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Contact structures, excisions and sutured monopoles

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Abstract

In this paper we explore some of the interplay between contact structures and sutured monopoles. We first study the behavior of contact elements defined by Baldwin and Sivek [1] under Floer excisions, which was introduced to sutured monopoles by Kronheimer and Mrowka [15]. Then we do some computations in sutured monopoles and in particular, we obtain an exact triangle for oriented Skein relation for knot monopole Floer homology and derive the connected sum formula for sutured monopoles. A similar argument also leads to the connected sum formula for sutured instatons and framed instanton Floer homology.

Contents

1 Introduction

The sutured monopole and instanton Floer homology was introduced by Kronheimer and Mrowka in [15]. They were designed to be the counter parts of Juhász's sutured Heegaard Floer homology [11] in monopole and instanton settings respectively.

It has been shown by works of Kutluhan, Lee and Taubes [17] and subsequent papers, Baldwin and Sivek [3] that the sutured monopole Floer homology and sutured (Heegaard) Floer homology are isomorphic to each other. So if we simply aim at computing monopole Floer homologies, then we could make use of the isomorphism and look at the Heegaard Floer side, which is known to be more computable. However, the computations and constructions in this paper will be restricted to be with in the monopole setting and will not make use the isomorphism to Heegaard Floer theories.

This is not only for fun but also for the following three reasons. The first is that we would like to develop a theory within the monopole settings so that it might be possible some day, when equipped with enough tools, we could derive a new proof of the isomorphism between monopole and Heegaard Floer theory, by looking at basic building blocks for the two theories. The second is that though the isomorphism between the two Floer theories have been proven, the morphisms within each theory have not been identified. The third reason is that the constructions in sutured monopoles would also shed some light on sutured instantons, as these two objects are constructed in a similar way.

A sutured manifold is a compact oriented 3-manifold M whose boundary is divided by an embedded 1-submanifold γ, which is called the suture, into two parts of the same Euler characteristics. To define the monopole Floer homology, we construct a closed 3-manifold Y together with a closed surface $R \subset Y$ out of (M, γ), by first gluing $T \times [-1, 1]$ to M along the suture and then identifying the remaining boundaries. Here T is a choice of auxiliary surface so that ∂T has the same number of components as γ. The pair (Y, R) is called a closure. We can also choose a non-separating curve $\eta \subset R$ for the use of local coefficients. Then we define

$$\mathrm{SHM}(M, \gamma) := HM(Y|R; \Gamma_\eta) := \bigoplus_{c_1(\mathfrak{s})[R]=2g(R)-2} \widetilde{HM}(Y, \mathfrak{s}; \Gamma_\eta).$$

If (M, γ) is equipped with a contact structure ξ so that ∂M is convex and γ is the dividing set, then Baldwin and Sivek in [1] found a way to

extend ξ to a contact structure $\bar{\xi}$ on all of Y. Then by work of Kronheimer, Mrowka, Ozsváth and Szabó [16], one can define a contact invariant

$$\phi_\xi = \phi_{\bar{\xi}} \in HM(-Y|-R;\Gamma_{-\eta}) = \mathrm{SHM}(-M,-\gamma).$$

for sutured monopoles.

Contact structures and contact elements have played very important roles in sutured (Heegaard) Floer theory. The construction of gluing maps and cobordism maps both need contact structures (see [12, 9]). The reconstruction of HFK^- using direct limit systems of sutured manifolds by Etnyre, Vela-Vick and Zarev in [6] also involves contact structures in an essential way. Besides, in [13] Kálmán and Mathews provided some examples so that the generators of the sutured (Heegaard) Floer homologies of some family of balanced sutured manifolds are in one-to-one correspondence to the tight contact structures on those manifolds.

In this paper we will explore more about the interplay between contact structures and sutured monopoles. We have two main topics.

1.1 Contact element through Floer excisions

We will first look at contact elements and Floer excisions. In [15], Kronheimer and Mrowka first uses connected auxiliary surfaces to get closures of a balanced sutured manifold but then disconnected surfaces were used to prove some important results. The isomorphism between using connected and disconnected surfaces were constructed through Floer excision maps. Later Baldwin and Sivek constructed the contact invariants by also using connected auxiliary surface. So it would be interesting to ask whether the construction can be extended to the case of disconnected auxiliary surfaces and how those contact elements are related by Floer excisions. The answer to these questions may help us understand more about trace, co-trace cobordisms and the behavior of contact elements under suitable sutured manifold decompositions.

To be more specific, suppose for $i = 1, 2$, (M_i, γ_i) is a balanced sutured manifold and T_i is a connected auxiliary surface which leads to a closure (Y_i, R_i) of (M_i, γ_i). If we cut T_1 and T_2 along non-separating simple closed curves and re-glue to get a connected surface T, we can use T to close up $(M_1 \sqcup M_2, \gamma_1 \cup \gamma_2)$ and get a large connected closure (Y, R). In [14], Kronheimer and Mrowka constructed a cobordism W from $(Y_1 \sqcup Y_2)$ to Y

and after choosing some suitable local coefficients this cobordism induces a map

$$F = HM(-W) : HM(-(Y_1 \sqcup Y_2)|-(R_1 \cup R_2); \Gamma_{-(\eta_1 \cup \eta_2)}) \to HM(-Y|-R; \Gamma_{-\eta}).$$

Suppose further that for $i = 1, 2$, (M_i, γ_i) is equipped with a contact structure ξ_i so that ∂M_i is convex and γ_i is the dividing set. Then as done by Baldwin and Sivek [1], there are corresponding contact structures $\bar{\xi}_1$, $\bar{\xi}_2$ and $\bar{\xi}$ on Y_1, Y_2 and Y respectively.

In this paper, we prove the following.

Theorem 1.1. *Under the above settings, the map F preserves the contact elements up to multiplication by a unit. That is,*

$$F(\phi_{\bar{\xi}_1 \cup \bar{\xi}_2}) \doteq \phi_{\bar{\xi}},$$

where $\doteq$ means equal up to multiplication by a unit.

However, the result in the above theorem is not fully satisfactory. Suppose (M, γ) is a large connected sutured manifold so that

$$\partial M \cong \partial M_1 \sqcup \partial M_2,$$

and under the isomorphism, γ is identified with $\gamma_1 \cup \gamma_2$. Then we can still use $T_1 \sqcup T_2$ or T to close up (M, γ). The two resulting closures are still related by a Floer excision and still there is a map between the corresponding monopole Floer homologies. The proof of the theorem 1.1 in this paper, however, does not apply to the case when (M, γ) is connected. Though we still make the following conjecture:

Conjecture 1.2. *Theorem 1.1 still holds if we replace $(M_1 \sqcup M_2, \gamma_1 \cup \gamma_2)$ by a connected (M, γ) described as above.*

Some evidence or idea of the proof lies in [23] by Niederkruger and Wendl. In the paper they defined an operation called slicing which coincides with the procedure of doing Floer excision and an operation of attaching torus 1-handles which coincide with the cobordism W constructed by Kronheimer and Mrowka in [15] for Floer excisions. Hence the cobordism W is equipped with a weak symplectic structure. Compared with the previous results by Hutchings and Taubes [10] and by Echeverria [5] that exact symplectic or strong symplectic cobordisms preserve contact elements, we make the following conjecture.

Conjecture 1.3. *Suppose (W,ω) is a weakly symplectic cobordism from (Y_1,ξ_1) and (Y_2,ξ_2). Suppose that for $i=1,2$, there is a 1-cycles $\eta_i\subset Y_i$, so that η_i is dual to $\omega|_{Y_i}$. Suppose $\nu\subset W$ is a 2-cycle so that $\partial\nu=-\eta_1\cup\eta_2$, then the map*

$$\widetilde{HM}(W,\mathfrak{s}_\omega;\Gamma_\nu):\widetilde{HM}(-Y_2,\mathfrak{s}_{\xi_2};\Gamma_{-\eta_2})\to\widetilde{HM}(-Y_1,\mathfrak{s}_{\xi_1};\Gamma_{-\eta_1})$$

will preserve the contact elements:

$$\widetilde{HM}(W,\mathfrak{s}_\omega;\Gamma_\nu)(\phi_{\xi_2})\doteq\phi_{\xi_1}.$$

The confirmation of conjecture 1.3 would possibly provide a proof of conjecture 1.2.

1.2 Connected sum formula

The second topic is motivated by the connected sum formula for sutured monopoles. In particular, we prove the following theorem.

Theorem 1.4. *When using $\mathbb{Z}_2$ coefficients, suppose (M_1,γ_1) and (M_2,γ_2) are two balanced sutured manifolds, then we have*

$$\mathrm{SHM}(M_1\sharp M_2,\gamma_1\cup\gamma_2)\cong\mathrm{SHM}(M_1,\gamma_1)\otimes\mathrm{SHM}(M_2,\gamma_2)\otimes(\mathbb{Z}_2)^2.$$

Furthermore, the same result holds for sutured instantons with $\mathbb{C}$ coefficients. As a consequences we also get a connected sum formula for the framed instanton Floer homologies of two closed manifolds Y_1 and Y_2:

$$I^\sharp(Y_1\sharp Y_2)\cong I^\sharp(Y_1)\otimes I^\sharp(Y_2).$$

The connected sum formula relies on the balanced sutured manifold $(S^3(2),\delta^2)$, where $S^3(2)$ is the sutured manifold obtained from S^3 by digging out two disjoint 3-balls and pick one simple closed curve on each spherical boundary as the suture. The computation for sutured instantons was done by Baldwin and Sivek in [2] using an oriented Skein relation for sutured instantons. In this paper, we follow the idea of Kronheimer and Mrowka [14] and prove the same result for sutured monopoles.

Theorem 1.5. *When using $\mathbb{Z}_2$ coefficients, there is an exact triangle associated to the oriented Skein relation for knot monopole Floer homology.*

In the proof of the above theorem, another important sutured manifold (V, λ^4) arises. Here V is a framed solid torus and the suture λ^4 consists of four longitudes on ∂V. The computation relies ultimately on the surgery exact triangle for monopole Floer homology, which was proved by Kronheimer, Mrowka, Ozsváth and Szabó [16]. However their proof only applied to $\mathbb{Z}_2$ coefficients so we need also work with that coefficients. The usage of $\mathbb{Z}_2$ coefficient is guaranteed by Sivek [24].

Along the computation, there is an interesting observation. In order to bound the rank of some relative balanced sutured manifold (V, λ^6), which is a solid torus with six longitudes as the suture, we need to decompose it along an oriented meridian disk D. However, we can also decompose (V, λ^6) along $-D$ and the spin^c structures associated to decomposing along D and $-D$, as discussed in [15], are different: thus we know that $\mathrm{SHM}(V, \lambda^6)$ has rank at least 2. This observation is related to a similar construction done by Baldwin and Sivek in [4], where they used a surface with only one boundary component and having two transverse intersections with the suture to define a grading for sutured monopole Floer homologies. The argument above for (V, γ^6) is a naive version of generalization of their work and a more systematic treatment would be helpful for further researches.

