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SOME REFINED RESULTS ON THE MIXED LITTLEWOOD CONJECTURE FOR PSEUDO-ABSOLUTE VALUES

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Abstract

In this paper, we study the mixed Littlewood conjecture with pseudo-absolute values. For any pseudo-absolute-value sequence $\mathcal{D}$, we obtain a sharp criterion such that for almost every α the inequality

$$|n|_{\mathcal{D}}|n\alpha - p| \le \psi(n)$$

has infinitely many coprime solutions $(n, p) \in \mathbb{N} \times \mathbb{Z}$ for a certain one-parameter family of ψ. Also, under a minor condition on pseudo-absolute-value sequences $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_k$, we obtain a sharp criterion on a general sequence $\psi(n)$ such that for almost every α the inequality

$$|n|_{\mathcal{D}_1}|n|_{\mathcal{D}_2}\cdots|n|_{\mathcal{D}_k}|n\alpha - p| \le \psi(n)$$

has infinitely many coprime solutions $(n, p) \in \mathbb{N} \times \mathbb{Z}$.

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1. Introduction

The *Littlewood conjecture* states that for every pair (α, β) of real numbers,

$$\liminf_{n\to\infty} n\|n\alpha\|\,\|n\beta\| = 0, \tag{1.1}$$

where $\|x\| = \mathrm{dist}(x, \mathbb{Z})$. We refer the reader to [4, 6] for recent progress. By a fundamental result of Einsiedler *et al.* [9], the set of pairs (α, β) for which (1.1) does not hold is a zero Hausdorff dimension set.

From the metrical point of view, (1.1) can be strengthened. Gallagher [13] established that if $\psi : \mathbb{N} \to \mathbb{R}$ is a nonnegative decreasing function, then for almost every (α, β) the inequality

$$\|n\alpha\|\,\|n\beta\| \le \psi(n)$$

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has infinitely many solutions for $n \in \mathbb{N}$ if and only if $\sum_{n\in\mathbb{N}} \psi(n) \log n = \infty$. In particular,

$$\liminf_{n\to\infty} n\,(\log n)^2 \|n\alpha\|\,\|n\beta\| = 0$$

for almost every pair (α, β) of real numbers. By a method of [18], Bugeaud and Moshchevitin [6] showed that there exist pairs (α, β) such that

$$\liminf_{n\to\infty} n\,(\log n)^2 \|n\alpha\|\,\|n\beta\| > 0.$$

This result has been improved by Badziahin [1] and states that the set of pairs (α, β) satisfying

$$\liminf_{n\to\infty} n \log n \log\log n \|n\alpha\|\,\|n\beta\| > 0$$

has full Hausdorff dimension in $\mathbb{R}^2$. It is conjectured that the Littlewood conjecture can be strengthened to

$$\liminf_{n\to\infty} n \log n \|n\alpha\|\,\|n\beta\| = 0$$

for all $(\alpha, \beta) \in \mathbb{R}^2$.

In [7], de Mathan and Teulié formulated another conjecture – known as the *mixed Littlewood conjecture*. Let $\mathcal{D} = \{n_k\}_{k\geq 0}$ be an increasing sequence of positive integers with $n_0 = 1$ and $n_k | n_{k+1}$ for all k. We refer to such a sequence as a *pseudo-absolute-value sequence* and we define the $\mathcal{D}$-adic pseudo-norm $|\cdot|_{\mathcal{D}} : \mathbb{N} \to \{n_k^{-1} : k \geq 0\}$ by

$$|n|_{\mathcal{D}} = \min\{n_k^{-1} : n \in n_k\mathbb{Z}\}.$$

In the case $\mathcal{D} = \{p^k\}_{k=0}^{\infty}$ for some integer $p \geq 2$, we also write $|\cdot|_{\mathcal{D}} = |\cdot|_p$. de Mathan and Teulié [7] conjectured that for any real number α and any pseudo-absolute-value sequence $\mathcal{D}$,

$$\liminf_{n\to\infty} n|n|_{\mathcal{D}}\|n\alpha\| = 0.$$

In particular, the statement that $\liminf_{n\to\infty} n|n|_p\|n\alpha\| = 0$ for every real number α and prime number p is referred to as the p-adic Littlewood conjecture.

