NON-COMMUTATIVE FUNCTIONAL CALCULUS

By

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Abstract. We develop a functional calculus for d-tuples of non-commuting elements in a Banach algebra. The functions we apply are free analytic functions, that is, nc-functions that are bounded on certain polynomial polyhedra.

1 Introduction

1.1 Overview. The purpose of this note is to develop an approach to functional calculus and spectral theory for d-tuples of elements of a Banach algebra, with no assumption that the elements commute.

In [28], J. L. Taylor considered this problem for d-tuples in $\mathcal{L}(X)$, the bounded linear operators on a Banach space X. His idea was to start with the algebra $\mathbb{P}^d$ of free polynomials[1] in d variables over the complex numbers and consider what he called "satellite algebras", that is, algebras $\mathcal{A}$ that contain $\mathbb{P}^d$ with the property that every representation from $\mathbb{P}^d$ to $\mathcal{L}(X)$ that extends to a representation of $\mathcal{A}$ has a unique extension. As a representation of $\mathbb{P}^d$ is determined by choosing the images of the generators, i.e., choosing $T = (T^1, \dots, T^d) \in \mathcal{L}(X)^d$, the extension of the representation to $\mathcal{A}$, when it exists, constitutes an $\mathcal{A}$-functional calculus for T. The classes of satellite algebras that Taylor considered, which he called free analytic algebras, were intended to be non-commutative generalizations of the algebras $O(U)$, the algebra of holomorphic functions on a domain U in $\mathbb{C}^d$ (and indeed he proved in [28, Prop 3.3] that when $d = 1$, these constitute all the free analytic algebras). Taylor had already developed a successful $O(U)$ functional calculus for d-tuples T of *commuting* operators on X for which a certain spectrum (now called the Taylor spectrum) is contained in U; see [25, 26] for the original articles, and the article [21] by M. Putinar, which shows uniqueness. An excellent treatment is

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[1] We use the terms **free polynomial** and **non-commuting polynomial** in d variables interchangeably to mean an element of the algebra over the free monoid with d generators.

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in [6] by R. Curto. However, in the non-commutative case, Taylor's approach in [27, 28] using homological algebra was only partially successful.

What constitutes a successful theory? This is of course subjective, but we argue that it should contain some of the following ingredients, and one has to make trade-offs between them. The functional calculus should use algebras $\mathcal{A}$ that one knows something about–the better the algebras are understood, the more useful the theory. Secondly, the condition under which a given T has an $\mathcal{A}$-functional calculus should be related as simply as possible to the way in which T is presented. Thirdly, the more explicit the map that sends ϕ in $\mathcal{A}$ to $\phi(T)$ in $\mathcal{L}(X)$, the easier it is to use the theory. Finally, one should have a theory which, when restricted to the commutative case, agrees with the normal idea of a functional calculus.

The approach that we advocate in this note is to replace the universal set $\mathbb{C}^d$ with the nc-universe $\mathbb{M}^{[d]} := \bigcup_{n=1}^{\infty} \mathbb{M}_n^d$, where $\mathbb{M}_n$ denotes the set of n-by-n matrices over $\mathbb{C}$, with the induced operator norm from ℓ_n^2. In other words, we look at d-tuples of n-by-n matrices; but, instead of fixing n, we allow all values of n. We look at certain special open sets in $\mathbb{M}^{[d]}$.

Let δ be a matrix of free polynomials in d variables, and define

$$G_\delta = \{x \in \mathbb{M}^{[d]} : \|\delta(x)\| < 1\}. \tag{1.1}$$

The algebras we work with are algebras of the form $H^\infty(G_\delta)$. We define $H^\infty(G_\delta)$ in Definition 1.3. For now, think of it as some sort of non-commutative analogue of the set of bounded analytic functions defined on G_δ. We develop conditions for a d-tuple in $\mathcal{L}(X)$ to have an $H^\infty(G_\delta)$ functional calculus, in other words, for a particular $T \in \mathcal{L}(X)^d$ to have the property that there is a unique extension of the polynomial functional calculus to all of $H^\infty(G_\delta)$.

1.2 Non-commutative functions. Let $\mathbb{M}^{[d]} = \bigcup_{n=1}^{\infty} \mathbb{M}_n^d$. A **graded function** defined on a subset of $\mathbb{M}^{[d]}$ is a function ϕ with the property that $\phi(x) \in \mathbb{M}_n$ if $x \in \mathbb{M}_n^d$. If $x \in \mathbb{M}_n^d$ and $y \in \mathbb{M}_m^d$, we let $x \oplus y = (x^1 \oplus y^1, \ldots, x^d \oplus y^d) \in \mathbb{M}_{n+m}^d$; and, if $s \in \mathbb{M}_n$, we let sx (respectively, xs) denote the tuple $(sx^1, \ldots, sx^d)$ (respectively, $(x^1 s, \ldots, x^d s)$).

Definition 1.1. An **nc-function** is a graded function ϕ defined on a set $D \subseteq \mathbb{M}^{[d]}$ such that

i) if $x, y, x \oplus y \in D$, then $\phi(x \oplus y) = \phi(x) \oplus \phi(y)$;
ii) if $s \in \mathbb{M}_n$ is invertible and $x, s^{-1}xs \in D \cap \mathbb{M}_n^d$, then $\phi(s^{-1}xs) = s^{-1}\phi(x)s$.

Observe that every non-commutative polynomial is an nc-function on all of $\mathbb{M}^{[d]}$. Subject to being locally bounded with respect to an appropriate topology, nc-functions are holomorphic [2, 9, 11], and can be thought of as bearing an analogous relationship to non-commutative polynomials as holomorphic functions do to regular polynomials.

Nc-functions have been studied by, among others, G. Popescu [15, 16, 17, 18, 19, 20]; J. Ball, G. Groenewald, and T. Malakorn [5]; D. Alpay and D. Kaliuzhnyi-Verbovetzkyi [4]; and J.W. Helton, I. Klep and S. McCullough [8, 9] and Helton and McCullough [10]. We refer to the book [11] by Kaliuzhnyi-Verbovetskyi and V. Vinnikov on nc-functions.

We define matrix- or operator-valued nc-functions in the natural way, and use upper-case letters to denote them.

Definition 1.2. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be Hilbert spaces, and $D \subseteq \mathbb{M}^{[d]}$. We say a function F is an $\mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$**-valued nc-function** on D if

$$\begin{gathered}\forall_n\ \forall_{x \in D \cap \mathbb{M}_n^d}\ F(x) \in \mathcal{L}(\mathbb{C}^n \otimes \mathcal{K}_1, \mathbb{C}^n \otimes \mathcal{K}_2),\\ \forall_{x,y,x\oplus y \in D}\ F(x \oplus y) = F(x) \oplus F(y),\ \text{ and}\\ \forall_n\ \forall_{x \in D \cap \mathbb{M}_n^d}\ \forall_{s \in \mathbb{M}_n}\ s^{-1}xs \in D \Rightarrow F(s^{-1}xs) = (s^{-1} \otimes \mathrm{id}_{\mathcal{K}_1})F(x)(s \otimes \mathrm{id}_{\mathcal{K}_2}).\end{gathered}$$