One direct result using this sort of grading is the following.

Theorem 1.6. *Let (V, λ^{2n}) be a solid torus with $2n$ longitudes as sutures. We will use $\mathbb{Q}$ coefficients, and suppose $n = 2k + 1$ is odd. Then there is a grading induced by a meridian disk of V and under this grading the sutured monopole Floer homology of (V, λ^{2n}) can be described as follows:*

$$\mathrm{SHM}(V, \gamma^{2n}, i) \cong \begin{cases} H_{i+k}(T^{n-1}), & -k \leqslant i < k. \\ 0 & i > k \text{ or } i < -k. \end{cases}$$

The conclusion also holds for sutured instantons with $\mathbb{C}$ coefficients.

Remark 1.7. It is commented by Yi, Xie that for sutured instantons and for odd n, the representation variety of a suitable closure of (V, γ^{2n}) is precisely the $(n-1)$-dimensional torus T^{n-1}.

As we will explain more in subsection 4.1, the following question might be interesting:

Question 1.8. *Is the homology group (or module) $\mathrm{SHM}(V, \gamma^{2n})$ fully generated by the contact elements of some tight contact structures on (V, γ^{2n})?*

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2 Prelimilaries

2.1 Sutured monopole Floer homology

The definitions and notations shall be in consistent with the author's previous paper [20]. For more details readers are referred to that paper. We shall start with the definition of sutured manifolds.

Definition 2.1. Suppose M is a compact oriented 3-manifold with boundary. Suppose γ is a collection of oriented simple closed curves on ∂M so that

(1). M has no closed components and any component of ∂M contains at least one component of γ.

(2). The surface $\partial M \backslash \gamma$ can be oriented so that the induced boundary orientation is the same as the chosen one on γ. The unique orientation satisfying this requirement is called the *canonical orientation*.

(3). Let $A(\gamma) = \gamma \times [-1, 1] \subset \partial M$ be an annular neighborhood of $\gamma \subset \partial M$, and let $R(\gamma) = \partial M \backslash \text{int}(A(\gamma))$. Let $R_+(\gamma)$ be the part of $R(\gamma)$ so that the canonical orientation coincide with the boundary orientation induced by M, and $R_-(\gamma) = R(\gamma) \backslash R_+(\gamma)$. Then we shall require that

$$\chi(R_+(\gamma)) = \chi(R_-(\gamma)).$$

The pair (M, γ) is called a *balanced sutured manifold*.

To define the monopole Floer homology, we need to construct a closed 3-manifold out of the sutured data. Suppose (M, γ) is a balanced sutured manifold. Let T be a connected surface so that

(1). There exists an orientation reversing diffeomorphism $f : \partial T \to \gamma$.

(2). T contains a simple closed curve c, so that c represents a non-trivial class in $H_1(T)$.

(3). Let

$$\widetilde{M} = M \underset{f \times id}{\cup} T \times [-1, 1]$$

and suppose the two oriented boundary components of $\widetilde{M}$ are

$$\partial\widetilde{M} = R_+ \cup R_-.$$

We know that $c \times \{\pm 1\} \subset R_\pm$ is non-separating by assumption. Let

$$h : R_+ \to R_-$$

be an orientation preserving diffeomorphism so that

$$h(c \times \{1\}) = c \times \{-1\}.$$

We can use h to glue the two boundary components of $\widetilde{M}$ together. Alternatively we can define

$$Y = \widetilde{M} \underset{id\times\{-1\}\cup h\times\{1\}}{\cup} R_+ \times [-1, 1].$$

Let $R = R_+ \times \{0\} \subset Y$.

Definition 2.2. The pair (Y, R) is called a *closure* of (M, γ). The choices T, f, c, h are called the *auxiliary data*. In particular, T is called an *auxiliary surface*. Pick η to be a non-separating simple closed curve on R, and define

$$\mathfrak{S}(Y|R) = \{\mathfrak{s} \text{ spin}^c \text{ structures } Y|\ c_1(\mathfrak{s})[R] = 2g(R) - 2.\}$$

Then define the sutured monopole Floer homology of (M, γ) to be

$$\mathrm{SHM}(M, \gamma) = \bigoplus_{\mathfrak{s}\in\mathfrak{S}(Y|R)} \widetilde{HM}(Y, \mathfrak{s}; \Gamma_\eta).$$

Remark 2.3. The curve η may be absent, when it is convenient to use $\mathbb{Z}$ or $\mathbb{Z}_2$ coefficients. In general when η do exists, we will use the Novikov ring $\mathcal{R}$ or other suitable rings to construct local coefficient system. For the precise meaning of 'suitable', readers are referred to [15] and [24].

The well-definedness of sutured monopole Floer homology is proved by Kronheimer and Mrowka [15].

Theorem 2.4. *The isomorphism class of* $\mathrm{SHM}(M, \gamma)$ *is independent of all the auxiliary data and the curve* η *made in definition 2.2.*

Floer excisions will be used repeatedly in the paper so we would like to present it here. Floer excisions in sutured monopoles were originally introduced by Kronheimer and Mrowka [15].

Suppose Y_1, Y_2 are two closed oriented 3-manifolds. Suppose for $i = 1, 2$, there is an oriented closed surface $R_i \subset Y_i$ and an oriented torus $T_i \subset Y_i$, so that $R_i \cap T_i = c_i$. Here c_i is a simple closed curve such that there exits another simple closed curve $\eta_i \subset R_i$, intersecting c_i transversely once. We can cut Y_i along T_i to get a manifold with boundary $\widetilde{Y}_i$, so that

$$\widetilde{Y}_i = T_{i,+} \cup T_{i,-}.$$

Here $T_{i,\pm}$ are parallel copies of T_i. Let $c_{i,\pm} \subset T_{i,\pm}$ be parallel copies of c_i. Pick an orientation preserving diffeomorphism

$$h : T_{1,+} \to T_{2,-},$$

so that

$$h(c_{1_+}) = c_{2,-},\ h(\eta_1 \cap c_{1,+}) = \eta_2 \cap c_{2,-}.$$

Then we can use h to glue $\widetilde{Y}_1$ and $\widetilde{Y}_2$ together to get a large oriented connected 3-manifold Y with an oriented connected surface R obtained by gluing R_1 and R_2 together. Also η_1 and η_2 are glued together to result in a simple closed curve $\eta \subset R$.

Now we construct a cobordism from $Y_1 \sqcup Y_2$ to Y as follows. Let U be the surface as depicted in figure 1 and let $\mu_1, \mu_2, \mu_3, \mu_4$ be the four vertical arcs as part of the boundary of U. Suppose all μ_i are identified with the interval $[0, 1]$.

Let

$$W = (\widetilde{Y}_1 \times [0,1]) \underset{\phi}{\cup} (T_{1,+} \times U) \underset{\psi}{\cup} (\widetilde{Y}_2 \times [0,1])$$

be the 4-manifold obtained by gluing three pieces together. Here

$$\phi = (id \cup id) \times id : (T_{1,+} \cup T_{1,-}) \times [0,1] \to T_{1,+} \times (u_1 \cup u_2),$$

and

$$\psi = (h \cup h) \times id : T_{1,+} \times (u_3 \cup u_4) \to (T_{2,+} \cup T_{2,-}) \times [0,1]$$

are the gluing maps. Let $F_W = R_1 \cup R_2 \cup R$ and let

$$\nu = ((\eta_1 \cap \widetilde{Y}_1) \times [0,1]) \underset{\phi}{\cup} ((\eta_1 \cap c_{1,+}) \times U) \underset{\psi}{\cup} ((\eta_2 \cap \widetilde{Y}_2) \times [0,1]).$$

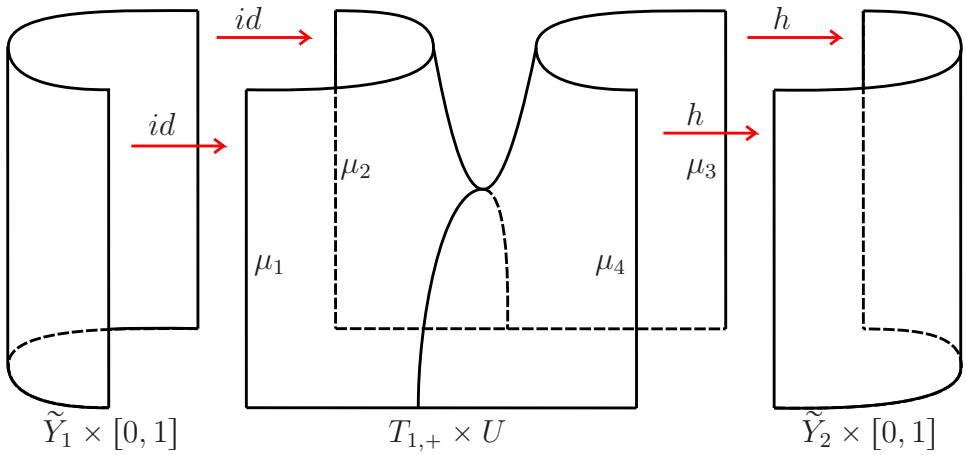


Figure 1: Gluing three parts together to get W. The middle part is $T_{1,+} \times U$, while the $T_{1,+}$ directions shrink to a point in the figure.

See figure 1. Then we can define a map

$$F = \widetilde{HM}(W|F_W;\Gamma_\nu) : \widetilde{HM}(Y_1 \sqcup Y_2|R_1 \cup R_2;\Gamma_{\eta_1\cup\eta_2}) \to \widetilde{HM}(Y|R;\Gamma_\eta).$$

In [15] Kronheimer and Mrowka proved the following theorem.

Theorem 2.5. *The map F is an isomorphism.*

Remark 2.6. In the rest of the paper, when the choices of the surface and the local coefficients are clear in the contents, we will omit them from the notation, and simply write

$$\widetilde{HM}(W) : \widetilde{HM}(Y_1 \sqcup Y_2|R_1 \cup R_2) \to \widetilde{HM}(Y|R).$$

2.2 Arc configurations and contact elements

In this subsection we will review Baldwin and Sivek's work in [1] on constructing the contact elements for balanced sutured manifolds.

Definition 2.7. Suppose (M,γ) is a balanced sutured manifold. A contact structure ξ on M is said to be *compatible* if ∂M is convex and γ is (isotopic to) the dividing set.

Definition 2.8. Suppose T is a connected compact oriented surface with boundary. An *arc configuration* $\mathcal{A}$ on T consists of the following data.

(1). A finite collection of pairwise disjoint simple closed curves $\{c_1, ...c_m\}$ so that for any j, $[c_j] \neq 0 \in H_1(T)$.

(2). A finite collection of pairwise disjoint simple arcs $\{a_1, ..., a_n\}$ so that

(a). For any i, j, $\text{int}(a_i) \cap c_j = \varnothing$.

