Direct systems and knot Floer homology

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Abstract

In this paper we construct possible candidates for the minus version of monopole or instanton knot Floer homology. We use a direct system which was introduced by Etnyre, Vela-Vick and Zarev [7]. If $K \subset Y$ is a knot then we can construct a direct system based on a sequence of sutures on $\partial Y(K)$ and the direct limit is of our interests. We prove that a Seifert surface of the knot will induce an Alexander grading and there is a U map on the direct limit shifting the degree down by 1. We prove some basic properties and compute the case of unknots. We also use the techniques developed in this paper to compute the sutured monopole and instanton Floer homology of a solid torus with any valid sutures.

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1 Introduction

1.1 Statement of result

Floer homologies have become very important tools in the study of 3-manifolds, since the first construction by Floer [8]. Among them two major branches are the monopole, which was introduced by Kronheimer and Mrowka [15] and the Heegaard Floer, which was introduced by Oszváth and Szabó [23] or Rasmussen [24]. For a closed oriented 3-manifold Y, there are four flavors of homologies associated to Y in each of the two theories, and they have been known to be isomorphic by works of Kutluhan, Lee and Taubes in [18] and subsequent papers. If there is a knot K inside a 3-manifold Y, then there are corresponding four flavors of homologies of the pair (Y, K) in the Heegaard Floer theory. See Oszváth and Szabó [22]. However, some corresponding constructions in the monopole and instanton theory are missing. The only monopole or (non-singular) instanton Floer homology for knots in 3-manifolds is the sutured version constructed by Kronheimer and Mrowka [16], and is refined by Baldwin and Sivek [2]. The monopole version is proved to be isomorphic to the hat version of the knot Heegaard Floer homology by Baldwin and Sivek [5] or Lekili [19]. In this paper, we are going to construct homologies associated to a based non-homologous knot, which are candidates for the monopole and instanton correspondences of the minus version of the knot (Heegaard) Floer homology.

Theorem 1.1. *Suppose Y is a closed connected oriented 3-manifold and $K \subset Y$ is an oriented knot. Suppose S is a Seifert surface of K and $p \in K$*

is a fixed base point. Then we can associate the triple (Y, K, p) *a module* $\underline{\mathrm{KHM}}^-(Y, K, p)$ *over the mod 2 Novikov Ring* $\mathcal{R}$*. It is well defined up to multiplication by a unit in* $\mathcal{R}$*. The Seifert surface* S *induces a* $\mathbb{Z}$ *grading on* $\underline{\mathrm{KHM}}^-(Y, K, p)$*, which we denote by* $\underline{\mathrm{KHM}}^-(Y, K, P, S, i)$*. Moreover, the following properties hold:*

(1). For $i > g = g(S)$*,* $\underline{\mathrm{KHM}}^-(Y, K, p, S, i) = 0$*.*

(2). There is a map

$$U : \underline{\mathrm{KHM}}^-(Y, K, p) \to \underline{\mathrm{KHM}}^-(Y, K, p)$$

which is of degree -1*.*

(3). There exists an $N_0 \in \mathbb{Z}$*, such that if* $i < N_0$*, then*

$$U : \underline{\mathrm{KHM}}^-(Y, K, p, S, i) \cong \underline{\mathrm{KHM}}^-(Y, K, p, S, i-1).$$

(4). There exists an exact triangle

$$\begin{tikzcd} \underline{\mathrm{KHM}}^-(Y,K,p) \arrow[rr, "U"] & & \underline{\mathrm{KHM}}^-(Y,K,p) \arrow[dl, "\psi"] \\ & \underline{\mathrm{KHM}}(Y,K,p) \arrow[ul, "\psi'"] & \end{tikzcd}$$

(5). If $Y = S^3$ *and* S *realizes the genus of the knot, then we have*

$$\underline{\mathrm{KHM}}^-(Y, K, p, S, i) \neq 0$$

for $i = g(S)$ *and* $i < N_0$ *with the same* N_0 *as in (3).*

Theorem 1.2. *With the same setting as in theorem 1.1, we can construct* $\underline{\mathrm{KHI}}^-(Y, K, p)$ *using instanton Floer homology so that all the properties (1)-(5) in the above theorem hold.*

It worth mentioning here that Kutluhan [17] constructed another minus version of knot monopole Floer homology in a different way. He used the holonomy filtration for the construction.

1.2 Outline of the proof

We shall only present in this susbeciton with the monopole case. The construction of $\underline{\mathrm{KHM}}^-(Y, K, p)$ is based on the sutured monopole Floer homology. A sutured manifold (M, γ) is a compact oriented 3 manifold with an oriented 1-submanifold γ on ∂M which we call the suture. The suture divides ∂M into two parts, according to the orientations of the suture and the

3-manifold, which we call $R_-(\gamma)$ and $R_+(\gamma)$ respectively. Sutured manifolds were first introduced by Gabai [9]. Kronheimer and Mrowka then carried out the construction of the monopole Floer homology on balanced sutured manifolds in [16].

A sutured manifold (M,γ) is called balanced if $M, R_{\pm}(\gamma)$ have no closed components and $\chi(R_-(\gamma)) = \chi(R_+(\gamma))$. To define the sutured monopole Floer homology for such an (M,γ), Kronheimer and Mrowka constructed a closed 3-manifold Y together with a distinguishing surface R out of (M,γ). The pair (Y,R) is called a closure of (M,γ). Sometimes we simply call Y a closure. The genus of the closure refers to the genus of the surface R. To construct the closure, one need to first find an oriented surface T whose boundary is diffeomorphic to γ, and then glue $T\times[-1,1]$ to M with $\partial T\times[-1,1]$ identified with an annular neighborhood of $\gamma\subset\partial M$. The surface T is called an auxiliary surface. The new 3-manifold after the gluing is called a pre-closure and has two boundary components, R_+ and R_-, of the same genus. So we can pick a diffeomorphism h from R_+ to R_- to glue the two boundary components together to get the closure (Y,R). We call h a gluing diffeomorphism.

To study the naturality of the sutured monopole Floer homology, Baldwin and Sivek [1] constructed canonical maps between two different closures of a same balanced sutured manifold (M,γ). Their construction is only well-defined up to multiplication by a unit, so the closures and canonical maps form a projective transitive system and will result in a canonical module $\underline{\mathrm{SHM}}(M,\gamma)$, whose elements are well defined only up to a unit.

The construction of the (canonical) module $\underline{\mathrm{KHM}}^-(Y,K,p)$ was inspired by Etnyre, Vela-Vick and Zarev [7], where they use a sequence of balanced sutured manifolds $(Y(K),\Gamma_n)$, and gluing maps in sutured (Heegaard) Floer theory, which was introduced by Honda, Kazez and Matić [12], to construct a direct system. They proved that the direct limit is isomorphic to the classical minus version of knot Heegaard Floer homology. Here $Y(K)=Y\backslash\mathrm{int}(N(K))$ is the knot complement, and Γ_n consists of two curves on $\partial Y(K)\cong T^2$, which are of class $\pm(-n,1)$ under a framing induced by some Seifert surface. We are going to construct the same direct system in sutured monopoles. In details, there is a commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{SHM}}(-Y(K),-\Gamma_n) \arrow[r, "\psi^n_{-,n+1}"] \arrow[d, "\psi^n_{+,n+1}"] & \underline{\mathrm{SHM}}(-Y(K),-\Gamma_{n+1}) \arrow[d, "\psi^{n+1}_{+,n+2}"] \\
\underline{\mathrm{SHM}}(-Y(K),-\Gamma_{n+1}) \arrow[r, "\psi^{n+1}_{-,n+2}"] & \underline{\mathrm{SHM}}(-Y(K),-\Gamma_{n+2})
\end{tikzcd} \tag{1}$$

where the balanced sutured manifolds are described as above, and the maps come from gluing maps in sutured monopoles constructed by the author in [20].

The knot Floer homology $\underline{\mathrm{KHM}}(Y,K,p)$ introduced by Kronheimer and Mrowka [16] is based on the balanced sutured manifold $(Y(K),\Gamma_\infty)$, where Γ_∞ consists of two meridians on $\partial Y(K)$.

The commutativity of the above diagram (1) is guaranteed by the functoriality of the gluing map. The crucial difference from Etnyre, Vela-Vick and Zarev [7] is that because of the involvement of closures, the construction of the grading in sutured monopoles is a delicate issue. We construct the grading in the direct limit in two steps.

The first step is to construct a grading on each $\underline{\mathrm{SHM}}(Y(K),\Gamma_n)$, using a Seifert surface S, for any n. For a fixed n, the boundary of the Seifert surface S intersects Γ_n at least $2n$ times. To construct such a grading, we work with the general case where (M,γ) is an arbitrary balanced sutured manifold, S is a properly embedded surface whose boundary has only one component, and ∂S intersects γ transversely at $2n$ points.

For the case $n=1$, the construction has already been carried out by Baldwin and Sivek [6]. When $n=1$, we can pick a properly embedded arc $\alpha\subset T$, where T is an auxiliary surface of (M,γ). When glue $T\times[-1,1]$ to M, we shall require that the end points of α are glued to the two intersection points $\partial S\cap\gamma$ and hence $\alpha\times[-1,1]$ is glued to S along $\partial\alpha\times[-1,1]$. Then S becomes a new surface $\widetilde{S}$ inside the pre-closure $\widetilde{M}$. Note $\widetilde{M}$ has two boundary components $R_\pm$ and the two boundary components of $\widetilde{S}$ are contained in two different boundary components of $\widetilde{M}$. Then we shall pick a gluing diffeomorphism $h:R_+\to R_-$ which also identifies the two boundary components of $\widetilde{S}$. Hence $\widetilde{S}$ becomes a closed surface $\bar{S}$ inside the closure Y of (M,γ). The grading can thus be defined by looking at the pairing of the first Chern classes of the spin^c structures on Y with the fundamental class of $\bar{S}$. This idea was first introduced by Kronheimer and Mrowka [16] and Baldwin and Sivek [6] proved that the definition of the grading is independent of all choices made in the construction, and hence is well defined in $\underline{\mathrm{SHM}}(M,\gamma)$.

For a general n, the basic idea is the same but there are more choices to be made and thus many new issues arise. For example, now we will need n arcs $\alpha_1,...,\alpha_n$ instead of just one and we shall determine which arc is going to connect which two intersection points of ∂S with γ, and thus leading to an interesting combinatorial problem. We will deal with it in subsection 3.3. Along with the proof, we will also need to use a new interpretation of Baldwin and Sivek's canonical maps between different closures. We will use just the Floer excisions introduced by Kronheimer and Mrowka [16] to construct an equivalent canonical map. This will be covered in subsection 3.2.

In the above construction, actually the closed surface $\bar{S}$ could only be constructed out of S when n is odd. If n is even, then we need to perturb S to create a new pair of intersection points. There are two different ways of perturbations, which we call positive and negative stabilizations, and write S^+ and S^- respectively. Based on S^+ and S^-, we can construct two different gradings on $\underline{\mathrm{SHM}}(Y(K),\Gamma_n)$. The relation between the two gradings will be the key to the second step of constructing the grading for the direct limit. Also using the degree shifting property, we can compute the sutured monopole Floer homology of a solid torus with any valid suture.

Proposition 1.3. *Suppose V is a solid torus and γ is a suture on ∂V with $2n$ components and slope $\frac{p}{q}$, then*

$$\underline{\mathrm{SHM}}(-V,-\gamma)\cong\mathcal{R}^{(2^{n-1}\cdot|p|)}.$$

Similarly, in instanton theory, we can get

Proposition 1.4. *Suppose V is a solid torus and γ is a suture on ∂V with $2n$ components and slope $\frac{p}{q}$, then*

$$\underline{\mathrm{SHI}}(-V,-\gamma)\cong\mathbb{C}^{(2^{n-1}\cdot|p|)}.$$

The second step of constructing the grading for the direct limit is to prove that maps in the commutative diagram (1) will shift the grading in a desired way. To be explicit, $\psi^n_{-,n+1}$ shall be of degree 0 while $\psi^n_{+,n+1}$ shall be of degree -1. The construction of the maps $\psi^n_{\pm,n+1}$ rely on the by-pass attachments, which are realized by contact handle attachments in sutured monopoles, as introduced by Baldwin and Sivek [3].

It is a basic observation that the region we attach contact handles is disjoint from the Seifert surface S, hence if we look at the grading associated to the 'correct' surfaces, then $\psi^n_{-,n+1}$ and $\psi^n_{+,n+1}$ will both preserve the degree. However the 'correct' surfaces involves both positive and negative stabilizations, while in order to define a canonical grading on $\underline{\mathrm{SHM}}(Y(K),\Gamma_n)$, we only use negative stabilizations. Hence the problem is reduced to understanding the degree shifting between S^+ and S^-.

To understand this degree shifting property, we need a better understanding of the construction of the closures, and how spin^c structures on different closures are related by canonical maps. In particular, we prove the following result.

Proposition 1.5. *Suppose $(Y(K),\Gamma_n)$ is the balanced sutured manifold described as above, and Y_n is a closure of $(Y(K),\Gamma_n)$. Suppose $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are two spin^c structures on Y_n, so that both of them support the sutured monopole Floer homology of $(Y(K),\Gamma_n)$, then in terms of Poincáre duals of first Chern classes of the spin^c structures, the difference between $\mathfrak{s}_1$ and $\mathfrak{s}_2$ lies in $H_1(Y(K))$. More precisely, there is a 1-cycle $x\subset Y(K)$, so that*

$$P.D.(c_1(\mathfrak{s}_1)-c_1(\mathfrak{s}_2))=[x]\in H_1(Y).$$

We will deal with the basic properties of the direct limit in subsection 5.2. Most of them have been stated in theorem 1.1. Besides them, we can also prove that the direct system stabilizes:

Proposition 1.6. *For a fixed $i\in\mathbb{Z}$, there exists $N_1\in\mathbb{Z}$, such that for $n>N_1$, we have an isomorphism:*

$$\psi^n_{-,n+1}:\underline{\mathrm{SHM}}(-Y(K),-\Gamma_n,i)\cong\underline{\mathrm{SHM}}(-Y(K),-\Gamma_{n+1},i).$$

Moreover, a similar result in instanton theory also holds.

The techniques used in computing the sutured Floer homology of a solid torus can also be applied to knot complements. As a result, we obtain the following.

Proposition 1.7. *Suppose $K\subset Y$ is a knot and $S\subset Y$ is a Seifert surface of K. Suppose Y_ϕ is the manifold obtain from Y by doing a Dehn surgery along K with slope $-\frac{p}{q}$ with $p,q>0$. We also have the dual knot $K_\phi\subset Y$. Then for any fixed i, there exists $N\in\mathbb{R}$, such that if the surgery slope $-\frac{p}{q}<N$, then we have*

$$\underline{\mathrm{KHM}}^-(-Y_\phi,K_\phi,S,i)\cong\underline{\mathrm{KHM}}^-(-Y,K,S,i).$$

Moreover, a similar result in instanton theory also holds.

1.3 Future questions

The first to be asked is how the projective module $\underline{\mathrm{KHM}}^-(Y,K,p)$ is related to $HFK^-(Y,K)$. In [7] Etnyre, Vele-Vick and Zarev used a similar direct system and they have proved that the direct system is isomorphic to HFK^-. The sutured monopoles and sutured Heegaard Floer homology homologies of $(Y(K),\Gamma_n)$ are isomorphic, and the gluing maps in sutured monopole and Heegaard Floer settings also share many similarities in their constructions. See the author's previous paper [20] and Juhász and Zemke [14]. Hence we would like to make the following conjecture:

Conjecture 1.8. *There is an isomorphism*

$$\underline{\mathrm{KHM}}^-(Y,K,p)\cong HFK^-(Y,K)\otimes\mathcal{R},$$

where HFK^- uses $\mathbb{Z}_2$ coefficients.

In the paper we construct an Alexander $\mathbb{Z}$-grading, but it is still unknown whether there are other gradings. In particular, we would like to ask the following.

Question 1.9. *Can we construct a $\mathbb{Z}_2$ grading on $\underline{\mathrm{KHM}}^-$ based on the canonical $\mathbb{Z}_2$ grading on the monopole Floer homology?*

Throughout the paper we use mod 2 Novikov rings for local coefficients. It might be interesting to ask whether we could use other coefficients. There are two directions to think about. The first is try to work in characteristic 0. The reason why we need to work in characteristic 2 is that the current version of surgery exact triangle in the monopole theory is only proved without taking orientations into account. However, the construction of by-pass exact triangles in [3] relies on surgery exact triangles and without by-pass exact triangles, we cannot obtain stabilization properties as well as exact triangles relating $\underline{\mathrm{KHM}}^-(Y,K,p)$ and $\underline{\mathrm{KHM}}(Y,K,p)$. The second direction is to try to deal with the situation when local coefficients are absent. For technical reasons, if we want to construct the canonical maps between closures of different genus, then local coefficients are necessary. However, our present construction for the grading involves the usage of closures with arbitrarily large genus. In summary, we would like to ask the following question:

Question 1.10. *Can we construct the same direct system with grading, and having the same nice properties (1)-(5) as in theorem 1.1, but using $\mathbb{Z}$ coefficients?*

For knot monopole Floer homologies, there is currently no construction for the cobordism maps. The cobordism maps for $\underline{\mathrm{KHM}}(Y,K,p)$ or $\underline{\mathrm{KHI}}(Y,K,p)$ can be obtained directly from the cobordism maps for sutured monopole Floer homology and the same construction in sutured Heegaard Floer homology, but the construction of cobordism maps for $\underline{\mathrm{KHM}}^-(Y,K,p)$ or $\underline{\mathrm{KHI}}^-(Y,K,p)$ are different, since we need to construct a sequence of maps related to the direct system to induce one on the direct limit.

Question 1.11. *Can we construct cobordism maps for $\underline{\mathrm{KHM}}^-(Y,K,p)$ and $\underline{\mathrm{KHI}}^-(Y,K,p)$?*

In the Heegaard Floer theory, we have surgery formulas relating the knot Floer homology with the Heegaard Floer homology of the surgery manifold when surge along a knot. It might be useful to develop a similar formula in the monopole theory and the instanton theory. The latter might be of more interests as the instanton theory is closely related to representations of fundamental groups. We would like to ask

Question 1.12. *Can we develop a surgery formula for $\underline{\mathrm{KHM}}^-$ or $\underline{\mathrm{SHI}}^-$?*

In the paper we analyze spin^c structures of closures of knot complements. It is natural to ask whether the same conclusion holds for general balanced sutured manifolds, and what if we look directly at spin^c structures, not just their first Chern classes. Recall if the first homology of the closure does not have 2-torsions, then the spin^c structures and their first Chern classes are in one to one correspondence. However this is not true if 2-torsion do exist.

Question 1.13. *Suppose (M,γ) is a balanced sutured manifold, and Y is a closure of Y. Suppose $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are two spin^c structures, both supporting the sutured monopole Floer homology of (M,γ). Then their difference can be interpreted as a line bundle $\mathcal{L}$ over Y. Is it always true that $\mathcal{L}$ can be trivialized on $Y\backslash\mathrm{int}(M)$?*

If the answer to the above question is affirmative, or we could at least deal with knot complements, then we can further study the following question. Suppose (M,γ) is a balanced sutured manifold and Y_1 and Y_2 are two different closures between them. Kronheimer and Mrowka [16] proved that the sutured monopole Floer homology of (M,γ) defined using Y_1 and Y_2 are isomorphic. The above question serves as a refinement of their proof of isomorphism: not only the total homologies are isomorphic, but also the spin^c structures have a one-to-one correspondence. So we can ask further:

Question 1.14. *Can we see this isomorphism in the chain level? In details, if we choose suitable auxiliary data for both Y_1 and Y_2, can we directly relate the solution of Seiberg-Witten equations and flow lines between them?*

Kronheimer and Mrowka's proof that the sutured monopole Floer homology is independent of the closures gives some intuition that the essential data are all contained in the original balanced sutured manifold (M,γ). The above question might offer more evidence for this intuition. For a knot complement, there is a particular closure described as follows. Suppose $(Y(K),\Gamma_n)$ is described as above, and Σ is a closed connected oriented surface of large enough genus. Let α be a non-separating simple closed curve on Σ. In $\Sigma\times S^1$, identify α with $\alpha\times\{t\}$ for some $t\in S^1$, and we can remove a tubular neighborhood of $\alpha\subset\Sigma\times S^1$ and glue it to $Y(K)$:

$$Y=Y(K)\underset{\phi}{\cup}\Sigma\times S^1\backslash\mathrm{int}(N(\alpha)).$$

Proposition 1.5 implies that any spin^c structures on Y which support the sutured monopole Floer homology of $(Y(K),\Gamma_n)$ would restrict to a unique spin^c structure on $\Sigma\times S^1\backslash\mathrm{int}(N(\alpha))$. We know that $H_1(\Sigma\times S^1)$ has no 2-torsions so there is a unique spin^c structure on $\Sigma\times S^1$ whose first Chern class is the Poincáre dual of $(2g(\Sigma)-2)$ many copies of the curve $\{s\}\times S^1\subset\Sigma\times S^1$. We call this spin^c structure also $\mathfrak{s}_0$ and the unique one on $\Sigma\times S^1\backslash\mathrm{int}(N(\alpha))$ is just the restriction of $\mathfrak{s}_0$.

We could guess that the sutured monopole Floer homology of $(Y(K),\Gamma_n)$ might be obtained by glue the solutions to Seiberg-Witten equations on $Y(K)$ and $(\Sigma\times S^1\backslash\mathrm{int}(N(\alpha)),\mathfrak{s}_0)$ together along the boundary torus. If one can describe what happens explicitly, then it would shed some light on a more analytical way of constructing sutured monopole Floer homology (which may have better naturality), all flavors of knot monopole Floer homology and even a Bordered theory in monopole Floer homology.

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2 Prelimilaries

2.1 Balanced sutured manifolds and monopoles

We will start with the definition of balanced sutured manifolds.

Definition 2.1. A *balanced sutured manifold* is a pair (M,γ) of a compact oriented 3-manifold M with non-trivial boundary and an oriented 1-submanifold

$\gamma \subset \partial M$. On ∂M, let $A(\gamma) = \gamma \times [-1, 1]$ be an annular neighborhood of γ, and let

$$R(\gamma) = \partial M \backslash \mathrm{int}(A(\gamma)).$$

They shall satisfy the following requirements:

(1). Both M and $R(\gamma)$ have no closed components.

(2). If we orient $\partial R(\gamma) = \partial A(\gamma) = \gamma \times \{\pm 1\}$ in the same way as γ, then the orientation on $\partial R(\gamma)$ shall induce a unique orientation on $R(\gamma)$. This orientation is called the *canonical orientation* on $R(\gamma)$. Use $R_+(\gamma)$ to denote the part of $R(\gamma)$ whose canonical orientation coincides with the boundary orientation of ∂M and $R_-(\gamma)$ the rest.

(3). We have that

$$\chi(R_+(\gamma)) = \chi(R_-(\gamma)).$$

To define the sutured monopole Floer homology, we need to construct a closed 3-manifold out of a balanced sutured manifold (M, γ). Let T be a connected oriented surface, so that:

(1). There is an orientation reversing diffeomorphism

$$f : \partial T \to \gamma.$$

(2). There is a simple closed curve $c \subset T$ so that $[c] \neq 0 \in H_1(T)$.

(3). T has genus at least 2.

When we choose such a T, we can use f to glue T to M:

$$\widetilde{M} = M \underset{f}{\cup} T \times [-1, 1].$$

The manifold $\widetilde{M}$ is called a *pre-closure* of (M, γ) and it has two boundary components:

$$\partial \widetilde{M} = R_+ \cup R_-,$$

where

$$R_\pm = R_\pm(\gamma) \underset{f}{\cup} T \times \{\pm 1\}.$$

Let $h : R_+ \to R_-$ be an orientation preserving diffeomorphism, then we can form a closed 3-manifold as

$$Y = \widetilde{M} \underset{id \cup h}{\cup} R_+ \times [-1, 1],$$

where $h : R_+ \times \{1\} \to R_- \subset \partial \widetilde{M}$ is the map just defined and $id : R_+ \times \{-1\} \to R_+ \subset \partial \widetilde{M}$ is the identity on R_+. Let $R = R_+ \times \{0\} \subset Y$, and we make the following definition:

Definition 2.2. The pair (Y, R) is called a *closure* of the balanced sutured manifold (M, γ). The choices T, f, c, h are called the auxiliary data. In particular, the surface T is called an *auxiliary surface* and h is a *gluing diffeomorphism.*

Remark 2.3. Throughout this paper, we shall require that T is connected and has large enough genus. However, in general, the choice of auxiliary surface shall have more freedoms. See [16].

To use local coefficients, we shall also need to choose a non-separating simple closed curve $\eta \subset R$. The coefficient ring we use for the present paper will be the mod 2 Novikov ring. For detailed definitions, readers are referred to [3].

Definition 2.4. Suppose (Y, R) is a closure of (M, γ) as above. If R is connected, we define the set of *top* spin^c structures as follows:

$$\mathfrak{S}(Y|R) = \{\mathrm{spin^c\ structure\ }\mathfrak{s}\mathrm{\ on\ Y}|\mathrm{c_1}(\mathfrak{s})[\mathrm{R}] = 2\mathrm{g}(\mathrm{R}) - 2.\}$$

If in any case R is disconnected and let $R_1, ..., R_n$ be its components, then we define

$$\mathfrak{S}(Y|R) = \bigcap_{i=1}^{n} \mathfrak{S}(Y|R_i).$$

For later references, we also define the set of spin^c structures which *support the sutured monopole Floer homology* as follows:

$$\mathfrak{S}^*(Y|R) = \{\mathfrak{s} \in \mathfrak{S}(Y|R)|\widetilde{HM}_{\bullet}(Y, \mathfrak{s}; \Gamma_\eta) \neq 0\}.$$

For monopoles on closed 3-manifolds, readers are referred to [15].