Einsiedler and Kleinbock have shown that any exceptional set to the de Mathan–Teulié conjecture has to be of zero Hausdorff dimension [10]. By a theorem of Furstenberg [11], one has that for any two prime numbers p, q and every real number α,

$$\liminf_{n\to\infty} n|n|_p|n|_q\|n\alpha\| = 0. \tag{1.2}$$

This result can be made quantitative [3], that is,

$$\liminf_{n\to\infty} n(\log\log\log n)^{\kappa}|n|_p|n|_q\|n\alpha\| = 0$$

for some $\kappa > 0$. The statement (1.2) can be strengthened from a metrical point of view [5], that is, suppose that $p_1, \ldots, p_k$ are distinct prime numbers and $\psi : \mathbb{N} \to \mathbb{R}$ is a nonnegative decreasing function; then, for almost every real number α, the inequality

$$|n|_{p_1} \cdots |n|_{p_k} |n\alpha - p| \leq \psi(n)$$

has infinitely many coprime solutions $(n, p) \in \mathbb{N} \times \mathbb{Z}$ if and only if

$$\sum_{n\in\mathbb{N}} (\log n)^k \psi(n) = \infty. \tag{1.3}$$

As a corollary, it is true that

$$\liminf_{n\to\infty} n\,(\log n)^{k+1} |n|_{p_1} \cdots |n|_{p_k} \|n\alpha\| = 0 \tag{1.4}$$

for almost every $\alpha \in \mathbb{R}$.

In [14], Harrap and Haynes considered the $\mathcal{D}$*-adic pseudo-absolute value*. Given a pseudo-absolute-value sequence $\mathcal{D}$ with some minor restriction, let $\mathcal{M} : \mathbb{N} \to \mathbb{N} \cup \{0\}$ be

$$\mathcal{M}(N) = \max\{k : n_k \leq N\}.$$

Suppose that $\psi : \mathbb{N} \to \mathbb{R}$ is nonnegative and decreasing and that $\mathcal{D} = \{n_k\}$ is a pseudo-absolute-value sequence satisfying

$$\sum_{k=1}^{m} \frac{\varphi(n_k)}{n_k} \geq cm \quad \text{for all } m \in \mathbb{N} \text{ and for some } c > 0, \tag{1.5}$$

where φ is the Euler phi function. Then, for almost every $\alpha \in \mathbb{R}$, the inequality

$$|n|_{\mathcal{D}} |n\alpha - p| \leq \psi(n)$$

has infinitely many coprime solutions $(n, p) \in \mathbb{N} \times \mathbb{Z}$ if and only if

$$\sum_{n=1}^{\infty} \mathcal{M}(n)\psi(n) = \infty. \tag{1.6}$$

Note that when $\mathcal{D} = \{p^k\}$ for some positive integer p, we have that $\mathcal{M}(N) \asymp \log N$. Thus, Harrap–Haynes' result implies (1.3) for $k = 1$. The first goal of this paper is to extend (1.3) to the class of finitely many pseudo-absolute-value sequences.

As pointed out in [14], such generalization depends on the overlap among pseudo-absolute-value sequences. For example[1], if $\mathcal{D}_1 = \{2^k\}$ and $\mathcal{D}_2 = \{3^k\}$, (1.4) yields that inequality

$$|n|_{\mathcal{D}_1} |n|_{\mathcal{D}_2} \|n\alpha\| \leq \psi(n)$$

has infinitely many solutions for almost every α if and only if

$$\sum_{n\in\mathbb{N}} (\log n)^2 \psi(n) = \infty.$$

However, if $\mathcal{D}_1 = \mathcal{D}_2 = \{2^k\}$, by [5, Theorem 2], the inequality has infinitely many solutions for almost every α if and only if

$$\sum_{n\in\mathbb{N}} n\psi(n) = \infty.$$

[1] The present example and the following one are from [14].

Basically, the proof of (1.3) and (1.6) follows from the Duffin–Schaeffer theorem [8] (see Theorem 2.3), which is a weaker version of the Duffin–Schaeffer conjecture.

Duffin–Schaeffer conjecture. Let $\psi : \mathbb{N} \to \mathbb{R}$ be a nonnegative function and define

$$\mathcal{E}_n = \mathcal{E}_n(\psi) = \bigcup_{\substack{p=1\\(p,n)=1}}^{n} \Big(\frac{p-\psi(n)}{n}, \frac{p+\psi(n)}{n}\Big),$$

where (p, n) is the largest common divisor between p and n. Then $\lambda(\limsup \mathcal{E}_n) = 1$ if and only if $\sum_n \lambda(\mathcal{E}_n) = \infty$, where λ denotes the Lebesgue measure on $\mathbb{R}/\mathbb{Z}$.

One side of the Duffin–Schaeffer conjecture is trivial. If $\sum_n \lambda(\mathcal{E}_n) < \infty$, by the Borel–Cantelli lemma, $\lambda(\limsup \mathcal{E}_n) = 0$. Since it has been posted, the Duffin–Schaeffer conjecture was heavily investigated in [2, 15–17, 19, 20]. We should mention that the Duffin–Schaeffer conjecture is equivalent to the following statement: suppose that $\psi : \mathbb{N} \to \mathbb{R}$ is a nonnegative function and satisfies

$$\sum_n \frac{\varphi(n)\psi(n)}{n} = \infty,$$

where φ is the Euler phi function. Then, for almost every $\alpha \in \mathbb{R}$, the inequality

$$|n\alpha - p| \le \psi(n)$$

has infinitely many coprime solutions $(n, p) \in \mathbb{N} \times \mathbb{Z}$.

