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BARRIER LYAPUNOV FUNCTION BASED CONTROL OF A FLEXIBLE LINK CO-ROBOT WITH SAFETY CONSTRAINTS

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ABSTRACT
In this paper, a control design for a flexible link co-robot with safety constraints is proposed. The safety constraints are converted to the constraints on the tip position and velocity. To handle this constrained control problem, a barrier Lyapunov function (BLF) is employed in the control design. The derivative of the logarithmic BLF is more complicated compared with the derivative of a quadratic Lyapunov function, which makes the problem of "explosion of terms" more serious. Thus, the dynamic surface control is used to deal with the problem. Furthermore, an extended state observer is adopted to estimate and compensate the uncertainty and disturbance in the system. The stability analysis via the singular perturbation theory shows the local practical exponential stability of the system. Simulation results indicate that the control performance is guaranteed without violation of the constraints.

1 INTRODUCTION
In the past decade, the attention paid to corobots [1] increases rapidly. Unlike the conventional robots which usually have independent workspace isolated from human interactions, the corobots usually share a workspace with human workers. Thus, safety becomes an important concern for corobots [2].

To address the safety concerns, mechanical flexibility is often introduced in the design of corobots, such as the flexible joint robots [3] and flexible link robots [4] [5]. The flexibility enables the low-inertia design of the arms' link and decouples the actuators' rotor inertia from the links' inertia, which increases the safety of physical human-robot interaction (pHRI) [2]. However, the flexibility makes the dynamics of the robots more complicated than the conventional rigid robots. Involving with the safety requirements, the control designs for the flexible corobots can be challenging.

In this paper, we will discuss the control design for flexible link corobots whose main goals can be summarized as follows.

1) To make the tip motion tracks a desired trajectory.

2) To keep safety during the motion (limits on position and velocity).

In recent years, several techniques for motion control of flexible link robots have been proposed, such as the boundary control [6] [7], PD control [8], feedback linearization [9], backstepping [10], neural networks [11], adaptive control [12]. These approaches mentioned above mainly aim to the motion control of the flexible link robots (goal 1) without explicit consideration of the safety requirements (goal 2). To achieve the performance on motion control and safety simultaneously, we propose a barrier Lyapunov function (BLF) based control design for the flexible link corobots here.

In this design, we treat the safety requirements as constraints on the system. Thus, the control problem of the flexible link corobots is a constrained control problem. Significant efforts have been made to address the constrained control problems for electromechanical systems. Some representative methods are predictive control [13] and reference governors [14]. Besides these methods, the barrier Lyapunov function (BLF) has been proposed to handle constraints in strict-feedback systems [15]. Combined with the backstepping method, the design procedure based on BLF can be systematic and intuitively. However, besides those advantages, the backstepping method suffers from the so-called "explosion of terms" issue [16]. The BLF usually has a complex form, compared with the quadratic Lyapunov function (QLF). This makes the derivatives of the virtual controls more complicated than the QLF design. This problem is more serious for high-order systems because of the repeated differentiations of the virtual control. Another problem is that the backstepping design depends on the information of the model. In

*Previously affiliated with Ohio State University the real situation, the information from the plant is always limited and uncertainties because unknown dynamics and inaccurate parameters exist. The neglect of uncertainty in the design procedure may lead to a lack of robustness.

To overcome the two problems mentioned above, the dynamic surface control and the extended state observer are employed to improve the backstepping based controller.

The dynamic surface control (DSC) [16] technique is a practical way to deal with the problem of "explosion of terms". To avoid the derivations of the virtual control, the DSC design makes the virtual control pass through a first-order filter. For this sake, we combine the DSC technique with the BLF and backstepping design to prevent the "explosion of terms".

The extended state observer (ESO) in [17] is widely used to estimate and compensate for the system uncertainty and disturbance. The idea of the ESO is to extend the disturbance and uncertainty to a new system state and then design an observer for the extended system. The ESO estimates the generalized disturbance between the plant and the nominal model by observing the extended state. In this process, it does not need much information from the model to design the ESO which makes the ESO easy to implement. The modeling of the flexible link robots usually involves uncertainties coming from the reduction of the model and inaccurate parameters. Besides the uncertainties, disturbances like friction and unknown payload are common in robotic systems. Therefore, it is necessary to have a specific design to improve the robustness of the system. In this paper, we propose to use a reduced-order ESO to deal with the uncertainty and disturbance in the flexible link robot model.

In this paper, we integrate the DSC and ESO with the BLF based backstepping and propose a novel control design for the flexible link corobots. With this novel controller, both the performance on motion control and safety are ensured simultaneously.

The rest of this paper is organized as follows. In Section 2, preliminaries and the model of flexible link robots are introduced. The proposed control design including BLF design, ESO design and DSC design, are described in Section 3. The stability analysis of the closed-loop system is given in Section 4, where it is shown that the local practical exponential stability of the closed-loop system. The simulation results are presented in Section 5. The conclusion is given in Section 6.

2 PRELIMINARIES AND PROBLEM FORMULATION

2.1 MATHEMATICAL MODEL OF FLEXIBLE LINK ROBOT

For flexible link robots, a partial differential equation (PDE) by the Euler-Bernoulli beam theory is usually employed to describe their behaviors. Such models usually involve infinite vibration modes of the flexible link robots, and the control design problem based on these models are usually infinite dimension problems. However, with the increasing of the frequency of these modes, the amplitudes of these modes become insignificant. This feature enables the reduction of the model by ignoring the high-order modes of vibration. A popular way to reduce the model is using the lumped parameters model. In this paper, a lumped linear model of the flexible link robots is used as the one in [8] and [11]. Fig. 1 illustrates the lumping process.

