_i)$ is given by Eq. (7)

So, considering the dependency of replacement cost on number of aged and failed components the average long-run maintenance cost per unit is given as:

$$CR(\tau, \mathbf{H}^2) = \left(C_I \sum_{k=1}^{\infty} k(F_T^{\mathbf{H}^2}(k\tau) - F_T^{\mathbf{H}^2}((k-1)\tau)) + C_\rho \sum_{k=1}^{\infty} \left((F_T^{\mathbf{H}^2}(k\tau) - F_T^{\mathbf{H}^2}((k-1)\tau)) \int_{(k-1)\tau}^{k\tau} (k\tau - t) dF_T^{\mathbf{H}^1}(t) \right) + \sum_{l=1}^{n} (C_{Rf} + lC_{Rc})P_l \right) \Big/ \sum_{k=1}^{\infty} k\tau(F_T^{\mathbf{H}^2}(k\tau) - F_T^{\mathbf{H}^2}((k-1)\tau)) \tag{18}$$

4.1.2. Maintenance model for parallel configuration

Song et al. [6] developed reliability models for multi-components systems subjected to multi-dependent failure processes for different configurations, such a parallel and series-parallel. Although, the equations become more complicated, the new proposed model can be extended to different configurations. For example, for parallel configuration we have the following reliability model.

$$R_P(t) = 1 - \sum_{m=0}^{\infty} \prod_{i=1}^{n} \left[1 - P(W_i < D_i)^m P\left(X_i(t) + \sum_{j=1}^{m} Y_{ij} < H_i^1 \right) \right] \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$
$$= 1 - \sum_{m=0}^{\infty} \prod_{i=1}^{n} \left[1 - P(W_i < D_i)^m \int_0^{H_i^1} G_i(H_i^1 - u, t) f_{Y_i}^{<m>}(u) du \right] \frac{\exp(-\lambda t)(\lambda t)^m}{m!} \tag{19}$$

Consequently, the probability that failure can occur by time t can be calculated as follow:

$$F_T^{H^1}(t) = 1 - R_P(t)$$
$$= \sum_{m=0}^{\infty} \prod_{i=1}^{n} \left[1 - P(W_i < D_i)^m \int_0^{H_i^1} G_i(H_i^1 - u, t) f_{Y_i}^{<m>}(u) du \right] \frac{\exp(-\lambda t)(\lambda t)^m}{m!} \tag{20}$$

The probability of having no replacement by time t is:

$$P_{NR-P}(t) = 1 - \sum_{m=0}^{\infty} \prod_{i=1}^{n} \left[1 - P(W_i < D_i)^m P\left(X_i(t) + \sum_{j=1}^{m} Y_{ij} < H_i^2 \right) \right] \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$
$$= 1 - \sum_{m=0}^{\infty} \prod_{i=1}^{n} \left[1 - P(W_i < D_i)^m \int_0^{H_i^2} G_i(H_i^2 - u, t) f_{Y_i}^{<m>}(u) du \right] \frac{\exp(-\lambda t)(\lambda t)^m}{m!} \tag{21}$$

$P_{NR-P}(t)$ refers to the probability of having no replacement for a parallel system, and the probability of having replacement by time t for parallel configuration is:

$$F_T^{H^2}(t) = 1 - P_{NR-P}(t)$$
$$= \sum_{m=0}^{\infty} \prod_{i=1}^{n} \left[1 - P(W_i < D_i)^m \int_0^{H_i^2} G_i(H_i^2 - u, t) f_{Y_i}^{<m>}(u) du \right] \frac{\exp(-\lambda t)(\lambda t)^m}{m!} \tag{22}$$

The total cost function presented in Eqs. (15) and (18) can still be applied, but by substituting these new functions which are specific for a parallel system. Equations for a series-parallel or other configuration could also be developed, although they would be highly complex.