(b). For each i, one end point of a_i lies on ∂T and the other on some c_j.

(c). Each boundary component of T has a non-trivial intersection with some a_i.

See figure 2. It is called *reduced* if there is only one simple closed curve.

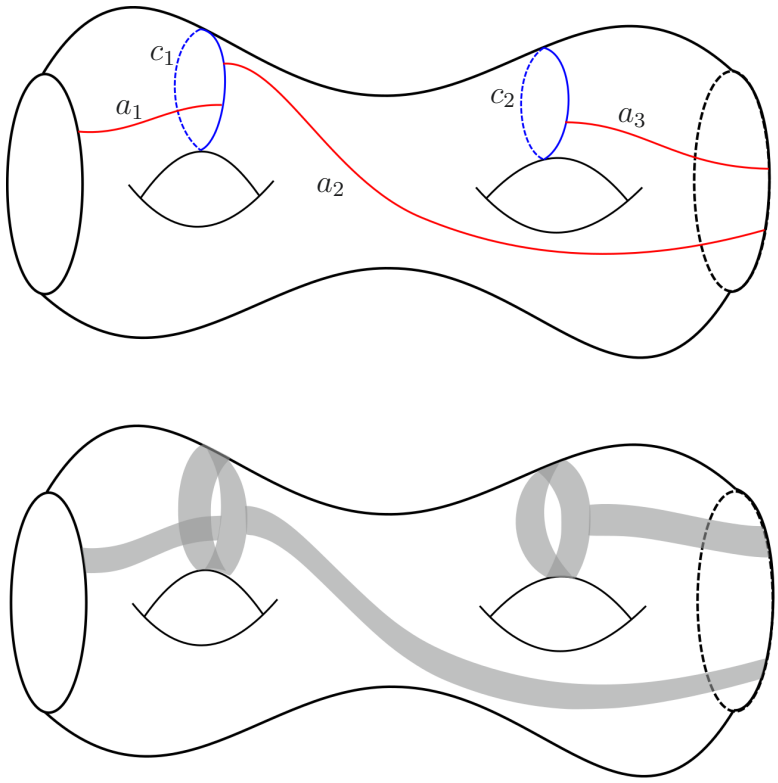


Figure 2: Above: an arc configuration on T. Below: the shaded region corresponds to the negative region on $T \times \{t\} \subset T \times [-1, 1]$ with respect to the contact structure induced by the arc configuration. Its boundary is the dividing set on on $T \times \{t\}$.

Now let (M, γ, ξ) be a balanced sutured manifold with a compatible

contact structure. Suppose T is a connected auxiliary surface of (M, γ) and $\mathcal{A}$ is a reduced arc configuration on T. Baldwin and Sivek constructed a suitable contact structure $\widetilde{\xi}$ on

$$\widetilde{M} = M \cup T \times [-1, 1]$$

as follows. First the arc configuration $\mathcal{A}$ gave rise to an $[-1, 1]$-invariant contact structure on $T \times [-1, 1]$. The negative region on any piece $T \times \{t\}$ is shown as in figure 2. Then they perturbed the contact structure on M in a neighborhood of $\gamma \subset M$ so that the dividing set in $A(\gamma)$ can be identified with that on $\partial T \times [-1, 1]$. So they were able to choose a diffeomorphism $f : \partial T \times [-1, 1] \to A(\gamma)$ which also identifies the contact structures. After rounding the corners, they derived $\widetilde{\xi}$ on $\widetilde{M}$. Suppose

$$\partial \widetilde{M} = R_+ \cup R_-,$$

then $R_\pm$ are convex and the dividing set on each of $R_\pm$ consists of two parallel non-separating simple closed curves. Finally they chose a diffeomorphism $h : R_+ \to R_-$ preserving the contact structures to get a closure (Y, R) with a contact structure $\bar{\xi}$, so that R is convex and the negative region on R is just an annulus. They also chose a simple closed curve $\eta \subset R$ intersecting each dividing set transversely once to support the local coefficients. From the construction

$$c_1(\bar{\xi})[R] = 2 - 2g(R),$$

and by work of Kronheimer, Mrowka, Ozsváth and Szabó [16], there is a contact element

$$\phi_{\bar{\xi}} \in \widetilde{HM}(-Y, \mathfrak{s}_{\bar{\xi}}; \Gamma_{-\eta}) \subset \mathrm{SHM}(-M, -\gamma).$$

Remark 2.9. In [1] Baldwin and Sivek only used reduced arc configurations to construct contact elements. However, the same construction on $\widetilde{M}$ can be made with a general arc configuration as defined in definition 2.8. The new dividing set on $R_\pm$ consists of m many pairs of parallel non-separating simple closed curves, where m is the number of simple closed curves in that arc configuration. However, in this case, the diffeomorphism h preserving contact structures may not always exists (as it shall identify the dividing sets). The reason why we want to make this more general definition is that we will see in the later section that a general arc configuration do exists during Floer excision, and the diffeomorphism h can indeed be chosen so that we can construct a contact structure on the closure Y.

At last we want to introduce the definition of contact handle attachment for the references in section 4.2.

Definition 2.10. A *contact handle* attached to a balanced sutured manifold (M, γ) with compatible contact structure ξ is a quadruple $h = (\phi, S, D^3, \delta)$ so that:

(1). D^3 is a 3-ball equipped with the standard tight contact structure and δ is the dividing set on ∂D^3.

(2). $S \subset \partial D^3$ is a compact submanifold and $\phi : S \to \partial M$ is an embedding so that $\phi(S \cap \delta) \subset \gamma$. S has different descriptions due to the index of the gluing:

(a). In index 0 case, $S = \varnothing$.

(b). In index 1 case, S is a disjoint union of two disks, and each disk intersects δ in an arc.

(c). In index 2 case, S is an annulus intersecting δ in two arcs. Also we require that each component of ∂S intersects each arc transversely once.

(d). In index 3 case, $S = \partial D^3$.

3 Contact element and excision

Suppose now for $i = 1, 2$, (M_i, γ_i) is a balanced sutured manifold. Suppose T_i, f_i, c_i, h_i are the auxiliary data to construct a closure (Y_i, R_i) as in definition 2.2. Now R_i contains a circle corresponding to $c_i \subset T_i$ which, by a little abuse of notation, we also denote by c_i. We can choose a 1-cycle η_i having exactly one transverse intersection with c_i.

Let $M = M_1 \sqcup M_2$ and $\gamma = \gamma_1 \cup \gamma_2$. Then (M, γ) is also a balanced sutured manifold and we can use auxiliary data (T, f, h) described as below to close up (M, γ). We cut T_i along c_i and re-glue the newly created boundary with respect to the orientation. Then T_1 and T_2 become a connected surface T so that

$$g(T) = g(T_1) + g(T_2) - 1, \ \partial T = \partial T_1 \cup \partial T_2.$$

We also choose $f = f_1 \cup f_2$ and $h = h_1 \cup h_2$. When doing the cut and paste along c_1 and c_2, the two curves η_1 and η_2 can also be glued together to get a curve η. See figure 3.

As in the subsection 2.1 we can construct a Floer excision map

$$F : \widetilde{HM}(-Y|-R) \to \widetilde{HM}(-(Y_1 \sqcup Y_2)|-(R_1 \cup R_2)).$$

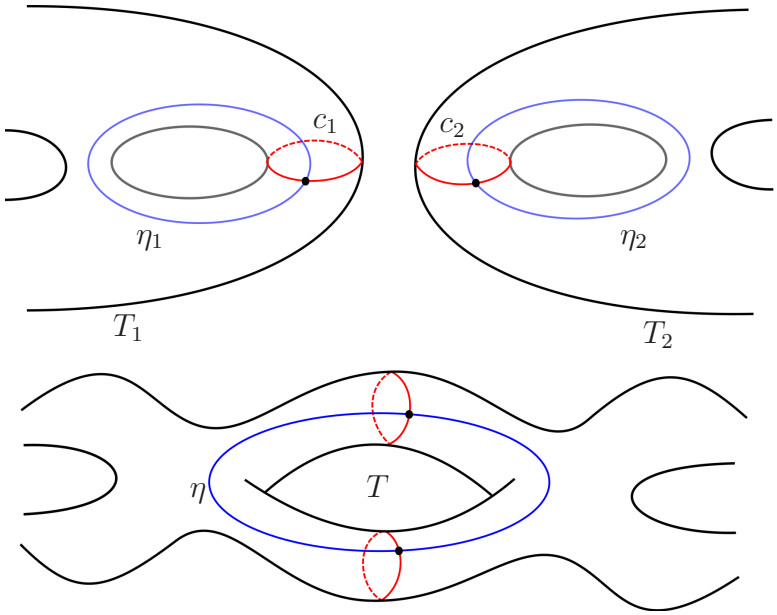


Figure 3: Above: The two auxiliary surfaces T_1 and T_2. Below: the connected auxiliary surface T.

We have the following theorem.

Theorem 3.1. *Under the above settings, suppose the genus of T_1 and T_2 are large enough, and suppose for $i = 1, 2$, (M_i, γ_i) is equipped with a compatible contact structure ξ_i. Then we can find suitable arc configurations $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}$ on T_1, T_2 and T respectively, so that there are corresponding contact structures $\bar{\xi}_1, \bar{\xi}_2$ and $\bar{\xi}$ on Y_1, Y_2 and Y respectively, as described in subsection 2.2. Then the map F above will preserve the contact elements:*

$$F(\phi_{\bar{\xi}}) \doteq \phi_{\bar{\xi}_1 \cup \bar{\xi}_2}.$$

Here $\doteq$ means equal up to multiplication by a unit.

Proof. We will choose some special arc configurations. For $i = 1, 2$ assume that we have a reduced arc configuration $\mathcal{A}_i$ on T_i so that the simple closed curve is just c_i and all arcs are attached to only one side of $c_i \subset T_i$. See figure 4. Recall c_i is the curve on the auxiliary surface T_i which is required as in definition 2.2. Then the induced contact structure on $T_i \times [-1, 1]$ has dividing set on $T_i \times \{t\}$ consisting of a few arcs, whose end points are both on $\partial T_i \times \{t\}$, and a simple closed curve which we shall also denote by

c_i. We then pick a gluing diffeomorphism h_i which identifies the contact structures and also preserves c_i.