A special case of G_δ in (1.1) is when $d = IJ$ and δ is the I-by-J rectangular matrix whose (i, j) entry is the $[(i-1)J + j]^{\text{th}}$ coordinate function. We give this the special symbol $\mathcal{E}$:

$$\mathcal{E}(x^1, \dots, x^{IJ}) = \begin{pmatrix} x^1 & x^2 & \cdots & x^J \\ x^{J+1} & x^{J+2} & \cdots & x^{2J} \\ \vdots & \vdots & \ddots & \vdots \\ x^{(I-1)J+1} & x^{(I-1)J+2} & \cdots & x^{IJ} \end{pmatrix}.$$

We denote the set $G_{\mathcal{E}}$ by $\mathbb{B}_{I\times J}$:

$$\mathbb{B}_{I\times J} = \bigcup_{n=1}^{\infty} \left\{ x = (x^1, \dots, x^{IJ}) \in \mathbb{M}_n^{IJ} : \|\mathcal{E}(x)\| < 1 \right\}.$$

Definition 1.3. We denote by $H^\infty(G_\delta)$ the set of bounded nc-functions on G_δ, and by $H^\infty_{\mathcal{L}(\mathcal{K}_1,\mathcal{K}_2)}(G_\delta)$ the set of bounded $\mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$-valued nc-functions on D.

Functions in these sets were studied in [1] and [2]. When $\mathcal{K}_1 = \mathcal{K}_2 = \mathbb{C}$, we identify $H^\infty(G_\delta)$ with $H^\infty_{\mathcal{L}(\mathcal{K}_1,\mathcal{K}_2)}(G_\delta)$. By a matrix-valued $H^\infty(G_\delta)$ function, we mean an element of some $H^\infty_{\mathcal{L}(\mathcal{K}_1,\mathcal{K}_2)}(G_\delta)$ with both $\mathcal{K}_1$ and $\mathcal{K}_2$ finite dimensional.

2 Hilbert tensor norms

We wish to define norms on matrices of elements of $\mathcal{L}(X)$. Were X restricted to be a Hilbert space $\mathcal{H}$, there would be a natural way to do this by thinking of an I-by-J matrix in $\mathcal{L}(\mathcal{H})$ as a linear map from the (Hilbert space) tensor product $\mathcal{H} \otimes \mathbb{C}^J$ to $\mathcal{H} \otimes \mathbb{C}^I$. We would like to do this in general.

Note first that although every Banach space can be embedded in an operator space (see e.g., [14, Chap. 3]), which in turn can be realized as a subset of some $\mathcal{L}(\mathcal{H})$, we would lose the multiplicative structure of $\mathcal{L}(X)$, so this approach does not work in general for our purpose.

Let us recall some definitions from the theory of tensor products on Banach spaces [7, 23]. A **reasonable cross norm** on the algebraic tensor product $X \otimes Y$ of two Banach spaces is a norm τ satisfying

i) for every $x \in X$, and $y \in Y$,

$$\tau(x \otimes y) = \|x\| \, \|y\|;$$

ii) for every $x^* \in X^*$ and $y^* \in Y^*$,

$$\|x^* \otimes y^*\|_{(X \otimes Y, \tau)^*} = \|x^*\| \, \|y^*\|.$$

A **uniform cross norm** is an assignment to each pair of Banach spaces X, Y a reasonable cross-norm on $X \otimes Y$ such that if $R : X_1 \to X_2$ and $S : Y_1 \to Y_2$ are bounded linear operators, then

$$\|R \otimes S\|_{X_1 \otimes Y_1 \to X_2 \otimes Y_2} \leq \|R\| \, \|S\|.$$

A uniform cross norm τ is **finitely generated** if, for every pair of Banach spaces X, Y and every $u \in X \otimes Y$,

$$\tau(u; X \otimes Y) = \inf\{\tau(u; M \otimes N),\ u \in M \otimes N,\ \dim M < \infty, \dim N < \infty\}.$$

A finitely generated uniform cross norm is called a **tensor norm**. Both the injective and projective tensor products are tensor norms [7, Propositions 1.2.1, 1.3.2], [23, Section 6.1]; there are also other tensor products [7, 23]. When τ is a reasonable cross norm, we write $X \otimes_\tau Y$ for the Banach space that is the completion of $X \otimes Y$ with respect to the norm given by τ.

Definition 2.1. Let X be a Banach space. A **Hilbert tensor norm** on X is an assignment of a reasonable cross norm h to $X \otimes \mathcal{K}$ for every Hilbert space $\mathcal{K}$ such that if $R : X \to X$ and $S : \mathcal{K}_1 \to \mathcal{K}_2$ are bounded linear operators and $\mathcal{K}_1$ and $\mathcal{K}_2$ are Hilbert spaces, then

$$\|R \otimes S\|_{\mathcal{L}(X \otimes_h \mathcal{K}_1,\ X \otimes_h \mathcal{K}_2)} \leq \|R\|_{\mathcal{L}(X)} \|S\|_{\mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)}. \tag{2.1}$$

Every uniform cross norm is a Hilbert tensor norm, but there are other Hilbert tensor norms. Most importantly, if X is a Hilbert space, then the Hilbert space tensor product is a Hilbert tensor norm.

In what follows, we denote by $\otimes$ without a subscript the Hilbert space tensor product of two Hilbert spaces, and by $\otimes_h$ a Hilbert tensor norm.

Let X be a Banach space, and let h be a Hilbert tensor norm on X. Let $R = (R_{ij})$ be an I-by-J matrix with entries in $\mathcal{L}(X)$. Thinking of R as a linear operator from $X \otimes \mathbb{C}^J$ to $X \otimes \mathbb{C}^I$, we use h to define a norm for R. Formally, let $E_{ij} : \mathbb{C}^J \to \mathbb{C}^I$ be the matrix with 1 in the (i, j) slot and 0 elsewhere. Let $\mathcal{K}$ be a Hilbert space. We define

$$\begin{aligned} &R_{h,\mathcal{K}} : X \otimes_h (\mathbb{C}^J \otimes \mathcal{K}) \to X \otimes_h (\mathbb{C}^I \otimes \mathcal{K}) \\ &R_{h,\mathcal{K}} = \sum_{i=1}^{I} \sum_{j=1}^{J} R_{ij} \otimes_h (E_{ij} \otimes \mathrm{id}_{\mathcal{K}}) \end{aligned} \tag{2.2}$$

Then we define

$$\|R\|_h = \sup\{ \|R_{h,\mathcal{K}}\| : \mathcal{K} \text{ is a Hilbert space}\}, \tag{2.3}$$

and (borrowing notation from the Irish use of a dot or *séimhiú* for an “h”)

$$\|R\|_{\bullet} = \inf\{ \|R\|_h \ : \ h \text{ is a Hilbert tensor norm}\}. \tag{2.4}$$

Let us record the following lemma for future use.