4.2. Maintenance cost optimization

For our maintenance optimization problem, if there are n components in a series system, there are $n + 1$ decision variables; namely n on-condition thresholds for all components and the periodic inspection interval for the whole system. Our objective is to minimize maintenance cost rate, and constraints are that on-condition thresholds for all components should be less than or equal to their critical failure thresholds, and inspection interval should be a positive value. Therefore, our maintenance optimization problem can be formulated as follows:

$$\begin{aligned} \min \quad & CR(\tau, \mathbf{H}^2) \\ \text{s.t.} \quad & 0 \le H_1^2 \le H_1^1, \\ & 0 \le H_2^2 \le H_2^1, \\ & \cdots \\ & 0 \le H_n^2 \le H_n^1, \\ & \tau \ge 0, \end{aligned} \tag{23}$$

It is a difficult non-linear optimization problem but with continuous decision variables and a convex feasible region. For constrained nonlinear optimization problems, there are many available algorithms to obtain optimal solutions. Interior point methods have proved to be very successful in solving many nonlinear problems [37–39]. The interior point method consists of a self-concordant barrier function used to encode the convex set. It reaches an optimal solution by traversing the interior of the feasible region using one of two main types of steps at each iteration [40]. The algorithm first attempts to take a direct step within the feasible region to solve the Karush Kuhn Tucker (KKT) equations for the approximate problem by a linear approximation, which is also called a Newton step. By solving the KKT equations, we can obtain the direct step and the solution for the next iteration. If a direct step cannot be completed, it attempts a conjugate gradient step, and minimizes a quadratic approximation to the approximate problem in a trust region, subject to linearized constraints. It does not take a direct step when the problem is not locally convex near the current iteration. At each iteration, the algorithm decreases a merit function. We reach a new solution point after taking the step and start a new iteration. It continues until a defined stopping criterion is met.

In this paper, to solve the optimization problem, an interior point method is used (as implemented as the *fmincon* algorithm in the MATLAB optimization toolbox). *fmincon* in Matlab is easy to use, robust and has wide variety of options. The built-in parallel computing support in *fmincon* accelerates the estimation of gradients. There have been some studies that demonstrate the preference of using *fmincon* in solving nonlinear optimization problems. Cohen et al. [41] compared different algorithms to solve scheduling optimization, and investigated that interior point using *fmincon* function in Matlab has the best performance such as fastest optimization time, minimum optimization cost and the robustness to noise. Chuan et al. [42] showed that the *fmincon* function in Matlab is faster than other methods such as genetic algorithm (GA) while they all have the same optimal results for their radiological worker allocation nonlinear problem. Erentok et al. [43] shows that using *fmincon* function has the same accuracy as GA method to obtain optimal value while it is faster than using GA methods. Samavati et al. [44] also compared GA and *fmincon* function in Matlab optimization toolbox to solve a cooperative grasp planning problem. They showed that *fmincon* converged faster than GA.

5. Numerical examples

We consider several numerical examples; the first one is a series system with four components where component 1 and 2 have the same parameters and component 3 and 4 have the same parameters. We conduct the optimization with both a fixed inspection interval, and then inspection interval as a decision variable. The second example is for a system with four different components, and the third one is a series system with four identical components with replacement cost dependent on the number of aged and failed component.

The parameters for reliability analysis of these examples are provided in Tables 2 and 3. Y_{ij} follows gamma distributions and W_{ij} follows normal distributions in both examples. The first example is a conceptual example to demonstrate the reliability function and maintenance models. However, although the example is conceptual, H_i^1 and D_i are estimated based on documented degradation trends [3]. To provide some interesting comparisons, we perform maintenance optimization for the series system and also, all the individual components making up the system separately, and we discuss the results.

5.1. Example 1

For the first example, we consider the maintenance policy for the whole series system with four components and a predetermined inspection interval, i.e., we inspect the whole system at one interval of τ and replace the system when the observed degradation is above H_i^2 for any component. For some actual applications, there is a fixed or known inspection interval that is imposed by the decision-maker or availability of the system for inspection. The system can only be inspected at those fixed intervals, which could be far from optimal. Therefore, to compare these cases with proposed model, we selected two fixed values as possible inspection intervals and the optimal on-condition thresholds and cost rate functions are found for these cases.