Definition 2.5. The *sutured monopole Floer homology* of (M, γ) is defined to be

$$SHM(M, \gamma) = HM(Y|R; \Gamma_\eta),$$

where

$$HM(Y|R; \Gamma_\eta) = \bigoplus_{\mathfrak{s} \in \mathfrak{S}(Y|R)} \widetilde{HM}_{\bullet}(Y, \mathfrak{s}; \Gamma_\eta)$$

The following lemmas from Kronheimer and Mrowka [16] will be useful.

Lemma 2.6. *Suppose Y is a surface bundle over S^1 whose fibres are closed connected oriented surfaces of genus at least 2. Let R be a fibre and $\eta \subset R$ be a non-separating simple closed curve. Then there is a unique spin^c structure $\mathfrak{s}$ on Y so that*

(1). We have $c_1(\mathfrak{s})[R] = 2g(R) - 2$.

(2). We have $\widetilde{HM}_{\bullet}(Y,\mathfrak{s};\Gamma_\eta)\neq 0$.
Moreover, for this spin^c *structure we have*

$$\widetilde{HM}_{\bullet}(Y,\mathfrak{s};\Gamma_\eta)\cong\mathcal{R},$$

where $\mathcal{R}$ *is the mod* 2 *Novikov ring which we use for the local coefficients.*

Lemma 2.7. *Suppose* Y *is a closed oriented* 3*-manifold and* $R\subset Y$ *is an embedded closed connected oriented surface of genus at least one. Suppose* $\mathfrak{s}$ *is a* spin^c *structure such that*

$$|c_1(\mathfrak{s})[R]|>2g(R)-2,$$

then we have

$$\widetilde{HM}_{\bullet}(Y,\mathfrak{s};\Gamma_\eta)=0,$$

for any choice of local coefficients.

Floer excisions were introduced into sutured monopoles by Kronheimer and Mrowka [16]. We will summarize the results we need in the rest of the subsection.

For $i=1,2$, suppose Y_i is a closed connected oreinted 3-manifold and $R_i\subset Y_i$ is an embedded closed connected oriented homologically essential surface of genus at least 2. Let $\eta_i\subset R_i$ be a non-separating simple closed curve. When cutting Y_i open along R_i, we get

$$\widetilde{Y}_i=Y_i\backslash\mathrm{int}(N(R_i)),$$

where $N(R_i)$ is a product neighborhood of $R_i\subset Y_i$. The manifold $\widetilde{Y}_i$ has two boundary components

$$\partial\widetilde{Y}_i=R_{i,+}\cup R_{i,-}.$$

We orient $R_{i,\pm}$ in the same way as R_i. There are parallel copies of η_i, which we call $\eta_{i,\pm}$, on the surfaces $R_{i,\pm}$. Pick an orientation preserving diffeomorphism

$$h:R_1\to R_2,$$

so that $h(\eta_1)=\eta_2$. We can use h to glue $R_{1,+}$ to $R_{2,-}$ and also $R_{1,-}$ to $R_{2,+}$. Then $\widetilde{Y}_1$ and $\widetilde{Y}_2$ are glued together to become a connected 3-manifold which we call Y. Let $R\subset Y$ be the disjoint union of surfaces $R_{1,+}$ and $R_{2,+}$ in Y. Let $\eta\subset R$ be the disjoint union of curves $\eta_{1,+}$ and $\eta_{2,+}$.

There is a 4-dimensional cobordism W from $Y_1\sqcup Y_2$ to Y as follows. Let U be the surface as depicted in figure 1. It has four vertical arcs as part of the boundary, and we can assume that each one of them is identified with

$[0, 1]$. Now we can use the identity and h to glue three pieces $\widetilde{Y}_1$, $\widetilde{Y}_2$ and $R_1 \times U$ together to get the desired cobordism. The cobordism W induces a map as in [16]

$$HM(W) : HM(Y_1 \sqcup Y_2 | R_1 \cup R_2; \Gamma_{\eta_1 \cup \eta_2}) \to HM(Y|R; \Gamma_\eta). \qquad (2)$$

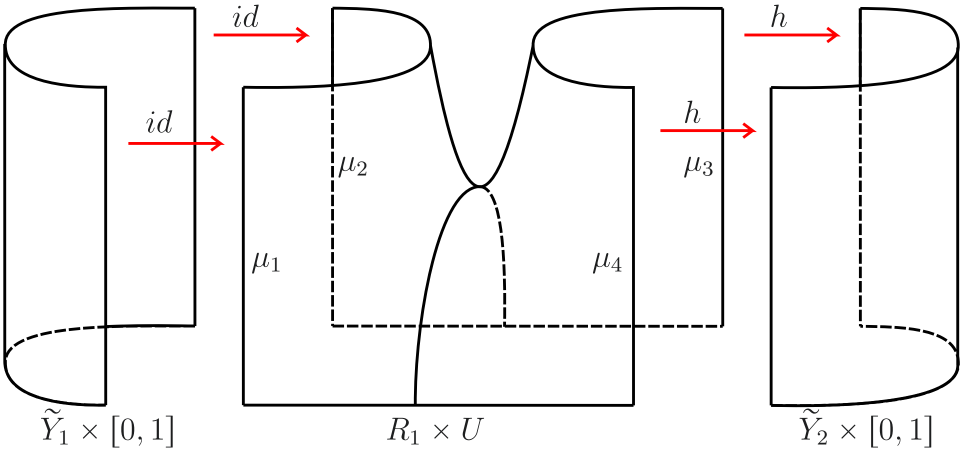


Figure 1: Gluing three parts together to get W. The middle part is $R_1 \times U$, while the $R_{1,+}$ directions shrink to a point in the figure.

We can also cut along tori. For $i = 1, 2$, let Y_i be as above. Let $T_i \subset Y_i$ be a torus and $R_i \subset Y_i$ be a closed connected oriented surface, so that R_i intersects T_i transversely along a circle c_i. Suppose $\eta_i \subset R_i$ is a simple closed curve so that η_i intersects c_i transversely at a point p_i. Let

$$h : T_1 \to T_2$$

be an orientation preserving diffeomorphism so that $h(c_1) = c_2$ and $h(p_1) = p_2$. As above, we can cut Y_i open along T_i and re-glue using h to get a connected 3-manifold Y. There is a distinguishing surface R, obtained by cutting R_i open along c_i and re-glue using h. The curve η_1 and η_2 are also cut and re-glued to result in a simple closed curve $\eta \subset R \subset Y$. As above, there is a cobordism map

$$HM(W) : HM(Y_1 \sqcup Y_2 | R_1 \cup R_2; \Gamma_{\eta_1 \cup \eta_2}) \to HM(Y|R; \Gamma_\eta). \qquad (3)$$

For more details of the excision process, readers are referred to Kronheimer and Mrowka [16]. In that paper, the follow theorem is proved.

Theorem 2.8. *(Kronheimer, Mrowka, [16].) The maps (2) and (3) are both isomorphisms.*

2.2 The naturality of sutured monopoles

Baldwin and Sivek [1] constructed canonical maps for two different closures of the same balanced sutured manifold. In order to do this, they also refined the definition of the closure.

Definition 2.9. A *marked closure* $\mathcal{D}=(Y,R,r,m,\eta)$ of a balanced sutured manifold (M,γ) consists of the following:

(1). A closed connected oriented 3-manifold Y.
(2). A closed connected oriented surface R.
(3). An orientation preserving embedding

$$r:R\times[-1,1]\hookrightarrow Y.$$

(4). An orientation preserving embedding

$$m:M\hookrightarrow Y\backslash\mathrm{int}(\mathrm{im}(r)).$$

(5). A non-separating simple closed curve $\eta\subset R$.
They shall satisfy following requirements:
(a). The embedding m extends to a diffeomorphism

$$M\underset{f}{\cup}T\times[-1,1]\rightarrow Y\backslash\mathrm{int}(\mathrm{im}(r)),$$

for some auxiliary data (T,f).

(b). The embedding m restricts to an orientation preserving embedding

$$R_+(\gamma)\hookrightarrow r(R\times\{-1\}).$$

The *genus* of the closure is referred to the genus of the surface R. We define

$$SHM(\mathcal{D})=\bigoplus_{\mathfrak{s}\in\mathfrak{S}(Y|r(R\times\{0\}))}\widetilde{HM}_\bullet(Y,\mathfrak{s};\Gamma_{r(\eta\times\{0\})}).$$

Theorem 2.10. *(Baldwin and Sivek, [2]) Suppose (M,γ) is a balanced sutured manifold, then for any two marked closures $\mathcal{D}_1$ and $\mathcal{D}_2$ of (M,γ), there is a canonical map $\Phi_{\mathcal{D}_1,\mathcal{D}_2}$, well defined up to a unit, from $SHM(\mathcal{D}_1)$ to $SHM(\mathcal{D}_2)$. The canonical maps satisfy following properties.*

(1). If $\mathcal{D}_1=\mathcal{D}_2$, then

$$\Phi_{\mathcal{D}_1,\mathcal{D}_2}\doteq id.$$

Here $\doteq$ means equal up multiplication by a unit.

(2). Suppose we have a third marked closure $\mathcal{D}_3$ for (M,γ), then we have

$$\Phi_{\mathcal{D}_1,\mathcal{D}_3}\doteq\Phi_{\mathcal{D}_2,\mathcal{D}_3}\circ\Phi_{\mathcal{D}_1,\mathcal{D}_2}.$$

Hence for a balanced sutured manifold (M, γ), marked closures $\mathcal{D}$ and canonical maps Φ fits into a projective transitive system, which is defined in [2]. The projective system determines a canonical module, which we shall denote by

$$\underline{\mathrm{SHM}}(M, \gamma).$$

We can then talk about elements (up to multiplication by a unit) in that canonical module.

Remark 2.11. There are two ways to think about $\underline{\mathrm{SHM}}(M, \gamma)$. The first is to think of it as a module over $\mathcal{R}$ but whose elements are only well defined up to a unit. The second way is to think it as a well defined set, obtained by a module over $\mathcal{R}$ quotient by $\mathcal{R}^{\times}$. We will not distinguish between the two descriptions.

We will have an extra complexity if we deal with knots in 3-manifolds. Let $K \subset Y$ be a knot. This extra complexity comes from the choices of tubular neighborhoods of $K \subset Y$ to remove to get knot complements. Fix a point $p \in K$. Suppose

$$\varphi : S^1 \times D^2 \hookrightarrow Y$$

is an embedding, where D^2 is the unit sphere in the complex plane, and $S^1 = \partial D^2$. We shall require that

$$\varphi(S^1 \times \{0\}) = K \text{ and } \varphi(\{1\} \times \{0\}) = p.$$

Now let $Y_\varphi = Y \backslash \mathrm{int}(\mathrm{im}(\phi))$, and let $\gamma_\varphi = \varphi(\{\pm 1\} \times \partial D^2)$, with opposite orientations on two components. For each fixed φ, we have a well defined canonical module $\underline{\mathrm{SHM}}(Y(\varphi), \gamma_\varphi)$, and we want also relate different choices of φ.

Suppose φ' is another embedding $S^1 \times D^2 \hookrightarrow Y$, satisfying the same conditions as φ. Pick a tubular neighborhood N of $K \subset Y$ such that $\mathrm{im}(\varphi), \mathrm{im}(\varphi') \subset N$ and an ambient isotopy

$$f_t : Y \to Y,\ t \in [0, 1],$$

such that:

(1). For any $t \in [0, 1]$, $f_t(p) = p$.

(2). For any $t \in [0, 1]$, f_t restricts to identity outside $N \subset Y$.

(3). We have $f_1(\mathrm{im}(\varphi)) = \mathrm{im}(\varphi')$.

(4). We have $f_1(\varphi(\{\pm 1\} \times \partial D^2)) = \varphi'(\{\pm 1\} \times D^2)$.

It is clear that $f_1 : (Y_\varphi, \gamma_\varphi) \to (Y_{\varphi'}, \gamma_{\varphi'})$ is a diffeomorphism between balanced sutured manifolds. Hence we can define

$$\Psi_{\varphi, \varphi'} = \underline{\mathrm{SHM}}(f_1) : \underline{\mathrm{SHM}}(Y_\varphi, \gamma_\varphi) \to \underline{\mathrm{SHM}}(Y_{\varphi'}, \gamma_{\varphi'}).$$

Theorem 2.12. *(Baldwin, Sivek, [2]) The map $\Psi_{\varphi,\varphi'}$ is well defined, i.e., is independent of choices of the tubular neighborhood N and the ambient isotopy f_t. Also it has the following properties:*

(1). We have $\Psi_{\varphi,\varphi} = id$.

(2). If we have a third embedding φ'', then

$$\Psi_{\varphi,\varphi''} = \Psi_{\varphi',\varphi''} \circ \Psi_{\varphi,\varphi'}.$$

Thus we know that $\{\underline{\mathrm{SHM}}(Y_\varphi, \gamma_\varphi)\}$ and $\{\Psi_{\varphi,\varphi'}\}$ actually form a transitive system of projective transitive systems. They then lead to a larger projective transitive system and hence the knot monopole Floer homology $\underline{\mathrm{KHM}}(Y, K, p)$ is well defined (as a projective transitive system).

2.3 Contact structures and contact elements

In this subsection we summarize the results related to contact geometry which we will use in later sections.

Definition 2.13. A *contact sutured manifold* (M, γ, ξ) is a triple where (M, γ) is a balanced sutured manifold and ξ is a contact structure on (M, γ) so that ∂M is convex and γ is the dividing set. The contact structure is said to be *compatible* with the balanced sutured manifold (M, γ).

Theorem 2.14. *(Baldwin, Sivek, [3]) Suppose (M, γ, ξ) is a contact sutured manifold, then we can associate an element*

$$\phi_\xi \in \underline{\mathrm{SHM}}(-M, -\gamma)$$

to it. This element is called the **contact element***.*

Definition 2.15. Suppose (M', γ') is a balanced sutured manifold. A *sutured submanifold* (M, γ) of (M', γ') is another balanced sutured manifold so that $M \subset \mathrm{int}(M')$.

The gluing maps in sutured monopoles were define by the author in [20], and it will be crucial in the construction of the direct system in section 5.

Theorem 2.16. *Suppose (M, γ) is a sutured submanifold of (M', γ') and suppose $Z = M' \backslash \mathrm{int}(M)$. Suppose ξ is a contact structure on Z so that $(Z, \gamma \cup \gamma', \xi)$ is a contact sutured manifold. Then there is a well defined map*

$$\Phi_\xi : \underline{\mathrm{SHM}}(-M, -\gamma) \to \underline{\mathrm{SHM}}(-M', -\gamma'),$$

so that

(1). If (M', γ') is a sutured submanifold of (M'', γ'') and there is a contact structure on $M''\backslash\mathrm{int}(M')$, making it a contact sutured manifold, then we have the composition

$$\Phi_{\xi'} \circ \Phi_{\xi} = \Phi_{\xi \cup \xi'} : \underline{\mathrm{SHM}}(-M, -\gamma) \to \underline{\mathrm{SHM}}(-M'', -\gamma'').$$

Here $\doteq$ means equal up to multiplication by a unit.

(2). Suppose (M', γ', ξ') is a contact sutured manifold and $\xi'|_Z = \xi$, then we have

$$\Phi_{\xi}(\phi_{\xi'|_M}) = \phi_{\xi'}.$$

Suppose we have three balanced sutured manifold (M, γ_1), (M, γ_2) and (M, γ_3), so that the underlining 3-manifold are the same but the sutures are different. Suppose further that (M, γ_1), (M, γ_2) and (M, γ_3) are only different with in a disk $D \subset \partial M$, and within the disk D, they are depicted as in figure 2. We say that (M, γ_2) is obtained from (M, γ_1) by a by-pass attachment along the arc α. Similarly, (M, γ_3) is obtained from a by-pass attachment from (M, γ_2) and (M, γ_1) from (M, γ_3). Then we have the following theorem.

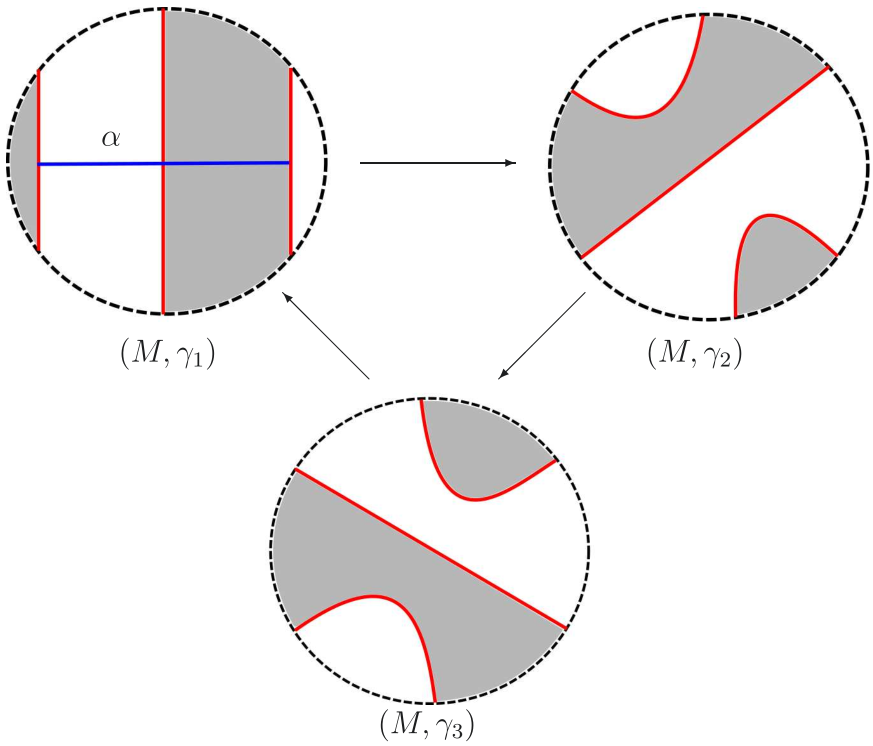


Figure 2: The by-pass exact triangle.

Theorem 2.17. *[Baldwin, Sivek [3]] There is an exact triangle relating the sutured monopole Floer homologies of the three balanced sutured manifolds as follows:*

$$\begin{tikzcd}
\underline{\mathrm{SHM}}(-M,-\gamma_1) \arrow[rr, "\psi_{12}"] & & \underline{\mathrm{SHM}}(-M,-\gamma_2) \arrow[dl, "\psi_{23}"] \\
& \underline{\mathrm{SHM}}(-M,-\gamma_3) \arrow[ul, "\psi_{31}"] &
\end{tikzcd}$$

In contact geometry, a by-pass is a half disk carrying some particular contact structure attached along a Legendrian arc to a convex surface. For details, see Honda [10]. There is a description of the maps in the above by-pass exact triangle as follows. We deal with the map ψ_{12}, and the other two are the same. Let $Z = \partial M \times [0,1]$ and we can pick the suture γ_1 on $\partial M \times \{0\}$ as well as the suture γ_2 on $\partial M \times \{1\}$. Then there is a particular contact structure ξ_{12} on Z which corresponds to the by-pass attachment and makes $(Z, \gamma_1 \cup \gamma_2)$ a contact sutured manifold. Hence we can attach Z to M by the identification $\partial M \times \{0\} = \partial M \subset M$. The result $(M \cup Z, \gamma_2)$ is just diffeomorphic to (M, γ_2) and we have

$$\psi_{12} = \Phi_{\xi_{12}}.$$

Here $\Phi_{\xi_{12}}$ is the gluing map associated to ξ_{12} as in theorem 2.16.

In section 5, we will use the by-passes on knot complements to construct the direct system. Let $K \subset Y$ be an oriented knot. Let λ and μ be the longitude and meridian according to some framing of the knot. Let Γ_n be a suture on $\partial Y(K)$ which consists of two curves of class $\pm(\lambda - n\mu)$ and Γ_∞ consists of two meridians. In this case $\partial Y(K)$ is a torus, and we have the following theorem due to Honda [10].

Theorem 2.18. *There are two tight and minimal-twisting contact structures on $T^2 \times [0,1]$ so that for $i = 1, 2$, $T^2 \times \{i\}$ is convex with dividing set being Γ_{n+i}. These two contact structures correspond to two different by-pass attachments on $(Y(K), \Gamma_n)$.*

Definition 2.19. We denote the two contact structures as in theorem 2.18 by ξ_{+n} and $\xi_{-,n}$ respectively and call the corresponding two by-passes *positive* and *negative* respectively. The positiveness and the negativeness of the two by-passes are defined as in figure 3.

Remark 2.20. This definition of the sign is in a way different from the original one in [10]. However this is the most direct way for us to develop the theory.

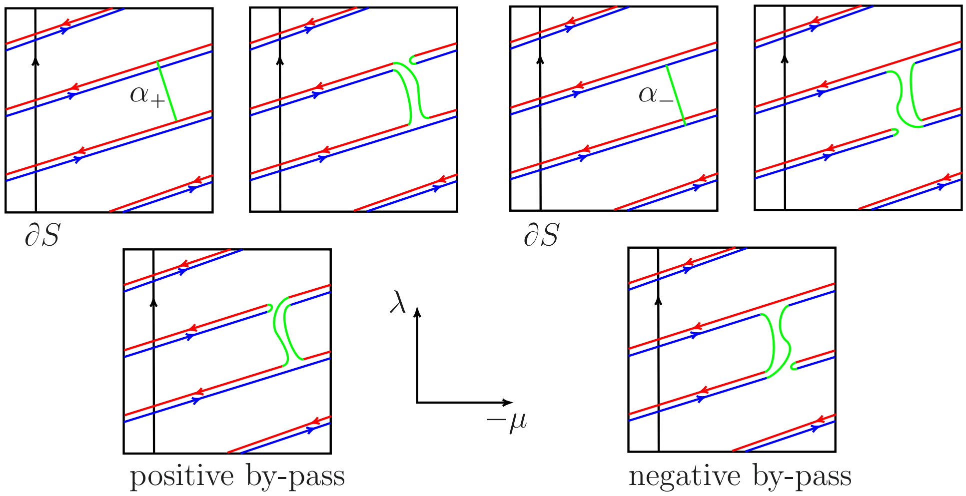


Figure 3: The positive and negative by-pass attachments for $(Y(K),\Gamma_3))$. The squares represent the toroidal boundary of $Y(K)$. Note the contact structures $\xi_{\pm,2}$ correspond to the by-passes from the bottom one to the top left one in each by-pass triangle.

There are by-pass exact triangles associated to the positive and negative by-passes defined as above:

$$\begin{tikzcd}
\underline{\mathrm{SHM}}(-Y(K),-\Gamma_{n+1}) \arrow[rr, "\psi^{n+1}_{\pm,\infty}"] & & \underline{\mathrm{SHM}}(-Y(K),-\Gamma_{\infty}) \arrow[dl, "\psi^{\infty}_{\pm,n}"] \\
& \underline{\mathrm{SHM}}(-Y(K),-\Gamma_n) \arrow[ul, "\psi^{n+1}_{\pm,n}"] &
\end{tikzcd} \tag{4}$$

Note as the above discussion, $\psi^{n+1}_{\pm,n} = \Phi_{\xi_{\pm,n}}$. To construct the direct system, we have the following fact.

Proposition 2.21. *[Honda, [10]] On $T^2\times[0,2]$, the two contact structures $\xi_{-,n}\cup\xi_{+,n+1}$ and $\xi_{+.n}\cup\xi_{-,n+1}$ are the same.*

Corollary 2.22. *We have a commutative diagram*

$$\begin{tikzcd}
\underline{\mathrm{SHM}}(Y(K),\Gamma_n) \arrow[r, "\psi^n_{-,n+1}"] \arrow[d, "\psi^n_{+,n+1}"] & \underline{\mathrm{SHM}}(Y(K),\Gamma_{n+1}) \arrow[d, "\psi^{n+1}_{+,n+2}"] \\
\underline{\mathrm{SHM}}(Y(K),\Gamma_{n+1}) \arrow[r, "\psi^{n+1}_{-,n+2}"] & \underline{\mathrm{SHM}}(Y(K),\Gamma_{n+2})
\end{tikzcd}$$

Proof. The corollary follows from proposition 2.21 and theorem 2.16. □

There is a second way to interpret the maps $\psi_\pm$ associated to by-pass attachments by Ozbagci [21]. He proved that a by-pass attachment can be realized by attaching a contact 1-handle followed by a contact 2-handle. In sutured monopoles, we have maps associated to the contact handle attachments due to Baldwin and Sivek [3] so we can composite those contact handle attaching maps to define $\psi_\pm$. This is actually the original way Baldwin and Sivek constructed the by-pass maps (when they define by-pass maps, there was no construction of gluing maps) and proved the existence of the exact triangle. The two interpretations are the same because of the functoriality of the gluing maps, and their relation with the contact handle attaching maps. For details see the author's previous paper [20]. We will use this second point of view in the proof of proposition 5.5.

3 An Alexander grading

3.1 Basic constructions

Definition 3.1. Suppose (M,γ) is a balanced sutured manifold and S is a properly embedded oriented surface. A *stabilization* of S is an isotopy of S to a surface S', so that the isotopy creates a new pair of intersection points:

$$\partial S'\cap\gamma=(\partial S\cap\gamma)\cup\{p_+,p_-\}.$$

We shall require that there are arcs $\alpha\subset\partial S'$ and $\beta\subset\gamma$ oriented in the same way as $\partial S'$ and γ respectively, such that

(1). We have $\partial\alpha=\partial\beta=\{p_+,p_-\}$.