Lemma 2.2. *Let $R = (R_{ij})$ be an I-by-J matrix with entries in $\mathcal{L}(X)$. Then*

$$\|R\|_{\bullet} \geq \max_{i,j} \|R_{ij}\|_{\mathcal{L}(X)}. \tag{2.5}$$

Proof. Let B_i be the 1-by-I matrix with i^{th} entry id_X, and the other entries the 0 element of $\mathcal{L}(X)$. Let C_j be the J-by-1 column matrix with j^{th} entry id_X and the other entries 0. Let h be a Hilbert tensor norm on X. By (2.1), $\|B_i\|_h \leq 1$ and $\|C_j\|_h \leq 1$; and, since h is a reasonable cross norm, $\|B_i\|_h = \|C_j\|_h = 1$. Then

$$\|R_{ij}\|_{\mathcal{L}(X)} = \|B_i R C_j\|_{\mathcal{L}(X)} \leq \|R\|_{\mathcal{L}(X \otimes_h \mathbb{C}^J, X \otimes_h \mathbb{C}^I)} \leq \|R\|_h.$$

Since this holds for every h, (2.5) follows. □

3 Free analytic functions

Here are some of the primary results of [2]. When δ is an I-by-J rectangular matrix with entries in $\mathbb{P}^d$, and $x \in \mathbb{M}_n^d$, we think of $\delta(x)$ as an element of $\mathcal{L}(\mathbb{C}^n \otimes$

$\mathbb{C}^J, \mathbb{C}^n \otimes \mathbb{C}^I$). When $\mathcal{M}$ is a Hilbert space, we write $\delta_{\mathcal{M}}(x)$ for $\delta(x) \otimes \mathrm{id}_{\mathcal{M}}$ and think of it as an element of

$$\mathcal{L}(\mathbb{C}^n \otimes \mathcal{M}^J, \mathbb{C}^n \otimes \mathcal{M}^I) = \mathcal{L}(\mathbb{C}^n \otimes (\mathbb{C}^J \otimes \mathcal{M}), \mathbb{C}^n \otimes (\mathbb{C}^I \otimes \mathcal{M})).$$

Theorem 3.1. *Let δ be an I-by-J rectangular matrix of free polynomials, and G_δ be non-empty. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be finite-dimensional Hilbert spaces. A function Φ is in $H^\infty_{\mathcal{L}(\mathcal{K}_1,\mathcal{K}_2)}(G_\delta)$ if and only if there is a function F in $H^\infty_{\mathcal{L}(\mathcal{K}_1,\mathcal{K}_2)}(\mathbb{B}_{I\times J})$, with $\|F\| \leq \|\Phi\|$, such that $\Phi = F \circ \delta$.*

Theorem 3.2. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be finite-dimensional Hilbert spaces. If F is in $H^\infty_{\mathcal{L}(\mathcal{K}_1,\mathcal{K}_2)}(\mathbb{B}_{I\times J})$ and $\|F\| \leq 1$, there exist an auxiliary Hilbert space $\mathcal{M}$ and an isometry*

$$V = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathcal{K}_1 \oplus \mathcal{M}^{(I)} \to \mathcal{K}_2 \oplus \mathcal{M}^{(J)} \tag{3.1}$$

such that

$$F(x) = \mathrm{id}_{\mathbb{C}^n} \otimes A + (\mathrm{id}_{\mathbb{C}^n} \otimes B)\mathcal{E}_{\mathcal{M}}(x)[\mathrm{id}_{\mathbb{C}^n} \otimes \mathrm{id}_{\mathcal{M}^{(J)}} - (\mathrm{id}_{\mathbb{C}^n} \otimes D)\mathcal{E}_{\mathcal{M}}(x)]^{-1}(\mathrm{id}_{\mathbb{C}^n} \otimes C) \tag{3.2a}$$

for $x \in \mathbb{B}_{I\times J} \cap \mathbb{M}^d_n$. Consequently, F has the series expansion

$$F(x) = \mathrm{id}_{\mathbb{C}^n} \otimes A + \sum_{k=1}^{\infty} (\mathrm{id}_{\mathbb{C}^n} \otimes B)\mathcal{E}_{\mathcal{M}}(x)[(\mathrm{id}_{\mathbb{C}^n} \otimes D)\mathcal{E}_{\mathcal{M}}(x)]^{k-1}(\mathrm{id}_{\mathbb{C}^n} \otimes C), \tag{3.2b}$$

which is absolutely convergent on G_δ.

If we write ${}_{\mathbb{C}^n}A$ for $\mathrm{id}_{\mathbb{C}^n} \otimes A$, equations (3.2a) and (3.2b) have the more easily readable form

$$F = {}_{\mathbb{C}^n} A + {}_{\mathbb{C}^n}B \; \mathcal{E}_{\mathcal{M}} \, [I - {}_{\mathbb{C}^n}D \; \mathcal{E}_{\mathcal{M}}]^{-1} \; {}_{\mathbb{C}^n}C \tag{3.3a}$$

$$F(x) = {}_{\mathbb{C}^n} A + \sum_{k=1}^{\infty} {}_{\mathbb{C}^n}B \; \mathcal{E}_{\mathcal{M}}(x) \, [{}_{\mathbb{C}^n}D \; \mathcal{E}_{\mathcal{M}}(x)]^{k-1} \; {}_{\mathbb{C}^n}C. \tag{3.3b}$$

We call (3.2a) a **free realization** of F. The isometry V is not unique, but each term on the right-hand side of (3.3b) is a free matrix-valued polynomial, each of whose non-zero entries is homogeneous of degree k. Thus we can rewrite (3.3b) as

$$F(x) = \sum_{k=0}^{\infty} P_k(x) \tag{3.3c}$$

where each P_k is a homogeneous $\mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$-valued free polynomial, and which satisfies

$$\|P_k(x)\| \leq \|x\|^k \quad \text{for all } x \in \mathbb{B}_{I\times J}, \text{for all } k \geq 1. \tag{3.4}$$

Formulas (3.3a) or (3.3c) allow us to extend the domain of F from d-tuples of matrices to d-tuples in $\mathcal{L}(X)$.

Let X be a Banach space, with a Hilbert tensor norm h. Let $T = (T_{ij})$ be an I-by-J matrix of elements of $\mathcal{L}(X)$. If $\|T\|_h < 1$, where $\|T\|_h$ is defined by (2.3), we can replace $\mathcal{E}_{\mathcal{M}}(x)$ in (3.2a) with $\sum_{i,j} T_{ij} \otimes_h (E_{ij} \otimes \mathrm{id}_{\mathcal{M}})$ and get a bounded operator from $X \otimes_h \mathcal{K}_1$ to $X \otimes_h \mathcal{K}_2$, provided we tensor with id_X.

Definition 3.3. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be finite-dimensional Hilbert spaces, and let F be a matrix-valued nc-function on $\mathbb{B}_{I\times J}$, bounded in norm by 1, with a free realization given by (3.2a) and an expansion into homogeneous $\mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$-valued free polynomials given by (3.3c). Let $T = (T_{ij})_{i=1,j=1}^{i=I,j=J}$ be an I-by-J matrix of bounded operators on a Banach space X. Let h be a Hilbert tensor norm on X. Then we define $F_h^\sharp(T) \in \mathcal{L}(X \otimes_h \mathcal{K}_1, X \otimes_h \mathcal{K}_2)$ by

$$F_h^\sharp(T) = \sum_{k=0}^{\infty} P_k(T), \tag{3.5}$$

provided that the right-hand side converges absolutely.

We extend the definition of $F^\sharp$ to functions of norm greater than 1 by scaling.