The first case has a very long inspection interval of $\tau = 120$ h, choosing $C_I = \$1$, $C_p = \$20{,}000$ and $C_R = \$100$, we can find the minimum average long-run maintenance cost rate for system is $\$3.054 \times 10^2$ and on-condition degradation threshold are ${H_1}^{2*} = {H_2}^{2*} = 0.0001556$, ${H_3}^{2*} = {H_4}^{2*} = 0.0001370$. Moreover, by considering a shorter fixed inspection interval of $\tau = 24$ h, the minimum average long-run maintenance cost rate for system reduces to $\$2.2796 \times 10^2$ and on-condition degradation threshold are ${H_1}^{2*} = {H_2}^{2*} = 0.0004637$, ${H_3}^{2*} = {H_4}^{2*} = 0.0004204$. When the system is inspected more frequently, we have higher on-condition degradation thresholds, i.e., closer to the failure threshold. Since the system status is detected more often, it can be replaced preventively, so on-condition degradation thresholds are closer to failure thresholds.

The contribution of this paper is to now simultaneously determine the optimal on condition thresholds and inspection interval. The minimum average long run maintenance cost rate for the system is $\$1.9023 \times 10^2$ found after 22 steps of iteration. The inspection interval is $\tau^* = 44.7129$ h, and on-condition degradation thresholds are ${H_1}^{2*} = {H_2}^{2*} = 0.0003055$ and ${H_3}^{2*} = {H_4}^{2*} = 0.0002728$. Fig. 5 illustrates the iteration process of decision variables: inspection interval, on-condition degradation threshold for component 1 and 2, and on-condition degradation threshold for component 3 and 4. Fig. 6 shows the iteration for our objective function, i.e., the system maintenance cost rate. From Iteration 10 on Fig. 5 and 6 the optimal values do not change; however, the algorithm continued to confirm that there is no additional improvement and the optimal solutions are converged.

To show the preference of the proposed model, the optimal maintenance cost rate of this example is compared to optimal cost rate values for different maintenance policies such as time-based maintenance and replace-on-failure maintenance. In fact, both these policies are

Table 2
Parameter values for multi-component system reliability analysis for the first example.

Parameter	Component 1 & 2	Component 3 & 4	Sources
H_i^1	0.00125 μm^3	0.00127 μm^3	Tanner and Dugger [3]
D_i	1.5 Gpa	1.4 Gpa	Tanner and Dugger [3]
α_i	0.7	0.8	Assumption
β_i	0.3	0.3	Assumption
λ	2.5×10^{-5}	2.5×10^{-5}	Assumption
Y_{ij}	$Y_{ij} \sim gamma(\alpha_{Y_i}, \beta_{Y_i})$ $\alpha_{Y_i} = 0.4$, $\beta_{Y_i} = 1$	$Y_{ij} \sim gamma(\alpha_{Y_i}, \beta_{Y_i})$ $\alpha_{Y_i} = 0.5$, $\beta_{Y_i} = 1$	Assumption
W_{ij}	$W_{ij} \sim N(\mu_{Wi}, {\sigma_{Wi}}^2)$ $\mu_{Wi} = 1.2$ GPa, $\sigma_{Wi} = 0.2$ GPa	$W_{ij} \sim N(\mu_{Wi}, {\sigma_{Wi}}^2)$ $\mu_{Wi} = 1.22$ GPa, $\sigma_{Wi} = 0.18$ GPa	Assumption

Table 3
Parameter values for multi-component system reliability analysis for a system with four different components.