(2). The curves α and β cobounds a disk D so that $\mathrm{int}(D)\cap(\gamma\cup\partial S')=\emptyset$.

The stabilization is called *negative* if D can be oriented so that $\partial D=\alpha\cup\beta$ as oriented curves. it is called *positive* if $\partial D=(-\alpha)\cup\beta$. See figure 4.

We will denote by $S^{\pm k}$ the result of doing k many positive or negative stabilizations of S.

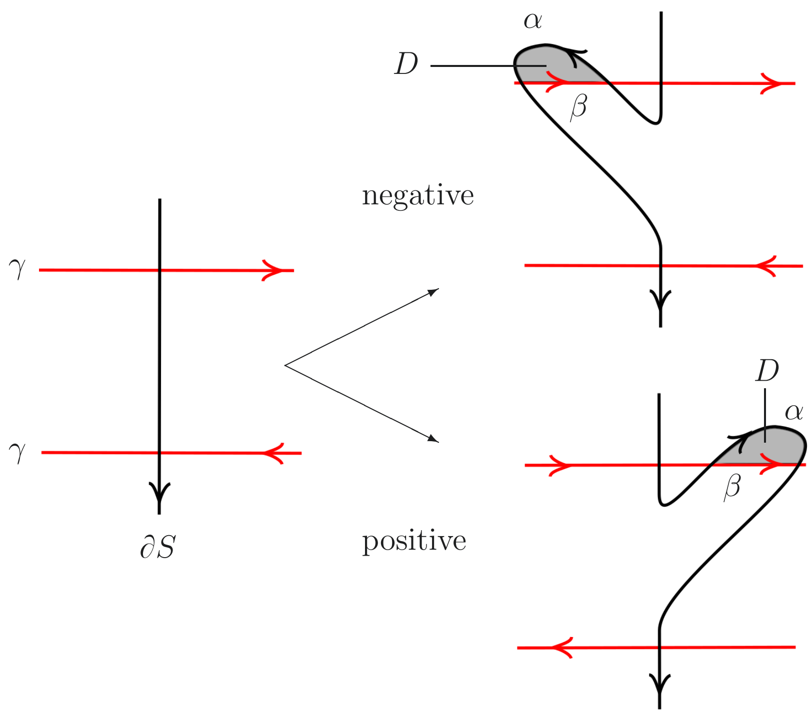


Figure 4: The positive and negative stabilizations of S.

The next lemma is straightforward.

Lemma 3.2. *Suppose (M,γ) is a balanced sutured manifold and S is a properly embedded oriented surface. Suppose $S^{\pm}$ is the result of doing a positive or negative stabilization on S. Then we have:*

(1). If we decompose $(-M,-\gamma)$ along S or S^-, then the resulting two balanced sutured manifolds are diffeomorphic.

(2). If we decompose $(-M,-\gamma)$ along S^+, then the resulting balanced sutured manifold (M',γ') is not taut, as $R_{\pm}(\gamma')$ would both become compressible.

Suppose (M,γ) is a balanced sutured manifold and S is a properly embedded oriented surface. Suppose further that S has precisely one boundary component and ∂S intersects γ at $2n$ points. Since γ is parallel to the boundary of $R_+(\gamma)$, it is non-homologous and hence the algebraic intersection number of ∂S with γ on ∂M must be zero. We shall also assume that $n = 2k+1$ is odd as this can be achieved by a negative stabilization of S if needed. Suppose the intersection points are $p_1, ..., p_{2n}$, and they are indexed so that if we travel along the oriented curve ∂S, starting from p_1, then we will always meet p_i before meeting p_{i+1}.

Now pick a connected auxiliary surface T of large enough genus. Let $f : \partial T \to \gamma$ be an orientation reversing diffeomorphism and let $p_i' = f^{-1}(p_i)$. Suppose $\alpha_1, ..., \alpha_n$ be n pair-wise disjoint simple arcs on T, so that

(1). We have that $[\alpha_1], ..., [\alpha_n]$ are linearly independent in $H_1(T, \partial T)$.

(2). We have that $\partial\alpha_1 = \{p_1', p_2'\}$, and for all $1 \leqslant i \leqslant k$, we have

$$\partial\alpha_{2i} = \{p'_{4i-1}, p'_{4i+2}\},\ \partial\alpha_{2i+1} = \{p'_{4i}, p'_{4i+1}\}.$$

Let

$$\widetilde{M} = M \underset{f\times id}{\cup} T \times [-1,1],\ \widetilde{S} = S \underset{f\times id}{\cup} (\bigcup_{i=1}^{n} \alpha_i \times [-1,1]).$$

We know that

$$\partial\widetilde{M} = R_+ \cup R_-,\ \partial\widetilde{S} \cap R_\pm = \bigcup_{i=1}^{k+1} C_{i,\pm}.$$

Here we require that for $i = 1, ..., k+1$,

$$\alpha_{2i-1} \times \{\pm 1\} \subset C_{i,\pm}.$$

Pick an orientation preserving diffeomorphism $h : R_+ \to R_-$ so that for $i = 1, ..., k+1$,

$$h(C_{i,+}) = C_{i,-}.$$

Then we can use h to get a closure (Y, R) of (M, γ). The boundary components of the surface $\widetilde{S}$ are glued with each other under h so $\widetilde{S}$ results in a closed surface $\bar{S} \subset Y$. From the construction we know that

$$\chi(\bar{S}) = \chi(S) - n.$$

We pick a non-separating simple closed curve $\eta \subset R$, so that η is disjoint from $\bar{S} \cap R$ and also represents a class which is linearly independent from the classes represented by the components of $\bar{S} \cap R$ in $H_1(R)$.

Definition 3.3. We say that the surface $\bar{S} \subset Y$ is *associated to* the surface $S \subset M$. We can use $\bar{S}$ to define a grading on $\mathrm{SHM}(M, \gamma)$ as follows.

$$\mathrm{SHM}(M, \gamma, S, i) = \bigoplus_{\substack{\mathfrak{s}\in\mathfrak{S}(Y|R) \\ c_1(\mathfrak{s})[\bar{S}]=2i}} \widetilde{HM}_\bullet(Y, \mathfrak{s}; \Gamma_\eta).$$

We say that this grading is *associated to* the surface $S \subset M$. When we use the language of marked closures, the closure (Y, R) corresponds to a marked closure $\mathcal{D} = (Y, R, m, r, \eta)$ and we write the grading as

$$\mathrm{SHM}(\mathcal{D}, S, i).$$

The grading on $\mathrm{SHM}(\mathcal{D})$ will also induces a grading on $\underline{\mathrm{SHM}}(M,\gamma)$ as in the following theorem. We also say it is *associated to S* and write

$$\underline{\mathrm{SHM}}(M,\gamma,S,i).$$

Theorem 3.4. *When $S\subset M$ is fixed, and the number of intersection points of S with γ is $2n$ with n odd. Then the grading on $\underline{\mathrm{SHM}}(M,\gamma)$ associated to S is well-defined. That is, it is independent of all choices made in the construction of the grading.*

Proof. There are four types of choices we made:

I. The starting point p_1.

II. The choice of the linearly independent arcs $\alpha_1,...,\alpha_n$ on T.

III. The choice of the gluing diffeomorphism h.

IV. The genus of the closure.

The proof of the independence will be the contents of the rest of the current section. Particularly the results are stated in corollary 3.21, corollary 3.7, proposition 3.9, and lemma 3.5. ☐

In [6], Baldwin and Sivek have already dealt with the choices of type II, III and IV. Among them the idea for type IV can be adapted to the setting of the current paper directly, so we will not write the proof again.

Lemma 3.5. *The definition of the grading on $\underline{\mathrm{SHM}}(M,\gamma)$ associated to the surface $S\subset M$ is independent of choices of type IV.*

We will deal with choices of type II right now.

Lemma 3.6. *Suppose T is a compact connected oriented surface with boundary and of large enough genus. Suppose $\{\alpha_1,...,\alpha_n\}$ is a set of properly embedded simple arcs on T, so that*

(1). The arcs $\alpha_1,...,\alpha_n$ are pair-wise disjoint.

(2). The arcs represent linearly independent classes $[\alpha_1],...,[\alpha_n]$ in $H_1(T,\partial T)$.

Suppose $\{\alpha_1',...,\alpha_n'\}$ is another set of properly embedded simple arcs so that

(3). For $i=1,...,n$, we have $\partial\alpha_i=\partial\alpha_i'$.

(4). The set of arcs $\{\alpha_1',...,\alpha_n'\}$ also satisfies the above conditions (1) and (2).

Then there is an orientation preserving diffeomorphism $h:T\to T$ so that h fixes the boundary of T and for $i=1,...,n$, we have

$$h(\alpha_i)=\alpha_i'.$$

Proof. Suppose N is a product neighborhood of

$$\alpha_1 \cup ... \cup \alpha_n \subset T$$

Let

$$\widetilde{T} = T \backslash \mathrm{int}(N).$$

The boundary $\partial \widetilde{T}$ consists of the following:

$$\partial \widetilde{T} = (\partial T \cap \widetilde{T}) \cup (\bigcup_{i=1}^{n} \alpha_{i,+} \cup \alpha_{i,-}).$$

Here $\alpha_{i,\pm}$ are parallel copies of α_i, being part of the boundary of the product neighborhood N. From condition (2) we know that $\widetilde{T}$ is connected. Also

$$\chi(\widetilde{T}) = \chi(T) + n.$$

Similarly we can pick N' to be a product neighborhood of

$$\alpha_1' \cup ... \cup \alpha_n' \subset T,$$

and have

$$\widetilde{T}' = T \backslash \mathrm{int}(N'),\ \partial \widetilde{T}' = (\partial T \cap \widetilde{T}') \cup (\bigcup_{i=1}^{n} \alpha_{i,+}' \cup \alpha_{i,-}').$$

By condition (3) we can assume that $N \cap \partial T = N' \cap \partial T$, and so there is an orientation preserving diffeomorphism

$$f : \partial \widetilde{T} \to \partial \widetilde{T}',$$

so that

$$f|_{\partial T \cap \widetilde{T}} = id,\ f(\alpha_{i,\pm}) = \alpha_{i,\pm}'$$

for all $i = 1, ..., n$. Since we have

$$\chi(\widetilde{T}') = \chi(T) + n = \chi(\widetilde{T}),$$

the diffeomorphism f extends to a diffeomorphism

$$g : \widetilde{T} \to \widetilde{T}'.$$

After a small perturbation, we can glue $\widetilde{T}$ and $\widetilde{T}'$ along $\alpha_{i,\pm}$ and $\alpha_{i,\pm}'$, and g is glued to become a diffeomorphism

$$h : T \to T$$

which is the desired one. $\square$

As discussed in [6], the above lemma together with proposition 3.9 will result in the following corollary.

Corollary 3.7. *The definition of the grading on* $\underline{\mathrm{SHM}}(M,\gamma)$ *induced by the surface* $S \subset M$ *is independent of choices of type II.*

We will deal with choices of type III in subsection 3.2 and choices of type I in subsection 3.3.

3.2 A reformulation of Canonical maps

In this subsection we will give a simpler description of the canonical maps $\Phi_{\mathcal{D},\mathcal{D}'}$ constructed by Baldwin and Sivek in [2] for two different marked closures of the same genus. For our convenience, we only study the following special case. It would be essentially the same to deal with a general canonical map.

Suppose (M,γ) is a balanced sutured manifold and T is a connected auxiliary surface. Suppose

$$\widetilde{M} = M \cup T \times [-1,1],\ \partial M = R_+ \cup R_-.$$

Suppose h_1, h_2 are two different gluing diffeomorphisms and using them we can get two marked closure $\mathcal{D}_1 = (Y_1, R_+, r_1, m, \eta)$ and $\mathcal{D}_2 = (Y_2, R_+, r_2, m, \eta)$. Here we choose the same non-separating simple closed curve supporting the local coefficients.

Let $h = h_1^{-1} \circ h_2$ and Y^h be the mapping torus of h, or to be more precise, the manifold obtained from $R_+ \times [-1,1]$ by identifying $R_+ \times \{1\}$ with $R_+ \times \{-1\}$ via h. Then we can obtain Y_2 from Y_1 and Y^h as follows. Cut Y_1 open along $R_+ \times \{0\}$ and cut Y^h along $R_+ \times \{0\}$. We can re-glue them via the identity on R_+ to get a large connected manifold. This resulting manifold is precisely Y_2. As in theorem 2.8, there is a cobordism W from $Y_1 \sqcup Y^h$ to Y_2. Hence W induces a map

$$HM(W) : HM(Y_1 \sqcup Y^h | R_+ \cup R_+) \to HM(Y_2 | R_+).$$

Note from lemma 2.6, we know that

$$HM(Y^h | R_+) \cong \mathcal{R}.$$

Let a be a generator of $HM(Y^h|R_+)$ and let ι be the map

$$\iota : HM(Y_1|R_+) \to HM(Y_1|R_+) \otimes HM(Y^h|R_+) \cong HM(Y_1 \sqcup Y^h|R_+ \cup R_+)$$

defined as

$$\iota(x) = x \otimes a.$$

We have the following proposition.

Proposition 3.8. *Under above notations, the canonical map $\Phi_{\mathcal{D}_1,\mathcal{D}_2}$ can be re-interpreted as*

$$\Phi_{\mathcal{D}_1,\mathcal{D}_2} \doteq HM(W) \circ \iota.$$

Before proving the proposition, we first use it to prove that the definition of the grading is independent of choices of type III. Suppose (M,γ) is a balanced sutured manifold and $S \subset M$ is a properly embedded surface with precisely one boundary component, so that ∂S intersects γ at $2n$ points for some odd $n = 2k+1$. Suppose in the construction of the grading induced by S, the choices of type I, II, IV are fixed. This means that there is a connected auxiliary surface T for (M,γ) and n arcs $\alpha_1, ..., \alpha_n$ so that

(1). We have

$$\partial(\alpha_1 \cup ... \cup \alpha_n) = \partial S \cap \gamma.$$

(2). If we let

$$\partial M \cup T \times [-1,1] = R_+ \cup R_-,\ \widetilde{S} = S \bigcup_{i=1^n} (\alpha_i \times [-1,1]),$$

then we have

$$\partial\widetilde{S} \cap R_{\pm} = C_{1,\pm}, ..., C_{k+1,\pm}.$$

Then there are two gluing diffeomorphisms h_1 and h_2 so that for $i = 1,2$

$$h_i(C_{1,+} \cup ... \cup C_{k+1,+}) = C_{1,-} \cup ... \cup C_{k+1,-}.$$

We can use h_1 or h_2 to glue $R_+ \times [-1,1]$ to $M \cup T \times [-1,1]$, to get marked closures $\mathcal{D}_1 = (Y_1, R_+, m, r_1, \eta)$ or $\mathcal{D}_2 = (Y_2, R_+, m, r_2, \eta)$. Here we choose the same non-separating simple closed curve $\eta \subset R_+$ for simplicity. We have the following proposition.

Proposition 3.9. *Under the above settings, we have for any $i \in \mathbb{Z}$*

$$\Phi_{\mathcal{D}_1,\mathcal{D}_2} : \mathrm{SHM}(\mathcal{D}_1, S, i) \xrightarrow{\cong} \mathrm{SHM}(\mathcal{D}_2, S, i).$$

As a result, the definition of grading in the projective transitive system $\underline{\mathrm{SHM}}(M,\gamma)$ is independent of the choices of type III.

Proof. Let $h = h_1^{-1} \circ h_2$, and form Y^h as in proposition 3.8. From lemma 2.6, there is a unique spin^c structure $\mathfrak{s}_0$ so that

$$HM(Y^h|R_+) = \widetilde{HM}_{\bullet}(Y^h, \mathfrak{s}_0; \Gamma_\eta) \cong \mathcal{R}.$$

There are tori inside Y^h: the cylinders $C_{i,+} \times [-1,1] \subset R_+ \times [-1,1]$ are glued via h to become a union of tori T. Lemma 2.7 tells us that

$$c_1(\mathfrak{s}_0)[T] = 0.$$

Let $\bar{S}_1 \subset Y_1$ and $\bar{S}_2 \subset Y_2$ be the surfaces induced by $S \subset M$ in the construction of the grading. We know that there is a 3-dimensional cobordism from $S_1 \sqcup T$ to S_2 inside the the cobordism W. The construction of this (3-dimensional) cobordism is just the same as that of Floer excisions but is done with the dimension reduced by 1. If $\mathfrak{s}$ is a spin^c structure on W which contributes non-trivially to the cobordism map $HM(W)$, then $\mathfrak{s}$ should restricts to $\mathfrak{s}_0$ on Y^h hence we know that

$$c_1(\mathfrak{s})[\bar{S}_2] = c_1(\mathfrak{s})([\bar{S}_1] + [T]) = c_1(\mathfrak{s})([\bar{S}_1]) + c_1(\mathfrak{s}_0)([T]) = c_1(\mathfrak{s})([\bar{S}_1]).$$

Hence $HM(W)$ preserves the grading and so does $\Phi^g_{\mathcal{D}_1,\mathcal{D}_2}$ by proposition 3.8. □

Now we are going to prove proposition 3.8. There are a few preparations we will need.

Lemma 3.10. *In the settings of proposition 3.8, suppose we have a third gluing diffeomorphism h_3, and let $h' = h_2^{-1} \circ h_3$ and $h'' = h \circ h' = h_1^{-1} \circ h_3$. Construct W', W'', ι' and ι'' just in the same ways as we construct W and ι. Then we have the identity:*

$$HM(W'') \circ \iota'' \doteq HM(W') \circ \iota' \circ HM(W) \circ \iota. \tag{5}$$

Proof. Let $Y_{h'}, Y_{h''}$ be the mapping tori of h' and h'' repsectively. Since $h'' = h \circ h'$, there is an excision cobordism from $Y_h \sqcup Y_{h''}$ to $Y_{h''}$ just as we construct W, W' and W''. Call this cobordism $-W_e^{\vee}$ and let W_e be the cobordism from $Y_{h''}$ to $Y_h \sqcup Y_{h'}$, obtained by putting $-W_e^{\vee}$ upset down and also reverse the orientation. By theorem 2.8 and lemma 2.6, it is straightforward to see that

$$HM(W \cup W' \cup W_e) \circ \iota_3 \doteq HM(W') \circ \iota' \circ HM(W) \circ \iota.$$

Hence to prove (5), it is enough to show that

$$HM(W \cup W' \cup W_e) \doteq HM(W''). \tag{6}$$

However, we can cut $W' \cup W' \cup W_e$ open along a 3-manifold $R_+ \times S^1$, as depicted in figure 5 and glue back two copies of $R_+ \times D^2$, and the resulting manifold is exactly W''. Hence from proposition 2.5 in [16], (6) must hold and we are done. □

Corollary 3.11. *If $h_1 = h_2$, then Y^h is just the product and we know that*

$$HM(W) \circ \iota \doteq id.$$

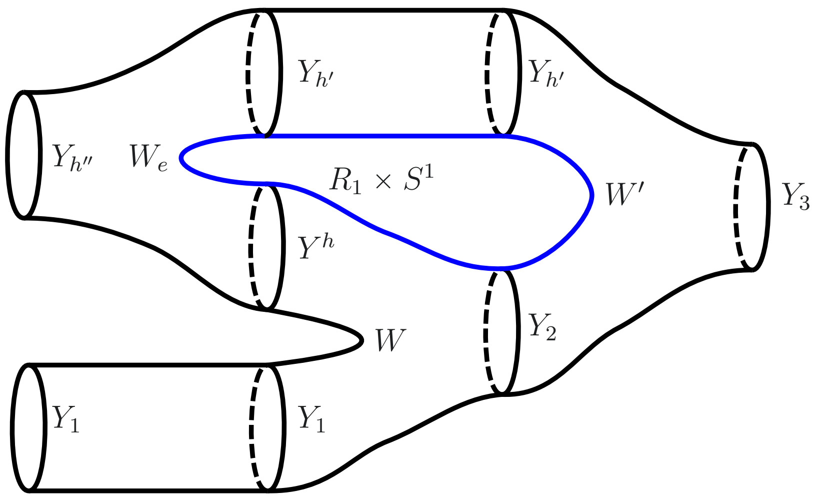


Figure 5: The union $W \cup W' \cup W_e$. The (blue) curve in the middle represents the 3-manifold $R_1 \times S^1$ to cut along.

Proof. From theorem 2.8 We know that

$$HM(W) \circ \iota$$

is an isomorphism. From lemma 3.10, we know that

$$HM(W) \circ \iota \circ HM(W) \circ \iota \doteq HM(W) \circ \iota.$$

Hence we are done. □

Proof of proposition 3.8. Suppose h is decomposed into Dehn twists:

$$h \sim D_{a_1}^{e_1} \circ ... \circ D_{a_n}^{e_n}$$

as in Baldwin and Sivek [2]. From theorem 2.10 and lemma 3.10, it is suffice to deal with the case when $n = 1$, i.e., there is only one Dehn twist involved.

When $e_1 = 1$, the Dehn twist is positive. In this case the canonical map $\Phi^g_{\mathcal{D}_1,\mathcal{D}_2}$ is constructed using the cobordism W as in the hypothesis of proposition 3.8, with the boundary component Y^h capped off by the total space of a relative minimal Lefschetz fibration, see lemma 4.9 in [2]. Also such a Lefschetz fibration would have relative monopole invariant being a unit in $\mathcal{R}$, as in proposition B1 in [2]. Hence we conclude

$$\Phi^g_{\mathcal{D}_1,\mathcal{D}_2} \doteq HM(W) \circ \iota.$$

When $e_1 = -1$, the Dehn twist is negative. We can also look at the canonical map $\Phi^g_{\mathcal{D}_2,\mathcal{D}_1}$. It corresponds to h^{-1} and is constructed using a positive Dehn twist. Suppose we construct W' and ι' out of h^{-1} just as we construct W and ι out of h. Then from the above argument we know that

$$\Phi^g_{\mathcal{D}_2,\mathcal{D}_1} \doteq HM(W') \circ \iota'.$$

Then the identity

$$\Phi^g_{\mathcal{D}_1,\mathcal{D}_2} \doteq HM(W) \circ \iota.$$

follows from theorem 2.10, lemma 3.10 and corollary 3.11. □

3.3 Pairing of the intersection points

In this subsection, we will deal with type I choices, i.e., the starting point p_1.

Let us first pick any intersection point of ∂S with γ as p_1. We shall first relax the requirement in the construction of the grading that $\partial \alpha_i$ shall be a pair of special points.

Definition 3.12. Suppose we have a collection of n pair of numbers

$$\mathcal{P} = \{(i_1, j_1), ..., (i_n, j_n)\},$$

so that

$$\{i_1, j_1, ..., i_n, j_n\} = \{1, 2, ..., 2n\},$$

and for all $l = 1, ..., n$, we have

$$i_l \not\equiv j_l \text{ (mod 2)}.$$

Then we call such a collection $\mathcal{P}$ a *pairing* of *size* n.

Suppose (M, γ) is a balanced sutured manifold and $S \subset M$ is a properly embedded oriented surface. Suppose S has only one boundary component and it intersects γ at $2n = 4k + 2$ points, and those points are labeled by $p_1, ..., p_{4k+2}$ in the same way as described in definition 3.3, with an arbitrary chosen starting point p_1. Suppose $\mathcal{P} = \{(i_l, j_l)\}$ is a pairing of size n, T is an auxiliary surface of M and $\alpha_1, ..., \alpha_n$ are pair-wise disjoint simple arcs so that

(1). The arcs α_1,..., α_n represents linearly independent classes in $H_1(T, \partial T)$.

(2). For $l = 1, ..., n$, we have

$$\partial \alpha_l = \{p_{i_l}, p_{j_l}\}.$$

Then as in the definition 3.3, we can construct

$$\widetilde{M} = M \cup T \times [-1,1],\ \widetilde{S}_{\mathcal{P}} = S \cup (\bigcup_{l=1}^{n} \alpha_l \times [-1,1]).$$

We have

$$\partial\widetilde{M} = R_+ \cup R_-, \partial\widetilde{S}_{\mathcal{P}} \cap R_{\pm} = C_{1,\pm} \cup C_{s_{\pm},\pm}.$$

In general, the number of intersection circles s_+ and s_- are not equal to each other, and we make the following definition.

Definition 3.13. A pairing $\mathcal{P}$ is called balanced if $s_- = s_+$.

Remark 3.14. Although in order to define balancedness, we need to go through the construction of pre-closurs of balanced sutured manifolds, it is well defined on its own (and is independent of all the other choices, such as $(M,\gamma), S, T, p_1$, in definition 3.13. Actually the set $\{s_+, s_-\}$ only depends on $\mathcal{P}$.)

Another thing to notice is that a pairing could be balanced only if its size n is odd.

Example 3.15. Here are some examples of the pairings. Assume $n = 2k+1$ is odd.

(1). The simplest pairing

$$\mathcal{P} = \{(1,2),(3,4),...,(4k+1,4k+2)\}$$

has $s_- = 1$ and $s_+ = n$, or $s_- = n$ and $s_+ = 1$, depending on the choice of the starting point p_1, so it is never a balanced paring when $n > 1$.

(2). In definition 3.3, we have a paring arising from the construction of the grading:

$$\mathcal{P}^g = \{(1,2),(3,6),(4,5),...,(4k-1,4k+2),(4k,4k+1)\}.$$

This is an example of a balanced pairing, with $s_+ = s_- = k+1$.