The definition of $F_h^\sharp(T)$ might seem to depend on the choice of free realization, but in fact does not, since the polynomials P_k do not depend on the free realization. However, the definition of $F_h^\sharp(T)$ does depend subtly on the choice of h, as $F_h^\sharp(T)$ is a bounded linear map in $\mathcal{L}(X \otimes_h \mathcal{K}_1, X \otimes_h \mathcal{K}_2)$, but these are all the same if $\mathcal{K}_1 = \mathcal{K}_2 = \mathbb{C}$. We write $F^\sharp(T)$ for the $\dim(\mathcal{K}_2)$-by-$\dim(\mathcal{K}_1)$ matrix

$$F^\sharp(T) = \sum_{k=0}^{\infty} P_k(T), \tag{3.6}$$

which is a matrix of elements of $\mathcal{L}(X)$.

In the following theorem, ${}_XA = \mathrm{id}_X \otimes_h A$ and $T_{\mathcal{M}} = \sum_{i,j} T_{ij} \otimes_h (E_{ij} \otimes \mathrm{id}_{\mathcal{M}})$, where we assume that h is understood.

Theorem 3.4. *Let X be a Banach space and T an I-by-J matrix of elements of $\mathcal{L}(X)$. Suppose F is as in Theorem 3.2 and $\|F\| \leq 1$.*

(i) *If h is a Hilbert tensor norm on X and $\|T\|_h < 1$, then*

$$F_h^\sharp(T) = {}_XA + ({}_XB)T_{\mathcal{M}}[I - ({}_XD)T_{\mathcal{M}}]^{-1}\,{}_XC, \tag{3.7}$$

and

$$\|F_h^\sharp(T)\| \leq \frac{1}{1-\|T\|_h}. \tag{3.8}$$

(ii) *If* $\|T\|_\bullet < 1$*, then*

$$\|F^\sharp(T)\|_\bullet \leq \frac{1}{1-\|T\|_\bullet}. \tag{3.9}$$

(iii) *If* X *is a Hilbert space,* H *is the Hilbert space tensor product, and* $\|T\|_H < 1$*, then*

$$\|F_H^\sharp(T)\| \leq 1. \tag{3.10}$$

Proof. (i) Let $\|T\|_h = r < 1$. Let us temporarily denote by $G(T)$ the right-hand side of (3.7). By 2.1, we have $\|{}_X D\| \leq 1$; and, by (2.3), $\|T_{\mathcal{M}}\| < 1$. Therefore, the Neumann series $[I - ({}_X D)T_{\mathcal{M}}]^{-1} = \sum_{k=0}^{\infty}[{}_X D\ T_{\mathcal{M}}]^k$ converges to a bounded linear operator in $\mathcal{L}(X \otimes_h (\mathbb{C}^J \otimes \mathcal{M}))$ of norm at most $1/1-r$. Using 2.1 again, we conclude that

$$\|G(T)\|_{\mathcal{L}(X\otimes_h \mathcal{K}_1, X\otimes_h \mathcal{K}_2)} \leq 1 + \frac{r}{1-r} = \frac{1}{1-r}. \tag{3.11}$$

Replacing T by $e^{i\theta}T$ and integrating $G(e^{i\theta}T)$ against $e^{-ik\theta}$, we get, for $k \geq 1$,

$$\frac{1}{2\pi}\int_0^{2\pi} G(e^{i\theta}T)e^{-ik\theta}d\theta = {}_X B\ T_{\mathcal{M}}[{}_X D\ T_{\mathcal{M}}]^{k-1}{}_X C = P_k(T),$$

where P_k is the homogeneous polynomial from (3.3c). Therefore $G(T)$ is given by the absolutely convergent series $\sum_{k=0}^{\infty} P_k(T)$, and hence equals $F^\sharp(T)$, proving (3.7), and, by (3.11), also proving (3.8).

(ii) follows from the definition (2.4).

(iii) Using the fact that $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is an isometry, and equation (3.7), after some algebraic rearrangements we obtain

$$I - F_H^\sharp(T)^* F_H^\sharp(T) = {}_X C^*[I - T_{\mathcal{M}}^*\ {}_X D]^{-1}[I - T_{\mathcal{M}}^* T_{\mathcal{M}}][I - ({}_X D)T_{\mathcal{M}}]^{-1}{}_X C. \tag{3.12}$$

Since $\|T_{\mathcal{M}}\| < 1$, the right-hand side of (3.12) is positive, and so the left-hand side is positive, which means $\|F_H^\sharp(T)\| \leq 1$. □

Suppose $\Phi(x^1, \ldots, x^d)$ is in $H^\infty_{\mathcal{L}(\mathcal{K}_1,\mathcal{K}_2)}(G_\delta)$. By Theorem 3.1, we can write $\Phi = F \circ \delta$, for some F in $H^\infty_{\mathcal{L}(\mathcal{K}_1,\mathcal{K}_2)}(\mathbb{B}_{I\times J})$. Let $T = (T^1, \ldots, T^d) \in \mathcal{L}(X)^d$. Then $\delta(T)$ is an I-by-J matrix with entries in $\mathcal{L}(X)$. If $\|\delta(T)\|_\bullet < 1$, one would like to define $\Phi^\sharp$ by

$$\Phi^\sharp(T) = F^\sharp(\delta(T)). \tag{3.13}$$

But, since F is not unique, this raises questions about whether $\Phi^\sharp$ is well-defined. We address this question in Section 4.

4 Existence of a functional calculus

Throughout this section, X is a Banach space, and $T = (T^1, \dots, T^d)$ is a d-tuple of bounded linear operators on X.

Let δ be an I-by-J matrix of free polynomials in $\mathbb{P}^d$, and let

$$G_\delta = \bigcup_{n=1}^{\infty} \{x \in \mathbb{M}_n^d : \|\delta(x)\| < 1\}.$$

We say that G_δ is a **spectral set** for T if

$$\|p(T)\|_{\mathcal{L}(X)} \leq \sup_{x \in G_\delta} \|p(x)\| \quad \text{for all } p \in \mathbb{P}^d. \tag{4.1}$$

When P is an I-by-J matrix of polynomials, we consider P to be an $\mathcal{L}(\mathbb{C}^J, \mathbb{C}^I)$ valued nc-function. We denote by $\mathbb{M}(\mathbb{P}^d)$ the vector space of all (finite) matrices of free polynomials, with the norm of $P(x)$ given as the operator norm in $\mathcal{L}(\mathbb{C}^n \otimes \mathbb{C}^J, \mathbb{C}^n \otimes \mathbb{C}^I)$ where $x = (x_{ij})$ is a matrix with each $x_{ij} \in \mathbb{M}_n$. If (4.1) holds for all matrices of polynomials, i.e.,

$$\|P(T)\|_\bullet \leq \sup_{x \in G_\delta} \|P(x)\| \quad \text{for all } n, \text{ for all } P \in \mathbb{M}(\mathbb{P}^d), \tag{4.2}$$

we say that G_δ is a **complete spectral set** for T. If inequalities (4.1) or (4.2) hold with the right-hand side multiplied by a constant K, we say G_δ is a K**-spectral set** (respectively, **complete** K**-spectral set**) for T.

Theorem 4.1. *The following are equivalent.*

(i) *There exists $s < 1$ such that $G_{\delta/s}$ is a K-spectral set for T.*

(ii) *There exists $r < 1$ such that the map $\pi : f \circ (\frac{1}{r}\delta) \mapsto f^\sharp(\frac{1}{r}\delta(T))$ is a well-defined bounded homomorphism from $H^\infty(G_{\delta/r})$ to $\mathcal{L}(X)$ with $\|\pi\| \leq K$ that extends the polynomial functional calculus on $\mathbb{P}^d \cap H^\infty(G_{\delta/r})$.*

Moreover, if these conditions hold, then π is the unique extension of the evaluation homomorphism on the polynomials to a bounded homomorphism from $H^\infty(G_{\delta/r})$ to $\mathcal{L}(X)$.