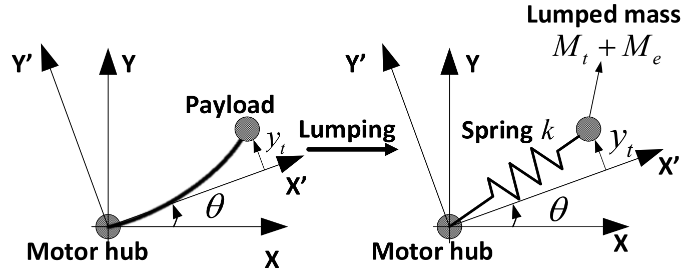

Figure 1. The lumped parameters model.

The distributed mass of the link is lumped to an equivalent tip mass. The flexibility is lumped to a spring with equivalent stiffness. The generalized coordinates are chosen as the motor hub angle θ and the arc tip position p. The arc position can be denoted by $p = y_t + \theta L$, in which y_t is the tip deflection, L is the length of the link. The model of the flexible link robot can be described by the following ordinary differential equations.

$$\dot{\mathbf{x}} = A\mathbf{x} + B\tau + Fd, \tag{1}$$

where $\mathbf{x} = [p, \dot{p}, \theta, \dot{\theta}]^{\mathrm{T}}$ are the state variables, p and θ are the tip position and the motor hub angle, respectively. d represents disturbances and modeling uncertainties, i.e., generalized disturbance. The matrices A, B and F are defined as follows:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ a_1 & 0 & a_2 & 0 \\ 0 & 0 & 0 & 1 \\ a_3 & 0 & a_4 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ b \end{pmatrix}, F = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$
$$a_1 = -k/M,\ a_2 = kL/M, \tag{2}$$
$$a_3 = kL/I_h,\ a_4 = -kL^2/I_h,\ b = 1/I_h.$$

in which, k is the equivalent stiffness, $M = M_t + M_e$, M_t and M_e are the mass of the payload and the equivalent mass of the link, respectively, I_h is the inertia of the motor hub, τ is the input torque and d is the generalized disturbance. Note that the matrices A and B are in strict-feedback form. This property will be used in the following design procedures. And we have a standard assumption for the generalized disturbance d.

Assumption 1 [18]: The generalized disturbance is bounded, continuous and differentiable.

2.2 SAFETY CONSTRAINTS

Safety is always the most critical issue in the physical human-robot interaction. Several types of safety criteria are defined in literature, for instance, the Head Injury Criterion (HIC) and the maximum impact force [19]. The analytical expressions of such criteria are usually multivariable functions involving the mass of the robot M_R, mass of the payload M_P, mass of human parts (where the impact happens) M_H, the stiffness of the robot k and the velocity of the robot v_R. The analytical expressions of HIC [2] and maximum impact force [20] are given below.

$$HIC = 1.016(\frac{k}{M_H})^{0.75}(\frac{M_R}{M_R+M_H})^{1.75}{v_R}^{2.5},$$
$$F_{max} = (\frac{M_R M_H}{M_R+M_H})^{0.5}k^{0.5}v_R. \tag{3}$$

Among the variables above, the mass and the stiffness are usually considered as constants for a specific task. Therefore, the functions of safety criteria can be simplified to single variable functions that only relate with the velocity of the robot. In this design, we use velocity constraint as a general safety criterion instead of a specific safety criterion like the HIC and impact force. Besides the velocity constraints, the position constraints such as limited workspace and obstacles are common as well. In this paper, we consider the safety constraints on position and velocity simultaneously.

2.3 LEMMAS AND DEFINITIONS

To facilitate the analysis, we introduce the following definition and lemma.

Definition 1 [15]**:** A BLF is a scalar function $V(x)$, defined with respect to the system $\dot{x} = f(x)$ on an open region $\mathbb{D}$ containing the origin, that is continuous, positive definite, has continuous first-order partial derivatives at every point of $\mathbb{D}$, has the property $V(x) \to \infty$ as x approaches the boundary of $\mathbb{D}$, and satisfies $V(x(t)) \le b, \forall t \ge 0$ along the solution of $\dot{x} = x(t)$ for $x(0) \in \mathbb{D}$ and some positive constant b.

Lemma 1: For any positive constants k_a and r satisfy $k_a > r > 0$, $\exists \bar{a} > \underline{a} > 0$, such that $\forall x \in [-r, r]$, the following inequality holds.

$$\underline{a}x^2 \le \frac{1}{2}\log\frac{{k_a}^2}{{k_a}^2 - x^2} \le \bar{a}x^2. \tag{4}$$

Proof: Let

$$f(x) = \frac{1}{2}\log\frac{k_a^2}{k_a^2 - x^2}. \tag{5}$$

It is obvious that $f(0) = 0$. Furthermore, we have

$$f'(x) = \frac{x}{k_a^2 - x^2}, \tag{6}$$

and

$$\frac{x}{k_a^2 - x^2} \le \frac{x}{k_a^2} = (\frac{1}{2}k_a^2 x^2)' =, x \in (-k_a, 0],$$
$$\frac{x}{k_a^2 - x^2} \ge \frac{x}{k_a^2} = (\frac{1}{2}k_a^2 x^2)', x \in (0, k_a). \tag{7}$$