(3). There is another very special balanced pairing with $s_+ = s_- = 1$:

$$\mathcal{P}^s = \{(1,2k+2),(2,2k+3),...,(2k+1,4k+2)\}.$$

Now if (M,γ), S and p_1 are chosen as above and we are equipped with a balanced pairing $\mathcal{P}$, then we can repeat the construction in definition 3.3, and define a grading on the projective transitive system $\underline{\mathrm{SHM}}(M,\gamma)$. Since we have had corollary 3.7, proposition 3.9 and lemma 3.5, the grading now depends only on the choice of S, p_1 and $\mathcal{P}$. As S and p_1 will actually be fixed almost throughout this subsection, we will omit them from the notation and write, in a moment, the grading as

$$\underline{\mathrm{SHM}}(M,\gamma,\mathcal{P},i).$$

There is a special operation we could do on balanced pairings. Suppose $\mathcal{P}$ is a balanced pairing and we pick two indices l_1 and l_2 so that the following two conditions hold:

(i). The two arcs $\alpha_{l_1}\times\{1\}$ and $\alpha_{l_2}\times\{1\}$ are not contained in the same boundary components of $\widetilde{S}_{\mathcal{P}}$.

(ii). The two arcs $\alpha_{l_1}\times\{-1\}$ and $\alpha_{l_2}\times\{-1\}$ are not contained in the same boundary components of $\partial\widetilde{S}$.

Then we can do the following operation on $\mathcal{P}$ as follows. Suppose in the two pairs (i_{l_1},j_{l_1}) and (i_{l_2},j_{l_2}), i_{l_1} and i_{l_2} are odd (and the two other numbers must be even), then we can obtain a new pairing $\mathcal{P}'$ out of $\mathcal{P}$ by removing the two pairs (i_{l_1},j_{l_1}) and (i_{l_2},j_{l_2}) from $\mathcal{P}$ and add two new pairings (i_{l_1},j_{l_2}) and (i_{l_2},j_{l_1}).

Definition 3.16. We call the above operation the *cut an glue* on parings. Two pairings are called *equivalent* if one is obtained from the other by a cut and glue operation.

Example 3.17. If $n=3$, $\mathcal{P}=\{(1,2),(3,6),(5,4)\}$ and $l_1=1,l_2=3$ ($l_1=1,l_2=2$ do not meet the requirements of doing the cut and glue operation), then the resulting pairing $\mathcal{P}'$ is

$$\mathcal{P}'=\{(1,4),(3,6),(2,5)\},$$

and it is balanced.

It is obvious that the equivalence is an equivalent relation. Also the result of a cut and glue operation on a balanced pairing is still a balanced pairing.

The significance of equivalent pairings is the following.

Lemma 3.18. *Suppose a cut and glue operation on a balanced pairing $\mathcal{P}$ with two indices l_1 and l_2 will result in $\mathcal{P}'$, then we have for all i,*

$$\underline{\mathrm{SHM}}(M,\gamma,\mathcal{P},i)=\underline{\mathrm{SHM}}(M,\gamma,\mathcal{P}',i).$$

Proof. At this point we have shown that choices of type II, III, an IV do not make difference on the definition of grading so when fixing $\mathcal{P}$ we can freely choose other auxiliary data to construct the grading. Now let T and $\alpha_1,...,\alpha_n$ be chosen and the pre-closure $\widetilde{M}$ as well as the properly embedded surface $\widetilde{S}_{\mathcal{P}}$ have been constructed. We can assume that they are chosen so that there is

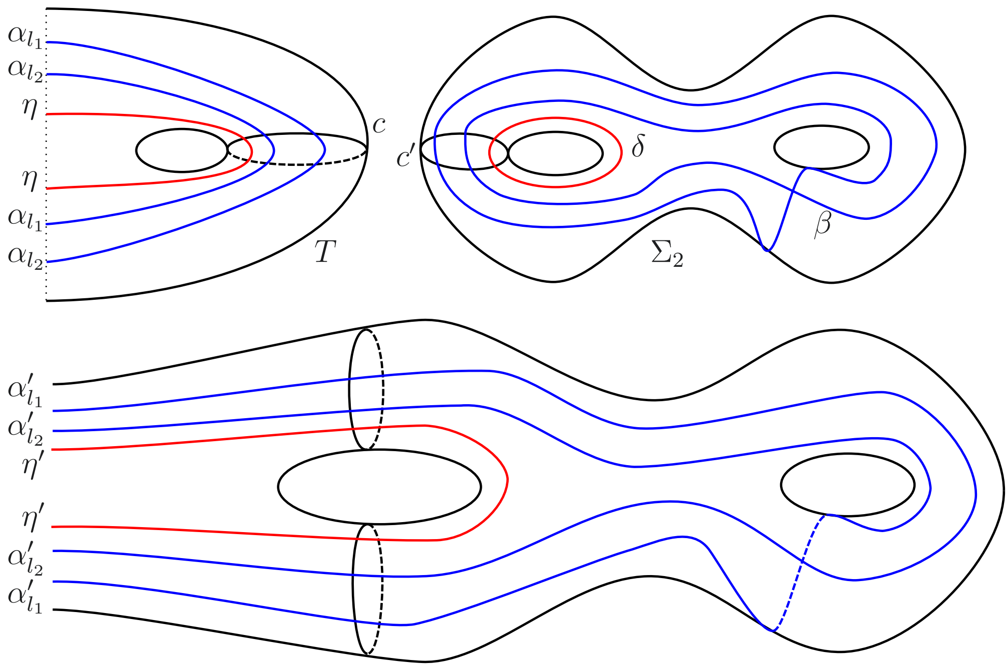


Figure 6: The auxiliary surface T and the surface Σ_2

a curve c intersecting each of α_{l_1} and α_{l_2} transversely at one point. See figure 6. The requirement (i) and (ii) make sure that $\alpha_{l_1} \times \{\pm 1\}$ and $\alpha_{l_2} \times \{\pm 1\}$ lie in four different boundary components of $\widetilde{S}_{\mathcal{P}}$. So we can choose an orientation preserving diffeomorphism $h : R_+ \rightarrow R_-$, where $\partial \widetilde{M} = R_+ \cup R_-$, so that

$$h(\partial \widetilde{S} \cap R_+) = \partial \widetilde{S} \cap R_-, \; h(c \times \{1\}) = c \times \{-1\}.$$

Also we can require that

$$h(\alpha_{l_1} \times \{1\}) = \alpha_{l_1} \times \{-1\} \text{ and } h(\alpha_{l_2} \times \{1\}) = \alpha_{l_2} \times \{-1\}.$$

Let

$$Y = \widetilde{M} \underset{id \cup h}{\cup} R_+ \times [-1, 1], \; R = R \times \{0\}$$

be a closure of (M, γ). The surface $\widetilde{S}_{\mathcal{P}}$ results in a closed surface $\bar{S}_{\mathcal{P}} \subset Y$. We can also choose a simple closed curve η on $R = R_+ \times \{0\}$, so that η intersects $c \times \{0\}$ transversely at one point. Hence we get a marked closure $\mathcal{D} = (Y, R, m, r, \eta)$, where m, r are both inclusion.

By definition, we have

$$SHM(\mathcal{D}, \mathcal{P}, i) = \bigoplus_{\substack{\mathfrak{s} \in \mathfrak{S}(Y|R) \\ c_1(\mathfrak{s})[\bar{S}_{\mathcal{P}}] = 2i}} \widetilde{HM}_{\bullet}(Y, \mathfrak{s}; \Gamma_{\eta}).$$

Let Σ_2 be a closed connected oriented surface of genus 2. Let c', δ and β be three simple closed curves on Σ_2 as depicted in figure 6.

Let Y_Σ be the 3-manifold $\Sigma_2 \times S^1$. There is a torus $\Sigma \subset Y$ being $\Sigma = c \times S^1$ and a torus $\Sigma' \subset Y_\Sigma$ being $\Sigma' = c' \times S^1$. We can choose an orientation preserving diffeomorphism $h' : \Sigma \to \Sigma'$ so that for all $t \in S^1$,we have $h'(c \times \{t\}) = c' \times \{t\}$ as well as

$$h'(((\alpha_{l_1} \cap c) \cup (\alpha_{l_2} \cap c)) \times \{t\}) = (\beta \cap c') \times \{t\}.$$

We can use Σ, Σ' and h' to do a Floer excision on $Y \sqcup Y_\Sigma$. The result is a 3-manifold Y', with a distinguishing surface R', obtained from $R \sqcup \Sigma_2$ by cutting and re-gluing along two curves c and c'. The surface $\bar{S}_\mathcal{P} \subset Y$ also becomes a new closed surface $\bar{S}_{\mathcal{P}'} \subset Y'$, obtained from $\bar{S} \sqcup (\beta \times S^1)$ by cutting and re-gluing along four curves $(\alpha_{l_1} \cap c) \times S^1$, $(\alpha_{l_2} \cap c) \times S^1$, and $(\beta \cap c') \times S^1$ (there are two intersection points of β with c'). The curve η together with $\delta \subset \Sigma_2$ will result in a simple closed curve $\eta' \subset R'$. See figure 6. Hence we get a new marked closure $\mathcal{D}' = (Y', R', m', r', \eta')$ where m', r' are both inclusions. The Floer excision results in a cobordism W from $Y \sqcup Y_\Sigma$ to Y' and then a map

$$HM(W) : HM(Y \sqcup Y_\Sigma | R \cup \Sigma_2; \Gamma_{\eta \cup \delta}) \to HM(Y'|R'; \Gamma_{\eta'}).$$

Let $a \in HM(Y_\Sigma|\Sigma_2; \Gamma_\delta) \cong \mathcal{R}$ be a generator. Then we can define

$$\iota : HM(Y|R; \Gamma_\eta) \to HM(Y'|R'; \Gamma_{\eta'})$$

as $\iota(x) = x \otimes a$ and we know that

$$\Phi_{\mathcal{D},\mathcal{D}'} = HM(W) \circ \iota,$$

as in [2].

The surface $\bar{S}_{\mathcal{P}'} \subset Y'$ actually arises from the balanced pairing $\mathcal{P}'$, which is obtained by doing a cut and glue operation on $\mathcal{P}$ with two indices l_1 and l_2. Just as we did in the proof of proposition 3.9, we can conclude that for all i,

$$\Phi_{\mathcal{D},\mathcal{D}'}(SHM(\mathcal{D}, \mathcal{P}, i)) = SHM(\mathcal{D}', \mathcal{P}', i).$$

Hence we are done. □

Definition 3.19. Two balanced pairings $\mathcal{P}, \mathcal{P}'$ are called *connected* if there is a sequence of balanced pairings

$$\mathcal{P}_0 = \mathcal{P}, \mathcal{P}_1, ..., \mathcal{P}_n = \mathcal{P}',$$

so that for all $i = 0, 1, ..., n-1$, $\mathcal{P}_i$ and $\mathcal{P}_{i+1}$ are equivalent.

Lemma 3.20. *The two special balanced pairings $\mathcal{P}^g$ and $\mathcal{P}^s$ in example 3.15 are connected to each other.*

Proof. In example 3.17, we have shown that

$$\{(12), (3,6), (4,5)\} \text{ and } \{(1,4), (2,5), (3,6)\}$$

are equivalent. In a similar way, we can also show that

$$\{(16), (2,4), (3,5)\} \text{ and } \{(1,4), (2,5), (3,6)\}$$

are equivalent. So

$$\{(12), (3,6), (4,5)\} \text{ and } \{(16), (2,4), (3,5)\}$$

are connected. The later one can be thought of as slide the arc α_1, which originally joined the points p_1 and p_2, over the two arcs α_2 and α_3.

If we skip the pairs $(2,4), (3,5)$ and look at $\{(1,6), (7,10), (8,9)\}$, then the above argument applies and we can connect it to $\{(1,10), (6,9), (7,8)\}$, and this can be thought of slide α_1 over α_4 and α_5. We can repeat this step for many times.

Case 1. If n is of the form $4k+1$. In this case, we can slide α_1 over to join p_1 and p_{4k+2}. Hence $\mathcal{P}^g$ is connected to a new balanced pairing

$$\begin{aligned}\mathcal{P}' =&\{(1, n+1 = 4k+2), (2,5), (3,4), ..., (4k-2, 4k+1), (4k-1, 4k),\\ &(4k+3, 4k+6), (4k+4, 4k+5), ..., (8k-1, 8k+2), (8k, 8k+1)\}.\end{aligned}$$

Then we can do cut an glue operations on pairs $(4l-2, 4l+1)$ and $(4l-2+n, 4l+1+n)$ as well as on pairs $(4l-1, 4l)$ and $4l-1+n, 4l+n$, for all $1 \leqslant l \leqslant k$. The result of these operations is just the special balanced paring $\mathcal{P}^s$ so we are done.

Case 2. If n is of the form $4k+3$. In this case, we can still slide α_1 to join p_1 with p_{4k+2}, so $\mathcal{P}^g$ is connected to

$$\begin{aligned}\mathcal{P}' =&\{(1, 4k+2), (2,5), (3,4), ..., (4k-2, 4k+1), (4k-1, 4k),\\ &(4k+3, 4k+6), (4k+4, 4k+5), ..., (8k+3, 8k+6), (8k+4, 8k+5)\}.\end{aligned}$$

Now do another cut and glue operation on pairs $(1, 4k+2)$ and $(4k+4, 4k+5)$, we will get a new balanced pairing

$$\begin{aligned}\mathcal{P}' =&\{(1, n+1 = 4k+4), (2,5), (3,4), ..., (4k-2, 4k+1), (4k-1, 4k),\\ &(4k+2, 4k+5), (4k+3, 4k+6), ..., (8k+3, 8k+6), (8k+4, 8k+5)\}.\end{aligned}$$

There is then an α arc joining p_{4k+2} and p_{4k+5}, we can slide it over to join p_{4k+5} and p_2. Similarly there is an α arc joining p_{4k+3} with p_{4k+6} and we can slide it over to join p_{4k+3} with p_{8k+6}. Then $\mathcal{P}^g$ is connected to a new balanced pairing

$$\begin{aligned}\mathcal{P}'' =&\{(1, n+1 = 4k+4), (2, n+2 = 4k+5), (n = 4k+3, 2n = 8k+6),\\ &(3,6), (4,5)...(4k-1, 4k+2), (4k, 4k+1)\\ &(4k+6, 4k+9), (4k+7, 4k+8), ..., (8k+2, 8k+5), (8k+3, 8k+4)\}.\end{aligned}$$

Finally, we can do cut and glue operations on pairs $(4l-1, 4l+2)$ and $(4l-1+n, 4l+2+n)$ as well as on $(4l, 4l+1)$ and $(4l+n, 4l+1+n)$ for all $1 \leqslant l \leqslant k$, then the final result is $\mathcal{P}^s$ and we are done. □

Corollary 3.21. *The definition of grading on* $\underline{\mathrm{SHM}}(M, \gamma)$ *is independent of choices of type I.*

Proof. It is straightforward to check if we use the special balanced pairing $\mathcal{P}^s$, then the surface $\widetilde{S}_{\mathcal{P}^s}$ is the same for all possible choices of the starting point p_1. Hence the corollary follows from lemma 3.18 and lemma 3.20. □

Remark 3.22. We want to use $\mathcal{P}^g$ in the definition of grading because it is more convenient to use this construction to discuss about the positive and negative stabilizations (see definition 3.1), as we will see in subsection **??**.

Though we only discussed some special pairings, we would like to make the following conjecture. Note the concept of balancedness, equivalence, connectedness defined above can be reached in a purely combinatorial way and is independent of all the topological input.

Conjecture 3.23. *Any two balanced pairings of the same size n, where n is odd, are connected.*

4 The degree shifting property

4.1 A naive version

Suppose (M, γ) is a balanced sutured manifold and suppose S is a properly embedded surface in M with only one boundary component. In definition 3.3, we constructed a grading on $\underline{\mathrm{SHM}}(M, \gamma)$ associated to S, when $|\partial S \cap \gamma| = 2n$ with n being odd. If n is even, then we introduced in definition 3.1 positive and negative stabilizations $S^{\pm}$ to increase n by 1. It is a natural question to ask how the gradings associated to S^+ and S^- are related to each other. The following proposition is a first answer to this question.

Proposition 4.1. *Suppose (M,γ) is a balanced sutured manifold, $S\subset M$ is a properly embedded surface with only one boundary component and that ∂S intersects γ transversely at $2n$ points with $n=2k>0$ even. Suppose that the balanced sutured manifold obtained by decomposing $(-M,-\gamma)$ along S is taut. Suppose $S^{\pm}$ are the positive and negative stabilizations of S. Suppose S is of genus g and let*

$$g_c=g+k.$$

Then we have

$$\underline{\mathrm{SHM}}(-M,-\gamma,S^-,g_c)\subset\underline{\mathrm{SHM}}(-M,-\gamma,S^+,g_c-1).$$

We need a lemma before the proof of the proposition.

Lemma 4.2. *Suppose (M,γ) is a balanced sutured manifold and S is properly embedded surface inside M so that ∂S is connected and $|\partial S\cap\gamma|=2n$ with n even. Let*

$$g_c=\frac{n-1}{2}+g(S),$$

then we know that

$$\underline{\mathrm{SHM}}(M,\gamma,S,i)=0$$

for all $i>g_c$ and

$$\underline{\mathrm{SHM}}(M,\gamma,S,g_c)\cong\underline{\mathrm{SHM}}(M',\gamma'),$$

where (M',γ') is the balanced sutured manifold obtained from (M,γ) by decomposing along S.

Proof. This follows from the construction of the grading in definition 3.3, the adjunction inequality in lemma 2.7 and the proof of proposition 6.9 in Kronheimer and Mrowka [16]. □

Proof of proposition 4.1. If we have two different negative stabilizations S_1^- and S_2^-, then we know from lemma 3.2 and lemma 4.2 that

$$\underline{\mathrm{SHM}}(-M,-\gamma,S_1^-,g_c)=\underline{\mathrm{SHM}}(-M',-\gamma')=\underline{\mathrm{SHM}}(-M,-\gamma,S_2^-,g_c),$$

where (M',γ') is obtained from $(-M,-\gamma)$ by performing a sutured manifold decomposition along S. Hence we can choose a special negative stabilization to deal with.

Suppose the intersection points of $\partial S\cap\gamma$ are labeled as $p_1,...,p_{2n}$ as we did in definition 3.3. We also pick a suitable p_1 so that the new pair of intersection points created by the positive or negative stabilization lie between p_3 and p_4. Let $\beta'\subset\partial S$ be part of ∂S so that $\partial\beta'=\{p_3,p_4\}$ and β' contains no other intersection points p_j for $j \neq 3, 4$. Let $\beta \subset S$ be a properly embedded arc so that $\partial\beta = \{p_3, p_4\}$, β and β' co-bound a disk on D, and when doing positive and negative stabilizations, the isotopies on S are fixed outside the disk D. Now if we use the same starting point p_1 to label $\partial S^{\pm} \cap \gamma$, then the new pair of intersection points are both p_4 and p_5 in the two cases. See figure 7.

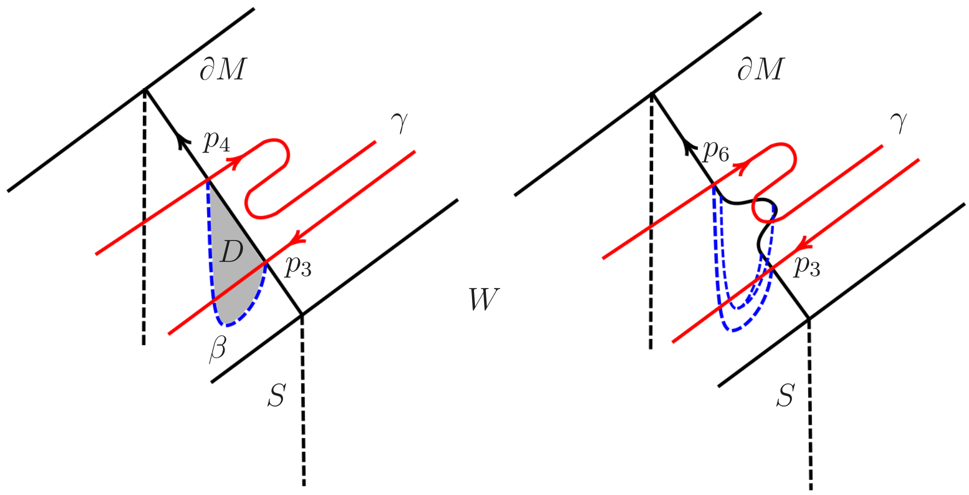


Figure 7: The negative stabilization of S. Positive stabilizations are similar.

Suppose T is an auxiliary surface of (M, γ) with large enough genus. When constructing the grading using $S^{\pm}$, we also need to choose linearly independent arcs $\alpha_1, \alpha_2, \alpha_3^{\pm}, \alpha_4 ..., \alpha_{n+1} \subset T$ to connect intersection points $\partial S^{\pm} \cap \gamma$, and the special pairing $\mathcal{P}^g$, as defined in example 3.15, to tell us what exactly are the end points of those arcs α_i. Here $\alpha_3^{\pm}$ correspond to the different surfaces $S^{\pm}$ while T and all other arcs α_i can be chosen the same for both S^+ and S^-. Now in the pre-closure $\widetilde{M} = M \cup T \times [-1, 1]$, we have surfaces $\widetilde{S}^{\pm} \subset \widetilde{M}$. After picking suitable gluing diffeomorphisms $h^{\pm}$, we get two marked closures

$$\mathcal{D}^{\pm} = (Y^{\pm}, R^{\pm}, r^{\pm}, m^{\pm}, \eta^{\pm})$$

so that there are closed surfaces $\bar{S}^{\pm} \subset Y^{\pm}$, and the gradings are defined by the pairings between first Chern classes of spin^c structures with fundamental classes of $\bar{S}^{\pm}$. Note the genuses of $\bar{S}^{\pm}$ are both $g_c = g + k + 1$.

From proposition 3.8, we know that the canonical map $\Phi_{-\mathcal{D}^-, -\mathcal{D}^+}$ can be interpreted in terms of a Floer excision cobordism W from $-Y^- \sqcup -Y^h$, where Y^h is the mapping torus of $h = (h^-)^{-1} \circ h^+$, to $-Y^+$.

Now we can construct a special closed surface of genus 2 as follows. Recall we have an arc $\beta \subset S$, and since the isotopies for positive or negative stabilizations are supported in the interior of the disk D, β also lies in $\bar{S}^{\pm}$. Let $\delta = \beta \cup (\alpha_2 \times \{0\}) \subset \bar{S}^{\pm}$ be a closed curve. Then the curve δ cuts each of $\bar{S}^{\pm}$ into two parts. One part contains $S\backslash\mathrm{int}(D)$ and the other part is a connected oriented surface $T^{\pm} \subset \bar{S}^{\pm}$ of genus 1 and with boundary δ. Inside W, we can define

$$\Sigma_2 = T^- \cup \delta \times [0,1] \cup -T^+ \subset W.$$

It is straightforward to see that in W,

$$[\bar{S}^-] = [\bar{S}^+] + [\Sigma_2].$$

Hence by the adjunction inequality in dimension 4, which is a 4-dimensional analogue to lemma 2.7, we have

$$\begin{aligned}\Phi_{-\mathcal{D}^-,-\mathcal{D}^+}(SHM(-\mathcal{D},S^-,g_c)) \subset \quad & SHM(-\mathcal{D},S^+,g_c+1) \\ & \oplus SHM(-\mathcal{D},S^+,g_c) \\ & \oplus SHM(-\mathcal{D},S^+,g_c-1).\end{aligned}$$

The adjunction inequality also implies that $SHM(-\mathcal{D},S^+,g_c+1) = 0$. If we decompose $(-M,-\gamma)$ along S^+ and suppose (M',γ') is the resulting balanced sutured manifold, then by lemma 3.2, $R_{\pm}(\gamma')$ is compressible and so

$$SHM(-\mathcal{D},S^+,g_c) \cong SHM(-M',-\gamma') = 0.$$

The first isomorphism follows from lemma 4.2 and the second equality follows again from the adjunction inequality in lemma 2.7.

Hence the only possibility left is

$$\Phi_{-\mathcal{D}^-,-\mathcal{D}^+}(SHM(-\mathcal{D},S^-,g_c)) \subset SHM(-\mathcal{D},S^+,g_c-1)$$

and we are done. □

4.2 Knot complement with two sutures

In this section we shall focus on the case when the balanced sutured manifold (M,γ) is the complement of a non-homologous knot, which means that $M = X(K) = X\backslash\mathrm{im}(N(K))$, where X is a closed connected oriented 3-manifold and $K \subset X$ is a non-homologous knot. Also we focus on the case where γ has only two components. Under these conditions, we can prove that the result of proposition 4.1 holds not only for the top grading but for all gradings.

Proposition 4.3. *Suppose $(M = X(K), \gamma)$ is the balanced sutured manifold described as above. Suppose S is a Seifert surface of the knot K, viewed as a properly embedded surface in M, so that $|\partial S \cap \gamma| = 2n$. Then for any $q, k, l \in \mathbb{Z}$ such that $n + q$ is odd, we have*

$$\underline{\mathrm{SHM}}(-M, -\gamma, S^q, l) = \underline{\mathrm{SHM}}(-M, -\gamma, S^{q+2k}, l - k).$$

Note S^q is defined as in definition 3.1 and in particular $S^0 = S$.

Before proving proposition 4.3, we will first deal with the following related proposition.

Proposition 4.4. *Suppose (Y, R) is a closure of $(-M, -\gamma)$, and let $\mathfrak{s}_1, \mathfrak{s}_2 \in \mathfrak{S}^*(Y|R)$ (see definition 2.4) be two spin^c structures on Y both supporting the sutured monopole Floer homology. Then there is a 1-cycle x inside M, so that*

$$P.D.c_1(\mathfrak{s}_1) - P.D.c_1(\mathfrak{s}_2) = [x] \in H_1(Y).$$

Note the cycle is contained in M but the identity is on the whole Y.