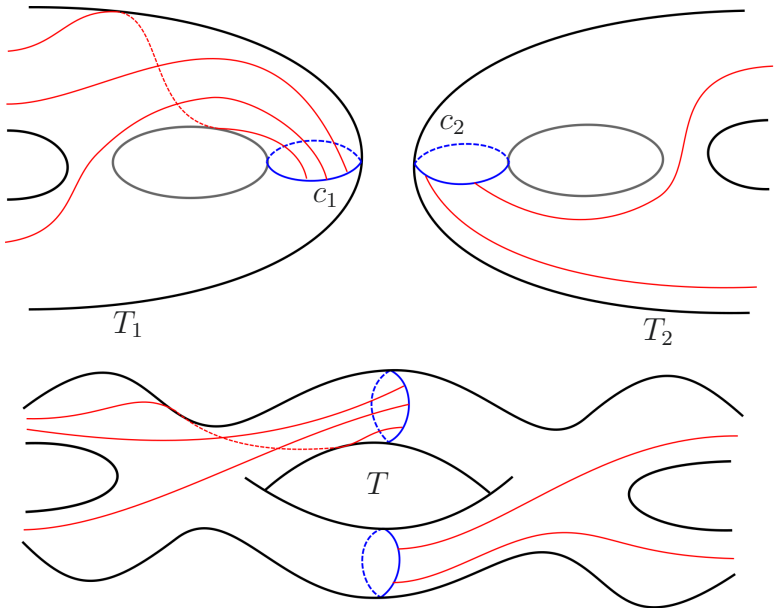


Figure 4: Above: The two reduced arc configuration on T_1 and T_2. Below: the resulting arc configuration on T from slicing. It has two simple closed curves instead of one.

When we extend ξ_i to $\bar{\xi}_i$, which is defined on all of Y_i, the new contact structure $\bar{\xi}_i$ will be S^1 invariant in a neighborhood of c_i. To describe this contact structure in coordinates, let $A_i \subset T_i$ be a neighborhood of $c_i \subset T_i$. In Y_i, $A_i \times S^1$ is a neighborhood of $c_i \subset Y_i$. In this neighborhood, we can write the contact form as

$$\alpha_i = \beta_i + u_i \cdot d\varphi_i,$$

where β_i is a 1-form on A_i, u_i is a function on A_i with

$$c_i = \{p \in A_i | u_i(p) = 0\},$$

and φ_i is the S^1 direction. See [8]. The non-degeneracy condition reads

$$0 \neq \alpha_i \wedge d\alpha_i = (u_i \cdot d\beta_i + \beta_i \wedge du_i) \wedge d\varphi_i.$$

Along c_i we know then $\beta_i \wedge du_i \neq 0$. Hence along c_i, $\beta = d\theta_i$ where φ is a coordinate for c_i and (u, θ_i) is a local coordinate for $[-\varepsilon, \varepsilon] \times c_i \subset A_i$ for

some small $\varepsilon > 0$. We shall also assume that $u_i > 0$ (or < 0) corresponds to the positive or negative regions. Then the slicing operation defined in [23] can be described as follows. Let $L_i = c_i \times S^1$ be the pre-Lagrangian torus (For definition see [21]) and $N_i = [-\varepsilon, \varepsilon] \times c_i \times S^1$ be a neighborhood of L_i with coordinates $(u_i, \theta_i, \varphi_i,)$ (The coordinates u_i corresponds to r in [23] and the other two coordinates are the same, while we didn't write them in the same order as in that paper.) We can cut N_i open along L_i so that N_i is cut into two parts $N_{i,\pm}$ corresponding to $\pm u_i \geqslant 0$. We can then re-glue $N_{1,+}$ to $N_{2,-}$ and $N_{1,-}$ to $N_{2,+}$ by identifying L_1 with L_2 so that (θ_1, φ_1) is identified with (θ_2, φ_2). Suppose the resulting 3-manifold is Y, then Y has a distinguished surface R obtained from cutting and re-gluing R_1 and R_2 along c_1 and c_2. Recall we also have a simple closed curve $\eta_i \subset R_i$ which intersects c_i transversely once. After a suitable isotopy, we can assume that under the above identification of L_1 with L_2, we can also identify the intersection point $\eta_1 \cap c_1$ with $\eta_2 \cap c_2$. Hence η_1 and η_2 are also glued together to get a curve $\eta \subset R$. This is exactly the same setting of doing Floer excision along tori L_1 and L_2. Hence (Y, R) is a closure of $(M_1 \sqcup M_2, \gamma_1 \cup \gamma_2)$ as we have discussed above. Also from theorem 2.5 there is an isomorphism

$$F : HM(-(Y_1 \sqcup Y_2)| - (R_1 \cup R_2); \Gamma_{-(\eta_1 \cup \eta_2)}) \to HM(-Y| - R; \Gamma_{-\eta}).$$

The process of slicing also glues the contact structures $\bar{\xi}_i$ on Y_i to get a contact structure $\bar{\xi}'$ on Y. The contact structure $\bar{\xi}'$, however, arises from an arc configuration $\mathcal{A}'$ which is not reduced. This is because with respect to $\bar{\xi}'$, the dividing set on R consists of two pairs of parallel non-separating simple closed curves instead of just one pair. See figure 4. Let $\bar{\xi}$ be a contact structure on Y obtained by extending ξ_i on Y_i using a reduced arc configuration $\mathcal{A}$. Here $\mathcal{A}$ is obtained by 'merging' the two simple closed curves of $\mathcal{A}'$ into one in a way that the curve $\eta \subset R$ still intersects the new simple closed curve transversely once. See figure 5. The proof of theorem 3.1 is then clearly the combination of the following two lemmas.

□

Lemma 3.2. *If the genus of T_1 and T_2 are large enough, then the spin^c structures associated to $\bar{\xi}$ and $\bar{\xi}'$ are the same. Furthermore, if we denote that spin^c structure by $\mathfrak{s}_0$, then we have*

$$\phi_{\bar{\xi}} \doteq \phi_{\bar{\xi}'} \in \widehat{HM}(-Y, \mathfrak{s}_0; \Gamma_{-\eta}).$$

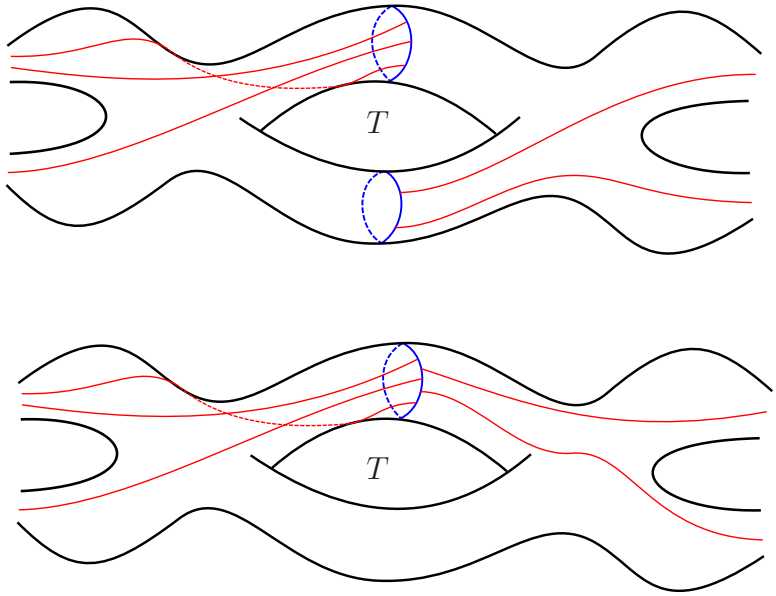


Figure 5: Above: the arc configuration on T obtained from slicing. Below: the reduced arc configuration after merginng the two simple closed curves.

Lemma 3.3. *If the genus of T_1 and T_2 are large enough, then we have*

$$F(\phi_{\bar{\xi}_1} \cup \phi_{\bar{\xi}_2}) \doteq \phi_{\bar{\xi}'}.$$

In order to prove the above two lemmas, we will need some preliminaries.

Lemma 3.4. *(Baldwin, Sivek [1]) Suppose (M, γ) is a balanced sutured manifold and ξ is a contact structure on M so that ∂M is convex and γ is the dividing set. Suppose (Y, R) is a closure of (M, γ) obtained by using a connected auxiliary surface with large enough genus. Suppose we use some (not necessarily) arc configuration on T to extend ξ to a contact structure $\bar{\xi}$ on Y. Then there exist a contact structure ξ_R on $R \times S^1$ and pair-wise disjoint simple closed curves $\alpha_1, ..., \alpha_n$, so that*

(1). The contact structure ξ_R is S^1-invariant so that each $R \times \{t\}$ is convex with dividing set being some pairs of parallel non-separating simple closed curves.

(2). Each α_i is Legendrian and is disjoint from the pre-Lagrangian tori of the form

$$(\text{Dividing set on } R) \times S^1.$$

(3). The result of doing $+1$ contact surgeries along all $a_i \subset R \times S^1$ is contactomorphic to Y equipped with $\bar{\xi}$.

Lemma 3.5. *(Niederkrüger, Wendl, [23]) Suppose R is the surface as above and ξ_R is an S^1-invariant contact structure on $R \times S^1$ so that each $R \times t$ is convex with dividing set being a few pairs of non-separating simple closed curves. Suppose that there is a curve $\eta \subset R$ so that η intersects every component of the dividing set transversely once. Then $(R \times S^1, \xi_R)$ is weakly fillable by (W, ω) and η is dual to $\omega|_{R\times S^1}$ up to a scalar.*

Lemma 3.6. *(Kronheimer, Mrowka, Ozsváth and Szabó, [16]) In the above lemma the contact element*

$$\phi_{\xi_R} \in \widetilde{HM}(-R \times S^1, \mathfrak{s}_{\xi_R}; \Gamma_{-\eta})$$

is primitive. Hence in particular it is non-vanishing.

Lemma 3.7. *(Kronheimer, Mrowka, [15]) In the above lemma, there is actually a unique spin^c structure $\mathfrak{s}_0$ so that*

(1). We have

$$c_1(\mathfrak{s}_0)[R] = 2 - 2g.$$

(2). The monopole Floer homology of $\widetilde{HM}(-R \times S^1, \mathfrak{s}_0; \Gamma_{-\eta})$ is non-zero. Furthermore, we actually have

$$\widetilde{HM}(-R \times S^1, \mathfrak{s}_0; \Gamma_{-\eta}) \cong \mathcal{R}.$$

Here $\mathcal{R}$ is the coefficient ring we use for local coefficients as in remark 2.3.

Lemma 3.8. *(Baldwin, Sivek, [1]) Suppose for $i = 1, 2$, Y_i is a closed oriented 3-manifold with contact structure ξ_i. Suppose (Y_2, ξ_2) is obtained from (Y_1, ξ_1) by performing a contact $+1$ surgery along a Legendrian curve. Then there is a cobordism W from Y_1 to Y_2 obtained from $Y_1 \times [0, 1]$ by attaching a 2-handle with suitable framing. Suppose for $i = 1, 2$, η_i is a 1-cycle in Y_i supporting local coefficients. Then the map*

$$\widetilde{HM}(-W) : \widetilde{HM}(-Y_1, \mathfrak{s}_{\xi_1}; \Gamma_{-\eta_1}) \to \widetilde{HM}(-Y_2, \mathfrak{s}_{\xi_2}; \Gamma_{-\eta_2})$$

preserves the contact elements (up to multiplication by a unit).