Thus, we can obtain

$$\frac{1}{2}k_a^2 x^2 \le f(x), x \in (-k_a, k_a). \tag{8}$$

For any given r satisfies $k_a > r > 0$, we have

$$\frac{x}{k_a^2 - x^2} \ge \frac{x}{k_a^2 - r^2} = (\frac{1}{2}(k_a^2 - r^2)x^2)' =, x \in (-k_a, 0],$$
$$\frac{x}{k_a^2 - x^2} \le \frac{x}{k_a^2 - r^2} = (\frac{1}{2}(k_a^2 - r^2)x^2)', x \in (0, k_a). \tag{9}$$

Similarly, we can obtain

$$\frac{1}{2}(k_a^2 - r^2)x^2 \ge f(x), x \in (-r, r). \tag{10}$$

Let $\underline{a} = \frac{1}{2}k_a^2$ and $\bar{a} = \frac{1}{2}(k_a^2 - r^2)$. From (8) and (10), we know the inequality (4) holds. ■

3 CONTROL DESIGN

3.1 ESO DESIGN

To estimate and compensate for the disturbance, we extend the disturbance d to a new system state x_5. For convenience, we denote the reduced-order extended system as

$$\begin{cases} \dot{x}_3 = x_4 \\ \dot{x}_4 = a_3 x_1 + a_4 x_3 + bu + x_5 \\ \dot{x}_5 = h(t) \end{cases}, \tag{11}$$

where $h(t)$ is the derivative of d and it is locally Lipschitz. For system (11), a linear type of the ESO (LESO) [21] is designed to observe the generalized disturbance.

$$\begin{cases} e_3 = x_3 - \hat{x}_3 \\ \dot{\hat{x}}_3 = \hat{x}_4 + 3\omega_0 e_3 \\ \dot{\hat{x}}_4 = a_3 x_1 + a_4 x_3 + bu + \hat{x}_5 + 3\omega_0^2 e_3 \\ \dot{\hat{x}}_5 = \omega_0^3 e_3 \end{cases}, \tag{12}$$

where ω_0 is the bandwidth of the observer. The estimation error can be obtained from (11) and (12).

$$\begin{cases} \dot{e}_3 = e_4 - 3\omega_0 e_3 \\ \dot{e}_4 = e_5 - 3\omega_0^2 e_3 \\ \dot{e}_5 = h(t) - \omega_0^3 e_3 \end{cases}. \tag{13}$$

Let $\varepsilon_i = e_i / \omega_0^{i-3}$ with $i = 3, 4, 5$ and $\boldsymbol{\varepsilon} = [\varepsilon_3, \varepsilon_4, \varepsilon_5]^{\mathrm{T}}$. Rewrite (13) into the matrix form

$$\dot{\boldsymbol{\varepsilon}} = \omega_0 \Lambda \boldsymbol{\varepsilon} + \omega_0^{-3} E h(t),$$
$$\Lambda = \begin{pmatrix} -3 & 1 & 0 \\ -3 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, E = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{14}$$

We will use this result in the stability analysis.

By Theorem 2 in [21], the estimation error system (14) is stable and the estimation error of the LESO can be made to be arbitrarily small by increasing the bandwidth ω_0. The generalized disturbance can be estimated by $\hat{x}_5$.

3.2 STEP 1: BARRIER LYAPUNOV FUNCTION BASED DESIGN

For a tracking problem, we hope the tip position can track the desired trajectory x_d and the tip velocity can track the desired trajectory of velocity $\dot{x}_d$. Meanwhile, the constraints on position and velocity are not violated. i.e., $|x_1|<k_{c1}, |x_2|<k_{c2}$, where k_{c1} and k_{c2} are positive constants.

The following standard assumption is exploited for the control design.

Assumption 2: $x_d(t) \in \mathcal{C}^4$ and $|x_d^{(i)}(t)|<M_i$ with $M_i>0$ and $i=0,1,2,3,4$.

The model (1) of the flexible link robots is a strict-feedback system. Therefore, we can propose a backstepping-based control design. Define the tracking error as $z_i = x_i - \alpha_{i-1}$ for $i=1,2,3,4$ and $\alpha_0(t)=x_d(t)$, where $\alpha_{i-1}(t)$ are virtual controls. Let $\mathbf{z}=[z_1,z_2,z_3,z_4]^{\mathrm{T}}$.

Considering the constraint on position, define the barrier Lyapunov function candidate (BLFC) as

$$V_1 = \frac{1}{2}\log\frac{k_a^2}{k_a^2 - z_1^2}, \tag{15}$$

where

$$0 \le k_a \le k_{c1} - M_0. \tag{16}$$

Then the Lie derivative of V_1 is

$$\dot{V}_1 = \frac{z_1 \dot{z}_1}{k_a^2 - z_1^2}, \tag{17}$$

where $\dot{z}_1 = z_2 + \alpha_1 - \dot{x}_d$. Design the α_1 as

$$\alpha_1 = -k_1(k_a^2 - z_1^2)z_1 + \dot{x}_d, \tag{18}$$

then the derivative of V_1 can be rewritten as

$$\dot{V}_1 = -k_1 z_1^2 + \frac{z_1 z_2}{k_a^2 - z_1^2}, \tag{19}$$

where $k_1>0$ is a constant. Similarly, to meet the velocity constraint, we define the BLFC as