We will also employ the Duffin–Schaeffer theorem to study the mixed Littlewood conjecture in the present paper and find a nice divergence condition for finite pseudo-absolute values.

THEOREM 1.1. *Let $\psi : \mathbb{N} \to \mathbb{R}$ be nonnegative and decreasing and let $\mathcal{D}_1 = \{n_k^1\}, \mathcal{D}_2 = \{n_k^2\}, \dots, \mathcal{D}_m = \{n_k^m\}$ be m pseudo-absolute-value sequences. Suppose that $\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_m$ satisfies the following condition: there exists some constant $c_1 > 0$ such that*

$$\frac{\varphi(n_{k_1}^1 n_{k_2}^2 \cdots n_{k_m}^m)}{n_{k_1}^1 n_{k_2}^2 \cdots n_{k_m}^m} \ge c_1, \tag{1.7}$$

where φ is the Euler phi function. Then, for almost every $\alpha \in \mathbb{R}$, the inequality

$$|n|_{\mathcal{D}_1} |n|_{\mathcal{D}_2} \cdots |n|_{\mathcal{D}_m} |n\alpha - p| \le \psi(n)$$

has infinitely many coprime solutions $(n, p) \in \mathbb{N} \times \mathbb{Z}$ if and only if

$$\sum_{n=1}^{\infty} \frac{\psi(n)}{|n|_{\mathcal{D}_1} |n|_{\mathcal{D}_2} \cdots |n|_{\mathcal{D}_m}} = \infty. \tag{1.8}$$

REMARK 1.2. Let $p_1, \dots, p_m$ be distinct prime numbers and $\mathcal{D}_i = \{p_i^k\}$, $i = 1, 2, \dots, m$. For such pseudo-absolute-value sequences $\mathcal{D}_i$, $i = 1, 2, \dots, m$, one has that (1.7) holds. By the fact that (see [5])

$$\sum_{n\in\mathbb{N}} (\log n)^m \psi(n) = \infty \quad \Longleftrightarrow \quad \sum_{n\in\mathbb{N}} \frac{\psi(n)}{|n|_{p_1} \cdots |n|_{p_m}} = \infty,$$

Theorem 1.1 implies (1.3).

We say that a pseudo-absolute-value sequence $\mathcal{D} = \{n_k\}$ is generated by finite integers if there exist prime numbers $p_1, p_2, \ldots, p_N$ such that every n_k can be written as $p_1^{k_1} p_2^{k_2} \cdots p_N^{k_N}$ for some proper positive integers $k_1, k_2, \ldots, k_N$. We call $p_1, p_2, \ldots, p_N$ the generators of $\mathcal{D}$.

COROLLARY 1.3. *Let $\psi : \mathbb{N} \to \mathbb{R}$ be nonnegative and decreasing and let $\mathcal{D}_1 = \{n_k^1\}, \mathcal{D}_2 = \{n_k^2\}, \ldots, \mathcal{D}_m = \{n_k^m\}$ be m pseudo-absolute-value sequences. Suppose that each $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_m$ is generated by finite integers. Then, for almost every $\alpha \in \mathbb{R}$, the inequality*

$$|n|_{\mathcal{D}_1} |n|_{\mathcal{D}_2} \cdots |n|_{\mathcal{D}_m} |n\alpha - p| \le \psi(n)$$

has infinitely many coprime solutions $(n, p) \in \mathbb{N} \times \mathbb{Z}$ if and only if

$$\sum_{n=1}^{\infty} \frac{\psi(n)}{|n|_{\mathcal{D}_1} |n|_{\mathcal{D}_2} \cdots |n|_{\mathcal{D}_m}} = \infty.$$

PROOF. If $\mathcal{D}_j$ is generated by finite integers for each $j = 1, 2, \ldots, m$, one has that (1.7) holds. Thus, Corollary 1.3 directly follows from Theorem 1.1. □

Suppose that there is no intersection between the pseudo-absolute-value sequences. Then we can get better results. We say that two pseudo-absolute-value sequences $\mathcal{D}_1 = \{n_k^1\}$ and $\mathcal{D}_2 = \{n_k^2\}$ are coprime if n_i^1 and n_j^2 are coprime for any $i, j \in \mathbb{N}$.

THEOREM 1.4. *Let $\psi : \mathbb{N} \to \mathbb{R}$ be nonnegative and decreasing. Suppose that the pseudo-absolute-value sequences $\mathcal{D}_1 = \{n_k^1\}, \mathcal{D}_2 = \{n_k^2\}, \ldots, \mathcal{D}_m = \{n_k^m\}$ are mutually coprime and*

$$\sum_{n_{k_1}^1 n_{k_2}^2 \cdots n_{k_m}^m \le N} \frac{\varphi(n_{k_1}^1 n_{k_2}^2 \cdots n_{k_m}^m)}{n_{k_1}^1 n_{k_2}^2 \cdots n_{k_m}^m} \ge c_2 \#\{(k_1, k_2, \ldots, k_m) : n_{k_1}^1 n_{k_2}^2 \cdots n_{k_m}^m \le N\} \tag{1.9}$$

for some constant $c_2 > 0$. Suppose that there exists some c_3 with $0 < c_3 < 1$ such that

$$\sum_{n_{k_1}^1 n_{k_2}^2 \cdots n_{k_m}^m \le N} n_{k_1}^1 n_{k_2}^2 \cdots n_{k_m}^m \le c_3 N \#\{(k_1, k_2, \ldots, k_m) : n_{k_1}^1 n_{k_2}^2 \cdots n_{k_m}^m \le N\} \tag{1.10}$$

for all large N.