Remark 3.9. The above lemma is stated in Baldwin and Sivek [1] as a corollary to results from Hutchings and Taubes [10].

Proof of lemma 3.2. As in the settings of theorem 3.1, $\bar{\xi}$ and $\bar{\xi}'$ are contact structures on Y which are obtained from contact structures $\xi_1 \cup \xi_2$ on $(M_1, \gamma_1) \cup (M_2, \gamma_2)$ and some particular arc configurations $\mathcal{A}$ and $\mathcal{A}'$ on T. From lemma 3.4 we know that there are contact structures ξ_R and ξ'_R on $R \times S^1$ and a set of pair-wise disjoint curves $\alpha_1, ..., \alpha_n \subset R \times S^1$ so that

(1). Both ξ_R and ξ'_R are S^1 invariant and each $R \times \{t\}$ is convex.

(2). We have $\xi_R = \xi'_R$ near a neighborhood of each α_i.

(3). All α_i are disjoint from the pre-Lagrangian tori of the form

$$(\text{Dividing set on } R) \times S^1$$

for the dividing sets with respect to both ξ_R and ξ'_R.

(4). If we do contact $+1$ surgery on all of α_i, then $(R \times S^1, \xi_R)$ (or $(R \times S^1, \xi'_R)$) will become a contact manifold contactomorphic to $(Y, \bar{\xi})$ (or $(Y, \bar{\xi}')$).

The condition (2) relies on the proof of lemma 3.4 (of the current paper) in [1]. The essential reason is that $\bar{\xi}$ and $\bar{\xi}'$ are only different in the part of Y coming from auxiliary surfaces while the curves α_i are contained in the interior of the original sutured manifold.

By lemma 3.5 and 3.6 we know that the contact invariants ϕ_{ξ_R} and $\phi_{\xi'_R}$ are both non-zero and primitive in the same monopole Floer homology. Then lemma 3.7 makes sure that ξ_R and ξ'_R correspond to the same spin^c structure $\mathfrak{s}_0$ on $R \times S^1$ (since there is only one candidate of possible spin^c structures). Then we have

$$\phi_{\xi_R} \doteq \phi_{\xi'_R} \in \widetilde{HM}(-R \times S^1, \mathfrak{s}_0; \Gamma_{-\eta}), \tag{1}$$

for suitable choice of local coefficients.

The surgery description above makes sure that on Y, $\bar{\xi}$ and $\bar{\xi}'$ also corresponds to the same spin^c structure. This fact, together with lemma 3.8 and equality (1), then imply the result of lemma 3.2. □

Proof of lemma 3.3. First apply lemma 3.4 to $(Y_i, \bar{\xi}_i)$ for $i = 1, 2$, we get a contact structure ξ_{R_i} on $R_i \times S^1$ and a set of Legendrian curves $\alpha_{i,1}, ..., \alpha_{i,n_i}$ satisfying the conclusions of the lemma. In particular, if we do contact $+1$ surgery on all of $\alpha_{i,j}$ we will arrive at $(Y_i, \bar{\xi}_i)$. If we pick a suitable connected component c_i of dividing set on $R_i \times t$ and do the slicing operation on $R_1 \times S^1$ and $R_2 \times S^1$ along the two pre-Lagrangian tori $c_1 \times S^1$ and $c_2 \times S^1$, then the result is the 3-manifold $R \times S^1$ with contact structure ξ'_R in the proof of

lemma 3.2 and the two sequences of curves $\alpha_{1,1}, ..., \alpha_{1,n_1}$ and $\alpha_{2,1}, ..., \alpha_{2,n_2}$ together form the set of curves $\alpha_1, ..., \alpha_n$ as in the proof of lemma 3.2. There is a cobordism associated to the slicing operation, or equivalently, doing a Floer excision, on $R_1 \times S^1$ and $R_2 \times S^1$. We call this cobordism W_e and it is from $(R_1 \times S^1) \sqcup (R_2 \times S^1)$ to $R \times S^1$. There is a second cobordism W_s, associated to the surgery along α_i as in lemma 3.8, from $R \times S^1$ to Y. Finally there is a third one W_F corresponding to F (also from Floer excision) from Y to $Y_1 \sqcup Y_2$.

As usual, we shall choose suitable surfaces and local coefficients to make precise the cobordism map but we omit them from the notation. The map $HM(-W_e)$ would preserve contact elements because it is an isomorphism between two copies of $\mathcal{R}$ and the contact elements are units in each copy of $\mathcal{R}$. The map $HM(-W_s)$ would preserve contact elements as in lemma 3.8. So if we could prove that the composition $HM(-(W_e \cup W_s \cup W_F))$ preserves the contact elements, then $HM(-W_F) = F$ would also do, since it is an isomorphism. Thus lemma 3.3 is proven.

To show that $HM(-(W_e \cup W_s \cup W_F))$ preserves contact elements, we observe that when we cut the cobordism $W_e \cup W_s \cup W_F$ along a $T_{1,+} \times S^1$ and glue back two copies of $T_{1,+} \times D^2$, the result is a disjoint union of two cobordism W_1 and W_2. See figure 6. For W_i is from $R_i \times S^1$ to Y_i and is associated to the surgeries along $\alpha_{i,j}$ as in lemma 3.8. Hence by that lemma, $HM(-(W_1 \cup W_2))$ would preserve the contact elements. Finally, by lemma 2.10 in [15], we know that

$$HM(-(W_e \cup W_s \cup W_F)) \doteq HM(-(W_1 \cup W_2)).$$

So we are done. □

4 Connected sum formula

We will derive the connected sum formula for sutured monopoles in this section. The formula relies on the computation of some particular balanced sutured manifold. We will explore how the contact structures and an Floer excision would help us in the calculation.

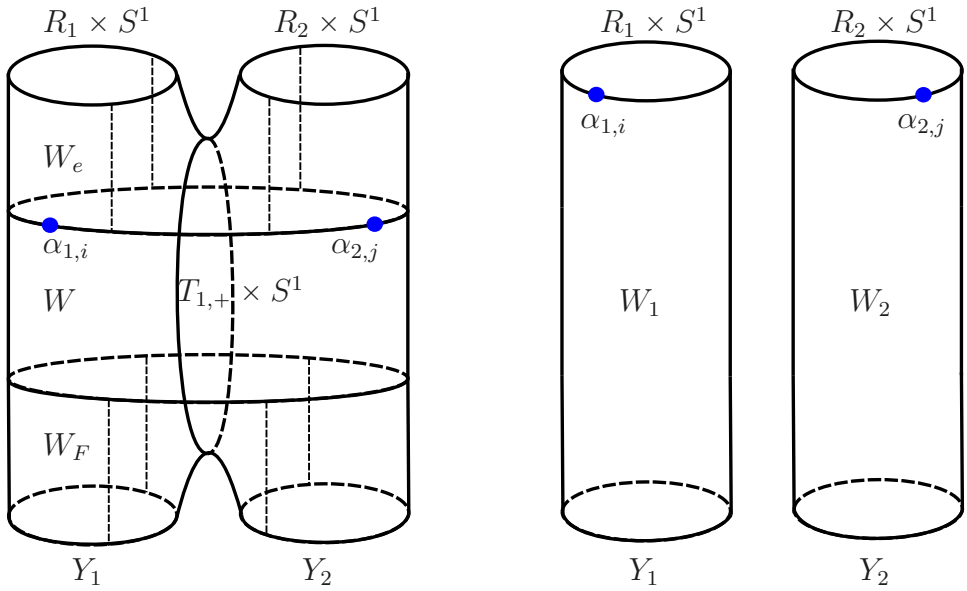


Figure 6: Left: the union of the three cobordisms, cut along the 3-torus $T_{1,+} \times S^1$. Right: the two disjoint cobordisms resulting from the cutting and pasting.

4.1 Computing $\mathrm{SHM}(V, \gamma^{4k+2})$

We will start with the family of balanced sutured manifolds (V, γ^{2n}). Suppose $V = S^1 \times D^2$ be a solid torus and $\gamma^{2n} \subset \partial V$ is a suture consists of $2n$ many longitudes (each of the form $S^1 \times \{t\}$ for $t \in \partial D$). Note adjacent longitudes should be oriented oppositely, and there should be in total an even number of longitudes in order to give $R(\gamma^{2n})$ a compatible orientation.

When $n > 2$, we can pick an annulus A properly embedded in V so that

(1). $\partial A \cap \gamma^{2n} = \varnothing$.

(2). On the boundary, $\partial V \backslash \partial A$ has two components so that one contains precisely three components of the suture γ^{2n} in the interior.

The result of (sutured manifold) decomposition of (V, γ^{2n}) along A consists of two components. One components is diffeomorphic to (V, γ^{2n-2}) and the other is diffeomorphic to (V, γ^4). See figure 7 for an example of decomposing (V, γ^8). By induction and proposition 6.7 in [15], we know that, when using $\mathbb{Q}$ coefficients and $n \geqslant 2$, we have

$$\mathrm{SHM}(V, \gamma^{2n}; \mathbb{Q}) \cong \mathrm{SHM}(V, \gamma^4; \mathbb{Q})^{\otimes(n-1)} \tag{2}$$

Now we have the following.

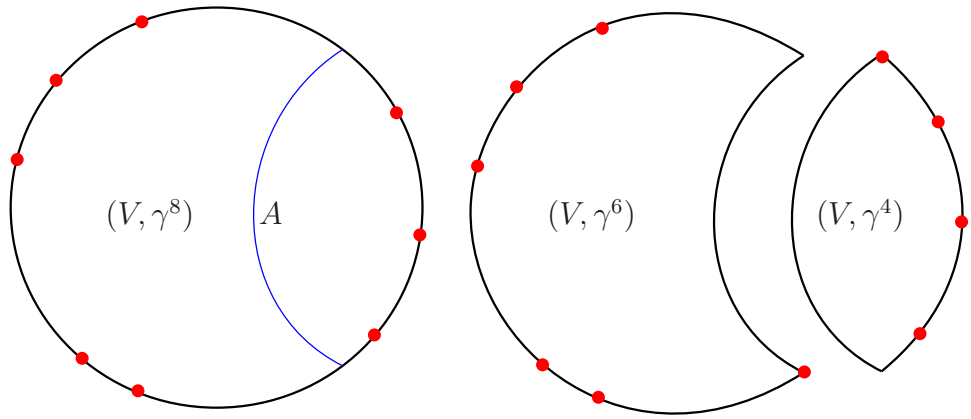


Figure 7: Everything is S^1-invariant so we look at a cross section, which is a disk $\{t\} \times D^2 \subset S^1 \times D^2$. The (red) dots represent the suture and the (blue) arc inside the disk represents the annulus A along which we do the decomposition.