Proof. (ii) $\Rightarrow$ (i). Let $s = r$. Let $q \in \mathbb{P}^d$. If $\|q\|_{G_{\delta/r}}$ is infinite, there is nothing to prove, so assume that $\|q\|_{G_{\delta/r}}$ is finite. By Theorem 3.1, there exists $f \in H^\infty(\mathbb{B}_{I\times J})$ such that $q = f \circ \frac{1}{r}\delta$ on $G_{\delta/r}$, and $\|f\| \leq \|q\|_{G_{\delta/r}}$. Since π is well-defined and extends the polynomial evaluation, $\pi(q) = q(T) = f^\sharp(\frac{1}{r}\delta(T))$. Therefore

$$\|q(T)\| \leq K \left\| f \circ \frac{1}{r}\delta \right\|_{G_{\delta/r}} \leq K \|q\|_{G_{\delta/r}}.$$

(i) $\Rightarrow$ (ii). Choose r in $(s,1)$. Let $\phi \in H^\infty(G_{\delta/r})$, and assume that there are functions f_1 and f_2 in $H^\infty(\mathbb{B}_{I\times J})$ such that $\phi(x) = f_1 \circ \frac{1}{r}\delta(x) = f_2 \circ \frac{1}{r}\delta(x)$ for all $x \in G_{\delta/r}$. Expand each f_l as in (3.3c) into a series of homogeneous polynomials, obtaining $f_l(x) = \sum_{k=0}^\infty p_k^l(x)$, $l = 1, 2$. By (3.4), we have $\|p_k^l(x)\| \le \|x\|^k$, so

$$\begin{aligned}\Big\| \sum_{k=0}^N p_k^1\Big(\frac{1}{r}\delta(x)\Big) - \sum_{k=0}^N p_k^2\Big(\frac{1}{r}\delta(x)\Big)\Big\|_{G_{\delta/s}} &= \Big\| \sum_{k=N+1}^\infty p_k^1\Big(\frac{1}{r}\delta(x)\Big) - \sum_{k=N+1}^\infty p_k^2\Big(\frac{1}{r}\delta(x)\Big)\Big\|_{G_{\delta/s}} \\ &\le 2\sum_{k=N+1}^\infty \Big(\frac{s}{r}\Big)^k = 2\frac{s^{N+1}}{r^N}\frac{1}{r-s}.\end{aligned}$$

Therefore

$$\Big\| \sum_{k=0}^N p_k^1\Big(\frac{1}{r}\delta(T)\Big) - \sum_{k=0}^N p_k^2\Big(\frac{1}{r}\delta(T)\Big)\Big\| \le 2K\frac{s^{N+1}}{r^N}\frac{1}{r-s}. \tag{4.3}$$

Both series $\sum_{k=0}^\infty p_k^1(\frac{1}{r}\delta(T))$ converge to the same limit, so $\pi(\phi)$ is well-defined.

Moreover, since $\sum_{k=0}^N p_k^1(\frac{1}{r}\delta(x))$ converges uniformly to $\phi(x)$ on $G_{\delta/s}$, we have

$$\limsup_{N\to\infty} \Big\| \sum_{k=0}^N p_k^1 \circ \frac{1}{r}\delta\Big\|_{G_{\delta/s}} \le \|\phi\|_{G_{\delta/s}} \le \|\phi\|_{G_{\delta/r}}.$$

Therefore

$$\|\pi(\phi)\|_{\mathcal{L}(X)} = \lim_{N\to\infty} \Big\| \sum_{k=0}^N p_k^1\Big(\frac{1}{r}\delta(T)\Big)\Big\| \le K\|\phi\|_{G_{\delta/r}}.$$

The fact that π is a homomorphism follows from it being well defined, as if $\phi = f \circ (\frac{1}{r}\delta)$ and $\psi = g \circ \frac{1}{r}\delta$, then $\phi\psi = (fg) \circ \frac{1}{r}\delta$.

Finally, to show that π extends the polynomial functional calculus, suppose q is a free polynomial in $H^\infty(G_{\delta/r})$, so $q = f \circ \frac{1}{r}\delta$. Expand $f(x) = \sum p_k(x)$ into its homogeneous parts. Then $\sum_{k=0}^N p_k(\frac{1}{r}\delta(x))$ converges uniformly to $q(x)$ on $G_{\delta/s}$. So, since $G_{\delta/s}$ is a K-spectral set for T,

$$\pi(q) = \lim_{N\to\infty} \sum_{k=0}^N p_k\Big(\frac{1}{r}\delta(T)\Big) = q(T).$$

This last argument shows that π is the unique continuous extension of the evaluation map on polynomials. $\square$

A similar result holds for complete K-spectral sets.

Theorem 4.2. *The following are equivalent.*

(i) *There exists $s < 1$ such that $G_{\delta/s}$ is a complete K-spectral set for T.*

(ii) *There exists $r < 1$ such that the map $\pi : F \circ \frac{1}{r}\delta \mapsto F^\sharp(\frac{1}{r}\delta(T))$ is a well-defined completely bounded homomorphism, satisfying*

$$\left\| F^\sharp\Big(\frac{1}{r}\delta(T)\Big)\right\|_\bullet \leq K \left\| F \circ \frac{1}{r}\delta \right\|_{G_{\delta/r}}$$

that extends the polynomial functional calculus on $\mathbb{P}^d \cap H^\infty(G_{\delta/r})$.

Moreover, if these conditions hold, then π is the unique extension of the evaluation homomorphism on the polynomials to a bounded homomorphism from $H^\infty(G_{\delta/r})$ to $\mathcal{L}(X)$.

The proof is very similar to the proof of Theorem 5.2. The only significant difference is that (4.3) becomes

$$\left\| \sum_{k=0}^{N} P_k^1\Big(\frac{1}{r}\delta(T)\Big) - \sum_{k=0}^{N} P_k^2\Big(\frac{1}{r}\delta(T)\Big)\right\|_\bullet \leq 2K \frac{s^{N+1}}{r^N} \frac{1}{r-s}.$$

We apply Lemma 2.2 to conclude that both series converge to the same limit matrix.

Definition 4.3. We say that T has a **contractive** (respectively, **completely contractive, bounded, completely bounded**) G_δ **functional calculus** if there exists $0 < r < 1$ such that $G_{\delta/r}$ is a spectral set (respectively, complete spectral set, K spectral set, complete K spectral set) for T.

Remark 4.4. Even in the case $d = 1$, $T \in \mathcal{L}(\mathcal{H})$, and $\delta(x) = x$, the question of when T has an $H^\infty(\mathbb{D})$ functional calculus becomes murky without the a priori requirement that $\|T\| < 1$. By von Neumann's inequality [29], T has a completely contractive G_δ functional calculus if $\|T\| < 1$. When $\|T\| = 1$, $p \mapsto p(T)$ extends contractively to $H^\infty(\mathbb{D})$ if T does not have a singular unitary summand [24, Theorem III.2.3]; but, to guarantee uniqueness, one usually imposes the standard extra assumption of continuity in the strong operator topology for functions that converge boundedly almost everywhere on the unit circle [24, Section III.2.2].