We shall start by describing the closures of $(-M, -\gamma)$. Note if (Y, R) is a closure of (M, γ), then $(-Y, -R)$ is a closure of $(-M, -\gamma)$. So in the following discussion, we shall describe the closures of (M, γ) and for $(-M, -\gamma)$, one can just reverse the orientations.

Let Σ_g be a closed oriented connected surface of genus g which is large enough. Its first homology is generated by the curves $a_1, b_1, ..., a_g, b_g$ as in figure 8.

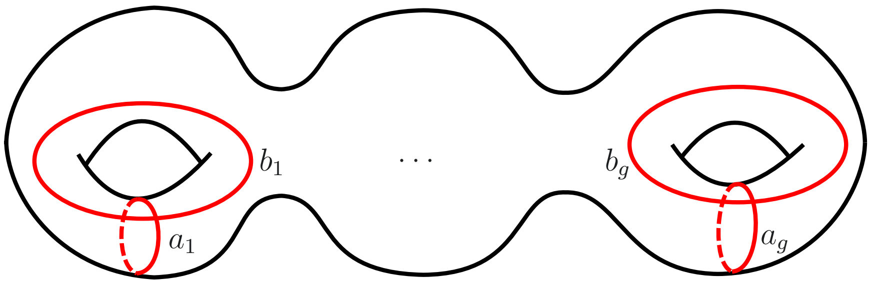


Figure 8: The surface Σ_g.

Let $T = \Sigma_g \backslash \mathrm{int}(N(a_1))$ be a surface obtained from Σ_g by cutting Σ_g open along a_1, then T can be viewed as an auxiliary surface for (M, γ). Let

$$\widetilde{M} = M \cup T \times [-1, 1]$$

be a pre-closure of (M,γ). Let

$$\partial\widetilde{M} = R_+ \cup R_-.$$

Remark 4.5. For different sutures γ and surfaces S, the genus g of the auxiliary surface T might be different in order to construct the grading.

If we choose a special gluing diffeomorphism $h^0 : R_+ \to R_-$ so that $h_{T\times\{1\}} = id$, then we get a special marked closure

$$\mathcal{D}^0 = (Y^0, R, r^0, m^0, \eta).$$

Similar to the closures described in section 5.1 in [16], the closure (Y^0, R) can be thought of being obtained as follows. Let Σ_g be described as above, and let $Y_\Sigma = \Sigma_g \times S^1$, where S^1 is identified with the unit circle in the complex plane. Let a_1 also denote the curve $a_1 \times \{1\} \subset Y_\Sigma$, and $N(a_1)$ is a tubular neighborhood of $a_1 \subset Y_\Sigma$. Note $a_1 \subset \Sigma_g$ so there is a framing on $\partial N(a_1)$ induced by Σ_g. Let λ_a, μ_a be the meridian and longitude respectively.

Then we actually have

$$Y^0 = M \underset{\phi}{\cup} (Y_\Sigma \backslash \mathrm{int}(N(a_1))).$$

Here

$$\phi : \partial N(a_1) \to \partial M$$

sends two copies of λ_a to the suture γ. Note there are canonical ways to identify $R_\pm$ with Σ_g. So in the marked closure $\mathcal{D}_0$, we have $R = \Sigma_g$.

Note (Y_0, Σ_g) is a closure of (M,γ) so $(-Y_0, -\Sigma_g)$ is one for $(-M, -\gamma)$.

Lemma 4.6. *Proposition 4.4 is true for $-Y^0$.*

Proof. From the Mayer-Vietoris sequece we know that there is an exact sequence

$$H_1(T^2) \to H_1(M) \oplus H_1(Y_\Sigma \backslash \mathrm{int}(N(a_1))) \to H_1(Y^0) \to 0,$$

where $T^2 = \partial M = \partial(Y_\Sigma \backslash \mathrm{int}(N(a_1)))$. Hence we conclude that

$$H_1(Y^0) = H_1(M) \oplus H_1(Y_\Sigma \backslash \mathrm{int}(N(a_1)))/\sim,$$

where $\sim$ is the relation induced by the gluing map ϕ :

$$[\lambda_a] \sim \phi_*([\lambda_a]),\ [\mu_a] \sim \phi_*([\mu_a]).$$

A direct calculation shows that

$$H_1(Y_\Sigma \backslash \mathrm{int}(N(a_1))) = \langle [\mu_a], [a_1], [b_1], ..., [a_g], [b_g], [s^0] \rangle,$$

where s^0 corresponds to the S^1 direction in $Y_\Sigma = \Sigma_g \times S^1$. Hence we can write

$$H_1(Y^0) = H_1(M) \oplus \langle [b_1], [a_2], [b_2], ..., [a_g], [b_g], [s^0] \rangle. \tag{7}$$

This is because a_1 and μ_a are absorbed into $H_1(M)$.

Suppose $\mathfrak{s} \in \mathfrak{S}^*(-Y^0| - \Sigma_g)$, then we can express $P.D.c_1(\mathfrak{s})$ in terms of the above basis. The coefficient for $[s]$ can be fixed by the evaluation

$$c_1(\mathfrak{s})[-\Sigma_g] = 2g - 2.$$

There are no $[b_1], [a_2], [b_2]...[a_g], [b_g]$ terms because we have tori $a_1 \times S^1, b_2 \times S^1..., a_g \times S^1 \subset Y^0$ and the adjunction inequality in lemma 2.7 rules out those possibilities. The rest terms must then lie in $H_1(M)$. So if further we look at the difference of two supporting spin^c structures, the difference (of the Poincaré dual of their first Chern class) must lie in M. $\square$

Now we want to deal with other closures of $(-M, -\gamma)$. As above, we have the pre-closure

$$\widetilde{M} = M \cup T \times [-1, 1],$$

where $T = \Sigma_g \backslash N(a_1)$. Also recall

$$\partial \widetilde{M} = R_+ \cup R_-.$$

Note as in the above discussion, there are canonical ways to identify R_+ and R_- with Σ_g. Now we can pick any orientation preserving diffeomorphism $h : R_+ \to R_-$ to get a closure (Y, Σ_g) of (M, γ), or a marked closure

$$\mathcal{D} = (Y, \Sigma_g, r, m, \eta).$$

In particular, the special marked closure $\mathcal{D}^0$ in lemma 4.6 corresponds to taking $h = h^0 = id$.

Let Y^h be the mapping torus of the diffeomorphism $h : \Sigma_g \to \Sigma_g$, then we can reinterpret Y as

$$Y = M \cup_\phi (Y^h \backslash \mathrm{int}(N(a_1))).$$

From proposition 3.8, we know that the canonical map $\Phi_{\mathcal{D}_0, \mathcal{D}}$ can be obtained from a cobordism W from $Y^0 \sqcup Y^h$ to Y. The cobordism W arises from the Floer excision as in subsection 2.2. The computation of the first homologies of Y, Y^h and W_1 are straightforward and we can describe them as follows

$$H_1(Y) = H_1(M) \oplus \langle [\mu_a], [a_1], [b_1], ..., [a_g], [b_g], [s] \rangle / \sim_{\phi,h} \tag{8}$$

$$H_1(Y^h) = \langle [a_1], [b_1]..., [a_g], [b_g], [s]^h \rangle / \sim_h \tag{9}$$

$$H_1(W) = H_1(M) \oplus \langle [\mu_a], [a_1], [b_1], ..., [a_g], [b_g], [s^0], [s^h] \rangle / \sim_{\phi,h} . \quad (10)$$

Here s is a circle intersecting Σ_g once. We can isotope h so that h has a fixed point $p \in \Sigma_g$, then inside Y, there is a circle $s = \{p\} \times S^1$. The class s^h is similar. The relations $\sim_{\phi,h}$ are

$$[a_1] \sim \phi_*([a_1]),\ [\mu_a] \sim \phi_*([\mu_a]),\ [a_i] \sim h([a_i]),\ [b_i] \sim h([b_i]).$$

The relations $\sim_h$ are

$$[a_i] \sim h([a_i]),\ [b_i] \sim h([b_i]).$$

From the above description, the following lemma is straightforward.

Lemma 4.7. *The inclusion $i : Y \hookrightarrow W$ induces injective maps*

$$i_* : H_1(Y) \hookrightarrow H_1(W).$$

Lemma 4.8. *Suppose (W, ν) is an oriented cobordism between two oriented 3-manifolds with local coefficients systems (Y, η) and (Y', η'). Suppose $\mathfrak{s}$ is a spin^c structure on Y and $\mathfrak{s}'$ is a spin^c structure on Y', so that*

$$\widetilde{HM}(W,\nu)(\widetilde{HM}_\bullet(Y,\mathfrak{s};\Gamma_\eta)) \cap \widetilde{HM}_\bullet(Y',\mathfrak{s}';\Gamma_{\eta'}) \neq \{0\},$$

then we know that

$$i_*(P.D.c_1(\mathfrak{s})) = i'_*(P.D.c_1(\mathfrak{s}')) \in H_1(W).$$

Here $i : Y \to W$ and $i' : Y' \to W'$ are the inclusions.

Proof. This is straightforward since the monopole cobordism map is constructed through spin^c structures on the cobordism W. So

$$\widetilde{HM}(W,\nu)(\widetilde{HM}_\bullet(Y,\mathfrak{s};\Gamma_\eta)) \cap \widetilde{HM}_\bullet(Y',\mathfrak{s}';\Gamma_{\eta'}) \neq \{0\}$$

means that there exists a spin^c structure $\mathfrak{s}_W$ which restricts to $\mathfrak{s}$ on Y and to $\mathfrak{s}'$ on Y'. Then the dual of $c_1(\mathfrak{s}) \in H^2(W)$ has boundary $P.D.c_1(\mathfrak{s}) - P.D.c_1(\mathfrak{s}') \in H_1(\partial W)$ which means that

$$i_*(P.D.c_1(\mathfrak{s})) = i'_*(P.D.c_1(\mathfrak{s}')) \in H_1(W).$$

□

Recall we have defined $\mathfrak{S}^*(-Y^0| - \Sigma_g)$ to be the set of supporting spin^c structures as in definition 2.4. We can also define

$$\mathfrak{PDS}^*(-Y^0| - \Sigma_g) = \{P.D.c_1(\mathfrak{s}) | \mathfrak{s} \in \mathfrak{S}^*(-Y^0| - \Sigma_g)\}.$$

We can define $\mathfrak{PDS}^*(-Y|-\Sigma_g)$ similarly. Then we have the following lemma.

Lemma 4.9. *Suppose we have the closures*

$$-\mathcal{D}_0 = (-Y^0, -\Sigma_g, r, m, -\eta), \quad -\mathcal{D} = (-Y, -\Sigma_g, r, m, -\eta)$$

for $(-M, -\gamma)$, the mapping torus Y^h and the cobordism W from $Y^0 \sqcup Y^h$ to Y defined as above. Suppose $\mathfrak{s}_h$ is the unique supporting spin^c structure on $-Y^h$ satisfying the conclusion of lemma 2.6. Then there exists a map

$$\rho : \mathfrak{PD}\mathfrak{S}^*(-Y^0| - \Sigma_g) \to H_1(Y)$$

so that $\mathfrak{PD}\mathfrak{S}^*(-Y| - \Sigma_g) \subset \mathrm{im}(\rho)$ and ρ satisfies the following property $(*)$: suppose we have spin^c structures $\mathfrak{s} \in \mathfrak{S}^*(-Y^0| - \Sigma_g)$ and $\mathfrak{s}' \in \mathfrak{S}^*(-Y| - \Sigma_g)$, so that

$$\widetilde{HM}(-W)(\widetilde{HM}_\bullet(-Y^0, \mathfrak{s}; \Gamma_{-\eta}) \otimes \widetilde{HM}_\bullet(-Y^h, \mathfrak{s}_h; \Gamma_{-\eta})) \cap \widetilde{HM}_\bullet(-Y, \rho(\mathfrak{s}); \Gamma_{-\eta}) \neq \varnothing,$$

then

$$P.D.c_1(\mathfrak{s}') = \rho(P.D.c_1(\mathfrak{s})).$$

Proof. Suppose $\mathfrak{s} \in \mathfrak{S}^*(-Y^0| - \Sigma_g)$ is any supporting spin^c structure. We define the image $\rho(P.D.c_1(\mathfrak{s}))$ as follows. Pick any spin^c structure $\mathfrak{s}_W$ on $-W$ so that

(1). We have $\widetilde{HM}(-W, \mathfrak{s}_W, \nu) \neq 0$.

(2). We have $\mathfrak{s}_W|_{-Y^0} = \mathfrak{s}$. Then we define $\rho(P.D.c_1(\mathfrak{s})) = P.D.c_1(\mathfrak{s}_W|_{-Y})$.

We now show that this map is well defined. Suppose we have another spin^c structure $\mathfrak{s}'_W$ on $-W$ so that condition (1) and (2) also satisfied, then we need to show that

$$P.D.c_1(\mathfrak{s}_W|_{-Y}) = P.D.c_1(\mathfrak{s}'_W|_{-Y}).$$

Let $i : Y \to W$ be the inclusion. We know that there is an exact sequence

$$H_2(W, Y) \xrightarrow{\partial} H_1(Y) \xrightarrow{i_*} H_1(W).$$

By lemma 4.7 and the exactness, we know that $\mathrm{im}(\partial) = \ker(i_*) = 0$. However, clearly we have

$$\partial(P.D.c_1(\mathfrak{s}_W) - P.D.c_1(\mathfrak{s}'_W)) = P.D.c_1(\mathfrak{s}_W|_{-Y}) - P.D.c_1(\mathfrak{s}'_W|_{-Y}),$$

thus we conclude that

$$P.D.c_1(\mathfrak{s}_W|_{-Y}) = P.D.c_1(\mathfrak{s}'_W|_{-Y}).$$

The property $(*)$ follows from the construction of ρ and lemma 4.8. The fact that $\mathfrak{PD}\mathfrak{S}^*(-Y| - \Sigma_g) \subset \mathrm{im}(\rho)$ follows directly from the fact that $-W$ induces an isomorphism as in theorem 2.8. □

Proof of proposition 4.4. We want a more explicit description of the map ρ in lemma 4.9. Using the notations in that lemma, we have a supporting spin^c structure $\mathfrak{s}$ on $-Y^0$ and a (unique) supporting spin^c structure $\mathfrak{s}_h$ on $-Y^h$. We can write

$$P.D.c_1(\mathfrak{s}) = [x] + (2-2g)[s^0],$$

where $[x] \in H_1(M) \subset H_1(Y^0)$ and s^0 is the class as in (7), by lemma 4.6. Also we can write

$$P.D.c_1(\mathfrak{s}_h) = [y^h] + (2-2g)[s^h],$$

where $[y^h]$ is a linear combination of the classes $[a_1], ..., [b_g]$ in $H_1(Y^h)$, which is described in (9).

Now we claim that

$$\rho(P.D.c_1(\mathfrak{s})) = [x] + [y^h] + (2-2g)[s] \in H_1(Y).$$

This is because the cycles $x \subset Y^0$ and $x \subset Y$ co-bound annuli $x \times [0,1]$ inside W, $y^h \subset Y^h$ and $y^h \subset Y$ co-bound annuli $y^h \times [0,1]$ inside W and $s^0 \subset Y^0$, $s^h \subset Y^h$, $s \subset Y$ co-bound a pair of pants in side W. Thus inside W we can find an explicit (relative) 2-cycle c so that

$$c \cap Y^0 = \partial c \cap Y^0 = P.D.c_1(\mathfrak{s}),\ c \cap Y^h = \partial c \cap Y^h = P.D.c_1(\mathfrak{s}_h).$$

Thus as in the proof of lemma 4.9, the injectivity in lemma 4.7 implies that

$$\rho(P.D.c_1(\mathfrak{s})) = [x] + [y^h] + (2-2g)[s]. \tag{11}$$

With this explicit formula, proposition 4.4 follows directly. $\square$

Corollary 4.10. *If the inclusion $j : M \to Y$ induces an injective homomorphism*

$$j_* : H_1(M) \to H_1(Y),$$

then the map ρ in lemma 4.9 is in fact a bijection:

$$\rho : \mathfrak{PDS}^*(-Y^0| - \Sigma_g) \to \mathfrak{PDS}^*(-Y| - \Sigma_g).$$

Proof. It is straightforward from (10) to check that when j_* is injective, the inclusion $j^0 : M \to W$ also induces an injective homomorphism

$$j^0_* : H_1(M) \to H_1(W).$$

Then the injectivity follows directly from (11), since $[y^h]$ and $(2-2g)[s]$ in that formula are fixed and the only variance is $[x]$ which is represented by a cycle in M. $\square$

Proof of proposition 4.3. Now we have a balanced sutured manifold (M, γ) where $M = X(K)$ is the complement of a non-homologous knot $K \subset X$ and γ has two components. Also we have a Seifert surface S of K which can be viewed as a properly embedded surface in M. Let $|\partial S \cap \gamma| = 2n$. For any p so that $n + p$ is odd, we can do p-stabilization as in definition 3.1 and apply the construction in definition 3.3 to construct a grading

$$\underline{\mathrm{SHM}}(-M, -\gamma, S^p, l)$$

on $\underline{\mathrm{SHM}}(-M, -\gamma)$. As in definition 3.3, we can construct a marked closure

$$\mathcal{D}_p = (Y_p, \Sigma_g, r_p, m_p, \eta)$$

so that $S^p \subset M$ extends to a closed surface $\bar{S}^p \subset Y_p$.

We claim that the inclusion $j : M \to Y_p$ for any p satisfies the condition in lemma 4.10, that is,

$$j_* : H_1(M) \to H_1(Y_p)$$

is injective. So then we can apply the corollary.

To prove this claim, first note that $M = X(K)$ so we can compute directly that

$$H_1(M) = H_1(X) \oplus \langle [\mu_K] \rangle,$$

where μ_K is a meridian circle of K inside $M = X(K)$. From the discussion above, we know that

$$Y_p = M \underset{\phi}{\cup} (Y^{h_p} \backslash \mathrm{int}(N(a_1))),$$

where $h_p : \Sigma_g \to \Sigma_g$ is some orientation preserving diffeomorphism and

$$H_1(Y_p) = H_1(M) \oplus \langle [\mu_a], [a_1], ..., [b_g], [s_p] \rangle / \sim_{\phi, h_p}$$

as in (8). Thus we know that the relations $\sim_{\phi, h_p}$ only affects $[\mu_k] \in H_1(M)$ but not anything in $H_1(X)$. Hence to show that j_* is injective, it is enough to show that $j_*([\mu_k])$ is of infinite order. Yet this last thing is obvious since inside Y_p, μ_K intersects $\bar{S}_p$ transversely at one point.

Thus we get a bijection

$$\rho_p : \mathfrak{PDS}^*(-Y^0 | -\Sigma_g) \to \mathfrak{PDS}^*(-Y_p | -\Sigma_g)$$

as in corollary 4.10. Here (Y^0, Σ_g) or $\mathcal{D}^0 = (Y^0, \Sigma_g, r^0, m^0, \eta)$ is the special (marked) closure of (M, γ) described above.

Similarly we have the surface $S^{p+2k}\subset M$, a marked closure $\mathcal{D}_{p+2k}=(Y_{p+2k},\Sigma_g,r_{p+2k},m_{p+2k},\eta)$, an extension $\bar{S}_{p+2k}$ of S inside Y_{p+2k} and a bijection

$$\rho_{p+2k}:\mathfrak{PDS}^*(-Y^0|-\Sigma_g)\rightarrow\mathfrak{PDS}^*(-Y_{p+2k}|-\Sigma_g).$$

Thus we can define

$$\rho^p_{p+2k}=\rho_{p+2k}\circ\rho_p^{-1}:\mathfrak{PDS}^*(-Y_p|-\Sigma_g)\rightarrow\mathfrak{PDS}^*(-Y_{p+2k}|-\Sigma_g).$$

Also from proposition 3.8, lemma 4.9 and the functoriality of the canonical maps, we know that ρ has the following significant property: if $\mathfrak{s}\in\mathfrak{S}^*(-Y_p|-\Sigma_g)$ and $\mathfrak{s}'\in\mathfrak{S}^*(-Y_{p+2k}|-\Sigma_g)$ are supporting spin^c structures so that

$$\Phi_{-\mathcal{D}_p,-\mathcal{D}_{p+2k}}(\widetilde{HM}_\bullet(-Y_p,\mathfrak{s};\Gamma_{-\eta}))\cap\widetilde{HM}_\bullet(-Y_{p+2k},\mathfrak{s}';\Gamma_{-\eta})\neq\varnothing,$$

then we must have

$$P.D.c_1(\mathfrak{s}')=\rho(P.D.c_1(\mathfrak{s})).$$

From the explicit description of ρ in (11), we know that $\mathfrak{s}_1,\mathfrak{s}_2\in\mathfrak{S}^*(-Y_i|-\Sigma_g)$ and $\mathfrak{s}_1',\mathfrak{s}_2'\in\mathfrak{S}^*(-Y_{i+2k}|-\Sigma_g)$ are supporting spin^c structures so that

$$\Phi_{-\mathcal{D}_p,-\mathcal{D}_{p+2k}}(\widetilde{HM}_\bullet(-Y_p,\mathfrak{s}_1;\Gamma_{-\eta}))\cap\widetilde{HM}_\bullet(-Y_{p+2k},\mathfrak{s}_1';\Gamma_{-\eta})\neq\varnothing$$

and

$$\Phi_{-\mathcal{D}_p,-\mathcal{D}_{p+2k}}(\widetilde{HM}_\bullet(-Y_p,\mathfrak{s}_2;\Gamma_{-\eta}))\cap\widetilde{HM}_\bullet(-Y_{p+2k},\mathfrak{s}_2';\Gamma_{-\eta})\neq\varnothing,$$

then there exists a 1-cycle $x\subset M$ so that

$$P.D.c_1(\mathfrak{s}_1)-P.D.c_1(\mathfrak{s}_2)=[x]\in H_1(Y_p)\tag{12}$$

and

$$P.D.c_1(\mathfrak{s}_1')-P.D.c_1(\mathfrak{s}_2')=[x]\in H_1(Y_{p+2k}).\tag{13}$$

Recall in definition 3.3, the grading is obtained by the evaluation of the first Chern classes of the supporting spin^c structures and by theorem 3.4, the grading should be preserved by the canonical map. Hence the above equalities (12) and (13 actually implies that there is a fixed integer l_0 so that for any $l\in\mathbb{Z}$, we have

$$\underline{\mathrm{SHM}}(-M,-\gamma,S^p,l)=\underline{\mathrm{SHM}}(-M,-\gamma,S^{p+2k},l-l_0).$$

If we go through the construction of ρ, we know that ρ is not only independent of $l\in\mathbb{Z}$, but also independent of the interior of M and S (and only related to the data ∂S, p, k and γ.) Thus in order to figure out the value of k, we can only look at the basic case where M is the complement of a trefoil inside S^3. The convenience is that when we decompose (M, γ) along S^p for $p \leqslant 0$ then the resulting sutured manifold is taut.

Case 1. If $p < 0$ and $p + 2k < 0$. From lemma 3.2 and lemma 4.2 we know that first non-vanishing degree of $\underline{\mathrm{SHM}}(-M, -\gamma, S^p)$ is

$$l = \frac{n - p - 1}{2} + g(S),$$

while the first non-vanishing degree of $\underline{\mathrm{SHM}}(-M, -\gamma, S^{p+2k})$ is

$$l' = \frac{n - p - 2k - 1}{2} + g(S).$$

However, from the above discussion we know that

$$l' = l - l_0$$

so $l_0 = k$.

Case 2. If $p = -1$ and $k = 1$ or $p = 1$ and $k = -1$. Then $l_0 = 1 = k$ from proposition 4.1.

Case 3. If $p > 0$ and $p + 2k > 0$. Then we can look at the surface $-S \subset M$. Note positive stabilizations of S are negative stabilizations of $-S$. Hence this is reduced to case 1 and we still have $l_0 = k$.

Case 4. If p and $p + 2k$ are of difference sign, and is not in case 2. We can apply case 1,2 and 3 above and conclude that we still have $l_0 = k$.

So in summary we always have $l_0 = k$ and we are done. □

4.3 Sutured monopoles on a solid torus

As a first application of the degree shifting property, we compute the sutured monopole Floer homology of any valid sutures on a solid torus. The same result in sutured Heegaard Floer homology can be found in Juhász [13].

Suppose $V = S^1 \times D^2$ is a solid torus. Let λ denote a longitude $S^1 \times \{t\}$ where $t \in \partial D^2$ and let μ denote a meridian $\{s\} \times \partial D^2$ where $s \in S^1$. Suppose γ is a choice of suture on V so that (V, γ) is a balanced sutured manifold. Then γ is parametrized by two quantities n and s where n is a positive even number being the number of components of γ and s is a rational number being the slope of the suture. Then the suture γ would be write as $\gamma^n_{(q,p)}$. We will usually write s as $\frac{p}{q}$ or (q, p). Here p and q are co-prime and $p \geqslant 0$. Note (q, p) means going around longitude p times and meridian q times.

Remark 4.11. The suture $\gamma^2_{(1,-p)}$ in this subsection would be the same as the suture Γ_p as in formula (4).

Also note that adjacent components of the suture have opposite orientations. So the slope s and $-s$ are exactly the same. In the present paper, we want to be in consistence with Honda [10] so we will write the slope as $s = -\frac{p}{q}$ or $(q,-p)$. But in any case it shall be understood that p denote a non-negative integer.