Lemma 4.1. *When using $\mathbb{Z}$ coefficients, we have*

$$\mathrm{SHM}(V, \gamma^4; \mathbb{Z}) \cong \mathbb{Z}^2 \oplus G_{tor},$$

where G_{tor} is a (finite) torsion group without any even-torsion.

Proof. We prove that the rank of the homology should be precisely 2. To get a lower bound, we first use $\mathbb{Q}$ coefficients and look at (V, γ^6). Recall $V = S^1 \times D^2$ is a solid torus. Let $t_0 \in S^1$ be a point and $D = \{t_0\} \times D^2 \subset V$ be a meridian disk of V. We have ∂D intersects γ^6 at six points:

$$\partial D \cap \gamma^6 = \{t_0\} \times \{p_1, ..., p_6\} \subset S^1 \times \partial D^2.$$

Let p_i be arranged in the way that if we travel along the oriented curve ∂D starting from p_1, then we will meet p_i before meeting p_{i+1}. Suppose $\gamma^6 = l_1 \cup ... \cup l_6$ so that for $i = 1, ..., 6$

$$\partial D \cap l_i = \{t_0\} \times \{p_i\}.$$

Now we can assume that the annular neighborhood $A(\gamma)$ of $\gamma \subset \partial V = S^1 \times \partial D^2$ is of the form

$$A(\gamma) = \bigcup_{i=1}^{6} S^1 \times [p_i - \varepsilon, p_i + \varepsilon],$$

for some small enough fixed constant $\varepsilon > 0$. Let T be an auxiliary surface of T consists of three disjoint annuli:

$$T = A_1 \cup A_2 \cup A_3,$$

where for $i = 1, 2, 3$, A_i has the form

$$A_i = S_i^1 \times [-1, 1].$$

We choose an orientation reversing diffeomorphism $f : \partial T \to \gamma$ so that

$$f(S_1^1 \times \{1\}) = l_1,\ f(S_1^1 \times \{-1\}) = l_2,\ f(S_2^1 \times \{1\}) = l_3,$$

and also

$$f(S_2^1 \times \{-1\}) = l_6,\ f(S_3^1 \times \{1\}) = l_4,\ f(S_3^1 \times \{-1\}) = l_5.$$

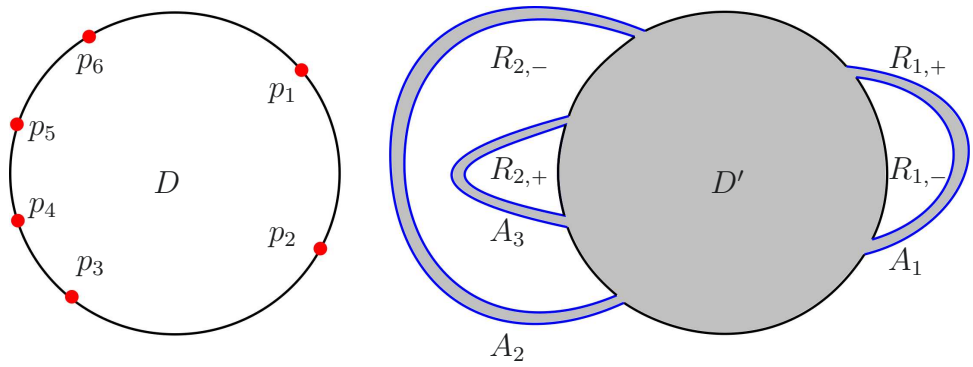


Figure 8: We still look at a cross section, which is the disk $D = \{t_0\} \times D^2 \subset S^1 \times D^2$. The (red) dots in the left sub-figure represent the suture and the stripes (with blue boundary) in the right sub-figure represent the three annuli A_1, A_2 and A_3. The shaded region is precisely the surface D'.

Let

$$\widetilde{V} = V \underset{f \times id}{\cup} T \times [-\varepsilon, \varepsilon],$$

then $\tilde{V}$ has four boundary components:

$$\widetilde{V} = R_{1,+} \cup R_{2,+} \cup R_{1,-} \cup R_{2,-},$$

so that $S_1^1 \times \{\pm\varepsilon\} \subset R_{1,\pm}$ and $S_3^1 \times \{\pm\varepsilon\} \subset R_{2,\pm}$. Suppose for $i = 1, 2, 3$, S_i^1 has coordinate t^i, and t_0^i is identified with $t_0 \in S^1$ by f.

$$D' = D \cup (\{t_0^1\} \cup \{t_0^2\} \cup \{t_0^3\}) \times [-1, 1] \times [-\varepsilon, \varepsilon].$$

Then for $j = 1, 2$, we have $D' \cap R_{j,\pm} = C_{j,\pm}$. See figure 8. Choose an orientation preserving diffeomorphism

$$h : (R_{1,+} \sqcup R_{2,+}) \to R_{1,-} \sqcup R_{2,-},$$

so that for $j = 1, 2$

$$h(C_{j,+}) = C_{j,-}.$$

Then we can close $\widetilde{V}$ up as we did in subsection 2.1 to get a closure $(Y^{(6)}, R^{(6)})$ of (V, γ^6). The surface D' becomes closed oriented surface $\bar{D}^{(6)}$ of genus 2 inside Y. Now define

$$\mathrm{SHM}(V, \gamma^6, i) = \bigoplus_{\substack{\mathfrak{s} \in \mathfrak{S}(Y|R), \\ c_1(\mathfrak{s})[\bar{D}^{(6)}] = 2i.}} \widetilde{HM}(Y, \mathfrak{s}; \mathbb{Q}).$$

We know that

$$\mathrm{SHM}(V, \gamma^6) \cong \bigoplus_{i \in \mathbb{Z}} \mathrm{SHM}(V, \gamma^6, i).$$

If we decompose the balanced sutured manifold (V, γ) along D, the result is a 3-ball with one simple closed curve as the suture. So by proposition 6.9 of [15], we know that

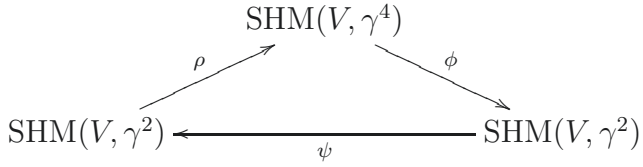
$$\mathrm{SHM}(V, \gamma^6, 1) \cong \mathbb{Q}.$$

On the other hand, we can also decompose (V, γ) along $-D$. A similar argument then shows that

$$\mathrm{SHM}(V, \gamma^6, -1) \cong \mathbb{Q}.$$

Hence with $\mathbb{Q}$ coefficients the rank of $\mathrm{SHM}(V, \gamma^6)$ is at least 2. From formula (2) and universal coefficient theorem we know that $\mathrm{SHM}(V, \gamma^4)$, with either $\mathbb{Q}$ or $\mathbb{Z}$ coefficients, has rank at least two.

To obtain a lower bound, we will need to work with $\mathbb{Z}_2$ coefficients and use by-pass attachment for sutured monopoles introduced by Baldwin and Sivek in [1]. The by-pass attachment, as depicted in figure 9 induces an exact triangle



We know that $\mathrm{SHM}(V,\gamma^2)\cong\mathbb{Z}_2$ so with $\mathbb{Z}_2$ coefficients the rank of $\mathrm{SHM}(V,\gamma^4)$ is at most two and so is with $\mathbb{Z}$ coefficients by the universal coefficient theorem. □

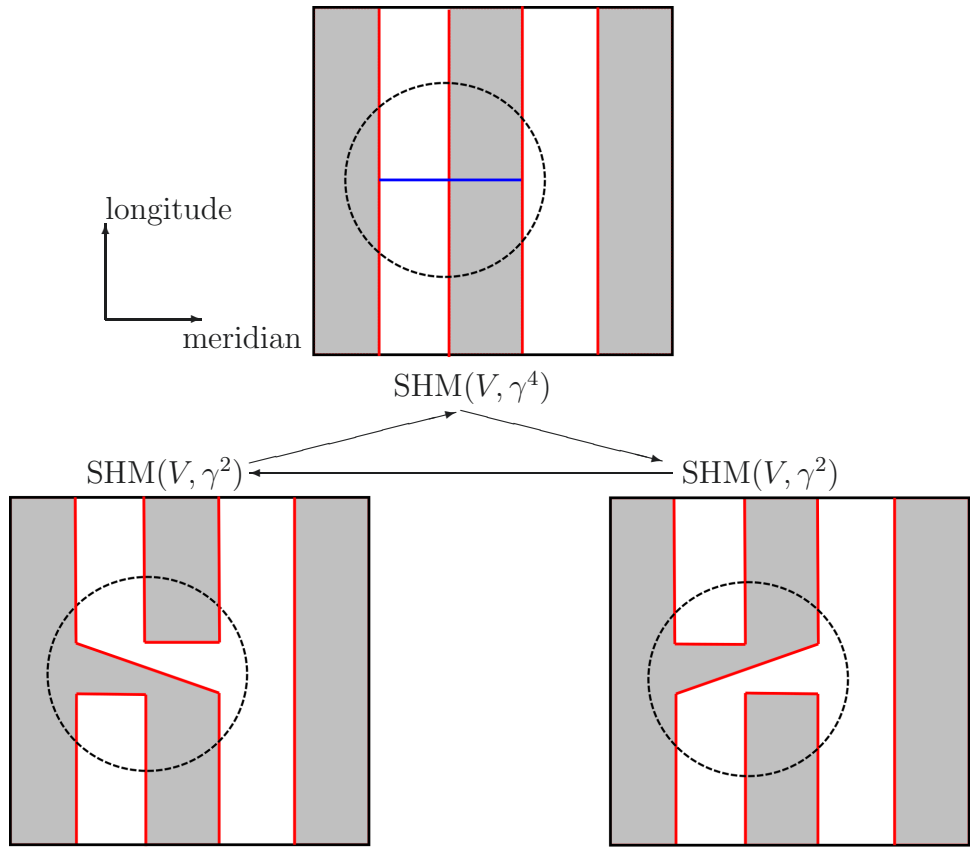


Figure 9: The by-pass attachment along the horizontal (blue) arc α. The change of sutures are limited in the dotted circles. The shaded region represents $R_-(\gamma)$.

Remark 4.2. In [14] a particular closure of the manifold $(V,\gamma^4)\sqcup(V,\gamma^4)$ was constructed by Kronheimer and Mrowka. One can try to compute the monopole Floer homology of that closure directly, and we expect the following the conjecture.

Conjecture 4.3. *The torsion group G_{tor} is actually 0 in lemma 4.1.*

Remark 4.4. In the proof of lemma 4.1 we go through (V, γ^6) instead of just looking at (V, γ^4). This is not only because we want to make some convenience for the following theorem 4.5 but also for some other subtleties. when dealing with (V, γ^4) directly, we cannot pick a meridian disk D intersecting γ^4 four times, as we will not be able to construct the closed surface $\bar{D}$, as we did for (V, γ^6), in any closure of (V, γ^4). There is another subtlety in the above construction. When pairing intersection points $p_1, ..., p_6$, we didn't just pair the adjacent points, but pair them in a particular way (we paired p_3 with p_6 and p_4 with p_5, not just simply pairing adjacent ones). We shall remark here that these two subtleties already existed in Kronheimer and Mrowka's paper [15], but they didn't discussed on those subtleties in that paper.