$$V_2 = \frac{1}{2}\log\frac{k_b^2}{k_b^2 - z_2^2}. \tag{20}$$

Because $z_2 = x_2 - \alpha_1$, we choose

$$0 \le k_b \le k_{c2} - M_1 - \frac{2k_a^3 k_1}{3\sqrt{3}}, \tag{21}$$

then we can obtain the derivative of V_2 as

$$\dot{V}_2 = \frac{z_2 \dot{z}_2}{k_b^2 - z_2^2}. \tag{22}$$

To cancel the nonlinear term in (19), we choose α_2 as

$$\alpha_2 = \left(-k_2(k_b^2 - z_2^2)z_2 - \frac{k_b^2 - z_2^2}{k_a^2 - z_1^2}z_1 + \dot{\alpha}_1 - a_1 x_1\right)/a_2, \tag{23}$$

in which $k_2>0$ is a constant. From (22) and (23), the derivative of $\dot{V}_2$ can be obtained.

$$\dot{V}_2 = -k_2 z_2^2 - \frac{z_1 z_2}{k_a^2 - z_1^2} + \frac{a_2 z_2 z_3}{k_b^2 - z_2^2}. \tag{24}$$

For z_3 and z_4, choose the quadratic Lyapunov function candidate (QLFC) as

$$\begin{aligned} V_3 &= \frac{1}{2}z_3^2, \\ V_4 &= \frac{1}{2}z_4^2. \end{aligned} \tag{25}$$

Repeat the recursive design procedures, we can choose the virtual control α_3 and the input u as follows

$$\begin{aligned} \alpha_3 &= -k_3 z_3 + \dot{\alpha}_2 - \frac{a_2 z_2}{k_b^2 - z_2^2}, \\ u &= (-a_3 x_1 - a_4 x_3 - k_4 z_4 - z_3 + \dot{\alpha}_3)/b, \end{aligned} \tag{26}$$

where k_3 and k_4 are positive constants. Then the derivatives of V_3 and V_4 are as follows

$$\begin{aligned} \dot{V}_3 &= -k_3 z_3^2 - \frac{a_2 z_2 z_3}{k_b^2 - z_2^2} + z_3 z_4, \\ \dot{V}_4 &= -k_4 z_4^2 - z_3 z_4. \end{aligned} \tag{27}$$

Combining (1), (18), (23) and (26), the dynamics of the tracking errors can be obtained.

$$\begin{cases} \dot{z}_1 = z_2 - k_1(k_a^2 - z_1^2)z_1 \\ \dot{z}_2 = a_3 z_3 - k_2(k_b^2 - z_2^2)z_2 - \dfrac{k_b^2 - z_2^2}{k_a^2 - z_1^2}z_1 \\ \dot{z}_3 = z_4 - k_3 z_3 - \dfrac{a_2 z_2}{k_b^2 - z_2^2} \\ \dot{z}_4 = -k_4 z_4 - z_3 \end{cases}. \tag{28}$$

The stability of the error dynamics (28) is given by the following theorem.

Theorem 1: For system (1) with *Assumption 2*, if the control input is determined by (18), (23) and (26), then $\exists r>0,\ r<k_a,\ r<k_b$, such that for any initial condition $\mathbf{z}(0) \in \mathbf{Z}_0 := \{\mathbf{z} \in \mathbb{R}^4 : |z_1|<r, |z_2|<r\}$, the following properties hold.

i. The origin of system (28) is locally exponentially stable on $\mathbf{Z}_0$.

ii. The constraints on position and velocity are never violated. i.e., $\mathbf{x}(t) \in \{\mathbf{x} \in \mathbb{R}^4 : |x_1| \le k_{c1}, |x_2| \le k_{c2}\}$.

iii. The tracking errors z_1 and z_2 are constrained with $|z_1| \le k_a$ and $|z_2| \le k_b$.

Proof: (i) Define the Lyapunov function candidate as

$$V = \sum_{i=1}^{4} V_i. \tag{29}$$

The derivative of V can be calculated from (19), (24) and (27).

$$\dot{V} = -\sum_{i=1}^{4} k_i z_i^2 \le -k_{min}\|\mathbf{z}\|_2^2 \le 0, \tag{30}$$

where $k_{min} = \min\{k_i,\ i = 1,2,3,4\}$. From *Lemma 1*, $\exists \underline{a}, \bar{a}, \underline{b}, \bar{b} > 0$, such that $\forall |z_1| < r$, $|z_2| < r$ the following inequalities hold.

$$\begin{aligned} &\underline{a} z_1^2 \le V_1 \le \bar{a} z_1^2, \\ &\underline{b} z_2^2 \le V_2 \le \bar{b} z_2^2. \end{aligned} \tag{31}$$

Selecting $a_{min} = \min\{\underline{a}, \underline{b}, 1/2\}$, $a_{max} = \max\{\bar{a}, \bar{b}, 1/2\}$, then we have the following inequality.

$$a_{\min} \|\mathbf{z}\|_2^2 \le V(\mathbf{z}) \le a_{max} \|\mathbf{z}\|_2^2 . \tag{32}$$

From (30) and (32), the origin of the system (28) is locally exponentially stable by Theorem 4.10 in [22].