Then, for almost every $\alpha \in \mathbb{R}$, the inequality

$$|n|_{\mathcal{D}_1} |n|_{\mathcal{D}_2} \cdots |n|_{\mathcal{D}_m} |n\alpha - p| \le \psi(n)$$

has infinitely many coprime solutions $(n, p) \in \mathbb{N} \times \mathbb{Z}$ if and only if

$$\sum_{n=1}^{\infty} \psi(n) \#\{(k_1, k_2, \ldots, k_m) : n_{k_1}^1 n_{k_2}^2 \cdots n_{k_m}^m \le n\} = \infty.$$

The Duffin–Schaeffer theorem is crucial to the proof of Theorems 1.1 and 1.4. However, the Duffin–Schaeffer theorem requires a good match between the sequence $\psi(n)$ and the Euler function $\varphi(n)$, so that hypotheses (1.5), (1.7) and (1.9) are very important. For some nice functions $\psi(n)$, the Duffin–Schaeffer theorem can be improved [2, 15–17]. We will use [17, Theorem 1.17] to study the mixed Littlewood conjecture and find that the restriction (1.5) is not necessary in some sense.

Given $n \in \mathbb{N}$ and $x \in \mathbb{R}$, define

$$\|nx\|' = \min\{|nx - p| : p \in \mathbb{Z}, (n, p) = 1\}.$$

THEOREM 1.5. *Let $\mathcal{D} = \{n_k\}$ be a pseudo-absolute-value sequence and define*

$$\mathfrak{M}(n) = \sum_{n_k \le n} \frac{\varphi(n_k)}{n_k}. \tag{1.11}$$

Suppose that $\epsilon \ge 0$. Then, for almost every $\alpha \in \mathbb{R}$,

$$\liminf_{n\to\infty} n\mathfrak{M}(n)(\log n)^{1+\epsilon}|n|_{\mathcal{D}}\|n\alpha\|' = 0$$

if and only if $\epsilon = 0$.

2. Proof of Theorem 1.1

In this paper, we always assume that C (c) is a large (small) constant, which is different even in the same equation. We should mention that the constant C (c) also depends on c_1, c_2 and c_3 in the theorems.

Before we give the proof of Theorem 1.1, some preparations are necessary.

LEMMA 2.1 [5, Lemma 2]. *Let $p_1, \dots, p_k$ be distinct prime numbers and $N \in \mathbb{N}$. Then*

$$\sum_{\substack{n\le N\\ p_1,\dots,p_k\nmid n}} \frac{\varphi(n)}{n} = \frac{6N}{\pi^2}\prod_{i=1}^{k}\frac{p_i}{p_i+1} + O(\log N).$$

Obviously, Lemma 2.1 implies the following lemma.

LEMMA 2.2. *Suppose that $d_1, d_2, \dots, d_m \ge 2$. Then there exists some $d > 0$ depending only on m such that*

$$\sum_{\substack{n=1\\ d_1\nmid n, d_2\nmid n,\dots,d_m\nmid n}}^{N} \frac{\varphi(n)}{n} \ge dN \quad \text{for any } N \in \mathbb{N}.$$

THEOREM 2.3 (Duffin–Schaeffer [8]). *Suppose that $\sum_{n=1}^{\infty}\psi(n) = \infty$ and*

$$\limsup_{N\to\infty}\Big(\sum_{n=1}^{N}\frac{\varphi(n)}{n}\psi(n)\Big)\Big(\sum_{n=1}^{N}\psi(n)\Big)^{-1} > 0.$$

Then, for almost every α, the inequality

$$|n\alpha - p| \le \psi(n)$$

has infinitely many coprime solutions $(n, p) \in \mathbb{N}\times\mathbb{Z}$.