As mentioned in the introduction, the above construction is a naive version of the generalization of the grading defined by Baldwin and Sivek [4] for knot instanton Floer homology. We plan to develop a more systematical treatment in the author's following paper [19].

We are now able to prove the following theorem.

Theorem 4.5. *Suppose $n = 2k + 1$ is odd. With $\mathbb{Q}$ coefficients, there is a grading on $\mathrm{SHM}(V, \gamma^{2n})$ induced by a meridian disk of V, so that with respect to this grading. we have for $-k \leqslant i \leqslant k$,*

$$\mathrm{SHM}(V, \gamma^{2n}, i) \cong H_{i+k}(T^{n-1}; \mathbb{Q}),$$

and $\mathrm{SHM}(V, \gamma^{2n}, i) = 0$ for $|i| > k$. Here T^{n-1} is the $(n-1)$-dimensional torus.

Proof. The basic case is trivial: if $k = 0$, then we have

$$\mathrm{SHM}(V, \gamma^2) = \mathrm{SHM}(V, \gamma^2, 0) \cong \mathbb{Q} \cong H_0(T^0 = \{pt\}; \mathbb{Q}).$$

When $k = 1$, the grading was already constructed in the proof of lemma 4.1, and we have

$$\mathrm{SHM}(V, \gamma^6, \pm 1) \cong \mathbb{Q} \cong H_0(T^2; \mathbb{Q}) \cong H_2(T_2; \mathbb{Q}).$$

From the adjunction inequality (see subsection 2.4 in [15]), we know that for $|i| > 1$

$$\mathrm{SHM}(V, \gamma^6, i) = 0,$$

while from lemma 4.1 and formula (2), we know that $\mathrm{SHM}(V,\gamma^6) \cong \mathbb{Q}^4$, hence we have

$$\mathrm{SHM}(V,\gamma^6,0) \cong \mathbb{Q}^2 \cong H_1(T^2;\mathbb{Q}).$$

Now for a general k, we argue in a similar way as we did for (V,γ^6). Let $D = \{t_0\} \times D^2$ be the meridian disk and

$$\partial D \cap \gamma = \{p_1, ..., p_{2n}\}.$$

The points are indexed in an order so that if we travel along the oriented circle ∂D and start from p_1, then we will meet p_i before p_{i+1}. The suture γ^{2n} can now be described as

$$\gamma^{2n} = \bigcup_{i=1}^{2n} S^1 \times \{p_i\}.$$

We pick an auxiliary surface T for (V,γ^{2n}) so that T consists of n many disjoint annuli:

$$T = \bigcup_{i=1}^{n} A_i.$$

We choose an orientation reversing diffeomorphism $f : \partial T \to \gamma$ so that

$$f(\partial A_1) = S^1 \times \{p_1, p_2\}$$

and for $j = 1, ..., k$, we have

$$f(\partial A_{2j}) = S^1 \times \{p_{4k-1}, p_{4k+2}\},\ f(\partial A_{2j+1}) = S^1 \times \{p_{4k}, p_{4k+1}\}.$$

Let

$$\widetilde{V} = V \underset{f\times id}{\cup} T \times [-\varepsilon, \varepsilon],$$

we know that

$$\partial\widetilde{V} = \bigcup_{i=1}^{k+1}(R_{i,+} \cup R_{i,-}),$$

so that for $j = 1, ..., k+1$,

$$A_{2j-1} \times \{\pm\varepsilon\} \subset R_{j,\pm}.$$

The meridian disk D becomes a surface $D' \subset \widetilde{V}$ so that for $j = 1, ..., k+1$

$$\partial D' \cap R_{j,\pm} = C_{j,\pm}.$$

Choose an orientation preserving diffeomorphism

$$h : (R_{1,+} \cup ... \cup R_{k+1,+}) \to R_{1,+} \cup ... \cup R_{k+1,+}$$

so that for $j = 1, ..., k+1$,

$$h(C_{j,+}) = C_{j,-}.$$

Then we get a closure $(Y^{(2n)}, R^{(2n)})$ for (V, γ^{2n}), so that D' becomes an oriented closed surface $\bar{D}^{(2n)} \subset Y^{(2n)}$.

Now we define a grading on $\mathrm{SHM}(V, \gamma^{2n})$ as follows:

$$\mathrm{SHM}(V, \gamma^{2n}, i) = \bigoplus_{\substack{\mathfrak{s} \in \mathfrak{S}(Y^{(2n)}|R^{(2n)}), \\ c_1(\mathfrak{s})[\bar{D}^{(2n)}]=2i.}} \widetilde{HM}(Y^{(2n)}, \mathfrak{s}; \mathbb{Q}).$$

Note D' is obtained from D by attaching $2k+1$ stripes so

$$\chi(\bar{D}^{2n}) = \chi(D') = \chi(D) - (2k+1) = -2k.$$

Hence from adjunction inequality, we know that if $|i| > k$ then

$$\mathrm{SHM}(V, \gamma^{2n}, i) = 0.$$

To compute the homology for each grading we need to use Floer excision again. Let $q_1, q_2 \in \partial D \cap C_{1,+} \subset \partial D'$ be a pair of points. Suppose $q_1' = h^{-1}(q_1)$ and $q_2' = h^{-1}(q_2)$ where h is the diffeomorphism we use to get the closure $(Y^{(2n)}, R^{(2n)})$ for (V, γ^{2n}). Suppose we choose an h so that

(1). We have $q_1', q_2' \in \partial D \cap C_{1,+} \subset D'$.

(2). We have that q_1' lies in between p_6 and p_7 and q_2' lies in between p_2 and p_3.

(3). We have for $i = 1, 2$,

$$h(S^1 \times \{q_i'\}) = S^1 \times \{q_i\}.$$

The two conditions can actually be achieved by an S^1-invariant h. Pick two arcs $\beta_1, \beta_2 \subset D$ so that for $i = 1, 2$

$$\beta_i \cap \partial D = \partial \beta_i = \{q_i, q_i'\}.$$

In the closure $(Y^{(2n)}, R^{(2n)})$, β_1 and β_2 becomes two circles and after crossing S^1, they become two tori T_1 and T_2. We pick a 1-cycle $\eta \subset R^{(2n)} \subset Y^{2n}$

to be union of all the images of $C_{i,+} \subset \partial\widetilde{V}$ in $Y^{(2n)}$. Clearly it intersects both T_1 and T_2 transversely once.

We can do Floer excision along T_1 and T_2, or to be more precise, the inverse operation of a Floer excision introduced in subsection 2.1. The result of this 'reversed' Floer excision is a disjoint union of two 3-manifold $Y^{(2n-4)}$ and $Y^{(6)}$. The surface $R^{(2n)}$ is cut into $R^{(2n-4)} \cup R^{(6)}$ and the surface $\bar{D}^{(2n)}$ is cut into $\bar{D}^{(2n-4)} \cup \bar{D}^{(6)}$ as well. See figure 10.

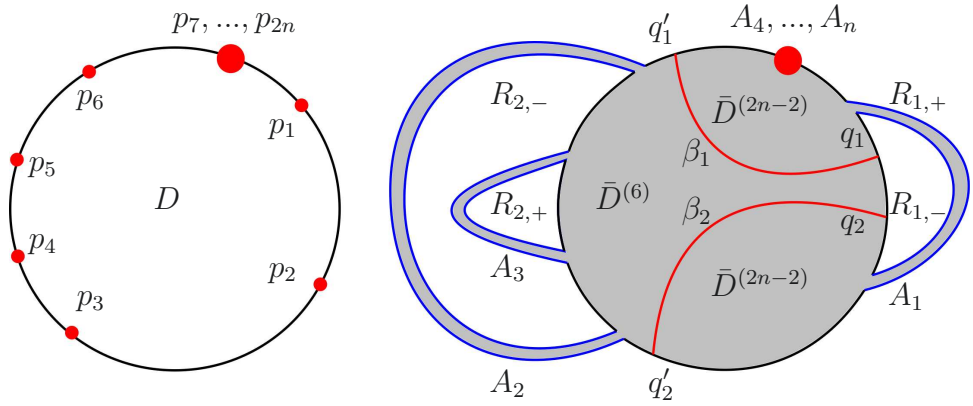


Figure 10: We still look at a cross section, which is the disk $D = \{t_0\} \times D^2 \subset S^1 \times D^2$. The (red) dots in the left sub-figure represent the suture and the stripes (with blue boundary) in the right sub-figure represent the three annuli A_1, A_2 and A_3. $A_4, ..., A_n$ are not depicted. The two (red) arcs inside D are β_1, β_2. The shaded region represents the surface D'.

As described in subsection 2.1 there is a cobordism W from $Y^{(2n-4)} \sqcup Y^{(6)}$ to $Y^{(2n)}$. Inside the cobordism W, there is another (3-dimensional) cobordism between $\bar{D}^{(2n-4)} \sqcup \bar{D}^{(6)} \subset Y^{(2n-4)} \sqcup Y^{(6)}$ and $\bar{D}^{(2n)} \subset Y^{(2n)}$. This cobordism is gotten from the same way we construct W from the Floer excision in dimension three but do it in dimension two. Hence if $\mathfrak{s}$ is a spin^c structure on W so that

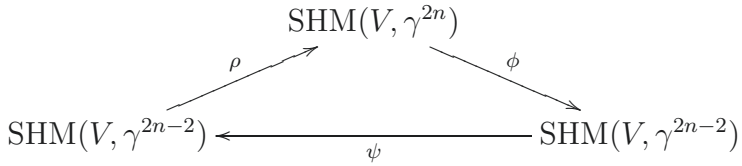
$$c_1(\mathfrak{s})[\bar{D}^{(2n-4)}] = x, \ c_1(\mathfrak{s})[\bar{D}^{(6)}] = y,$$

then we must have

$$c_1(\mathfrak{s})[\bar{D}^{(2n)}] = x + y.$$

So there is a product formula for computing $\mathrm{SHM}(V, \gamma^{2n})$ out of $\mathrm{SHM}(V, \gamma^{2n-4})$ and $\mathrm{SHM}(V, \gamma^6)$. After a degree shifting, this product formula is precisely the one we compute $H_*(T^{n-1})$ from $T^{n-1} = T^{n-3} \times T^2$ and hence we are done. □

One question arises in this argument. We shall first fix a suitable field $\mathcal{F}$ of characteristic 2. Then we have a by pass exact triangle just as in the proof of 4.1 for general (V, γ^{2n}):



From formula (2), we know that for $n > 1$,

$$\mathrm{SHM}(V, \gamma^{2n}) \cong (\mathcal{F})^{2^{n-2}}.$$

This force the map ψ being 0. Hence ρ is injective and ϕ is surjective. If we assume $n = 2$, then we know from [8] that there is a unique tight contact structure ξ_0 compatible with (V, γ^2). From [1] we know that the contact element of ξ_0 generates $\mathrm{SHM}(V, \gamma^2) \cong \mathcal{F}$. Since by-pass attachment preserves contact elements, we know that after the by-pass associated to ψ, ξ_0 becomes overtwisted, and after the by-pass associated to ρ, ξ_0 becomes a compatible contact structure ξ_1 on (V, γ^4), so that the contact element of ξ_1 generates $im(\rho) \cong \mathcal{F} \subset \mathrm{SHM}(V, \gamma^4)$. If there were another compatible contact structure ξ_2 on (V, γ^4) so that after the by-pass associated to ϕ, it becomes ξ_0 on (V, γ^2), then we know that $\mathrm{SHM}(V, \gamma^4)$ is simply generated by the two contact elements of ξ_1 and ξ_2. We can also try to use induction to look at general (V, γ^{2n}) then. However, by-pass attachments do not necessarily have inverses. So this lead to the following question:

Question 4.6. *Is* $\mathrm{SHM}(V, \gamma^{2n})$ *generated by contact elements of compatible contact structures?*

4.2 The connected sum formula

Now let us derive the connected sum formula for sutured monopoles. First we have the follow proposition.