(ii) From $\dot{V} \le 0$, it can be shown that $V(t) \le V(0)$. From Lemma 1 in [15], for $\mathbf{z}(0) \in \mathbf{Z}_0$, infers $|z_1(t)| < k_a$, $|z_2(t)| < k_b$, $\forall t > 0$. Since $x_1(t) = z_1(t) + x_d(t)$, $x_2(t) = z_2(t) + \alpha_1(t)$, and $x_d(t) \le M_0$, $\dot{x}_d(t) \le M_1$, we know $|x_1(t)| \le k_a + M_0 \le k_{c1}$, and $|x_2(t)| \le k_b + M_1 + \dfrac{2k_a^3 k_1}{3\sqrt{3}} \le k_{c2}$. i.e., all constraints are never violated. ■

We already have the estimation of the generalized disturbance, i.e., $\hat{x}_5$. To compensate for the disturbance, the input u is designed as

$$u = (-a_3 x_1 - a_4 x_3 - k_4 z_4 - z_3 + \dot{\alpha}_3 - \hat{x}_5)/b. \tag{33}$$

Then the error dynamics (28) is rewritten as

$$\begin{cases} \dot{z}_1 = z_2 - k_1(k_a^2 - z_1^2) z_1 \\ \dot{z}_2 = a_3 z_3 - k_2(k_b^2 - z_2^2) z_2 - \dfrac{k_b^2 - z_2^2}{k_a^2 - z_1^2} z_1 \\ \dot{z}_3 = z_4 - k_3 z_3 - \dfrac{a_2 z_2}{k_b^2 - z_2^2} \\ \dot{z}_4 = -k_4 z_4 - z_3 + \varepsilon_5 \end{cases}. \tag{34}$$

The closed-loop system (34) and (14) can be denoted as

$$\begin{cases} \dot{\mathbf{z}} = f_1(t, \mathbf{z}, \boldsymbol{\varepsilon}, \tau_{eso}) \\ \tau_{eso} \dot{\boldsymbol{\varepsilon}} = \Lambda \boldsymbol{\varepsilon} + \tau_{eso}^4 E h(t) \end{cases}, \tag{35}$$

in which, $\tau_{eso} = 1/\omega_0$ is a small parameter. To analyze the stability of (35), we refer to the method in [23]. We have the following theorem for the stability on the origin of system (35).

Theorem 2: For system (35), there exists a constant $\tau^* > 0$ such that $\forall \tau_{eso} < \tau^*$, the origin of system (35) is locally exponentially stable.

Proof: The local exponential stability of the reduced system (28) infers the stability of (35). The detailed proof is omitted due to the limited space. The reader can refer to the proof of Theorem 3.4 in [23].

3.3 STEP 2: DYNAMIC SURFACE CONTROL DESIGN

It is obvious that the control law (18), (23) and (26) involves the calculation of the derivatives of virtual control inputs $\dot{\alpha}_i$. Due to the using of logarithmic BLFC, the "explosion of terms" problem in this design is more serious than the one in traditional integrator backstepping control using the QLFC. To deal with this problem, we apply the dynamic surface control technique to the BLF based design.

Based on the idea of DSC [16], a set of first-order filters are applied to the virtual controls. Different from the traditional DSC, we also apply a filter to the input u to make the input smooth. Redefine the tracking errors as

$$\begin{aligned} &z_i = x_i - \lambda_{i-1},\ i = 1,2,3,4, \\ &\lambda_0(t) = x_d(t). \end{aligned} \tag{36}$$

The DSC control law can be denoted by

$$\begin{aligned} &\alpha_1 = -k_1(k_a^2 - z_1^2) z_1 + \dot{x}_d, \\ &\alpha_2 = (-k_2(k_b^2 - z_2^2) z_2 - \frac{k_b^2 - z_2^2}{k_a^2 - z_1^2} z_1 + \dot{\lambda}_1 - a_1 x_1)/a_2, \\ &\alpha_3 = -k_3 z_3 + \dot{\lambda}_2 - \frac{a_2 z_2}{k_b^2 - z_2^2}, \\ &\alpha_4 = (-a_3 x_1 - a_4 x_3 - k_4 z_4 - z_3 + \dot{\lambda}_3 - \hat{x}_5)/b, \\ &u = \lambda_4, \end{aligned} \tag{37}$$

where the first-order filters are

$$\tau \dot{\boldsymbol{\lambda}} = \Theta(\boldsymbol{\alpha} - \boldsymbol{\lambda}),\ i = 1,2,3,4, \tag{38}$$

in which $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \lambda_3, \lambda_4]^{\mathrm{T}}$, $\Theta = diag[\sigma_1, \sigma_2, \sigma_3, \sigma_4]$ and $\sigma_i > 0$ with $i = 1,2,3,4$, $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \alpha_3, \alpha_4]^{\mathrm{T}}$, $\tau \in (0,1)$. From (36) and (37), the dynamics of the tracking error can be obtained as

$$\begin{cases} \dot{z}_1 = z_2 - k_1(k_a^2 - z_1^2) z_1 + \tilde{\lambda}_1 \\ \dot{z}_2 = a_3 z_3 - k_2(k_b^2 - z_2^2) z_2 - \dfrac{k_b^2 - z_2^2}{k_a^2 - z_1^2} z_1 + a_2 \tilde{\lambda}_2 \\ \dot{z}_3 = z_4 - k_3 z_3 - \dfrac{a_2 z_2}{k_b^2 - z_2^2} + \tilde{\lambda}_3 \\ \dot{z}_4 = -k_4 z_4 - z_3 + \varepsilon_5 + b \tilde{\lambda}_4 \end{cases}, \tag{39}$$

where $\tilde{\lambda}_i = \lambda_i - \alpha_i$ with $i = 1,2,3,4$. The analysis of the closed-loop system (39) and (38) can be denoted as a standard singular perturbation problem

$$\begin{cases} \dot{\mathbf{z}} = f_2(\mathbf{z}, \boldsymbol{\lambda}, \tau) \\ \tau \dot{\boldsymbol{\lambda}} = g(\mathbf{z}, \boldsymbol{\lambda}, \tau) \end{cases}. \tag{40}$$

The stability analysis of system (38) and (39) is in the next section.