Parameter	Component 1	Component 2	Component 3	Component 4
H_i^1	0.00125 μm^3	0.00127 μm^3	0.0013 μm^3	0.00128 μm^3
D_i	1.5 Gpa	1.4 Gpa	1.2 Gpa	1.45 Gpa
α_i	0.7	0.8	0.6	0.2
β_i	0.3	0.3	0.25	0.25
λ	2.5×10^{-5}			
Y_{ij}	$Y_{ij} \sim gamma(\alpha_{Y_i}, \beta_{Y_i})$ $\alpha_{Y_i} = 0.45$, $\beta_{Y_i} = 1$	$Y_{ij} \sim gamma(\alpha_{Y_i}, \beta_{Y_i})$ $\alpha_{Y_i} = 0.5$, $\beta_{Y_i} = 1$	$Y_{ij} \sim gamma(\alpha_{Y_i}, \beta_{Y_i})$ $\alpha_{Y_i} = 0.48$, $\beta_{Y_i} = 1$	$Y_{ij} \sim gamma(\alpha_{Y_i}, \beta_{Y_i})$ $\alpha_{Y_i} = 0.4$, $\beta_{Y_i} = 1$
W_{ij}	$W_{ij} \sim N(\mu_{Wi}, \sigma_{Wi}^2)$ $\mu_{Wi} = 1.2$ GPa, $\sigma_{Wi} = 0.22$ GPa	$W_{ij} \sim N(\mu_{Wi}, \sigma_{Wi}^2)$ $\mu_{Wi} = 1.22$ GPa, $\sigma_{Wi} = 0.18$ GPa	$W_{ij} \sim N(\mu_{Wi}, \sigma_{Wi}^2)$ $\mu_{Wi} = 1.23$ GPa, $\sigma_{Wi} = 0.15$ GPa	$W_{ij} \sim N(\mu_{Wi}, \sigma_{Wi}^2)$ $\mu_{Wi} = 1.2$ GPa, $\sigma_{Wi} = 0.2$ GPa

special cases of our proposed model. For replace-on-failure model, failure is detected by inspection, and if failures are not detected promptly, there is costly downtime. Therefore, replace-on-failure still requires inspections, but by setting $H_i^2 = H_i^1$ for all i, we can generate the lowest cost replace-on-failure policy by solving an optimization problem where $CR(\tau)$ is the objective function with $H_i^2 = H_i^1$, and all the costs are the same. The optimal inspection interval is found as $\tau^* = 9.43$ and minimum average long run maintenance cost rate is \$$3.271 \times 10^2$. In this case, the inspection interval is small, because the only way to avoid costly downtime is to inspect frequently; while, when we have on-condition thresholds for each component to avoid failure and downtime, the minimum average long run maintenance cost rate for the system is \$$1.902 \times 10^2$ which shows the proposed method can provide a beneficial maintenance policy for cases with high downtime costs by replacing the system before failure and avoiding system downtime.

Similarly, time-based preventive maintenance is investigated by setting $H_i^2 = 0$ for all i, so, the whole system will be replaced on the first inspection. The optimal inspection interval for this case is $\tau^* = 52.45$ with the minimum average long run maintenance cost rate of \$$2.427 \times 10^2$ which shows this policy is costly compared to our proposed model.

To further evaluate the results, we also consider an inspection and maintenance policy for the individual components. That is, we treat four components as individual systems, and inspect individual four components at their own inspection intervals. Since component 1 and 2 share the same parameter, the maintenance optimization for them are the same. We can find the minimum average long-run maintenance cost rate for component 1 and 2 as \$$1.367 \times 10^2$ after 20 steps of iteration, with a solution of the periodic inspection interval $\tau_{1,2}^* = 65.044$ h, and

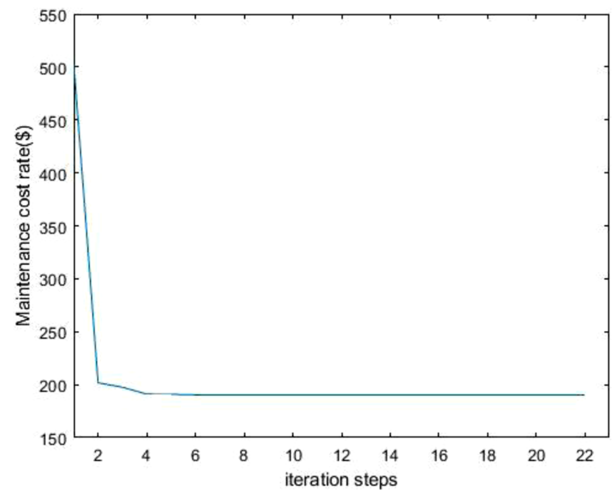

Fig. 6. Iteration process of maintenance cost rate for system with four components.

on-condition degradation threshold for components $H_1^{2*} = H_2^{2*} = 0.0002465$. Fig. 7 illustrates the iteration process of two decision variables, inspection interval and on-condition degradation threshold, for component 1 and 2. Fig. 8 shows the iteration for our objective function, that is, the maintenance cost rate.