By Rota's theorem [22], if $\sigma(T) \subseteq (\mathbb{D})$, T is similar to an operator which has a completely contractive $H^\infty(\mathbb{D})$ functional calculus. Again, the situation becomes more delicate if $\sigma(T)$ is not required to lie in $\mathbb{D}$. By Paulsen's theorem [13], T has a completely bounded polynomial functional calculus if and only if T is similar to a contraction.

5 Complete spectral sets

For $\Phi \in H^\infty(G_\delta)$, and T a d-tuple with $\|\delta(T)\|_\bullet < 1$, one wants to define $\Phi^\sharp(T)$ as $F^\sharp(\delta(T))$. But what if there are two different functions, F and F_1, both in

$H^\infty(\mathbb{B}_{I\times J})$, satisfying $\Phi(x) = F \circ \delta(x) = F_1 \circ \delta(x)$ for all $x \in G_\delta$. How does one know that $F^\sharp(\delta(T)) = F_1^\sharp(\delta(T))$? If it doesn't, is there a "best" choice?

Definition 5.1. We say G_δ is **bounded** if there exists M such that $\|x\| \leq M$ for all $x \in G_\delta$. This is the same as requiring that $\mathbb{P}^d \subseteq H^\infty(G_\delta)$. A stronger condition is that the algebra generated by the δ_{ij} is all of $\mathbb{P}^d$.

We say that δ is **separating** if every coordinate function x^r, $1 \leq r \leq d$, is in the algebra generated by the functions $\{\delta_{ij} : 1 \leq i \leq I, 1 \leq j \leq J\}$.

Theorem 5.2. *Assume $\|\delta(T)\|_\bullet < 1$. Then there exists $r < 1$ such that $G_{\delta/r}$ is a complete K-spectral set for T if and only if there exists s in the interval $(\|\delta(T)\|_\bullet, 1)$ such that*

$$F^\sharp\left(\frac{1}{s}\delta(T)\right) = P(T) \tag{5.1}$$

for every matrix-valued $H^\infty(\mathbb{B}_{I\times J})$ function F and matrix P of free polynomials satisfying

$$F \circ \left(\frac{1}{s}\delta\right)(x) = P(x) \quad \text{for all } x \in G_{(1/s)\delta}. \tag{5.2}$$

If δ is separating, then it suffices to check the condition for the case $P = 0$.

Proof. ($\Rightarrow$) By Theorem 4.2, (5.2) implies (5.1) whenever $G_{\delta/r}$ is a complete K-spectral set.

($\Leftarrow$) Suppose $\|\delta(T)\|_\bullet = t < 1$ and that $s \in (t, 1)$ has the property that (5.2) implies (5.1). Let $r = s$. We show that $G_{\delta/r}$ is a complete K-spectral set for T.

Let P be a matrix of polynomials. We wish to show that

$$\|P(T)\|_\bullet \leq K \sup\{\|P(x)\| : x \in \mathbb{M}_n^d, \|\delta(x)\| < r\}. \tag{5.3}$$

Without loss of generality, assume that the right-hand side of (5.3) is finite. By Theorem 3.1, we can find F and a matrix-valued function on $H^\infty(\mathbb{B}_{I\times J})$ such that $F \circ \frac{1}{r}\delta = P$ on $G_{\delta/r}$ and $\|F\| \leq \sup\{\|P(x)\| : x \in \mathbb{M}_n^d, \|\delta(x)\| < r\}$. By (5.1), we have $P(T) = F^\sharp \circ \frac{1}{r}\delta(T)$, and so, by Theorem 3.4, (5.3) holds in general, with $K = r/r - t$.

Now, suppose that

$$F \circ \left(\frac{1}{s}\delta\right) = 0 \quad \text{on } G_{(1/s)\delta} \tag{5.4}$$

implies

$$F^\sharp\left(\frac{1}{s}\delta(T)\right) = 0. \tag{5.5}$$

We wish to show that (5.2) implies (5.1). Since δ is separating, there is a matrix H of free polynomials such that $H \circ \frac{1}{s}\delta(x) = P(x)$. Then

$$(F - H) \circ \frac{1}{s}\delta(x) = 0 \quad \text{for all } x \in G_{\delta/s};$$

so, by hypothesis,

$$F^\sharp\Big(\frac{1}{s}\delta(T)\Big) = H^\sharp\Big(\frac{1}{s}\delta(T)\Big);$$

and, since H is a polynomial,

$$H^\sharp\Big(\frac{1}{s}\delta(T)\Big) = H\Big(\frac{1}{s}\delta(T)\Big) = P(T),$$

as required. □

Remark 5.3. To just check the case $P = 0$, we don't need to know that δ is separating; we just need to know that if a polynomial is bounded on $G_{\delta/r}$, it is expressible as a polynomial in the δ_{ij}.

Here is a checkable condition.

Theorem 5.4. *Suppose $\delta(0) = 0$ and that $T \in \mathcal{L}(X)^d$ satisfies*

$$\sup_{0 \le r \le 1} \|\delta(rT)\|_\bullet < 1.$$

Then T has a completely bounded G_δ functional calculus.

Proof. By Theorem 5.2, suffices to prove that (5.2) implies (5.1). Hence, assume (5.2) holds, i.e., $F \circ \frac{1}{s}\delta(x) = \sum_{k=0}^\infty P_k(\frac{1}{s}\delta(x)) = P(x)$ for all $x \in G_{(1/s)\delta}$. By Theorem 3.2, $F \circ \frac{1}{s}\delta - P$ has a power series expansion in a ball centered at 0 in $\mathbb{M}^{[d]}$. Since $\delta(0) = 0$, for each $m \in \mathbb{N}$, the number of terms in $F \circ \frac{1}{s}\delta(x) - P(x)$ that are of degree m in x is finite.

Expanding $P_k(\frac{1}{s}\delta(x))$, one gets $O((IJ)^k)$ terms, so if $\|\frac{1}{s}\delta(x)\| < \frac{1}{IJ}$, then the series expansion for $\sum_{k=0}^\infty P_k(\frac{1}{s}\delta(x))$ converges absolutely. We conclude therefore, by rearranging the absolutely convergent series, that if R is a d-tuple in $\mathcal{L}(X)$ satisfying $\|\delta(R)\|_\bullet < s/IJ$, then

$$\sum_{k=0}^\infty P_k\Big(\frac{1}{s}\delta(R)\Big) = P(R). \tag{5.6}$$

Since $\delta(0) = 0$, we can apply (5.6) to ζT, for all sufficiently small ζ. Now we analytically continue to $\zeta = 1$ and conclude that (5.6) also holds for T. □

6 Hilbert spaces

For d-tuples in $\mathcal{L}(\mathcal{H})^d$, it is natural to work with the Hilbert space tensor product and the Hilbert space norm instead of the norm $\|\cdot\|_\bullet$. Throughout this section, we assume that $S = (S^1, \dots, S^d) \in \mathcal{L}(\mathcal{H})^d$ and that all norms (including those used to define spectral and complete spectral sets) are Hilbert space norms. Many of our earlier results go through with essentially the same proofs; but, since we can use (3.10) instead of (3.9), we get better constants.

The following theorem is a sample result, proved like Theorem 5.2.