Proposition 4.12. *Suppose $(V, \gamma^2_{(q,-p)})$ is defined as above. Then we have*

$$\underline{\mathrm{SHM}}(-V, -\gamma^2_{(q,-p)}) = \mathcal{R}^p.$$

Proof. If $p = q$, then $p = q = 1$ because they are co-prime. Then $(V, \gamma^2_{(1,-1)})$ is diffeomorphic to a product sutured manifold $(A \times [-1,1], \partial A \times \{0\})$, where A is an annulus. Thus we know

$$\underline{\mathrm{SHM}}(-V, -\gamma^2_{(1,-1)}) \cong \mathcal{R}.$$

From now on we assume that $p > q$. We want to re-interpret the by-pass exact triangle as follows. We have two basic by-pass exact triangles

$$\begin{tikzcd}[column sep=small] & \underline{\mathrm{SHM}}(-V, -\gamma^2_{(1,-1)}) \arrow[dr, "\psi_{\pm,2}"] & \\ \underline{\mathrm{SHM}}(-V, -\gamma^2_{(1,0)}) \arrow[ur, "\psi_{\pm,1}"] & & \underline{\mathrm{SHM}}(-V, -\gamma^2_{(0,-1)}) \arrow[ll, "\psi_{\pm,0}"] \end{tikzcd} \tag{14}$$

Here $\psi_{\pm,0} = \psi^{\infty}_{\pm,0}$, $\psi_{\pm,1} = \psi^{0}_{\pm,1}$ and $\psi_{\pm,2} = \psi^{1}_{\pm,\infty}$ under the notations of (4).

Recall from subsection 2.3 that the maps $\psi_{-,1}$ (as well as the other two) is identified with a gluing map as follows. Suppose we have $\underline{\mathrm{SHM}}(-V, -\gamma^2_{(1,0)})$ and an identification $T^2 = S^1 \times \partial D^2$. We can glue $T^2 \times [0,1]$ to V by the identification $\partial V = S^1 \times \partial D^2 = T^2 \times \{0\}$. Suppose $T^2 \times \{0\}$ is equipped with the suture $\gamma^2_{(1,0)}$ and $T^2 \times \{1\}$ is equipped with the suture $\gamma^2_{(1,-1)}$, then we can identify $(V, \gamma^2_{(1,-1)})$ with $(V \cup T^2 \times [0,1], \gamma^2_{(1,-1)})$. There exists a compatible contact structure $\xi_{-,1}$ on $(T^2 \times [0,1], \gamma^2_{(1,0)} \cup \gamma^2_{(1,-1)})$ so that we have an equality

$$\psi_{-,0} = \Phi_{\xi_{-,0}} : \underline{\mathrm{SHM}}(-V, -\gamma^2_{(1,0)}) \to \underline{\mathrm{SHM}}(-V, -\gamma^2_{(1,-1)}).$$

Now if we are dealing with other sutures, we can still glue $(T^2\times[0,1],\gamma^2_{(1,0)}\cup\gamma^2_{(1,-1)})$ to V, but along a diffeomorphism

$$g:T^2\to\partial V$$

which is not the identity. Such a map also needs to be orientation preserving and hence is parametrized by an element in $SL_2(\mathbb{Z})$. We can pick the map corresponding to the matrix

$$A=\left(\begin{array}{cc} q-q' & -q' \\ p'-p & p' \end{array}\right),$$

where $p'q-pq'=1$, $p'\leqslant p$, $q'\leqslant q$, $q''=p-p'$, and $p''=p-p'$. (Such p',q',p'',q'' are unique.)

Then the suture $\gamma^2_{(1,0)}$ on $T^2\times\{0\}$ is glued to $\gamma^2_{(q,-p)}$ on ∂V and the suture $\gamma^2_{(1,-1)}$ on $T^2\times\{1\}$ now becomes the suture $\gamma^2_{(q',-p')}$. As in formula (14), they still fit into an exact triangle

$$\begin{tikzcd}[column sep=small] & \underline{\mathrm{SHM}}(-V,-\gamma^2_{(q,-p)}) \arrow[dr, "\psi_{-,2}"] & \\ \underline{\mathrm{SHM}}(-V,-\gamma^2_{(q'',-p'')}) \arrow[ur, "\psi_{-,1}"] & & \underline{\mathrm{SHM}}(-V,-\gamma^2_{(q',-p')}) \arrow[ll, "\psi_{-,0}"] \end{tikzcd} \tag{15}$$

We claim that $\psi_{-,0}=0$. Let D_p be a meridian disk of V which intersects $\gamma^2_{(q,-p)}$ at $2p$ points, then from a similar argument as in proposition 5.5, we have

$$\psi_{-,0}(\underline{\mathrm{SHM}}(-V,-\gamma^2_{(q',-p')},D_{p'}^{-(p-p')},i))\subset\underline{\mathrm{SHM}}(-V,-\gamma^2_{(q',-p')},D_{p'}^{+(p-p'')},i)$$

for any $i\in\mathbb{Z}$.

We will only deal with the case when p' is odd and p'' is even. Other cases are similar. From the construction of the grading in definition 3.3, we know that there is a suitable marked closure $\mathcal{D}_{p'}=(Y_{p'},R,r,m,\eta)$ and a closed surface $\bar{D}_{p'}\subset Y_{p'}$ so that the grading is defined via the evaluations of the first Chern classes of spin^c structures on the fundamental class of $\bar{D}_{p'}$. From the construction we know that

$$\chi(\bar{D}_{p'})=\chi(D_{p'})-p'=1-p'.$$

Hence the adjunction inequality in lemma 2.7 tells us that

$$\underline{\mathrm{SHM}}(-V,-\gamma^2_{(q',-p')},D_{p'},i)=0$$

if $i < \frac{1-p'}{2}$. Then from the degree shifting property in proposition 4.3, we know that

$$\underline{\mathrm{SHM}}(-V, -\gamma^2_{(q',-p')}, D^{-p''}_{p'}, i) = \underline{\mathrm{SHM}}(-V, -\gamma^2_{(q',-p')}, D_{p'}, i + (\frac{p''}{2})).$$

Thus we know

$$\underline{\mathrm{SHM}}(-V, -\gamma^2_{(q',-p')}, D^{-p''}_{p'}, i) = 0 \tag{16}$$

if $i < \frac{1-p'+p''}{2}$.

The above argument for $D_{p'}$ is the same for $D^{+}_{p''}$. Note p'' is assumed to be even, so we need to do a positive stabilization on $D_{p''}$ to construct the grading. The adjunction inequality tells us that

$$\underline{\mathrm{SHM}}(-V, -\gamma^2_{(q'',-p'')}, D^{+}_{p''}, i) = 0 \tag{17}$$

if $i > \frac{p''}{2}$. However, from proposition 6.9 in [16], we know that

$$\underline{\mathrm{SHM}}(-V, -\gamma^2_{(q'',-p'')}, D^{+}_{p''}, \frac{p''}{2}) \cong \underline{\mathrm{SHM}}(M', \gamma'),$$

where (M', γ') is the result of doing a sutured manifold decomposition on $(-V, -\gamma^2_{(q'',-p'')})$ along the surface $D^{+}_{p''}$. From lemma 3.2, we know that

$$\underline{\mathrm{SHM}}(-V, -\gamma^2_{(q'',-p'')}, D^{+}_{p''}, \frac{p''}{2}) \cong \underline{\mathrm{SHM}}(M', \gamma') = 0. \tag{18}$$

The degree shifting property in proposition 4.3 implies then

$$\underline{\mathrm{SHM}}(-V, -\gamma^2_{(q'',-p'')}, D^{+p'}_{p''}, i) = \underline{\mathrm{SHM}}(-V, -\gamma^2_{(q'',-p'')}, D^{+}_{p''}, i - \frac{p'-1}{2}).$$

The above equality, together with (17) and (18), implies that

$$\underline{\mathrm{SHM}}(-V, -\gamma^2_{(q'',-p'')}, D^{+p'}_{p''}, i) = 0$$

if $i \geqslant \frac{1-p'+p''}{2}$. Compare this with (16), we can see that $\psi_{-,0} = 0$.

Once we conclude that $\psi_{-,0} = 0$, we can compute $\underline{\mathrm{SHM}}(-V, -\gamma^2_{(q,-p)})$ by induction. Actually the other two slopes $(q', -p')$ and $(q'', -p'')$ can be written out explicitly in terms of the continued fraction of $(q, -p)$, as in Honda [10]. Note we have $p > q$. Suppose

$$-\frac{p}{q} = r_1 - \frac{1}{r_2 - \frac{1}{r_3 - \ldots}},$$

where it is a finite continued fraction, and $r_j < -1$ for all j. We can write

$$-\frac{p}{q} = [r_1, r_2, ..., r_k]. \tag{19}$$

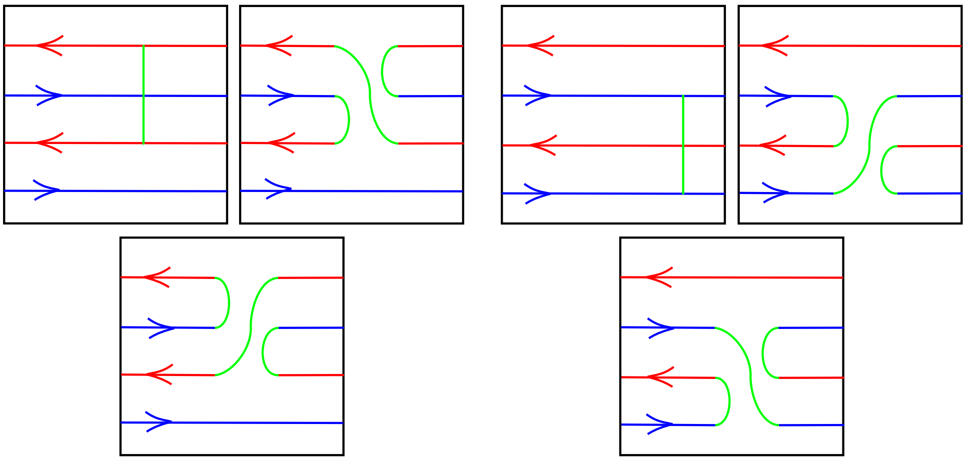
Under this notation, we have

$$-\frac{p'}{q'} = [r_1, r_2 ..., r_{k-1}], \quad -\frac{p''}{q''} = [r_1, r_2 ..., r_{k-1} + 1],$$

and we shall identify $[r_1, ..., r_{j-1}, r_j, -1]$ with $[r_1, ..., r_{j-1}, r_j + 1.]$ $\square$

Now we will deal with general sutures. There are two types by-passes relating $(V, \gamma^{2n+2}_{(q,-p)})$ and $(V, \gamma^{2n}_{(q,-p)})$. We call them positive and negative by-passes according to the figure 9. They give rise to by-pass exact triangles:

$$\begin{tikzcd}[column sep=small] & \underline{\mathrm{SHM}}(-V, -\gamma^{2n+2}_{(q,-p)}) \arrow[dr, "\psi^{n+1}_{\pm,n}"] & \\ \underline{\mathrm{SHM}}(-V, -\gamma^{2n}_{(q,-p)}) \arrow[ur, "\psi^{n}_{\pm,n+1}"] & & \underline{\mathrm{SHM}}(-V, -\gamma^{2n}_{(q,-p)}) \arrow[ll, "\psi^{n}_{\pm,n}"] \end{tikzcd} \tag{20}$$



Positive by-passes Negative by-passes

Figure 9: The positive and negative by-passes.

Remark 4.13. Unlike the case of two sutures, when there are exactly two different possibilities of by-passes, in the case when γ has more than two components, positive and negative by-passes are not unique. Here we just pick two specific by-passes so that they are 'adjacent' to each other. This is crucial to the proof of lemma 4.14 below.

Lemma 4.14. *For any n and slope $(q,-p)$, we have*

$$\psi^{n+1}_{-,n}\circ\psi^{n}_{+,n+1}=\psi^{n+1}_{+,n}\circ\psi^{n}_{-,n+1}=id:\underline{\mathrm{SHM}}(-V,-\gamma^{2n}_{(q,-p)})\rightarrow\underline{\mathrm{SHM}}(-V,-\gamma^{2n}_{(q,-p)}).$$

Proof. We will only prove that $\psi^{n+1}_{-,n}\circ\psi^{n}_{+,n+1}=id$. The other is the same.

From [3] or [21] we know that a by-pass attached along an arc α can be thought of as attaching a pair of contact 1-handle and 2-handle. The contact one handle is attached along the two end points $\partial\alpha$ while the contact two handle is attached along a Legendrian curve

$$\beta=\alpha\cup\alpha',$$

where α' is an arc on the contact 1-handle intersecting the dividing set once.

Now $\psi^{n+1}_{-,n}\circ\psi^{n}_{+,n+1}$ corresponds to first attaching a by-pass along α_+ and then attaching another one along α_-, as in figure 10. However, in terms of contact handle attachments, the two pairs of handles are disjoint from each other, so we can reverse the order of attachments: we can first attach a by-pass along α_- and then along α_+. If we attach a by-pass along α_- first, we will see as in figure 10 that this is actually a trivial by-pass as discussed in Honda [11]. In that paper it is proved that such a trivial by-pass would not change the contact structure. From theorem 2.16, we know that the induced map between sutured monopole Floer homologies must be the identity. Then the second by-pass attached along α_+ will also induces identity map for exactly the same reason and we conclude that $\psi^{n+1}_{-,n}\circ\psi^{n}_{+,n+1}=id$. □

Corollary 4.15. *We know that*

$$\underline{\mathrm{SHM}}(-V,-\gamma^{2n}_{(q,-p)})\cong\mathcal{R}^{(2^{n-1}\cdot p)}.$$

Proof. From lemma 4.14 we know that $\psi^{n+1}_{\pm,n}$ is surjective while $\psi^{n}_{\pm,n+1}$ is injective. Hence we can conclude the statement by using the by-pass exact triangles and the induction. □

Corollary 4.16. *We have*

$$|\pi_0(\mathrm{Tight}(V,\gamma^{2n}_{(q,-p)}))|\geqslant 2^{n-1}\cdot|r_1+1|\cdot...\cdot|r_{k-1}+1|\cdot|r_k|.$$

Proof. First assume $n=1$. In [10], Honda explained how can we construct any possible tight contact structures on a solid torus with convex boundary and dividing set $\gamma^2_{(q,-p)}$. First we shall start with the standard tight contact structure on $(V,\gamma^2_{(1,-1)})$. Then we can glue k different layers $T^2\times[i-1,i]$ for $1\leqslant i\leqslant k$ to V one by one, so that on $T^2\times[i-1,i]$, $T^2\times\{i-1\}$ has

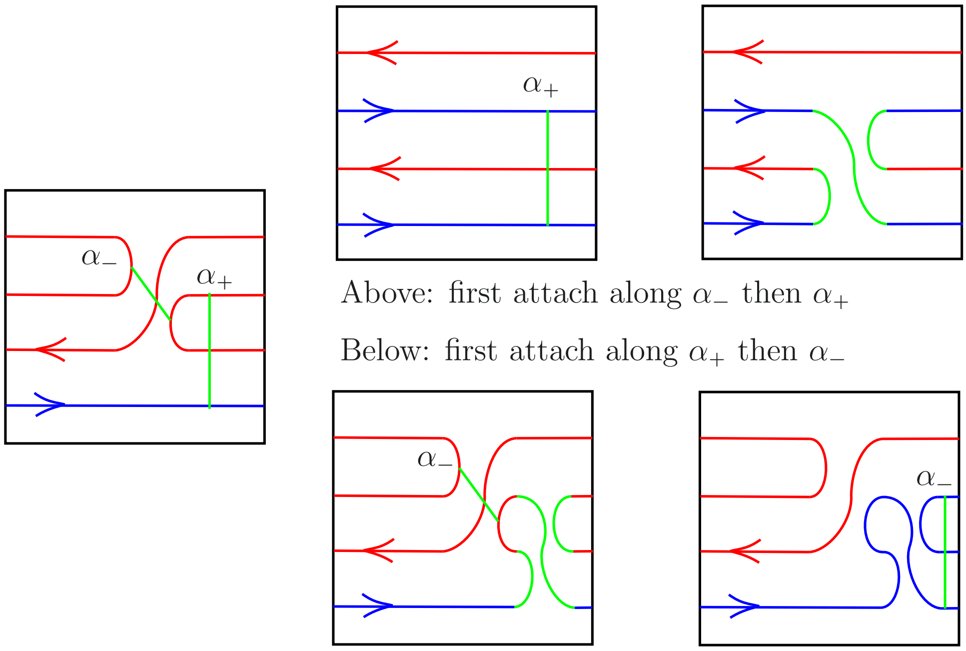


Figure 10: Reversing the order of doing by-pass attachments. Bottom right picture: we can isotope α_- to this new position where we can see directly that the by-pass is trivial.

the dividing set $\gamma^2_{(1,-1)}$, while $T^2\times\{i\}$ has the dividing set $\gamma^2_{(1,1-r_i)}$. We glue $T^2\times\{0\}$ to ∂V via identity, while glue $T^2\times\{i\}\subset T^2\times[i,i+1]$ to $T^2\times\{i\}\subset T^2\times[i-1,i]$ so that the dividing sets on these two surfaces are identified.

Each layer $T^2\times[i-1,i]$ is further decomposed into the combination of $-1-r_i$ (or $-r_k$ for the last layer) many by-passes. There are two by-passes, one corresponding to the map $\psi_{-,1}$ in formula (15) and the other corresponding to some other $\psi_{+,1}$ in a similar by-pass exact triangle and should be completely analogue to the one discussed above. Use the inductive step in [10] that Honda used to construct contact structures, we will see by above discussion that all such contact structures have distinct contact elements. Hence there are at least $|r_1+1|\cdot...\cdot|r_{k-1}+1|\cdot|r_k|$ many different contact structures.

When n is bigger than 1, we still proceed by induction. Suppose for $n=l$, there are $m_l=2^{l-1}\cdot|r_1+1|\cdot...\cdot|r_{k-1}+1|\cdot|r_k|$ many different non-zero contact elements $\psi_{\xi_1},...,\psi_{\xi_{m_l}}\in\underline{\mathrm{SHM}}(-V,\gamma^{2l}_{(q,-p)})$. From above discussion we know that $\psi^l_{\pm,l+1}$ are both injective, and

$$\psi^{l+1}_{\pm,l} \circ \psi^l_{\pm,l+1} = 0,\ \psi^{l+1}_{\mp,l} \circ \psi^l_{\pm,l+1} = id.$$

The first equality is the exactness of the by-pass triangle and the second is lemma 4.14. Hence we know that inside $\underline{\mathrm{SHM}}(-V, \gamma^{2l+2}_{(q,-p)})$, there are $m_{l+1} = 2^l \cdot |r_1 + 1| \cdot ... \cdot |r_{k-1} + 1| \cdot |r_k|$ many different contact elements

$$\psi^l_{\pm,l+1}(\phi_{\xi_1}), ..., \psi^l_{\pm,l+1}(\phi_{\xi_{m_l}})$$

as they are all distinct. Hence we are done. □

Remark 4.17. When $n = 1$, the above argument gives an alternative way to provide a tight lower bound of $|\pi_0(Tight(V, \gamma^2_{(q,-p)}))|$, which is originally done by Honda [10].

When $n > 1$, as we have mentioned before, there are not only two by-passes, so this lower bound in general should not be tight. However, one could try to study the impact of all other by-pass attachments to see if we could improve the lower bound.

Remark 4.18. We can use a meridian disk of the solid torus to define a grading on $\underline{\mathrm{SHM}}(-V, -\gamma^{2n}_{(q,-p)})$. The above method is also capable of computing the graded homology.

5 The direct system and the direct limit

5.1 The construction

Suppose Y is a closed oriented 3-manifold and $K \subset Y$ is an oriented knot with a Seifert surface $S \subset Y$, i.e., S is an embedded oriented surface so that $\partial S = K$. Suppose $p \in K$ is a fixed base point. Suppose $\varphi : S^1 \times D^2 \hookrightarrow Y$ be an embedding as in subsection 2.2, that is, we shall require

$$\varphi(S^1 \times \{0\}) = k, \text{ and } \varphi(\{1\} \times \{0\}) = p.$$

Then we have a 3-manifold with boundary $Y_\varphi = Y \backslash \mathrm{int}(\mathrm{im}(\varphi))$. The Seifert surface S induces a framing on ∂Y_φ. We call the meridian μ_φ and the longitude λ_φ. Let $\Gamma_{n,\varphi}$ be a collection of two disjoin parallel oppositely oriented simple closed curves on ∂Y_φ, each of class $\pm(\lambda_\varphi - n\mu_\varphi)$. Then we have a balanced sutured manifold $(Y_\varphi, \Gamma_{n,\varphi})$.

Suppose φ' is another embedding, then we also have $((Y_{\varphi'}, \Gamma_{n,\varphi'}))$. Suppose f_t is the ambient isotopy defined as in subsection 2.2, relating φ and φ'. We have the following lemma.

Lemma 5.1. *The diffeomorphism f_1 is a diffeomorphism from $(Y_\varphi, \Gamma_{n,\varphi})$ to $(Y_{\varphi'}, \Gamma_{n,\varphi'})$.*

Proof. It is enough to show that f_1 sends the framing $(\mu_\varphi, \lambda_\varphi)$ on ∂Y_φ to the framing $(\mu_{\varphi'}, \lambda_{\varphi'})$ on $\partial Y_{\varphi'}$.

By construction, f_1 sends μ_φ to $\mu_{\varphi'}$. So suppose f_1 sends λ_φ to $x\lambda_{\varphi'} + y\mu_{\varphi'}$ The fact that f_1 is a diffeomorphism implies that $x = 1$. Suppose $y \neq 0$, then we know that f_1 is a diffeomorphism between balanced sutured manifolds $(Y_\varphi, \Gamma_{n,\varphi})$ and $(Y_{\varphi'}, \Gamma_{n-y,\varphi'})$, for any $n \in \mathbb{Z}$. Pick n large enough so that $n > 0$ and $n - y > 0$. Note by construction f_1 restricts to identity outside a tubular neighborhood N of $K \subset Y$. So it is free to assume that $Y = S^3$ and K is the unknot. Then the above diffeomorphism actually gives us a diffeomorphism

$$f_1 : (V, \Gamma_n) \to (V, \Gamma_{n-y}),$$

where V is a solid torus and Γ_n is defined as in subsection 2.3. However, we know that $\underline{\mathrm{SHM}}(V, \Gamma_n)$ and $\underline{\mathrm{SHM}}(V, \Gamma_{n-y})$ are not isomorphic by proposition 4.12. So this gives a contradiction. $\square$

Corollary 5.2. *There is a transitive system (of projective transitive systems) $\{\underline{\mathrm{SHM}}(Y_\varphi, \Gamma_{n,\varphi})\}$ and $\{\Psi_{\varphi,\varphi'} = \underline{\mathrm{SHM}}(f_1)\}$. So it is valid to define the canonical module $\underline{\mathrm{SHM}}(Y, K, p, n)$ associated to the quadruple (Y, K, p, n).*

Pick a particular embedding φ and we can give $S^1_\theta \times D^2_{(x,y)}$ a standard tight contact structure. Let

$$\xi_{st} = \ker(\sin(\theta)dx + \cos(\theta)dy).$$

Under this contact structure, the boundary $S^1 \times \partial D^2$ is convex and the dividing set consists of two curves of slope -1. We can use φ to push forward this contact structure to a tubular neighborhood of $K \subset Y$. We can choose φ so that the curve $\{t\} \times \partial D^2$ is mapped to the longitude λ_φ defined as above. The the dividing set is mapped to two curves of slope -1 on ∂Y_φ under the framing $(\mu_\varphi, \lambda_\varphi)$.

The knot K is Legendrian under this local contact structure. Let K_n be the $(n-1)$-th negative stabilization of K, then we can remove a stardard neighborhood $\varphi'(S^1 \times D^2) \subset \mathrm{int}(\mathrm{im}(\varphi))$ of K_n, the new boundary torus is convex and having dividing set being two curves of class $\lambda_{\varphi'} - n\mu_{\varphi'}$. Thus we get the balanced sutured manifold $(Y_{\varphi'}, \Gamma_{n,\varphi'})$. From corollary 5.2 we can identify all $(Y_{\varphi'}, \Gamma_{n,\varphi'})$ as $(Y(K) = Y \backslash \mathrm{int}(N(K)), \Gamma_n)$ for a fixed tubular neighborhood $N(K)$ of K, and Γ_n consists of two curves of class $(\lambda - n\mu)$, where μ and λ are the meridian and longitude of the framing on $\partial Y(K)$ which is induced by S. Also let Γ_∞ denote two meridians μ. We can further assume that there is a contact structure defined in a collar of $\partial Y(K) \subset Y(K)$ so that $\partial Y(K)$ is convex with the dividing set Γ_n or Γ_∞. Then we can apply positive and negative by-pass attachments to $(Y(K), \Gamma_n)$, and there are exact triangles as in formula (4).

Lemma 5.3. *The maps $\psi_\pm$ in the exact triangle (4) induce maps between $\underline{\mathrm{SHM}}(-Y, K, p, n)$, $\underline{\mathrm{SHM}}(-Y, K, p, n-1)$ and $\underline{\mathrm{KHM}}(Y, K, p)$. Note for $n = \infty$, we are using a pair of meridians so we have $\underline{\mathrm{SHM}}(-Y, K, p, \infty) = \underline{\mathrm{KHM}}(-Y, K, p)$.*

Proof. Recall $(Y(K), \Gamma_n) = (Y_{\varphi'}, \Gamma_{n,\varphi'})$, and the by-pass attachments are realized by contact handle attachments. If we have a different embedding φ'', then the two balanced sutured manifolds $(Y_{\varphi'}, \Gamma_{n,\varphi'})$ and $(Y_{\varphi''}, \Gamma_{n,\varphi''})$ are related by an isotopy f_t. To prove the lemma, we need to show that $\underline{\mathrm{SHM}}(f_1)$ commute with the contact handle attaching maps. This follows from lemma 3.16 in [20] or a similar result in the instanton settings from [4]. $\square$

With the above lemma at hand, we can focus on $(Y(K), \Gamma_n)$ from now on.