Similarly, we inspect individual component 3 or component 4 at their own inspection intervals. The minimum average long-run maintenance cost rate for component 3 and 4 is \$$1.762 \times 10^2$ after 13 steps of iteration, with the periodic inspection interval $\tau_{3,4}^* = 71.55$ h, and

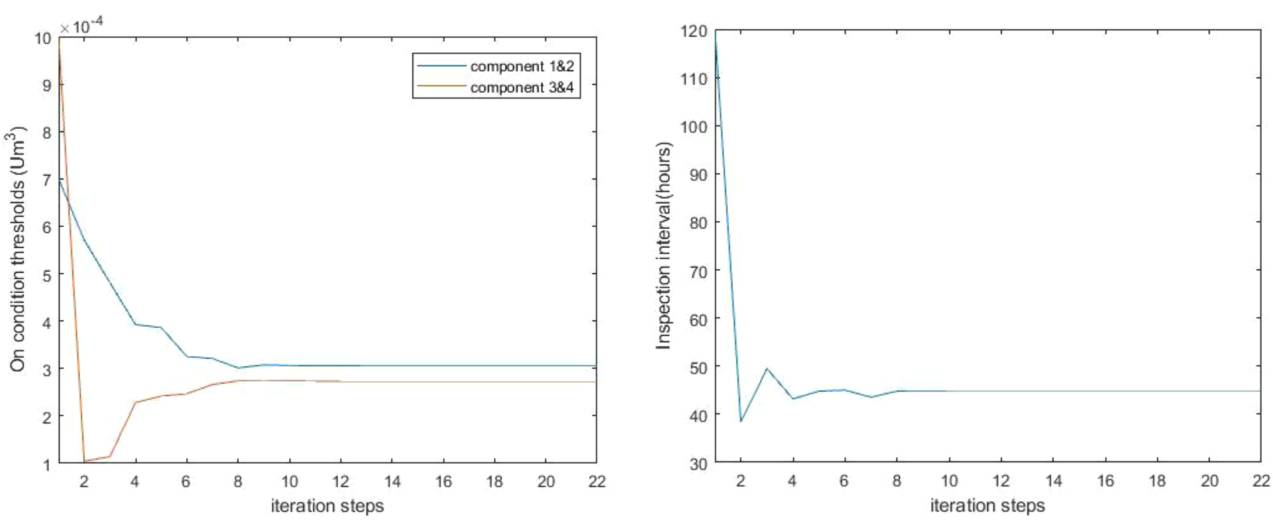

Fig. 5. Iteration process for these decision variables: inspection interval τ^*, and on-condition threshold for all components.

on-condition degradation threshold for components $H_3^{2*} = H_4^{2*} = 0.0002169$. Fig. 9 illustrates the iteration process of two decision variables: inspection interval and on-condition degradation threshold for component 3 and 4. Fig. 10 shows the iteration for our objective function, that is, the maintenance cost rate.

We can observe that inspection intervals for either component 1 and 2 or component 3 and 4 are greater than the inspection interval for the series system, which means we have to compromise to inspect the system more frequently if we have more components in the system. Since time to failure for all components are different, and series system reliability is less than the individual component reliability for all time, we should inspect system more often to increase probability of avoiding failure and relative high downtime cost.

5.2. Example 2

The second example is a series system with four different components. Table 3 presents the parameters of each component. Given the same cost $C_I = \$1$, $C_\rho = \$20{,}000$ and $C_R = \$100$, we find the minimum average long-run maintenance cost rate for the system as $\$1.8356 \times 10^2$, which is obtained at periodic inspection interval $\tau^* = 49.86$ h, and on-condition degradation threshold for components are $H_1^{2*} = 0.0002904$, $H_2^{2*} = 0.0002656$, $H_3^{2*} = 0.0007362$, $H_4^{2*} = 0.0012359$. As the results illustrate, component 4 has the highest optimal on-condition threshold that is very close to its failure threshold. This is because the degradation rate and shock load damage for component 4 is lower than other components which means its reliability is higher compared to the other three components. Accordingly, its optimal on-condition threshold is higher.