Theorem 6.1. *Let $S \in \mathcal{L}(\mathcal{H})^d$. Then there exists $r < 1$ such that*

$$\left\| F^\sharp\left(\frac{1}{r}\delta(S)\right)\right\| \leq \sup\left\{\left\| F\left(\frac{1}{r}\delta(x)\right)\right\| : x \in G_{\delta/r}\right\}$$

if and only if

(i) $\|\delta(S)\| < 1$, *and*

(ii) $F^\sharp(\frac{1}{s}\delta(S)) = P(S)$ *for every matrix-valued $H^\infty(\mathbb{B}_{I\times J})$ function F satisfying $F \circ \frac{1}{s}\delta(x) = P(x)$ for all $x \in G_{(1/s)\delta}$.*

Example 7.3 below shows that condition (i) in Theorem 6.1 does not imply (ii).

For the remainder of this section, $\{e_n\}_{n=1}^\infty$ is a fixed orthonoormal basis of $\mathcal{H}$. We can naturally identify $\mathbb{M}_n$ with the operators on $\mathcal{H}$ that map $\vee_{k=1}^n\{e_k\}$ to itself and vanish on the orthogonal complement. In this way, G_δ is a subset of $G_\delta^\sharp$, where

$$G_\delta^\sharp := \{S \in \mathcal{L}(\mathcal{H})^d : \|\delta(S)\| < 1\}.$$

Since multiplication is sequentially continuous in the strong operator topology, to get a functional calculus it suffices to determine that $S \in G_\delta^\sharp$ is the strong operator topology limit of a sequence of d-tuples in $G_{\delta/r}$. For a set $A \in B(\mathcal{H})^d$, we denote by $\mathrm{scl}_{\mathrm{SOT}}(\mathrm{A})$ the set of tuples in $B(\mathcal{H})^d$ that are strong operator topology limits of sequences from A.

Theorem 6.2. *Suppose $S \in \bigcup_{0<r<1} \mathrm{scl}_{\mathrm{SOT}}(\mathrm{G}_{\frac{1}{\mathrm{r}}\delta})$. Then S has a completely contractive G_δ functional calculus.*

Proof. By hypothesis, there exists a sequence (x_k) in $G_{(1/t)\delta}$ that converges to S in the strong operator topology, for some $t < 1$. Therefore $\delta(x_k)$ converges to $\delta(S)$ in the strong operator topology, so $\|\delta(S)\| = r \leq t < 1$. Let $s \in (t, 1)$. By Theorem 6.1, it suffices to prove that (5.2) implies (5.1). As in the proof of Theorem 4.1, we can approximate F uniformly on $\overline{\frac{t}{s}\mathbb{B}_{I\times J}}$ with a sequence Q_N, the sum of the first N homogeneous polynomials. So for all $\varepsilon > 0$, there exists N_0 such that

$$\left\|Q_N\left(\frac{1}{s}\delta(x_k)\right) - F\left(\frac{1}{s}\delta(x_k)\right)\right\| < \varepsilon \quad \text{and} \tag{6.1}$$

$$\left\|Q_N\left(\frac{1}{s}\delta(S)\right) - F^\sharp\left(\frac{1}{s}\delta(S)\right)\right\| < \varepsilon \tag{6.2}$$

if $N \geq N_0$. As $F \circ \frac{1}{s}\delta = P$ on $G_{(1/s)\delta}$, inequality (6.1) means

$$\left\|Q_N\left(\frac{1}{s}\delta(x_k)\right) - P(x_k)\right\| < \varepsilon \quad \text{for all } N \geq N_0. \tag{6.3}$$

Since multiplication is sequentially strong operator continuous and Q_N is a matrix of polynomials,

$$\text{S.O.T.} \lim_{k\to\infty} \left[Q_N\left(\frac{1}{s}\delta(x_k)\right) - P(x_k)\right] = Q_N\left(\frac{1}{s}\delta(S)\right) - P(S). \tag{6.4}$$

The norm of a strong operator topology sequential limit is less than or equal to the limit of the norms, so by (6.3), we get from (6.4) that

$$\left\|Q_N\left(\frac{1}{s}\delta(S)\right) - P(S)\right\| \leq \varepsilon \quad \text{for all } N \geq 0. \tag{6.5}$$

Using (6.5) in (6.2), we conclude that $\|F^\sharp \circ \frac{1}{s}\delta(S) - P(S)\| \leq 2\varepsilon$. Since ε was arbitrary, we conclude that (5.1) holds, i.e., $F^\sharp \circ \frac{1}{s}\delta(S) = P(S)$. □

Corollary 6.3. *Suppose each δ_{ij} is the sum of a scalar and a homogeneous polynomial of degree* 1. *Then S has a completely contractive G_δ functional calculus if and only if $\|\delta(S)\| < 1$.*

Proof. Let Π_N be the projection from $\mathcal{H}$ onto $\vee_{j=1}^n\{e_j\}$, and suppose $\|\delta(S)\| \leq r$. Let $x_N = \Pi_N S \Pi_N$. Then x_N converges to S in the strong operator topology. Moreover, $\delta(x_n) = \Pi_N \otimes \mathrm{id}_{\mathbb{C}^I}\delta(S)\Pi_N \otimes \mathrm{id}_{\mathbb{C}^J}$, so $\|\delta(x_N)\| \leq \|\delta(S)\|$.□

For Hilbert spaces, replacing completely bounded with completely contractive changes things only up to similarity. This follows from the following theorem of V. Paulsen. :

Theorem 6.4 ([13])**.** *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, and let A be a unital subalgebra of $\mathcal{L}(\mathcal{K})$. Let $\rho : A \to \mathcal{L}(\mathcal{H})$ be a completely bounded homomorphism. Then there exists an invertible operator a on $\mathcal{H}$, with $\|a\|\|a^{-1}\| = \|\rho\|_{cb}$, such that $a^{-1}\rho(\cdot)a$ is a completely contractive homomorphism.*

As a consequence, we get the following theorem.

Theorem 6.5. *Let S be a d-tuple of operators on $\mathcal{H}$. Then S has a completely bounded G_δ functional calculus if and only if there exists an invertible operator a on $\mathcal{H}$ such that $R = a^{-1}Sa$ has a completely contractive G_δ functional calculus.*

Proof. Sufficiency is clear. For necessity, suppose $0 < r < 1$, and that the map $H^\infty(G_{\delta/r}) \in \Phi \mapsto \Phi(S)$ is a completely bounded map, with completely bounded norm K, that extends polynomial evaluations for polynomials that are bounded on $G_{\delta/r}$. Then in particular, $G_{\delta/r}$ is a complete K-spectral set for S. Let $\{x_k\}_{k=1}^\infty$ be a countable dense set in $G_{\delta/r}$, and let $X = \oplus x_k$. Then, for every matrix-valued function P,

$$\|P\|_{G_{\delta/r}} = \sup\{\|P(x)\| : x \in G_{\delta/r}\} = \|P(X)\|.$$

By hypothesis, the map $\rho : P(X) \mapsto P(S)$ is completely bounded, with $\|\rho\|_{cb} \leq K$. By Theorem 6.4, there exists $a \in \mathcal{L}(\mathcal{H})$ such that the map $P(X) \mapsto P(a^{-1}Sa)$ is completely contractive. Therefore $G_{\delta/r}$ is a complete spectral set for $a^{-1}Sa$. □

Remark 6.6. We don't need $\mathcal{K}$ to be separable, so we could have taken X to be the direct sum over all of $G_{\delta/r}$. Indeed, we could sum over all $G_{\delta/r}$ which are complete K spectral sets, and get one similarity that works for all of them.