Proposition 4.7. *We use* $\mathbb{Z}_2$ *coefficients. Suppose three oriented links* K_0*,* K_1 *and* K_2 *are the same outside a* 3*-ball* B^3 *and inside* B^3 *they are depicted as in figure 11. We have the following.*

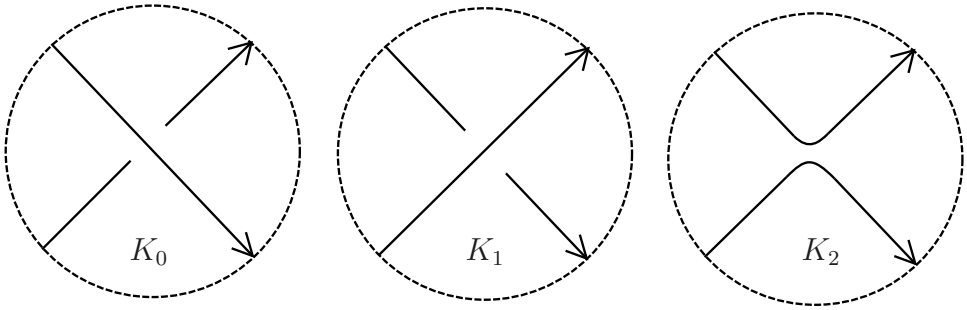


Figure 11: The oriented Skein relation.

(1). If K_2 has one more component than K and K_1, then there is an exact triangle:

$$\begin{array}{ccc} KHM(S^3,K) & \longrightarrow & KHM(S^3,K_1) \\ & \nwarrow \quad \swarrow & \\ & KHM(S^3,K_2) & \end{array}$$

(2). If K_2 has one less component than K and K_1, then there is an exact triangle:

$$\begin{array}{ccc} KHM(S^3,K) & \longrightarrow & KHM(S^3,K_1) \\ & \nwarrow \quad \swarrow & \\ & KHM(S^3,K_2)\otimes(\mathbb{Z}_2\oplus\mathbb{Z}_2) & \end{array}$$

Proof. It follows from an analogous argument in sutured instantons in [14]. The extra term in the second case actually rely on the sutured monopole Floer homology of the manifold (V,γ^4) with $\mathbb{Z}_2$ coefficients, and this is computed in lemma 4.1. □

As a corollary of the above proposition, we derive the following corollary independent of the work by [7] or [22].

Corollary 4.8. *With $\mathbb{Z}_2$ coefficients and the canonical $\mathbb{Z}_2$ grading of monopole Floer homology, the Euler characteristics of* $\mathrm{KHM}(S^3,K,i)$ *(for definition, see [15]) corresponds to the coefficients of a suitable version of Alexander polynomial of the knot $K\subset S^3$.*

Proof. It follows from an analogous argument in sutured instantons in [14]. □

Now we make the following notation.

Definition 4.9. Suppose Y is a closed oriented 3-manifold. Let $Y(n)$ denote the manifold obtained by removing n disjoint 3-balls from Y. We can make $Y(n)$ to be a balanced sutured manifold $(Y(n), \delta^n)$ by letting δ^n consisting of one simple closed curve on each boundary sphere of $Y(n)$.

The following two lemmas are straightforward:

Lemma 4.10. *Suppose Y is a closed oriented 3-manifold and $n \in \mathbb{Z}$ is no less than 2, then*

$$Y(n) \cong (Y(n-1) \sqcup S^3(2), \delta^{n-1} \cup \delta^2) \cup h,$$

where $h = (\phi, S, D^3, \delta)$ is a contact 1-handle so that ϕ send one component of S to $\partial Y(n-1)$ and the other component to $\partial S^3(2)$.

Lemma 4.11. *Suppose (M_1, γ_1) and (M_2, γ_2) are two balanced sutured manifolds (both of which has no empty sutures). Suppose $(S^3(2), \delta^2)$ is defined as in the defintion 4.9, and its two boundary components are*

$$\partial S^3(2) = S_1^2 \cup S_2^2.$$

Then we have

$$(M_1 \sharp M_2, \gamma \cup \gamma_2) \cong (M_1 \sqcup M_2 \sqcup S^3(2), \gamma_1 \cup \gamma_2 \cup \delta^2) \cup h_1 \cup h_2.$$

Here for $i = 1, 2$, $h_i = (\phi_i, S_i, D_i^3, \delta_i)$ is a contact 1-handle so that ϕ_i maps one component of S_i to ∂M_i and the other component of S_i to S_i^2.

Remark 4.12. In the above lemmas, we don't require a sutured manifold (M, γ) to have a global contact structure. However, we can identify a collar of the boundary to be identified with $\partial M \times [0, 1]$ and assume that there is an I-invariant contact structure in that collar so that ∂M is a convex surface with γ being the dividing set. Then the contact handle attachment makes sense.

From the above lemmas, we can see the significant role $(S^3(2), \delta^2)$ plays. So we will proceed to compute its sutured monopole Floer homology now.

Lemma 4.13. *For any closed 3-manifold Y and every positive integer n, there is an injective map*

$$\mathrm{SHM}(Y(n), \delta^n) \to \mathrm{SHM}(Y(n+1), \delta^{n+1}).$$

Proof. We can get $(Y(n+1), \delta^{n+1})$ from $(Y(n), \delta^n)$ by attaching a contact 2-handle and if we attach further a contact 3-handle, it will result in $(Y(n), \delta^n)$ again. The pair of handles form a 2-3 cancelation pair as in the paper [18] so the composition is the identity. □

Corollary 4.14. *We have* $\mathrm{SHM}(S^3(2), \delta^2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Proof. It follows from the proof of an analogous statement in sutured instantons in [2]. Some ingredients are different from their proof but are all discussed above.

□

Corollary 4.15. *Suppose (M_1, γ_1) and (M_2, γ_2) are balanced sutured manifolds. Then we have*

$$\mathrm{SHM}(M_1 \sharp M_2, \gamma_1 \cup \gamma_2) \cong \mathrm{SHM}(M_1 \sqcup M_2, \gamma_1 \cup \gamma_2) \otimes (\mathbb{Z}_2 \oplus \mathbb{Z}_2).$$

Corollary 4.16. *Suppose L is a link in S^3. Then for any coefficients,* $\mathrm{KHM}(S^3, L) \neq 0$.

There is another interesting observation. Suppose (M_1, γ_1) and (M_2, γ_2) are two balanced sutured manifolds and $h = (\phi, S, D^3, \delta)$ is a 1-handle so that ϕ maps one component of S to ∂M_1 and the other component to ∂M_2. Suppose $h' = (\phi', S', D^{3\prime}, \delta')$ is a 2-handle so that the core of S', which we denote by α', is mapped to a circle $\beta \subset \partial D^3$, so that it represents a generator of $H_1(\partial D^3 \backslash S)$. The result of first attaching h and then h' will resulting in a balanced sutured manifold (M, γ) which is diffeomorphic to $(M_1 \sharp M_2, \gamma_1 \cup \gamma_2)$. Hence we have a map:

$$C_{-h'} \circ C_{-h} : \underline{\mathrm{SHM}}(M_1 \sqcup M_2, \gamma_1 \cup \gamma_2) \to \underline{\mathrm{SHM}}(M_1 \sharp M_2, \gamma_1 \gamma_2),$$

and by the basic properties of gluing maps, we know that under the isomorphism

$$\underline{\mathrm{SHM}}(M_1 \sharp M_2, \gamma_1 \gamma_2) \cong \underline{\mathrm{SHM}}(M_1 \sqcup M_2, \gamma_1 \cup \gamma_2) \otimes \underline{\mathrm{SHM}}(S^3(2), \delta^2),$$

the above composition of handle gluing maps is just the identity on $\underline{\mathrm{SHM}}(M_1 \sqcup M_2, \gamma_1 \cup \gamma_2)$ tensor the contact element for a suitable contact structure $(-S^3(2), -\delta^2)$ in $\underline{\mathrm{SHM}}(S^3(2), \delta^2)$.

The discussion in Instanton settings would be completely analogous. We will use the field of complex numbers $\mathbb{C}$ as coefficients and have the following proposition:

Proposition 4.17. *Suppose (M_1, γ_1) and (M_2, γ_2) are two balanced sutured manifolds then*

$$\mathrm{SHI}(M_1 \sharp M_2, \gamma_1 \cup \gamma_2) \cong \mathrm{SHI}(M_1, \gamma_1) \otimes \mathrm{SHI}(M_2, \gamma_2) \otimes \mathbb{C}^2.$$

This formula can also be applied to the framed instanton Floer homology of closed 3-manifold. Suppose Y is a closed oriented 3-manifold, we can connect sum Y with T^3 and let ω be a circle which represent a generator of $H_1(T^3)$. The pair $(Y \sharp T^3, \omega)$ is then admissible and we can form the *framed instanton Floer homology* of Y:

$$I^\sharp(Y) = I^\omega(Y \sharp T^3).$$

In [15], Kronheimer and Mrowka discussed the relation between the framed instanton Floer homology of a closed 3-manifold and the sutured instanton Floer homology of $(Y(1), \delta^1)$. As a corollary to the connected sum formula for sutured instantons, we have the following.

Corollary 4.18. *Suppose Y_1 and Y_2 are two closed oriented 3-manifolds. Then as vector spaces over complex numbers, we have*

$$I^\sharp(Y_1) \otimes I^\sharp(Y_2) \cong I^\sharp(Y_1 \sharp Y_2).$$

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