5.3. Example 3

To evaluate the maintenance model in Section 4.1.1, a series system is considered with four identical components (Case 1). The parameter values for reliability analysis of these four components is the same as component 1 and 2 in Table 2. By using Eq. (18) as the objective function for the optimization problem and considering $C_I = \$1$, $C_\rho = \$20{,}000$, $C_{Rf} = \$20$ and $C_{Rc} = \$20$, we can find the minimum average long-run maintenance cost rate for system is $\$1.597 \times 10^2$ and on-condition degradation threshold are $H_1^{2*} = H_2^{2*} = H_3^{2*} = H_4^{2*} = 0.0002125$, and the optimal inspection interval is found as $\tau^* = 68.14$

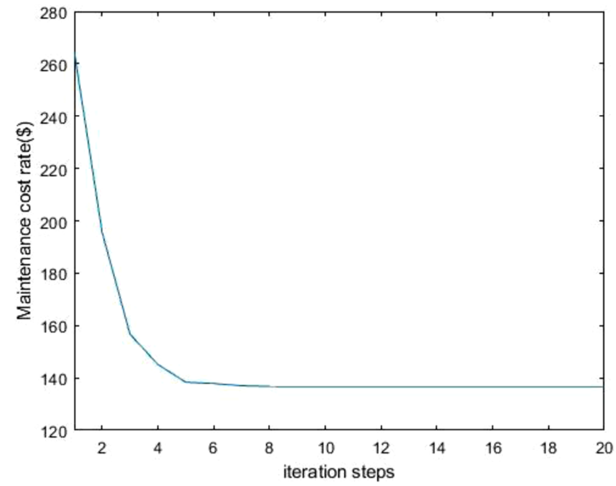

Fig. 8. Iteration process of maintenance cost rate for component 1 and 2.

6. Conclusions

In this paper, we propose a maintenance optimization model to determine on-condition failure thresholds and inspection intervals for systems with dependent degradation and dependent component failure times. For systems whose penalty cost due to downtime is high, this on-condition maintenance policy offers cost benefits over time-based preventive maintenance or replace-on-failure policies, because on-condition threshold increases the likelihood to detect system critical status and prevent failures. In this maintenance policy, the periodic inspection interval for the whole system and on-condition thresholds for all components are decision variables, and system maintenance cost rate is our optimization objective. The average long-run maintenance cost rate is evaluated and optimized. An interior point algorithm in MATLAB toolbox *fmincon* is used to solve the optimization problem. Numerical examples are provided and the results are discussed.

Acknowledgments

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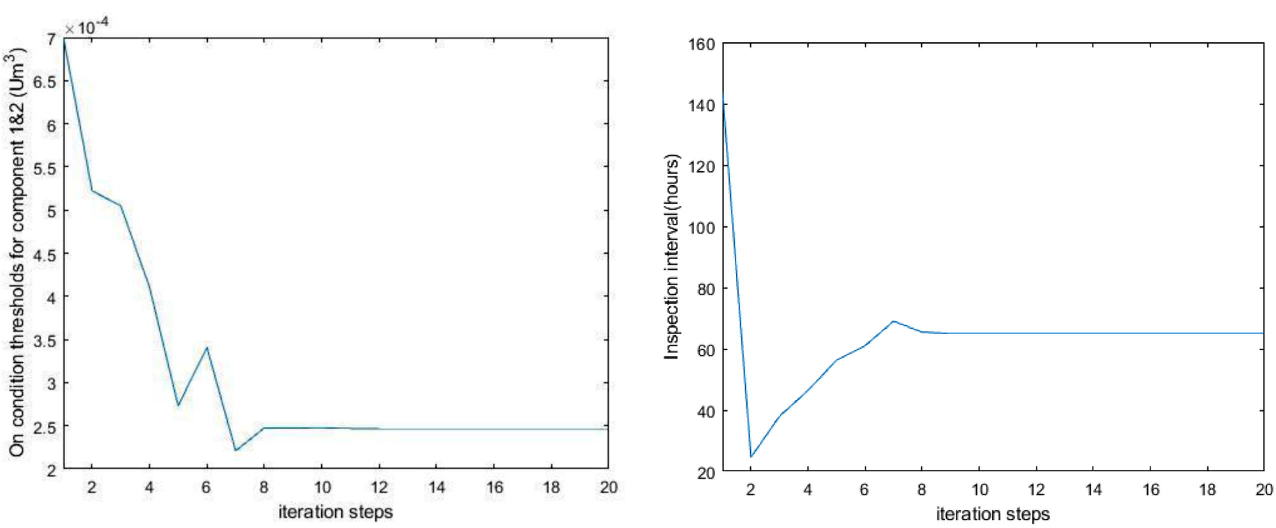