7 Examples

Example 7.1. Let $\delta(x) = (x^1, \dots, x^d)$ be a 1-by-d matrix. Then $H^\infty(G_\delta)$ is the algebra of all bounded nc-functions defined on the row contractions. Functions on the row contractions were studied by Popescu in [15]. Note that a function in $H^\infty(G_\delta)$ need not have an absolutely convergent power series. When we expand $f \in H^\infty(G_\delta)$ as in (3.3b) or (3.3c), we get $f(x) = \sum_{k=0}^\infty p_k(x)$, where each p_k is a homogeneous polynomial of degree k, having d^k terms. Knowing merely that all the coefficients are bounded, one would need $\|x^j\| < 1/d$ for each j to conclude that the series converged absolutely. However we do know that $\sum_{k=0}^\infty \|p_k(x)\|$ converges for all x in G_δ.

By Theorem 5.4 or Theorem 3.4, if $T \in \mathcal{L}(X)^d$ satisfies $\|\delta(T)\|_\bullet < 1$, the functional calculus $F \mapsto F^\sharp(T)$ is a completely bounded homomorphism from $H^\infty(G_\delta)$ to $\mathcal{L}(X)$, with completely bounded norm at most $1/1 - \|\delta(T)\|_\bullet$. Every function in the multiplier algebra of the Drury-Arveson space can be extended without increase of norm to a function in $H^\infty(G_\delta)$ [1]; so. in particular, one can then apply these functions to T.

Example 7.2. This is similar to Example 7.1. This time, let δ be the d-by-d diagonal matrix with the coordinate functions on the diagonal. Then $H^\infty(G_\delta)$

is the set of free analytic functions defined on d-tuples x with $\max \|x^j\| < 1$. Again, every function that is bounded on the commuting contractive d-tuples can be extended to all of G_δ without increasing its norm [1].

Let $T = (T^1, \ldots, T^d) \in \mathcal{L}(X)^d$. We can calculate $\|\delta(T)\|_\bullet$ by observing that defining $\|(x_1, x_2, \ldots, x_m)\| = \sqrt{\sum \|x_j\|_X^2}$ gives a Hilbert tensor norm on $X \otimes \ell_m^2$. It follows that $\|\delta(T)\|_\bullet \leq \max \|T^j\|$; and, since this is easily seen to be a lower bound, we conclude

$$\|\delta(T)\|_\bullet = \max_{1 \leq j \leq d} \|T^j\|_{\mathcal{L}(X)}. \tag{7.1}$$

So, one gets an $H^\infty(G_\delta)$ functional calculus whenever (7.1) is less than 1. Let us reiterate that if $f \in H^\infty(G_\delta)$ and we expand it in a power series, we have no guarantee that the resulting series converges absolutely, even if the norm of each T^j is less than one; we need to group the terms as in (3.6).

Example 7.3. Here is an example of a polynomial that has a different norm on G_δ and $G_\delta^\sharp$. Consequently, $\mathrm{scl}_{\mathrm{SOT}}(\mathrm{G}_\delta) \neq \mathrm{G}_\delta^\sharp$, proving that condition (i) in Theorem 6.1 does not imply (ii).

Let $0 < \varepsilon < 0.2$. For ease of reading, we write (x, y) instead of (x^1, x^2) to denote coordinates. Let

$$\delta(x, y) = \begin{pmatrix} \frac{1}{\varepsilon}(yx - I) & 0 & 0 \\ 0 & \frac{1}{1+\varepsilon}x & 0 \\ 0 & 0 & \frac{1}{1+\varepsilon}y \end{pmatrix}.$$

Let $p(x) = xy - I$. We claim that

$$\|p\|_{G_\delta} \leq \varepsilon + 4\varepsilon^2, \tag{7.2a}$$

$$\|p\|_{G_\delta^\sharp} \geq 1. \tag{7.2b}$$

To prove the claim, let $x \in G_\delta$. Then $\|y\| < 1 + \varepsilon$; and, since yx is bounded below by $1 - \varepsilon$, we conclude that x is bounded below by $(1-\varepsilon)/(1+\varepsilon)$. By this, we mean that for all vectors v, $\|xv\| \geq \frac{1-\varepsilon}{1+\varepsilon}\|v\|$. So x has an inverse z, and $\|z\| \leq \frac{1+\varepsilon}{1-\varepsilon}$. Let $e = yx - I$. Then $\|e\| < \varepsilon$, and $y = z + ez$. Therefore $p(x) = xz + xez - I = xez$, so $\|p(x)\| \leq \frac{\varepsilon(1+\varepsilon)^2}{1-\varepsilon} \leq \varepsilon + 4\varepsilon^2$, which yields (7.2a).

To prove (7.2b), let $T = (S, S^*)$, where S is the unilateral shift. Then $\|\delta(S, S^*)\| = 1/1 + \varepsilon < 1$, and $\|p(S, S^*)\| = 1$, which yields (7.2b).

Example 7.4. This is an example of our non-commutative approach applied to a single matrix. Let $U = \{z \in \mathbb{C} : |z| < 1 \text{ and } |z - 1| < 1\}$. Let X be a finite-dimensional Banach space, and $T \in \mathcal{L}(X)$ be such that $\sigma(T) \subset U$. Let

$$\delta(x) = \begin{pmatrix} x & 0 \\ 0 & x - 1 \end{pmatrix}.$$

Then $H^\infty(G_\delta)$ is a space of analytic functions on U, but the norm is not the sup-norm; it is the larger norm given by

$$\|\phi\| := \sup\{\|\phi(S)\| : S \in \mathcal{L}(\mathcal{H}),\ \|\delta(S)\| < 1\}.$$

Indeed, by Theorem 3.1, the norm can obtained as

$$\|\phi\| = \inf\{\|g\|_{H^\infty(\mathbb{D}^2)} : g(z, z-1) = \phi(z)\ \forall\, z \in U\}.$$

(It suffices to calculate the norm of g in the commutative case, since it always has an extension of the same norm to the non-commutative space, by [1]).

By [3, Theorem 4.9], every function analytic on a neighborhood of $\overline{U}$ is in $H^\infty(G_\delta)$. Since X is finite dimensional, T is similar to an operator on a Hilbert space; and, by the results of Smith and Paulsen, this can be taken to have U as a complete spectral set.

Putting all this together, we can write T as $a^{-1}Sa$, where S is a Hilbert space operator with $\|\delta(S)\| < 1$. For any ϕ in $H^\infty(G_\delta)$, we find a g of minimal norm in $H^\infty(\mathbb{D}^2)$ such that $g(z, z-1) = \phi(z)$ for all $z \in U$. Finally, we get the estimate

$$\|\phi(T)\|_{\mathcal{L}(X)} \leq \|a^{-1}\|\,\|a\|\,\|g\|_{H^\infty(\mathbb{D}^2)}.$$

If $\max(\|T\|, \|T-1\|) = r < 1$, we have the estimate (which works even if X is infinite dimensional)

$$\|\phi(T)\|_{\mathcal{L}(X)} \leq \frac{1}{1-r}\|g\|_{H^\infty(\mathbb{D}^2)}.$$

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