Suppose that $\mathcal{D}_1 = \{n_k^1\}, \mathcal{D}_2 = \{n_k^2\}, \ldots, \mathcal{D}_m = \{n_k^m\}$ are m pseudo-absolute-value sequences. Denote $d_{k+1}^j = n_{k+1}^j / n_k^j$ for $j = 1, 2, \ldots, m$. Define a subset $S(n)$ of $\mathbb{N}^m$ as follows:

$$S(n) = \{(k_1, k_2, \ldots, k_m) : (k_1, k_2, \ldots, k_m) \in \mathbb{N}^m \text{ and } \operatorname{lcm}(n_{k_1}^1, n_{k_2}^2, \ldots, n_{k_m}^m) \le n\},$$

where $\operatorname{lcm}(k_1, k_2, \ldots, k_m)$ means the least common multiple of $k_1, k_2, \ldots, k_m$. For any $(k_1, k_2, \ldots, k_m) \in S(n)$, we define $f(n; k_1, k_2, \ldots, k_m) \in \mathbb{N}$ as the largest positive integer such that

$$\operatorname{lcm}(n_{k_1}^1, n_{k_1}^2, \ldots, n_{k_m}^m) f(n; k_1, k_2, \ldots, k_m) \le n.$$

Proof of Theorem 1.1. Without of loss of generality, assume that $\alpha \in [0, 1)$. Define

$$\mathcal{E}_n = \mathcal{E}_n(\psi_0) = \bigcup_{\substack{p=1\\(p,n)=1}}^{n} \left(\frac{p - \psi_0(n)}{n}, \frac{p + \psi_0(n)}{n} \right),$$

where

$$\psi_0(n) = \frac{\psi(n)}{|n|_{\mathcal{D}_1} |n|_{\mathcal{D}_2} \cdots |n|_{\mathcal{D}_m}}.$$

The Lebesgue measure of $\mathcal{E}_n$ is obviously bounded above by $(2\psi_0(n)/n)\varphi(n)$. Obviously, the coprime pair $(n, p) \in \mathbb{N} \times \mathbb{Z}$ is a solution of $|n\alpha - p| \le \psi_0(n)$ if and only if $\alpha \in \mathcal{E}_n$.

If

$$\sum_{n=1}^{\infty} \frac{\psi(n)}{|n|_{\mathcal{D}_1} |n|_{\mathcal{D}_2} \cdots |n|_{\mathcal{D}_m}} < \infty,$$

$$\sum_{n} \lambda(\mathcal{E}_n) < \infty.$$

By the Borel–Cantelli lemma, the inequality

$$|n|_{\mathcal{D}_1} |n|_{\mathcal{D}_2} \cdots |n|_{\mathcal{D}_m} |n\alpha - p| \le \psi(n)$$

has infinitely many solutions $(n, p) \in \mathbb{N} \times \mathbb{Z}$ only for a zero Lebesgue measure set of α.

Now we start to prove the other side. First,

$$\begin{aligned}
&\sum_{n=1}^{N} \frac{\varphi(n)\psi(n)}{n |n|_{\mathcal{D}_1} |n|_{\mathcal{D}_2} \cdots |n|_{\mathcal{D}_m}} \\
&\quad = \sum_{n=1}^{N} (\psi(n) - \psi(n+1)) \sum_{j=1}^{n} \frac{\varphi(j)}{j |j|_{\mathcal{D}_1} |j|_{\mathcal{D}_2} \cdots |j|_{\mathcal{D}_m}} \\
&\quad\quad + \psi(N+1) \sum_{j=1}^{N} \frac{\varphi(j)}{j |j|_{\mathcal{D}_1} |j|_{\mathcal{D}_2} \cdots |j|_{\mathcal{D}_m}}.
\end{aligned} \tag{2.1}$$

Now we are in the position to estimate the inner sums. Direct computation implies that

$$\begin{aligned}
&\sum_{j=1}^{n}\frac{\varphi(j)}{j|j|_{\mathcal{D}_1}|j|_{\mathcal{D}_2}\cdots|j|_{\mathcal{D}_m}}\\
&\quad=\sum_{(k_1,k_2,\dots,k_m)\in S(n)}\ \sum_{\substack{j=1\\ n^1_{k_1}|j,n^2_{k_2}|j,\dots,n^m_{k_m}|j\\ n^1_{k_1+1}\nmid j,n^2_{k_2+1}\nmid j,\dots,n^m_{k_m+1}\nmid j}}^{n}\frac{\varphi(j)}{j|j|_{\mathcal{D}_1}|j|_{\mathcal{D}_2}\cdots|j|_{\mathcal{D}_m}}\\
&\quad=\sum_{(k_1,k_2,\dots,k_m)\in S(n)}\frac{n^1_{k_1}n^2_{k_2}\cdots n^m_{k_m}}{\operatorname{lcm}(n^1_{k_1},n^2_{k_2},\dots,n^m_{k_m})}\\
&\qquad\times\sum_{\substack{1\le j\le f(n;k_1,k_2,\dots,k_m)\\ d^1_{k_1+1}\nmid j,d^2_{k_2+1}\nmid j,\dots,d^m_{k_m+1}\nmid j}}\frac{\varphi(\operatorname{lcm}(n^1_{k_1},n^2_{k_2},\dots,n^m_{k_m})j)}{j}\\
&\quad\ge\sum_{(k_1,k_2,\dots,k_m)\in S(n)}n^1_{k_1}n^2_{k_2}\cdots n^m_{k_m}\frac{\varphi(\operatorname{lcm}(n^1_{k_1},n^2_{k_2},\dots,n^m_{k_m}))}{\operatorname{lcm}(n^1_{k_1},n^2_{k_2},\dots,n^m_{k_m})}\sum_{\substack{1\le j\le f(n;k_1,k_2,\dots,k_m)\\ d^1_{k_1+1}\nmid j,d^2_{k_2+1}\nmid j,\dots,d^m_{k_m+1}\nmid j}}\frac{\varphi(j)}{j}\\
&\quad\ge c\sum_{(k_1,k_2,\dots,k_m)\in S(n)}f(n;k_1,k_2,\dots,k_m)\varphi(n^1_{k_1}n^2_{k_2}\cdots n^m_{k_m}),
\end{aligned}\tag{2.2}$$