4 STABILITY ANALYSIS

Inspired by [24], the singular perturbation theory is applied to the stability analysis. Consider the singular perturbation problem (40). From the expression of (38), $g(z, \boldsymbol{\lambda}, 0) = 0$ has a unique isolated root $\boldsymbol{\lambda} = \mathbf{h}(\mathbf{z})$, $\mathbf{h}(\mathbf{z}) = \boldsymbol{\alpha}$. Substituting $\boldsymbol{\lambda} = \mathbf{h}(\mathbf{z})$ into $f_2(\cdot)$, the reduced system is denoted as

$$\dot{\mathbf{z}} = f_2(\mathbf{z}, \mathbf{h}(\mathbf{z}), 0). \tag{41}$$

Note that the reduced system is same as (35). We define a new state variable $\mathbf{y} := \boldsymbol{\lambda} - \mathbf{h}(\mathbf{z})$. The boundary-layer system can be derived from (38)

$$\mathbf{y}' = g(\mathbf{z}, \mathbf{y} + \mathbf{h}(\mathbf{z}), 0) = -\Theta \mathbf{y}. \tag{42}$$

in which $\mathbf{y}' = \mathrm{d}\mathbf{y} / \mathrm{d}\sigma$ and $\sigma = t / \tau$ is the stretched time scale. Let $\overline{\mathbf{z}}(t)$ and $\overline{\mathbf{y}}(\sigma)$ be the solutions of (41) and (42), respectively. The following definitions are introduced to facilitate the analysis.

Definition 2 [22]**:** The signal $\mathbf{z}(t, \tau)$ is said to be of the order τ, denoted as $\mathbf{z}(t, \tau) = \mathrm{O}(\tau)$, if there exist constant $k, \tau_0 \in \mathbb{R}^+$ such that $\| \mathbf{z}(t, \tau) \| \leq k\tau, \ \forall \tau \in (0, \tau_0)$.

Definition 3 [25]**:** The system $\dot{\mathbf{z}} = f(\mathbf{z}, \tau)$ has practical exponential stability as $\tau \to 0$ if for any given constant $\delta \in \mathbb{R}^+$ there exist $\tau_0, z_0, \mu \in \mathbb{R}^+$ such that

$$\begin{aligned} &\| \mathbf{z}(t) \| \leq \| \mathbf{z}(0) \| \exp(-\mu t) + \delta, \\ &\forall t \geq 0, \ \forall \tau \in (0, \tau_0), \ \forall \mathbf{z}(0) \in \mathbf{Z}_0. \end{aligned} \tag{43}$$

The theorem of the stability of (40) is given in here.

Theorem 3: For the system (40) with *Assumption 1 and Assumption 2*, if the control input is given from (37), then for any given control gain $k_i \in \mathbb{R}^+$ with $i = 1, 2, 3, 4$, there exists a small constant $\tau^{**} \in (0,1)$ such that the closed-loop system (40) is locally practically exponentially stable, $\forall \tau \in (0, \tau^{**})$, $\forall \mathbf{z}(0) \in \mathbf{Z}_0$. Meanwhile, the constraints in *Theorem 1* are never violated.

Proof: Let $\Omega_{cz} \subset \mathbf{Z}_0$ and $\Omega_{cy} \subset \mathbb{R}^4$ be compact sets of $\mathbf{z}$ and $\mathbf{y}$, respectively. To apply Theorem 11.2 in [22], we need to show that all the assumptions for the theorem are satisfied.

1. From (37), (38) and Assumption 2, $f_2(\cdot)$, $g(\cdot)$ and their partial derivatives with respect to $(\mathbf{z}, \boldsymbol{\lambda}, \tau)$ are bounded and continuous on compact sets $\Omega_{cz} \times \Omega_{cy} \subset \mathbb{R}^4$, $\mathbf{h}(\mathbf{z})$ and $[\partial g(\mathbf{z}, \boldsymbol{\lambda}, 0) / \partial \boldsymbol{\lambda}]$ have bounded first partial derivatives with respect to their arguments, and $[\partial f_2(\mathbf{z}, \mathbf{h}(\mathbf{z}), 0) / \partial \mathbf{z}]$ is Lipschitz in $\mathbf{z}$ and uniformly in t.

2. By Theorem 2, the origin of the reduced system (41) is exponentially stable. Als, there exists a Lyapunov function $V(\mathbf{z})$ that satisfies $W_1(\mathbf{z}) \leq V(\mathbf{z}) \leq W_2(\mathbf{z})$ and $\dot{V}(\mathbf{z}) \leq -W_3(\mathbf{z})$, where $W_i(\mathbf{z})$ with $i = 1, 2, 3$ are positive definite functions on $\mathbb{R}^4$.

3. Because (42) is independent of $(t, \mathbf{z})$, it infers that the origin of the boundary-layer system (42) is an exponentially stable equilibrium, uniformly in $(t, \mathbf{z})$. Let $\Omega_{cy0} \subset \Omega_{cy}$ be the region of attraction of (42).