Fig. 7. Iteration process two decision variables: inspection interval τ^*, and on-condition threshold for component 1 and 2.

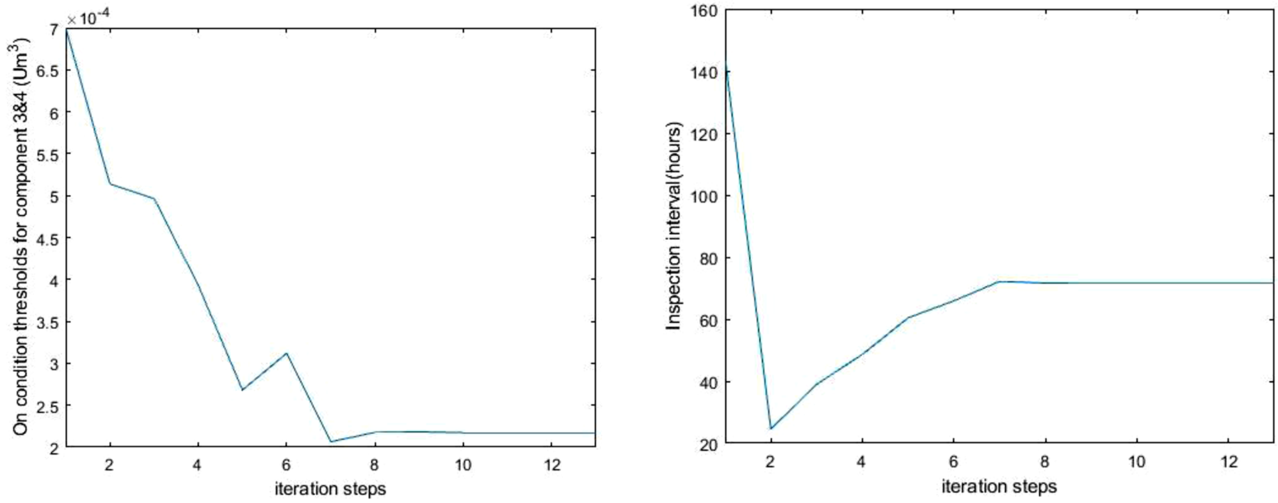

Fig. 9. Iteration process two decision variables: inspection interval τ^*, and on-condition threshold for component 3 and 4.

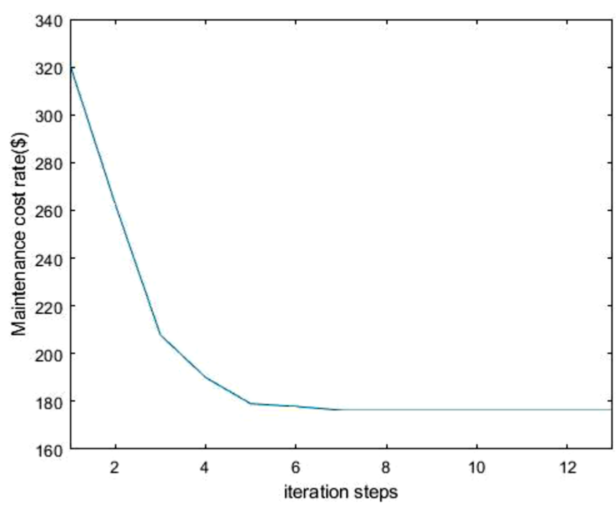

Fig. 10. Iteration process of maintenance cost rate for component 3 and 4.

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