where the first inequality holds by the fact that $\varphi(mn)\ge\varphi(m)\varphi(n)$ and the second inequality holds by Lemma 2.2 and the fact that

$$\frac{\varphi(\operatorname{lcm}(n^1_{k_1},n^2_{k_2},\dots,n^m_{k_m}))}{\operatorname{lcm}(n^1_{k_1},n^2_{k_2},\dots,n^m_{k_m})}=\frac{\varphi(n^1_{k_1}n^2_{k_2}\cdots n^m_{k_m})}{n^1_{k_1}n^2_{k_2}\cdots n^m_{k_m}}.$$

By (1.7) and (2.2),

$$\sum_{j=1}^{n}\frac{\varphi(j)}{j|j|_{\mathcal{D}_1}|j|_{\mathcal{D}_2}\cdots|j|_{\mathcal{D}_m}}\ge c\sum_{(k_1,k_2,\dots,k_m)\in S(n)}n^1_{k_1}n^2_{k_2}\cdots n^m_{k_m}f(n;k_1,k_2,\dots,k_m).\tag{2.3}$$

One the other hand,

$$\begin{aligned}
\sum_{j=1}^{n}\frac{1}{|j|_{\mathcal{D}_1}|j|_{\mathcal{D}_2}\cdots|j|_{\mathcal{D}_m}}&=\sum_{(k_1,k_2,\dots,k_m)\in S(n)}n^1_{k_1}n^2_{k_2}\cdots n^m_{k_m}\\
&\qquad\times\sum_{\substack{j=1\\ n^1_{k_1}|j,n^2_{k_2}|j,\dots,n^m_{k_m}|j\\ n^1_{k_1+1}\nmid j,n^2_{k_2+1}\nmid j,\dots,n^m_{k_m+1}\nmid j}}^{n}1 &(2.4)\\
&\le\sum_{(k_1,k_2,\dots,k_m)\in S(n)}n^1_{k_1}n^2_{k_2}\cdots n^m_{k_m}f(n;k_1,k_2,\dots,k_m). &(2.5)
\end{aligned}$$

Finally, putting (2.3) and (2.5) together,

$$\sum_{j=1}^{n} \frac{\varphi(j)}{j|j|_{\mathcal{D}_1}|j|_{\mathcal{D}_2}\cdots|j|_{\mathcal{D}_m}} \geq c\sum_{j=1}^{n} \frac{1}{|j|_{\mathcal{D}_1}|j|_{\mathcal{D}_2}\cdots|j|_{\mathcal{D}_m}}.$$

Combining with (2.1),

$$\begin{aligned}\sum_{n=1}^{N} \frac{\varphi(n)\psi(n)}{n|n|_{\mathcal{D}_1}|n|_{\mathcal{D}_2}\cdots|n|_{\mathcal{D}_m}} &\geq \sum_{n=1}^{N}(\psi(n)-\psi(n+1))\sum_{j=1}^{n} \frac{c}{|j|_{\mathcal{D}_1}|j|_{\mathcal{D}_2}\cdots|j|_{\mathcal{D}_m}}\\ &\quad + \psi(N+1)\sum_{j=1}^{N} \frac{c}{|j|_{\mathcal{D}_1}|j|_{\mathcal{D}_2}\cdots|j|_{\mathcal{D}_m}}\\ &\geq c\sum_{n=1}^{N} \frac{\psi(n)}{|n|_{\mathcal{D}_1}|n|_{\mathcal{D}_2}\cdots|n|_{\mathcal{D}_m}}.\end{aligned}$$

Now Theorem 1.1 follows from (1.8) and Theorem 2.3. □

3. Proof of Theorem 1.4

The proof of Theorem 1.4 is similar to the proof of Theorem 1.1 or (1.6). We need one lemma first. Denote

$$\mathcal{M}(n) = \#\{(k_1, k_2, \ldots, k_m) : n^1_{k_1} n^2_{k_2} \cdots n^m_{k_m} \leq n\} - 1.$$