So far, all conditions for applying Theorem 11.2 in [22] are satisfied. Let $c, r \in \mathbb{R}^+$ be constants satisfying $c < \min_{\|\mathbf{z}\|=r} W_2(\mathbf{z})$, $\Omega_{cz0} \subset \{W_2(\mathbf{z}) \leq \rho c, \rho \in (0,1)\}$ be the attraction region of $\mathbf{z}$. Then for each compact set Ω_{cz0}, there is a positive constant τ^{***} such that for all $t \geq 0$, $\mathbf{z} \in \Omega_{cz0}$, $\mathbf{y} \in \Omega_{cy0}$ and $0 < \tau < \tau^{***}$, the singular perturbation problem has a unique solution $\mathbf{z}(t, \tau)$, $\boldsymbol{\lambda}(t, \tau)$ on $t \in [0, \infty)$ and

$$\begin{cases} \mathbf{z}(t, \tau) - \overline{\mathbf{z}}(t) = O(\tau) \\ \boldsymbol{\lambda}(t, \tau) - \mathbf{h}(\overline{\mathbf{z}}(t)) - \overline{\mathbf{y}}(\sigma) = O(\tau) \end{cases} \tag{44}$$

holds uniformly for $t \in [0, \infty)$. Furthermore, for any given $t_a > 0$, there exists a $\tau^{**} \leq \tau^{***}$ such that

$$\boldsymbol{\lambda}(t, \tau) - \mathbf{h}(\overline{\mathbf{z}}(t)) = O(\tau) \tag{45}$$

holds uniformly for $t \in [t_a, \infty)$, $\tau \in (0, \tau^{**})$. As a further matter, since $\overline{\mathbf{z}}(t)$ and $\overline{\mathbf{y}}(\sigma)$ are the solutions of exponentially stable system (41) and (42) with $t \in [0, \infty)$, respectively. It infers that

$$\begin{aligned} &\lim_{t \to \infty} \overline{\mathbf{z}}(t) = 0 \\ &\lim_{t \to \infty} \overline{\mathbf{y}}(\sigma) = 0 \\ &\lim_{t \to \infty} \mathbf{h}(\overline{\mathbf{z}}(t)) = \boldsymbol{\alpha}(t). \end{aligned} \tag{46}$$

From (44), (45) and (46), we can obtain

$$\begin{aligned} &\lim_{\tau \to 0, t \to \infty} \| \mathbf{z}(t, \tau) \| = 0 \\ &\lim_{\tau \to 0, t \to \infty} \| \boldsymbol{\lambda}(t, \tau) - \boldsymbol{\alpha}(u) \| = 0 \end{aligned}, \tag{47}$$

which indicates the local practical exponential stability of the system (40) by the Definition 2. ■

5 SIMULATIONS

Numerical simulation results are presented in this section. The controller is tested with a planar single-link flexible manipulator model developed based on the finite element analysis (FEA). The parameters of the robot are $\rho = 0.1$ kg/m, $L = 0.4$ m, $I_h = 4 \times 10^{-4}$ kg$\cdot m^2$, $M_t = 0.1$ kg and $EI = 9.62 \mathrm{N} \cdot m^2$, in which ρ is the linear density of the link. The corresponding parameters for the nominal model (1) are $k = 450.94$, $M_e = 0.0411$. In the simulation, the desired position trajectory is a "smooth" step signal. The step signal passes through a fourth-order system to satisfy the Assumption 2. The transfer function of the fourth-order system is as follows

$$G(s) = \frac{1}{(Ts+1)^4}, \tag{48}$$

where the time constant $T = 0.05$.

The desired trajectory of velocity can be obtained by a differentiator. Two controllers are designed to demonstrate the performance of the proposed design. (1) The quadratic Lyapunov function based DSC control (QLF). (2) The barrier Lyapunov function based DSC control with ESO (BLFESO). The constraints for position and velocity are $x_1 < 1.05$ m and $x_2 < 4.705$ m/s, respectively. From (16) and(21), we can choose $k_a = 0.04$ and $k_b = 0.2$ to make sure the constraints will not be violated. The control gains for BLFESO are $k_1 = 1000$, $k_2 = 500$, $\mathrm{k}_3 = k_4 = 1$, the filter parameters for the DSC are selected as $\tau = 0.002$ and $\Theta = diag[1, 1, 0.1, 0.005]$. The bandwidth of the ESO is selected as $\omega_0 = 100$. Control gains for the QLF method are $k_1 = k_2 = \mathrm{k}_3 = k_4 = 3$.

The simulation results are shown in the following figures. From Figure 2 and Figure 3, both controllers reach the final set point. However, the QLF controller violates the position and the velocity constraints. In the practice, the unexpected overshoot on the position may cause a dangerous collision and higher speed may cause more severe injury to human operators. The violations of position and velocity constraints are clear to be seen from Figure 4 and Figure 5 as well. In fact, the BLF governs the tracking errors of positon and velocity. By constraining the tracking errors, the safety constraints on position and velocity are guaranteed. The torque input of the BLFESO controller is shown in Figure 6.

Although the performance of QLF based controller can be improved by increasing the gains, the constraints are not considered in the design procedures. In the design of the BLFESO, by adjusting the two parameters k_a and k_b, the tracking errors of position and velocity are governed to ensure the constraints are not violated.