Definition 5.4. Define the *minus version of monopole knot Floer homology* of a based knot $K \subset -Y$, which is denoted by $\underline{\mathrm{KHM}}^-(-Y, K, p)$, to be the direct limit of the direct system

$$... \to \underline{\mathrm{SHM}}(-Y(K), \Gamma_n) \xrightarrow{\psi^n_{-,n+1}} \underline{\mathrm{SHM}}(-Y(K), \Gamma_{n+1}) \to ...$$

where the maps $\psi^n_{-,n+1}$ are defined in the exact triangle (4). By corollary 2.22 the maps $\{\psi^n_{+,n+1}\}_{n\in\mathbb{Z}_+}$ induce a map on $\underline{\mathrm{KHM}}^-$, which we call U:

$$U : \underline{\mathrm{KHM}}^-(-Y, K, p) \to \underline{\mathrm{KHM}}^-(-Y, K, p).$$

We also want to construct a grading on the direct limit $\underline{\mathrm{KHM}}^-(-Y, K, p)$. Suppose S_n is the Seifert surface of K so that S_n intersects Γ_n at $2n$ points. Then we have the following proposition.

Proposition 5.5. *Suppose n is even, then we have for any $i \in \mathbb{Z}$*

$$\psi^n_{\pm,n+1}(\underline{\mathrm{SHM}}(-Y(K), -\Gamma_n, S^\pm_n, i)) \subset \underline{\mathrm{SHM}}(-Y(K), -\Gamma_{n+1}, S_{n+1}, i).$$

Suppose n is odd, then we have for any $i \in \mathbb{Z}$

$$\psi^n_{\pm,n+1}(\underline{\mathrm{SHM}}(-Y(K), -\Gamma_n, S^{\pm 2}_n, i)) \subset \underline{\mathrm{SHM}}(-Y(K), -\Gamma_{n+1}, S^\pm_{n+1}, i).$$

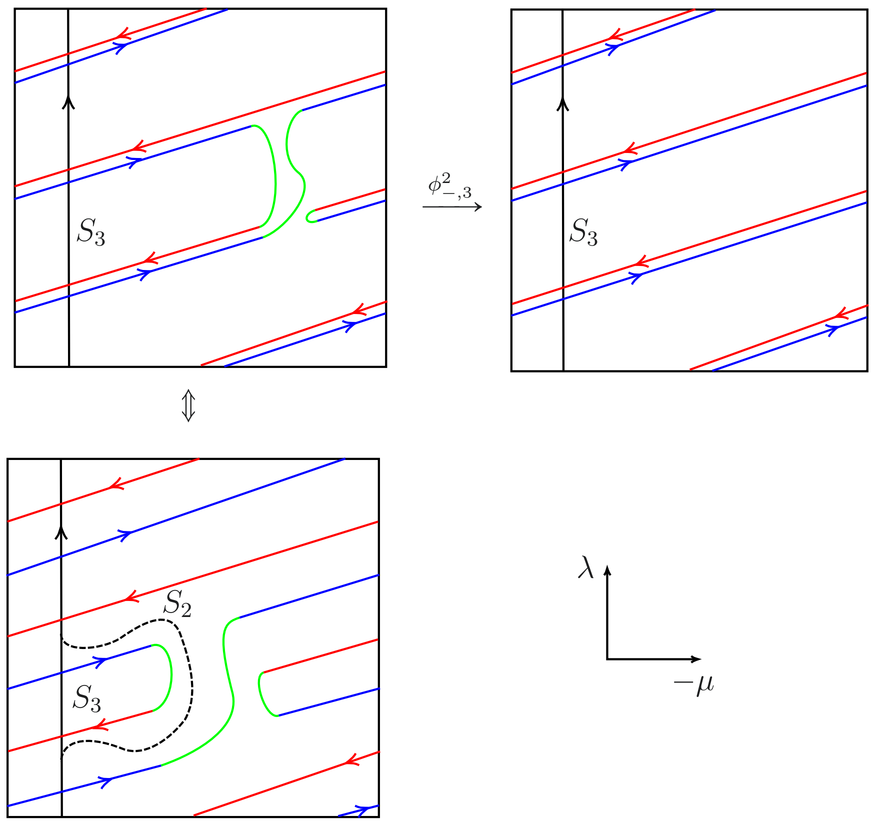


Figure 11: The solid vertical arc represents the surface $S_3 = S_2^-$ and the dashed arc represents S_2.

Proof. We only prove the case when n is even and we are dealing with $\phi^n_{-,n+1}$. Other cases are similar except for a possibly more complicated use of the degree shifting property. From figure 11, it is clear that the surface $S_{n+1} \subset (Y(K), \Gamma_{n+1})$ can also be obtained from the surface S_n by a negative stabilization:

$$S_{n+1} = S_n^-.$$

Thus we know that for any $i \in \mathbb{Z}$

$$\underline{\mathrm{SHM}}(-Y(K), -\Gamma_n, S_n^-, i) = \underline{\mathrm{SHM}}(-Y(K), -\Gamma_n, S_{n+1}, i).$$

Now for $S_n^- = S_{n+1} \subset (Y(K), \Gamma_n)$, we can choose some auxiliary data to construct a marked closure

$$\mathcal{D}_n^- = (Y_n^-, R, r_n, m_n, \eta),$$

so that there is a closed surface $\bar{S}_n^- \subset Y_n^-$ and it induces a grading on $\underline{\mathrm{SHM}}(Y(K),\Gamma_n)$ which is just the one associated to S_n^-. (See definition 3.3.)

We can obtain $(Y(K),\Gamma_{n+1})$ by attaching a by-pass disjoint from S_{n+1}. From [3], we know the map $\phi^n_{-,n+1}$ associated to the by-pass can be described as follows. There is a curve $\beta\subset(m_n(Y(K)))\subset Y_n^-$ so that a 0-framed Dehn surgery on β, with respect to the $\partial Y(K)$ framing, will result in a 3-manifold Y_{n+1}. Since β is disjoint from $\mathrm{im}(r_n)$, the data R, r_n and η can be copied and we get a marked closure

$$\mathcal{D}_{n+1}=(Y_{n+1},R,r_{n+1},m_{n+1},\eta)$$

which is a marked closure of $(Y(K),\Gamma_{n+1})$. The surgery description above results in a cobordism W from Y_n^- to Y_{n+1} and the cobordism map associated to this cobordism actually induces the by-pass attaching map $\phi^n_{-,n+1}$. This cobordism W is obtained from $Y_n^-\times[0,1]$ by attaching a 4-dimensional 2-handle along the curve $\beta\subset Y_n^-\times\{1\}$.

It is the key observation that $S_n^-=S_{n+1}$ is disjoint from the region we attach the by-pass and hence is disjoint from the curve β along which we do the Dehn surgery. As a result, the surface $\bar{S}_n^-$ remains as a closed surface $\bar{S}_{n+1}\subset Y_{n+1}$ and hence induces a grading on $\underline{\mathrm{SHM}}(Y(K),\Gamma_{n+1})$. If we check the definitions, then it is clear that this grading induced by $\bar{S}_{n+1}$ is just the one associated to the surface $S_{n+1}\subset(Y(K),\Gamma_{n+1})$.

There is a 3-dimensional cobordism $\bar{S}_n^-\times[0,1]\subset W$ from $\bar{S}_n^-\subset Y_n^-$ to $\bar{S}_{n+1}\subset Y_{n+1}$, hence we conclude that

$$\phi^n_{-,n+1}(\underline{\mathrm{SHM}}(Y(K),\Gamma_n,S_n^-,i))\subset\underline{\mathrm{SHM}}(Y(K),\Gamma_{n+1},S_{n+1},i).$$

So we are done. $\square$

The following figures 12 and 13 might be helpful for figuring out how does $\psi^n_{\pm,n+1}$ change the gradings. In the figures, $k'=k+g(S)$.

Now we can do a degree shifting as follows:

$$\underline{\mathrm{SHM}}(-Y(K),-\Gamma_n,S_n^{\tau(n)},i)[\sigma(n)]=\underline{\mathrm{SHM}}(-Y(K),-\Gamma_n,S_n^{\tau(n)},i+\sigma(n)).$$

Here $\tau(n)=-1$ if n is even and $\tau(n)=0$ if n is odd. Also

$$\sigma(n)=\frac{n-1+\tau(n)}{2}.$$

We will simply write

$$\underline{\mathrm{SHM}}(-Y(K),-\Gamma_n,S_n^{\tau})[\sigma]$$

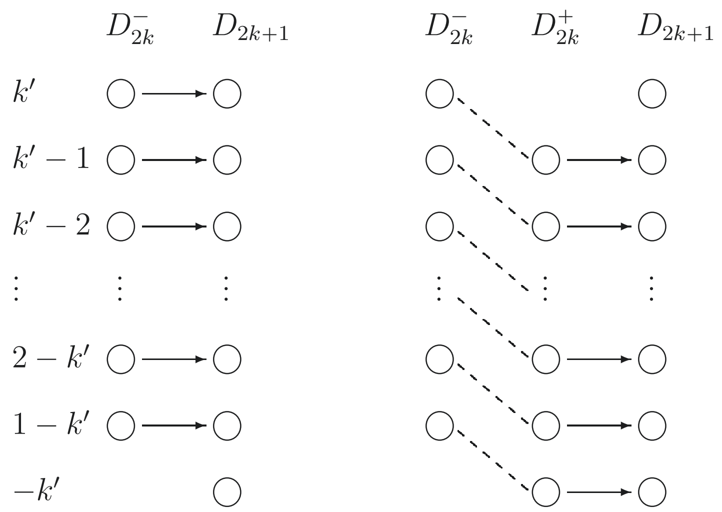


Figure 12: The map $\phi_{\pm}$ from $\underline{\mathrm{SHM}}(-Y(K),-\Gamma_{2k})$ to $\underline{\mathrm{SHM}}(-Y(K),-\Gamma_{2k+1})$. The map $\phi^{2k}_{-,2k+1}$ is depicted on the left and $\phi^{2k}_{+,2k+1}$ on the right. They are represented by the solid arrows. The circles ◯ denote the graded homologies. The dashed lines represent the degree shifting when using different surfaces to construct the grading.

and the direct system becomes

$$... \to \underline{\mathrm{SHM}}(-Y(K),-\Gamma_n,S^\tau_n)[\sigma] \xrightarrow{\phi^n_{-,n+1}} \underline{\mathrm{SHM}}(-Y(K),-\Gamma_{n+1},S^\tau_{n+1})[\sigma] \to ...$$

It is straight forward to prove that after the shifting $\phi^n_{-,n+1}$ is degree preserving and $\phi^n_{+,n+1}$ shifts the degree down by 1. Thus we conclude:

Proposition 5.6. *If S is a Seifert surface of $K \subset Y$, then S induces a grading on $\underline{\mathrm{KHM}}^-(-Y,K,p)$, which we write as*

$$\underline{\mathrm{KHM}}^-(-Y,K,p,S,i)$$

and the map U shift the degree down by 1.

5.2 Basic properties

Proposition 5.7. *Suppose Y is a closed oriented 3-manifold and $K \subset Y$ is a knot so that there exists an embedded disk $S = D^2$ with $\partial S = K$. Then*

$$\underline{\mathrm{KHM}}^-(-Y,K,p) \cong \underline{\mathrm{SHM}}(-Y(1),-\delta) \otimes_{\mathcal{R}} \mathcal{R}[U].$$

Here $p \in K$ is any choice of base point. $(Y(1),\delta)$ is the balanced sutured manifold obtained from Y by removing a 3-ball and pick one simple closed curve on the spherical boundary as the suture.

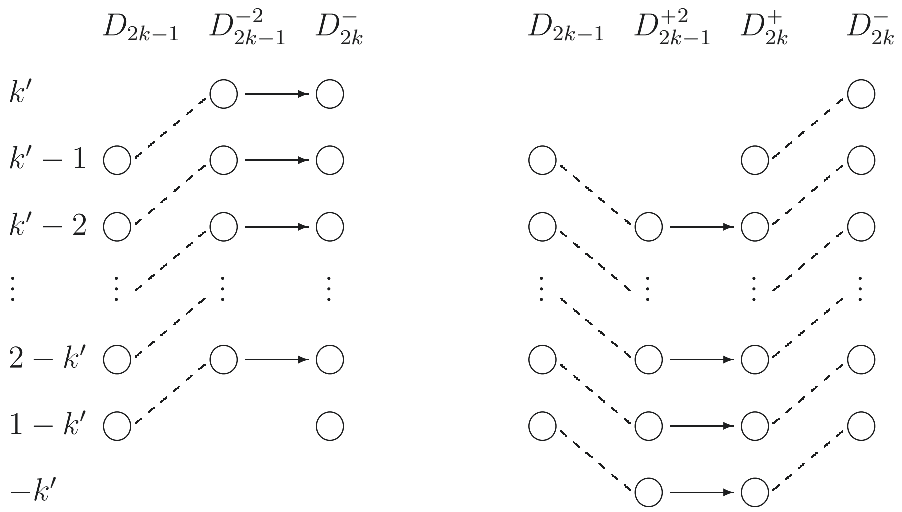


Figure 13: The maps $\phi_{\pm}$ from $\underline{\mathrm{SHM}}(-Y(K),-\Gamma_{2k-1})$ to $\underline{\mathrm{SHM}}(-Y(K),-\Gamma_{2k})$.

Proof. First assume that $Y=S^3$, then $(Y(1),\delta)$ is a product sutured manifold and $(Y(K),\Gamma_n)=(V,\gamma^2_{(1,-n)})$, where $(V,\gamma^2_{(1,-n)})$ is the balanced sutured manifold defined in subsection 4.3. From proposition 4.12, we know that

$$\underline{\mathrm{SHM}}(-V,-\gamma^2_{(1,-n)})\cong\mathcal{R}^n.$$

Suppose S_n is the Seifert surface of K surface intersecting $\Gamma_n=\gamma^2_{(1,-n)}$ at $2n$ points, then the argument in the proof of proposition 4.12 can also be applied to calculate the graded homology and we conclude

$$\underline{\mathrm{SHM}}(-V,-\gamma^2_{(1,-n)},S^{\tau}_n,i)[\sigma]\cong\mathcal{R}$$

for all i such that $1-n\leqslant i\leqslant 0$. Moreover, the map

$$\psi^n_{+,n+1}:\underline{\mathrm{SHM}}(-V,-\gamma^2_{(1,-n)},S^{\tau}_n)[\sigma]\to\underline{\mathrm{SHM}}(-V,-\gamma^2_{(1,-n-1)},S^{\tau}_{n+1})[\sigma]$$

is of degree -1 and is an isomorphism for all i such that $1-n\leqslant i\leqslant 0$. Thus we conclude that the direct limit

$$\underline{\mathrm{KHM}}^-(-S^3,K,p)\cong\mathcal{R}[U].$$

When Y is an arbitrary 3-manifold, we know that

$$(Y(K),\Gamma_n)=((Y(1),\delta)\sqcup(-S^3(K),-\gamma^2_{(1,-n)}))\cup h,$$

where h is a contact 1-handle, defined as in [1] or [20], which connects the two disjoint balanced sutured manifolds $((Y(1),\delta)$ and $(-S^3(K),-\gamma^2_{(1,-n)})$. Thus we know that

$$\underline{\mathrm{SHM}}(-Y(K),-\Gamma_n)\cong\underline{\mathrm{SHM}}(-Y(1),-\delta)\otimes(-S^3(K),-\gamma^2_{(1,-n)}).$$

Moreover, the the above isomorphism commute with the maps $\psi^n_{\pm,n+1}$ on $\underline{\mathrm{SHM}}(-Y(K),-\Gamma_n)$ and the maps $id\otimes\psi^n_{\pm,n+1}$ on $\underline{\mathrm{SHM}}(-Y(1),-\delta)\otimes(-S^3(K),-\gamma^2_{(1,-n)})$ as the corresponding contact handle attachments are disjoint from each other. Thus we conclude that

$$\underline{\mathrm{KHM}}^-(-Y,K,p)\cong\underline{\mathrm{SHM}}(-Y(1),-\delta)\otimes\mathcal{R}[U].$$

□

Proposition 5.8. *The direct system stabilizes, that is, for any fixed $i\in\mathbb{Z}$, there is a large enough N, so that for all $n>N$, we have an isomorphism*

$$\phi^n_{-,n+1}:\underline{\mathrm{SHM}}(-Y(K),-\Gamma_n,S^\tau_n,i)[\sigma]\cong\underline{\mathrm{SHM}}(-Y(K),-\Gamma_{n+1},S^\tau_{n+1},i)[\sigma].$$

Proof. We will need to use the following exact triangle again.

$$\begin{tikzcd}
\underline{\mathrm{SHM}}(-Y(K),-\Gamma_{n+1}) \arrow[rr, "\psi^{n+1}_{-,\infty}"] & & \underline{\mathrm{SHM}}(-Y(K),-\Gamma_\infty) \arrow[dl, "\psi^{\infty}_{-,n}"] \\
& \underline{\mathrm{SHM}}(-Y(K),-\Gamma_n) \arrow[ul, "\psi^{n+1}_{-,n}"] &
\end{tikzcd}$$

We will deal with the case that $n=2k$ is even here, and the other case is exactly the same. When n is even, we know from proposition 5.5 that

$$\phi^n_{-,n+1}(\underline{\mathrm{SHM}}(-Y(K),-\Gamma_n,S^-_n,i))\subset\underline{\mathrm{SHM}}(-Y(K),-\Gamma_{n+1},S_{n+1},i).$$

By a similar argument, we have

$$\phi^{n+1}_{-,\infty}(\underline{\mathrm{SHM}}(-Y(K),-\Gamma_{n+1},S_{n+1},i))\subset\underline{\mathrm{SHM}}(-Y(K),-\Gamma_\infty,S^{-n}_\infty,i)$$

where S_∞ intersects the suture Γ_∞ twice. Proposition 4.3 then implies that

$$\underline{\mathrm{SHM}}(-Y(K),\Gamma_\infty,-S^{-n}_\infty,i)=\underline{\mathrm{SHM}}(-Y(K),-\Gamma_\infty,S_\infty,i-k).$$

However, the adjunction inequality in lemma 2.7 tells us that if $i-k<-g(S)$, then

$$\underline{\mathrm{SHM}}(-Y(K),-\Gamma_\infty,S_\infty,i-k)=0.$$

Thus for large enough i, the map

$$\phi^n_{-,n+1}:\underline{\mathrm{SHM}}(-Y(K),-\Gamma_n,S^-_n,i)\to\underline{\mathrm{SHM}}(Y(K),\Gamma_{n+1},S_{n+1},i).$$

is injective by the exactness, and by a similar argument it is also surjective.

□

Corollary 5.9. *Under the above conditions, there exists an integer N_0, so that for any $i < N_0$, the U map induces an isomorphism:*

$$\underline{\mathrm{KHM}}^-(-Y, K, p, S, i) \cong \underline{\mathrm{KHM}}^-(-Y, K, p, S, i-1)$$

Proof. The proof is exactly the same as the above proposition. ☐

Corollary 5.10. *For a knot $K \subset Y$, a Seifert surface S of K and a fixed point $p \in K$, we have*

$$\underline{\mathrm{KHM}}^-(-Y, K, p, S, i) = 0$$

for $i > g$ and

$$\underline{\mathrm{KHM}}^-(-Y, K, p, S, g) \cong \underline{\mathrm{KHM}}(-Y, K, p, S, g).$$

Here g is the genus of the Seifert surface.

Proof. The first statement that

$$\underline{\mathrm{KHM}}^-(-Y, K, p, S, i) = 0$$

for $i > g$ follows from the adjunction inequality in lemma 2.7.

For the second part of the statement, suppose $n = 2k+1$ is odd and the other case is exactly the same. Suppose (M', γ') is obtained by a sutured manifold decomposition of $S_n \subset (Y(K), \Gamma_n)$. It is straight forward to check that if we decompose $\underline{\mathrm{SHM}}(Y(K), \Gamma_\infty)$ along S_∞, then we will get exactly the same balanced sutured manifold (M', γ'). Hence from proposition 6.9 in [16], we know that

$$\underline{\mathrm{SHM}}(-Y(K), -\Gamma_n, S_{n+1}, g(S)+k+1) = \underline{\mathrm{SHM}}(M', \gamma') \cong \underline{\mathrm{KHM}}(-Y, K, p, S_\infty, g(S)).$$

Then the corollary follows from proposition 5.8 the way we shift the degree in definition 5.4. ☐

Suppose $K \subset Y$ is a fibred knot with fibre S of genus g. Suppose (S, h) is an open book corresponding to the fibration of $K \subset Y$ so that it support a contact structure ξ on Y. We call h *not right-veering* if there is an arc $\alpha \subset S$ and one end point $p \in \partial\alpha$ so that near $p \subset S$, $h(\alpha)$ is to the left of α. See figure 14. See [6] for more details.

Corollary 5.11. *Under the above setting, if h is not right-veering, we have*

$$\underline{\mathrm{KHM}}^-(-Y, K, p, S, g) \cong \mathcal{R},$$

and the generator is in the kernel of the U map.

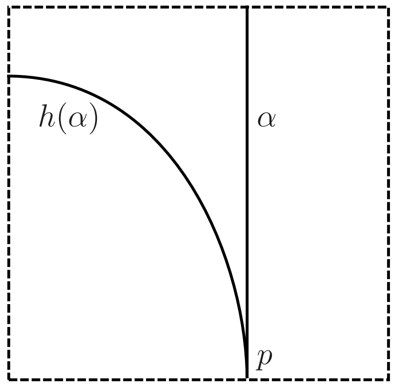


Figure 14: Not right-veering

Proof. This result is essentially the same as in Baldwin and Sivek [6]. The only difference is that we translate it into our language involving $\underline{\mathrm{KHM}}^-$. $\square$

Proposition 5.12. *We have an exact triangle:*

$$\begin{tikzcd}
\widehat{\mathrm{KHM}}(-Y,K,p) \arrow[rr, "U"] & & \widehat{\mathrm{KHM}}(-Y,K,p) \arrow[dl, "\psi"] \\
& \underline{\mathrm{KHM}}(-Y,K,p) \arrow[ul, "\psi'"] &
\end{tikzcd}$$

Proof. We will use the by-pass exact triangle

$$\begin{tikzcd}
\underline{\mathrm{SHM}}(-Y(K),-\Gamma_{n+1}) \arrow[rr, "\psi^{n+1}_{+,\infty}"] & & \underline{\mathrm{SHM}}(-Y(K),-\Gamma_{\infty}) \arrow[dl, "\psi^{\infty}_{+,n}"] \\
& \underline{\mathrm{SHM}}(-Y(K),-\Gamma_{n}) \arrow[ul, "\psi^{n+1}_{+,n}"] &
\end{tikzcd} \tag{21}$$

The maps $\{\phi^{n}_{+,n+1}\}_{n\in\mathbb{Z}_+}$ induce the U map. By a similar argument, the maps $\{\phi^{n+1}_{+,\infty}\}\}_{n\in\mathbb{Z}_+}$ and $\{\phi^{\infty}_{+,n}\}_{n\in\mathbb{Z}_+}$ induce the maps ψ and ψ' in the statement of the proposition. Then it is formal to check that the by-pass exact triangles (21) for all $n\in\mathbb{Z}_+$ will induce the desired one as stated in the proposition. $\square$

Suppose $K\subset Y$ is a knot and $\partial S=K$ is a Seifert surface. Let $Y(K)$ be a knot complement. Let λ and μ represent the longitude and meridian on $\partial Y(K)$ respectively, according to the framing induced S. We can do a Dehn surgery along the knot K and get a surgery manifold

$$Y_\phi = Y(K)\underset{\phi}{\cup} S^1\times D^2.$$

Suppose $\mu_\phi = \phi(\{1\} \times \partial D^2) = q_0\lambda - p_0\mu$ and $\lambda_\phi = \phi(S^1 \times \{1\}) = r_0\lambda - s_0\mu$. This result in a surgery of slope $-\frac{p_0}{q_0}$. Now λ_ϕ and μ_ϕ will form another framing on $\partial Y(K)$, so that μ_ϕ is the meridian of the knot $K_\phi = S^1 \times \{0\} \subset Y_\phi$. Note $Y(K)$ is also a knot complement of $K_\phi \subset Y_\phi$. Hence we can use the new framing to construct a minus version of knot monopole Floer homology $\underline{\mathrm{KHM}}^-(-Y_\phi, Y_\phi)$ of (Y_ϕ, K_ϕ). Here we will omit the choice of base points. The construction is exactly the same as in definition 5.4. The Seifert surface S for the original knot K will still induces a grading on $\underline{\mathrm{KHM}}^-(-Y_\phi, K_\phi)$. We can also shift the degree properly just as we did above.

Proposition 5.13. *For any fixed $i \in \mathbb{Z}$, there exists N so that for any surgery slope $-\frac{p_0}{q_0} < -N$, we have*

$$\underline{\mathrm{KHM}}^-(-Y, K, S, i) \cong \underline{\mathrm{KHM}}^-(-Y_\phi, K_\phi, S, i).$$

Proof. We will use the framing (λ, μ) intricately and write both the curve $q\lambda - p\mu$ or the slope $-\frac{p}{q}$ as $(q, -p)$. We will use $\gamma_{(q\lambda - p\mu)}$ $\gamma_{(q,-p)}$ to denote the suture consisting of two curves of slope $(q, -p)$. Again $\gamma_{(1,-p)} = \Gamma_n$ in the construction of knot monopole Floer homology.