Lemma 3.1. *Under the conditions of Theorem 1.4, the following estimate holds:*

$$N\mathcal{M}(N) \asymp \sum_{n=1}^{N} \mathcal{M}(n). \tag{3.1}$$

Proof. It suffices to show that

$$N\mathcal{M}(N) \leq O(1)\sum_{n=1}^{N} \mathcal{M}(n).$$

We rearrange $n^1_{k_1} n^2_{k_2} \cdots n^m_{k_m}$ as a monotone sequence $t_0 = 1, t_1, t_2, \ldots, t_k \ldots$. Then

$$\begin{aligned}\sum_{n=1}^{N} \mathcal{M}(n) &= \sum_{k=0}^{\mathcal{M}(N)-1} k(t_{k+1}-t_k) + \mathcal{M}(N)(N - t_{\mathcal{M}(N)} + 1)\\ &= (N+1)\mathcal{M}(N) - \sum_{k=0}^{\mathcal{M}(N)} t_k.\end{aligned} \tag{3.2}$$

By the assumption (1.10),

$$\sum_{k=0}^{\mathcal{M}(N)} t_k \leq c_3 N\mathcal{M}(N) \tag{3.3}$$

for some $0 < c_3 < 1$.

Now the lemma follows from (3.2) and (3.3). □

Proof of Theorem 1.4. We employ the same notation as in the proof of Theorem 1.1.

By the fact that the pseudo-absolute-value sequences are mutually coprime,

$$\mathcal{M}(n) + 1 = \#S(n).$$

Moreover,

$$\frac{n}{2} \le n^1_{k_1} n^2_{k_2} \cdots n^m_{k_m} f(n; k_1, k_2, \ldots, k_m) \le n.$$

By (2.2) and assumption (1.9),

$$\begin{aligned}\sum_{j=1}^{n} \frac{\varphi(j)}{j|j|_{\mathcal{D}_1}|j|_{\mathcal{D}_2}\cdots|j|_{\mathcal{D}_m}} &\ge c \sum_{(k_1,k_2,\ldots,k_m)\in S(n)} f(n; k_1, k_2, \ldots, k_m)\varphi(n^1_{k_1} n^2_{k_2} \cdots n^m_{k_m}) \\ &\ge cn \sum_{(k_1,k_2,\ldots,k_m)\in S(n)} \frac{\varphi(n^1_{k_1} n^2_{k_2} \cdots n^m_{k_m})}{n^1_{k_1} n^2_{k_2} \cdots n^m_{k_m}} \\ &\ge cn\mathcal{M}(n). \end{aligned} \tag{3.4}$$

By (3.4) and (2.4),

$$cn\mathcal{M}(n) \le \sum_{j=1}^{n} \frac{1}{|j|_{\mathcal{D}_1}|j|_{\mathcal{D}_2}\cdots|j|_{\mathcal{D}_m}} \le n\mathcal{M}(n). \tag{3.5}$$

Suppose that

$$\sum_{n} \psi(n)\mathcal{M}(n) < \infty.$$

In this case, by (3.1),

$$\begin{aligned}&\sum_{n=1}^{N} \frac{\psi(n)}{|n|_{\mathcal{D}_1}|n|_{\mathcal{D}_2}\cdots|n|_{\mathcal{D}_m}} \\ &\quad= \sum_{n=1}^{N}(\psi(n) - \psi(n+1)) \sum_{j=1}^{n} \frac{1}{|j|_{\mathcal{D}_1}|j|_{\mathcal{D}_2}\cdots|j|_{\mathcal{D}_m}} \\ &\qquad + \psi(N+1) \sum_{j=1}^{N} \frac{1}{|j|_{\mathcal{D}_1}|j|_{\mathcal{D}_2}\cdots|j|_{\mathcal{D}_m}} \\ &\quad\le \sum_{n=1}^{N}(\psi(n) - \psi(n+1))n\mathcal{M}(n) + \psi(N+1)N\mathcal{M}(N) \\ &\quad\le C\sum_{n=1}^{N}(\psi(n) - \psi(n+1)) \sum_{j=0}^{n} \mathcal{M}(j) + \psi(N+1)N\mathcal{M}(N) \\ &\quad\le C\sum_{n=1}^{N} \psi(n)\mathcal{M}(n) < \infty, \end{aligned} \tag{3.6}$$

where the first inequality holds by (3.5). By the Borel–Cantelli lemma, the inequality

$$|n|_{\mathcal{D}_1}|n|_{\mathcal{D}_2}\cdots|n|_{\mathcal{D}_m}|n\alpha-p|\le\psi(n)$$

has infinitely many coprime solutions $(n,p)\in\mathbb{N}\times\mathbb{Z}$ only for a zero Lebesgue measure set of α.

Now we are in the position to prove the other side.