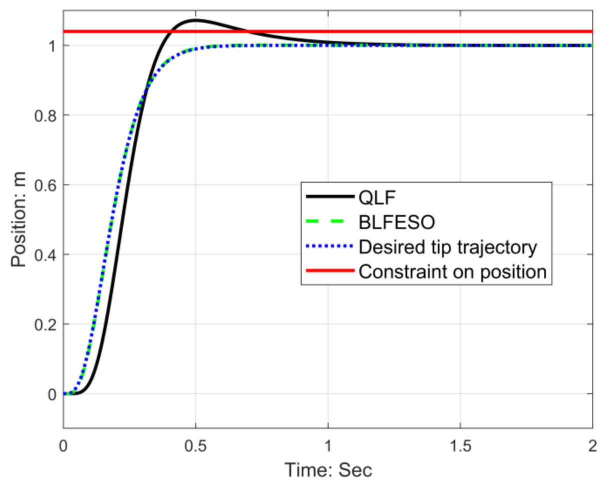

Figure 2 Tracking of tip position.

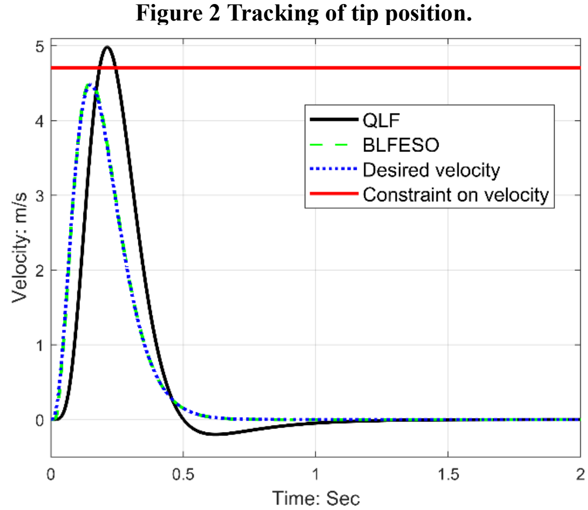

Figure 3 Tracking of velocity.

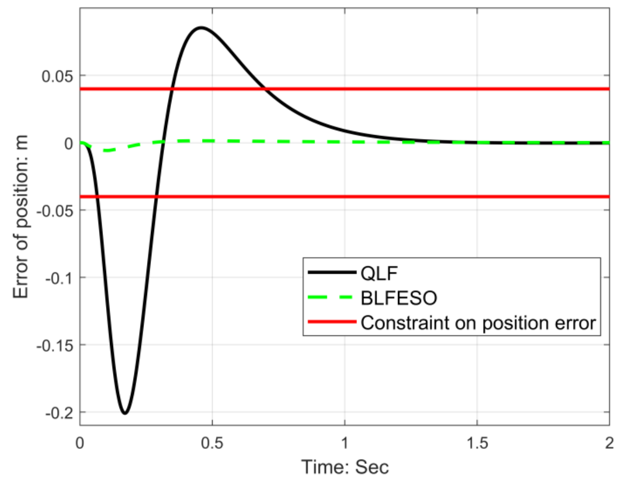

Figure 4 Tracking error of position.

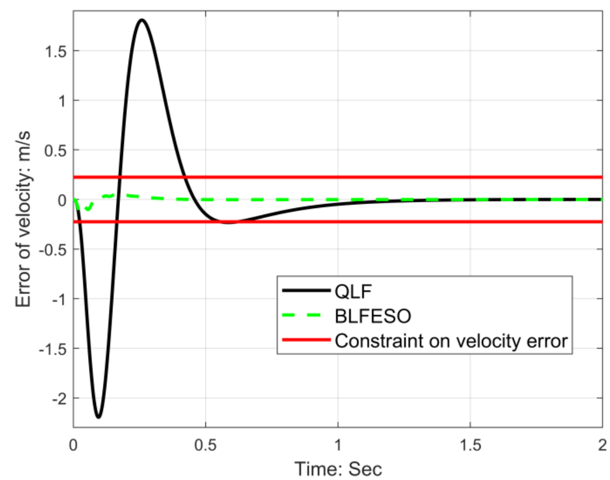

Figure 5 Tracking error of velocity.

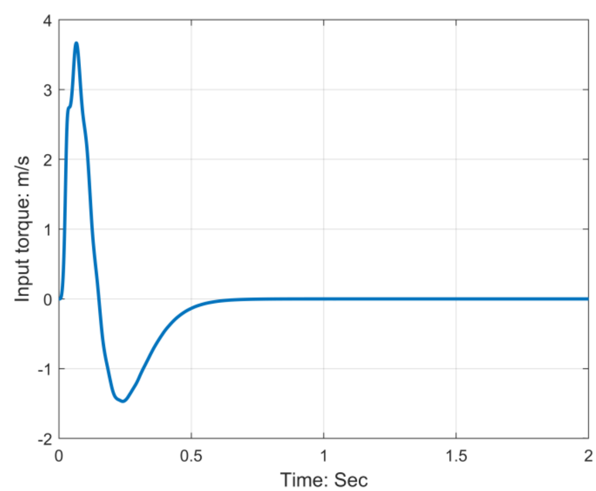

Figure 6 Input torque of BLFESO.

6 CONCLUSIONS

In this paper, we propose a control design for flexible link robot with safety constraints. The stability of the closed-loop system is proven via the singular perturbation theory. The simulation results show that the tracking of position is successful and constraints on both position and velocity are not violated. In this work, we use velocity constraint instead of specific safety criteria like the HIC and maximum impact force. We will design a controller with HIC or maximum impact force constraints in the future.

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