From the stabilization property in proposition 5.8, we know that there exists N_1 such that for any $n > N_1$, we have

$$\underline{\mathrm{KHM}}^-(-Y, K, S, i) \cong \underline{\mathrm{SHM}}(-Y(K), -\gamma_{(1,-n)}, S^\tau, i)[\sigma]. \tag{22}$$

Here S^τ and σ are defined as above. The degree shifted on $\underline{\mathrm{SHM}}(-Y(K), -\gamma_{(1,-n)})$ can be described more explicitly in the following way: there is a grading i_0 so that

$$\underline{\mathrm{SHM}}(-Y(K), -\gamma_{(1,-n)}, S^\tau, i_0)[\sigma] \cong \underline{\mathrm{SHM}}(M', \gamma'),$$

where (M', γ') is the balanced sutured manifold obtained from $(-Y(K), -\gamma_{(1,-n)})$ by performing a sutured manifold decomposition along S. This isomorphism is guaranteed by proposition 6.9 in [16]. We shift the degree so that $i_0 = g(S)$, the genus of S.

Remark 5.14. Here the exact value σ of the degree we shall shift down depends on n. In principle, it depends on the slope $(q, -p)$, or the p value, according to the frame (λ, μ). However, we will always omit n or p from the notation.

Now by a similar stabilization property, there exists N_2 so that for any $n > N_2$, we have

$$\underline{\mathrm{KHM}}^-(-Y_\phi, K_\phi, S, i) \cong \underline{\mathrm{SHM}}(-Y(K), -\gamma_{(\lambda_\phi - n\mu_\phi)}, S^\tau, i)[\sigma]. \tag{23}$$

Hence to prove the theorem, it is suffice to prove that for large enough n and large enough surgery slope, we have

$$\underline{\mathrm{SHM}}(-Y(K), -\gamma_{(1,-n)}, S^\tau, i)[\sigma] \cong \underline{\mathrm{SHM}}(-Y(K), -\gamma_{(\lambda_\phi - n\mu_\phi)}, S^\tau, i)[\sigma].$$

Now fix an $n_2 > N_2$, and write $\lambda_\phi - n_2\mu_\phi = q\lambda - p\mu$. From the proof of proposition 4.12, we can construct two sequences of slopes $\{(q'_j, -p'_j)\}$ and $\{(p''_j, -q''_j)\}$ inductively as follows. Let $(q'_0, -p'_0) = (q, -p)$. For any $j \geqslant 1$, suppose we have the continued fraction of $(q'_{j-1}, -p'_{j-1})$ to be

$$(q''_{j-1}, -p''_{j-1}) = [r_1, ..., r_{k-2}, r_{k-1}, r_k],$$

then define

$$(q''_j, -p''_j) = [r_1, ..., r_{k-2}, r_{k-1}, r_k + 1],\ (q'_j, -p'_j) = [r_1, ..., r_{k-2}, r_{k-1}].$$

Note we identify $[r_1, ...r_{k-2}, r_{k-1}, -1]$ as $[r_1, ..., r_{k-2}, r_{k-1} + 1]$. We end the sequence when

$$(q'_{k-1}, -p'_{k-1}) = [r_1] = (1, r_1). \tag{24}$$

Here $r_1 \leqslant -2$ is the first term in the continued fraction of $(q, -p) = (\lambda_\phi - n_2\mu_\phi)$.

The sequences of slopes fit into by-pass exact triangles:

$$\begin{tikzcd}[column sep=small] & \underline{\mathrm{SHM}}(-Y(K), -\gamma_{(q'_{j-1}, p'_{j-1})}) \arrow[dr, "\psi_{j,2}"] & \\ \underline{\mathrm{SHM}}(-Y(K), -\gamma_{(q''_j, p''_j)}) \arrow[ur, "\psi_{j,1}"] & & \underline{\mathrm{SHM}}(-Y(K), -\gamma_{(q'_j, p'_j)}) \arrow[ll, "\psi_{j,0}"] \end{tikzcd} \tag{25}$$

If $Y = S^3$ and K is the unknot, then $\psi_{j,k} = \psi_{-,k}$ for $k = 0, 1, 2$ in the previous exact triangle (15). As above, we know that

$$\psi_{j,0} : (\underline{\mathrm{SHM}}(-Y(K), -\gamma_{(q'_j, p'_j)}, S^{-p''_j}, i') \to \underline{\mathrm{SHM}}(-Y(K), -\gamma_{(q''_j, p''_j)}, S^{+p'_j}, i'),$$

$$\psi_{j,1} : \underline{\mathrm{SHM}}(-Y(K), -\gamma_{(q''_j, p''_j)}, S^{+p'_j}, i') \to \underline{\mathrm{SHM}}(-Y(K), -\gamma_{(q'_{j-1}, p'_{j-1})}, S, i'),$$

$$\psi_{j,2} : \underline{\mathrm{SHM}}(-Y(K), -\gamma_{(q'_{j-1}, p'_{j-1})}, S, i') \to (\underline{\mathrm{SHM}}(-Y(K), -\gamma_{(q'_j, p'_j)}, S^{-p''_j}, i').$$

Note in the above formula, we assume that p''_{j-1} is odd. When it is even, we shall use $\underline{\mathrm{SHM}}(-Y(K), -\gamma_{(q'_{j-1}, p'_{j-1})}, S^-, i')$ instead, and there shall be some adjustions on the other two terms but the argument is essentially the same. Also from the construction, we have an equality

$$p'_{j-1} = p'_j + p''_j \tag{26}$$

just as in the proof of proposition 4.12.

On this particular grading i', the three maps $\psi_{j,0}$, $\psi_{j,1}$ and $\psi_{j,2}$ also form an exact triangle. Note in the above formula, we use i' because we haven't apply the shifting σ as in (22) and (23). If we track the way we shift the degree via σ, we know that we shall look at the degree i' in this un-shifted version, so that

$$\frac{p'_{j-1}}{2} - i' = g(S) - i, \tag{27}$$

i,e., the grading whose difference from the top grading is the same as that of i. Here i is pre-fixed by the hypothesis of the proposition and i' actually depends on the indices j but we omit it from the notation.

Note in the sequence of slopes $\{(q'_j, -p'_j)\}$, we have

$$(q'_0, -p'_0) = (q, -p) = \lambda_\phi - n_2\mu_\phi, \text{ and } (q'_{k-1}, -p'_{k-1}) = [r_1] = (1, r_1).$$

If we could prove that for any j, the map $\psi_{j,2}$ is an isomorphism in at degree i' described as above, then we know that

$$\begin{aligned}
\underline{\mathrm{KHM}}^-(-Y_\phi, K_\phi, S, i) &\cong \underline{\mathrm{SHM}}(-Y(K), -\gamma_{(\lambda_\phi - n_2\mu_\phi)}, S^\tau, i)[\sigma] \\
&= \underline{\mathrm{SHM}}(-Y(K), -\gamma_{(\lambda_\phi - n_2\mu_\phi)}, S^\tau, i') \\
&\cong \underline{\mathrm{SHM}}(-Y(K), -\gamma_{(\lambda + r_1\mu)}, S^\tau, i') \\
&\cong \underline{\mathrm{SHM}}(-Y(K), -\gamma_{(\lambda + r_1\mu)}, S^\tau, i)[\sigma].
\end{aligned}$$

Here r_1 is defined as in (24). If further we had $r_1 < -N_1$, then we know from (22) that

$$\begin{aligned}
\underline{\mathrm{KHM}}^-(-Y_\phi, K_\phi, S, i) &\cong \underline{\mathrm{SHM}}(-Y(K), -\gamma_{(\lambda + r_1\mu)}, S^\tau, i)[\sigma] \\
&\cong \underline{\mathrm{KHM}}^-(-Y, K, S, i)
\end{aligned}$$

and we were done.

Hence there are two things to show:

(1). All the maps $\psi_{j,2}$ are isomorphisms.

(2). For 'small' enough surgery slopes $(q_0, -p_0)$, we have $r_1 < -N_1$.

We show the second statement first. By definition, we have

$$r_1 = -(\lfloor \frac{p}{q} \rfloor + 1) \text{ and } \frac{p}{q} = \frac{s_0 + n_2 p_0}{r_0 + n_2 q_0}. \tag{28}$$

If we choose large enough n_2 (we can freely make n_2 larger), then we know that

$$\lfloor \frac{p}{q} \rfloor \geqslant \lfloor \frac{p_0}{q_0} \rfloor - 1. \tag{29}$$

Hence for any surgery slope $-\frac{p_0}{q_0} < -N_1$, (2) is true.

To deal with (1), we apply the argument in the proof of proposition 4.12 again, and look at the map

$$\psi_{j,0} : (\underline{\mathrm{SHM}}(-Y(K), -\gamma_{(q'_j, p'_j)}, S^{-p''_j}, i') \to \underline{\mathrm{SHM}}(-Y(K), -\gamma_{(q''_j, p''_j)}, S^{+p'_j}, i').$$

The difference from previous argument is that now S is not always a disk, so the degree shifting property may not distinguish all the non-zero gradings of

$$\underline{\mathrm{SHM}}(-Y(K),-\gamma_{(q'_j,p'_j)},S^{-p''_j})$$

from that of

$$\underline{\mathrm{SHM}}(-Y(K),-\gamma_{(q''_j,p''_j)},S^{+p'_j}),$$

and as a result, ψ_{j_0} need not to be identically zero. However, the overlap only happens in the few bottom non-zero gradings in

$$\underline{\mathrm{SHM}}(-Y(K),-\gamma_{(q'_j,p'_j)},S^{-p''_j}),$$

while the desired grading i' is quite near the top as in (27). This idea is realized in details as follows.

From now on we still assume that p'_{j-1} and p'_j are both odd. Other cases are similar. To use the surface S to construct a grading in

$$\underline{\mathrm{SHM}}(-Y(K),-\gamma_{(q''_j,-p''_j)}),$$

we shall first perform a positive stabilization to get S^+ since $|S\cap\gamma_{(q''_j,-p''_j)}|=2p''_j$ and p''_j is even. There is a marked closure

$$\mathcal{D}=(Y',R,r,m,\eta)$$

of $(-Y(K),-\gamma_{(q''_j,-p''_j)})$ so that S extends to a closed surface $\bar{S}\subset Y'$. From definition 3.3, we know that

$$\chi(\bar{S})=2-2g(S)-1-(p''_j+1)=-2g(S)-p''_j.$$

Hence from the adjunction inequality, we know that for any $i''>g(S)+\frac{p''_j}{2}$,

$$\underline{\mathrm{SHM}}(-Y(K),-\gamma_{(q''_j,p''_j)},S^+,i'')=0.$$

The degree shifting property tells us that for any $i''>g(S)+\frac{p''_j}{2}+\frac{1-p'_j}{2}$,

$$\underline{\mathrm{SHM}}(-Y(K),-\gamma_{(q''_j,p''_j)},S^{+p'_j},i'')=0.$$

Hence to show that

$$\psi_{j,2}:\underline{\mathrm{SHM}}(-Y(K),-\gamma_{(q'_{j-1},p'_{j-1})},S,i')\rightarrow(\underline{\mathrm{SHM}}(-Y(K),-\gamma_{(q'_j,p'_j)},S^{-p''_j},i').$$

is an isomorphism, it is suffice to show that for the particular grading i' defined by (27), we have

$$i' > g(S) + \frac{p''_j}{2} + \frac{1 - p'_j}{2}.$$

This is equivalent to

$$\frac{p'_j + p'_{j-1} - p''_j}{2} > 2g(S) - i + 1.$$

Applying equality (26), this is also equivalent to

$$p'_j > 2g(S) - i + 1.$$

From (26), (28) and (29) we know that for any j,

$$p'_j \geqslant p'_{j+1} \geqslant p'_{k-1} = -r_1 \geqslant \lfloor \frac{p_0}{q_0} \rfloor.$$

Hence we can pick the constant

$$N = \max\{N_1, 2g(S) - i + 1\}$$

in the hypothesis of the proposition and we are done. $\square$

Remark 5.15. By a similar argument, we could prove that actually N_1 depends only on $g(S)$.

At last, we would like to introduce the following definition.

Definition 5.16. Suppose $K \subset Y$ is an oriented knot and S is a Seifert surface of K. We can define the *tau invariant* $\tau(Y, K, S)$ of $K \subset Y$ with respect to S as follows:

$$\tau(Y, K, S) = -\max\{i | \exists x \in \underline{\mathrm{KHM}}^-(Y, K, p, S, i),\ U^j x \neq 0 \text{ for any } j \geqslant 0.\}$$

Here the base point can be fixed arbitrarily.

Question 5.17. *What properties does $\tau(Y, K, S)$ have?*

6 Instantons and knot Floer homology

6.1 Instanton Floer homology and the generalized eigenspace decomposition

Suppose Y is a closed, connected and oriented 3-manifold. Suppose ω is a fixed Hermitian line bundle whose first Chern class $c_1(\omega)$ has an odd pairing with the fundamental class of some surface.

Suppose E is an $U(2)$-bundle whose determinant line bundle $\Lambda^2 E$ is isomorphic to ω. Let $\mathfrak{g}_E$ be the bundle of traceless skew-Hermitian endomorphisms of E, and let $\mathcal{A}_E$ be the $(SO(3))$ connections on $\mathfrak{g}_E$. Let $\mathcal{G}_E$ be the group of determinant one transformations and let $\mathcal{B}_E = \mathcal{A}_E \backslash \mathcal{G}_E$. Then we can use the Chern-Simons functional to construct a well defined $SO(3)$ instanton Floer homology over $\mathbb{C}$ which we denote by $I^\omega(Y)$.

If $x \in Y$ is a point, then there is an action $\mu(x)$ on $I^\omega(Y)$. The action $\mu(x)$ has eigenvalue 2 and -2. By slightly abusing the notation, from now on we use the same notation $I^\omega(Y)$ to denote only the generalized eigenspace of $\mu(x)$ with corresponding to eigenvalue 2.

Suppose $\Sigma \subset Y$ is a closed oriented embedded surface inside Y. Then there is also an action $\mu(\Sigma)$ on $I^\omega(Y)$. We have the following result about the eigenvalues:

Proposition 6.1 (Kronheimer, Mrowka, [16])**.** *The eigenvalues of the action $\mu(K)$ on $I^\omega(Y)$ belongs to the set of even integers ranged from $2-2g(\Sigma)$ to $2g(\Sigma)-2$.*

If Σ and Σ' are two such embedded surfaces, then the action $\mu(\Sigma)$ and $\mu(\Sigma')$ commute. Then we can look at the simultaneous generalized eigenspace. Similar to corollary 7.6 in Kronheimer and Mrowka [16], we can make the following definition.

Definition 6.2. Suppose we have a function $\lambda : H_2(Y;\mathbb{Z}) \to 2\mathbb{Z}$, then we can define

$$I^\omega(Y)_\lambda = \bigcap_{\sigma \in H_2(Y;Z)} \bigcup_{N \geqslant 0} ker(\mu(\sigma) - \lambda(\sigma))^N.$$

Such a function λ is a called an *eigenvalue function.*

If the embedded surface Σ represents a zero class in $H_2(Y;\mathbb{Q})$, then the action $\mu(\Sigma)$ is actually the zero action. This means that if $I^\omega(Y)_\lambda \neq 0$ then we can lift λ to a linear map (which we will use the same notation to denote)

$$\lambda : H_2(Y;\mathbb{Q}) \to \mathbb{Q}.$$

Thus we can think of λ as an element of $H^2(Y;\mathbb{Q})$. So from now on we will consider $\lambda \in H^2(Y;\mathbb{Q})$. We have a decomposition

$$I^{\omega}(Y) = \bigoplus_{\lambda \in H^2(Y;\mathbb{Q})} I^{\omega}(Y)_{\lambda}.$$

Suppose $R \subset Y$ is a closed oriented embedded surface inside Y, then as we did in definition 2.4, we can define the following.

Definition 6.3. Suppose the pair (Y, R) is as above. Then we can define the set

$$\mathfrak{H}^*(Y|R) = \{\lambda \in H^2(Y;\mathbb{Q}) | \lambda([R]) = 2g(R) - 2,\ I^{\omega}(Y)_{\lambda} \neq 0\},$$

The elements $\lambda \in \mathfrak{H}^*(Y|R)$ are called *supporting eigenspace functions.*

We have the following lemma which is the instanton correspondence to lemma 4.8 for monopole theory.

Lemma 6.4. *Suppose (W, ν) is a cobordism between (Y, ω) and (Y', ω'). Suppose $\lambda \in H^2(Y;\mathbb{Q})$ and $\lambda' \in H^2(Y';\mathbb{Q})$ are two eigenvalue functions. Suppose $i : Y \to W$ and $i' : Y' \to W$ are the inclusion map.*

$$I(W, \nu)(I^{\omega}(Y)_{\lambda}) \cap I^{\omega'}(Y')_{\lambda'} \neq \{0\},$$

then there must be an element $\tau \in H^2(W;\mathbb{Q})$ so that $i^*(\tau) = \lambda$ and $(i')^*(\tau) = \lambda'$.

Proof. For a second homology class σ and a rational number $r \in \mathbb{Q}$ we can define

$$I^{\omega}(Y, \sigma, r) = \bigcup_{N \geqslant 0} ker(\mu(\sigma) - r)^N.$$

By definition we know that

$$I^{\omega}(Y)_{\lambda} = \bigcap_{\sigma \in H_2(Y;\mathbb{Q})} I^{\omega}(Y, \sigma, \lambda(\sigma)).$$

Similarly we can define $I^{\omega'}(Y', \sigma', r')$.

Suppose there are no such τ as in the statement of the lemma, then we can regard an element $\tau \in H^2(W;\mathbb{Q})$ as a map

$$\tau : H_2(W;\mathbb{Q}) \to \mathbb{Q}$$

and thus the non-existence of τ implies that there is a class $\sigma_0 \in H_2(Y;\mathbb{Q})$ and a class $\sigma_0' \in H_2(Y';\mathbb{Q})$ so that

$$i_*(\sigma_0) = i'_*(\sigma_0') \in H_2(W),$$

while

$$\lambda(\sigma_0) \neq \lambda'(\sigma_0').$$

Thus we know that

$$I(W,\nu)(I^{\omega}(Y)_{\lambda}) \subset I(W,\nu)(I^{\omega}(Y,\sigma_0,\lambda(\sigma_0))) \subset I^{\omega'}(Y',\sigma_0',\lambda(\sigma_0)).$$

The last inclusion follows from lemma 2.6 in [6]. However, $\lambda(\sigma) \neq \lambda'(\sigma')$ so

$$I^{\omega'}(Y',\sigma_0',\lambda(\sigma_0)) \cap I^{\omega'}(Y',\sigma_0',\lambda'(\sigma_0')) = \{0\}.$$

Thus we conclude

$$I(W,\nu)(I^{\omega}(Y)_{\lambda}) \cap I^{\omega'}(Y')_{\lambda'} = \{0\}.$$

So we are done. $\square$

6.2 The sutured instanton Floer homology

Suppose (M,γ) is a balanced sutured manifold, then as we did for monopole theory, we can construct a closure of (M,γ) and apply the instanton Floer homology. Pick a connected auxiliary surface T of large enough genus, we can get a pre-closure

$$\widetilde{M} = M \cup T \times [-1,1],\ \partial\widetilde{M} = R_+ \sqcup R_-.$$

For the construction in instanton theory, we also need to pick a point $p \in T$ so that there are corresponding points $p_{\pm} \in R_{\pm}$. When choosing the gluing diffeomorphism $h: R_+ \to R_-$ so that $h(p_+) = p_-$. Thus we know that inside the closure (Y,R) there is a closed curve $p \times S^1 \subset Y$. Let ω be a complex line bundle over Y whose first Chern class is dual to the curve $p \times S^1$. Then we can make the following definition.

Definition 6.5 (Kronheimer, Mrowka [16])**.** Define the *sutured instanton Floer homology* of (M,γ) to be

$$SHI(M,\gamma) = I^{\omega}(Y|R) = \bigoplus_{\lambda\in\mathfrak{H}^*(Y|R)} I^{\omega}(Y)_{\lambda}.$$

Baldwin and Sivek [2] also made refinements of closures and constructed canonical maps for the sutured instanton Floer homology.

Definition 6.6. A marked odd closure $\mathcal{D} = (Y,R,r,m,\eta,\alpha)$ of (M,γ) is a tuple so that (Y,R,r,m,η) is a marked closure of (M,γ) as in definition 2.9, the simple closed curve α is disjoint from $\mathrm{im}(m)$, and $\alpha \cap r(R \times [-1,1])$ is of the form $r(p \times [-1,1])$.

We can pick a complex line bundle ω whose first Chern class is dual to $\alpha \sqcup \eta$. Then we can define

$$SHI(\mathcal{D}) = I^{\omega}(Y|r(R \times \{0\})).$$

Theorem 6.7 (Baldwin, Sivek [2]). *Suppose (M,γ) is a balanced sutured manifold and $\mathcal{D}$, $\mathcal{D}'$ are two marked odd closures of (M,γ). Then there is a canonical map*

$$\Phi_{\mathcal{D},\mathcal{D}'} : SHI(\mathcal{D}) \to SHM(\mathcal{D}'),$$

which is an isomorphism well defined up to multiplication by a non-zero element in $\mathbb{C}$. Furthermore, the canonical map satisfies the same functoriality properties as the canonical map for sutured monopole Floer homology in theorem 2.10.

Hence we have a well defined projective transitive system

$$\underline{\mathrm{SHI}}(M,\gamma)$$

associated to (M,γ). For a knot, there is a similar discussion as in subsection 2.2 and we have a well defined projective transitive system

$$\underline{\mathrm{KHI}}^{-}(Y,K,p)$$

associates to a triple (Y,K,p) for a knot $K \subset Y$ and a base point $p \in K$.

There are similar results for the contact gluing maps and by-pass exact triangles.

Theorem 6.8 (Li [20]). *There is a gluing map for sutured instanton Floer homology, satisfying the same properties as in theorem 2.16.*

Theorem 6.9 (Baldwin, Sivek [6]). *Suppose (M,γ_1), (M,γ_2) and (M,γ_3) are three balanced sutured manifolds which are related in the same way as in theorem 2.17. Then there is still a by-pass exact triangle*

$$\begin{tikzcd}
\underline{\mathrm{SHI}}(-M,-\gamma_1) \arrow[rr, "\psi_{12}"] & & \underline{\mathrm{SHI}}(-M,-\gamma_2) \arrow[dl, "\psi_{23}"] \\
& \underline{\mathrm{SHI}}(-M,-\gamma_3) \arrow[ul, "\psi_{31}"] &
\end{tikzcd}$$

where the maps ψ_{ij} comes from the gluing maps in sutured instanton Floer homology, just as the monopole case in subsection 2.3.

6.3 Statement of results

With lemma 6.4 and theorem 6.9 in place of lemma 4.8 and theorem 2.17, we can recover all results we did in this paper for sutured monopole Floer homology. We will present those results without further proofs.

Proposition 6.10. *Suppose (M,γ) is a balanced sutured manifold and $\mathcal{D}$ and $\mathcal{D}'$ are two marked odd closures of the same genus. Then the canonical map $\Phi_{\mathcal{D},\mathcal{D}'}$ for sutured instanton Floer homology can be interpreted in terms of the Floer excision cobordism, as in proposition 3.8 for sutured monopole Floer homology.*

Theorem 6.11. *Suppose (M,γ) is a balanced sutured manifold and S is a properly embedded surface inside M so that ∂S is connected and $|\partial S\cap\gamma| = 2n$ with n odd. Then S induces a grading on $\underline{\mathrm{SHI}}(M,\gamma)$ which we denote by*

$$\underline{\mathrm{SHI}}(M,\gamma,S,i).$$

Proposition 6.12. *Suppose (M,γ) is a balanced sutured manifold so that M is the complement of a non-homologous knot $K\subset X$ and γ has two components. Suppose S is a Seifert surface of K, viewed as a properly embedded surface in M, so that $|\partial S\cap\gamma| = 2n$. Then for any $p,l,k\in\mathbb{Z}$ such that $n+i$ is odd, we have*

$$\underline{\mathrm{SHI}}(-M,-\gamma,S^p,l) = \underline{\mathrm{SHI}}(-M,-\gamma,S^{p+2k},l-k).$$

Proposition 6.13. *Suppose V is a solid torus and γ is a suture on ∂V with $2n$ components and slope $\frac{p}{q}$, then*

$$\underline{\mathrm{SHI}}(-V,-\gamma)\cong\mathbb{C}^{(2^{n-1}\cdot|p|)}.$$

Theorem 6.14. *Suppose K is a non-homologous knot inside an closed connected oriented 3-manifold Y and $p\in K$ is a base point. Then there is a projective $\mathbb{C}$-vector space $\underline{\mathrm{KHI}}^-(Y,K,p)$, whose elements are well defined up to multiplication by a non-zero element in $\mathbb{C}$, associated to the triple (Y,K,p). Also there is a homomorphism*

$$U:\underline{\mathrm{KHI}}^-(Y,K,p)\to\underline{\mathrm{KHI}}^-(Y,K,p).$$

If S is a Seifert surface of K then S induces a $\mathbb{Z}$ grading on $\underline{\mathrm{KHI}}^-(Y,K,p)$ so that U is of degree -1. Furthermore, analogous results to proposition 5.7, proposition 5.8, corollary 5.9, corollary 5.10, proposition 5.12, proposition 5.13 also hold for $\underline{\mathrm{KHI}}^-(Y,K,p)$.

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