Suppose that

$$\sum_n\psi(n)\mathcal{M}(n)=\infty.$$

By (2.1) and (3.4),

$$\begin{aligned}\sum_{n=1}^{N}\frac{\varphi(n)\psi(n)}{n|n|_{\mathcal{D}_1}|n|_{\mathcal{D}_2}\cdots|n|_{\mathcal{D}_m}}&\ge c\sum_{n=1}^{N}(\psi(n)-\psi(n+1))nM(n)+c\psi(N+1)N\mathcal{M}(N)\\&\ge c\sum_{n=1}^{N}\psi(n)\mathcal{M}(n).\end{aligned}\tag{3.7}$$

Thus,

$$\sum_{n=1}^{\infty}\frac{\psi(n)}{|n|_{\mathcal{D}_1}|n|_{\mathcal{D}_2}\cdots|n|_{\mathcal{D}_m}}=\infty.\tag{3.8}$$

By (3.6) and (3.7),

$$\sum_{n=1}^{N}\frac{\varphi(n)\psi(n)}{n|n|_{\mathcal{D}_1}|n|_{\mathcal{D}_2}\cdots|n|_{\mathcal{D}_m}}\ge c\sum_{n=1}^{N}\frac{\psi(n)}{|n|_{\mathcal{D}_1}|n|_{\mathcal{D}_2}\cdots|n|_{\mathcal{D}_m}}.\tag{3.9}$$

Applying (3.8) and (3.9) to Theorem 2.3, we finish the proof. □

4. Proof of Theorem 1.5

Before we give the proof, one lemma is necessary.

LEMMA 4.1. *Let $\mathcal{D}=\{n_k\}$ be a pseudo-absolute-value sequence and $\mathfrak{M}(n)$ be given by (1.11). We have the following estimate:*

$$N\mathfrak{M}(N)\asymp\sum_{n=1}^{N}\mathfrak{M}(n).\tag{4.1}$$

PROOF. It is easy to see that (4.1) holds if the sequence $\mathfrak{M}(n)$ is bounded. Thus, we assume that $\mathfrak{M}(n)\to\infty$ as $n\to\infty$.

It suffices to show that

$$N\mathfrak{M}(N)\le O(1)\sum_{n=1}^{N}\mathfrak{M}(n).$$

As usual, let $\mathcal{M}(N)$ be the largest k such that $n_k\le N$.

By the definition of $\mathfrak{M}(n)$,

$$\sum_{n=1}^{N}\mathfrak{M}(n)=\sum_{k=0}^{\mathfrak{M}(N)}\Big(\sum_{j=0}^{k}\frac{\varphi(n_j)}{n_j}\Big)(n_{k+1}-n_k)+\Big(\sum_{j=0}^{\mathcal{M}(N)}\frac{\varphi(n_j)}{n_j}\Big)(N-n_{\mathcal{M}(N)}+1)$$
$$=(N+1)\Big(\sum_{j=0}^{\mathcal{M}(N)}\frac{\varphi(n_j)}{n_j}\Big)-\sum_{k=0}^{\mathcal{M}(N)}n_k\frac{\varphi(n_k)}{n_k}$$
$$=(N+1)\mathfrak{M}(N)-\sum_{k=0}^{\mathcal{M}(N)}\varphi(n_k). \tag{4.2}$$

By the fact that $n_{k+1}\geq 2n_k$,

$$\sum_{k=0}^{\mathcal{M}(N)}n_k\leq N\sum_{k=0}^{\mathcal{M}(N)}\frac{1}{2^k}\leq 2N.$$

This implies that

$$\sum_{k=0}^{\mathcal{M}(N)}\varphi(n_k)\leq 2N. \tag{4.3}$$

By (4.2) and (4.3),

$$N\mathfrak{M}(N)\leq O(1)\sum_{n=1}^{N}\mathfrak{M}(n).$$

We have finished the proof. □

We will split the proof of Theorem 1.5 into two parts.

THEOREM 4.2. *Let $\mathcal{D}=\{n_k\}$ be a pseudo-absolute-value sequence and $\mathfrak{M}(n)$ be given by (1.11). Suppose that $\psi:\mathbb{N}\to\mathbb{R}^+$ is nonincreasing and*

$$\sum_{n}\psi(n)\mathfrak{M}(n)<\infty. \tag{4.4}$$

Then, for almost every α, the inequality

$$|n|_{\mathcal{D}}|n\alpha-p|\leq\psi(n)$$

has finitely many coprime solutions $(n,p)\in\mathbb{N}\times\mathbb{Z}$. In particular, for any $\epsilon>0$,

$$\liminf_{n\to\infty} n\mathfrak{M}(n)(\log n)^{1+\epsilon}|n|_{\mathcal{D}}\|n\alpha\|'=0$$

holds for a zero Lebesgue measure set $\alpha\in\mathbb{R}$.

PROOF. The proof of Theorem 4.2 is based on the Borel–Cantelli lemma. Without loss of generality, assume that $\alpha\in[0,1)$. Define

$$\mathcal{E}_n = \mathcal{E}_n(\psi_0) = \bigcup_{\substack{p=1 \\ (p,n)=1}}^{n} \Big(\frac{p - \psi_0(n)}{n}, \frac{p + \psi_0(n)}{n} \Big),$$

where

$$\psi_0(n) = \frac{\psi(n)}{|n|_{\mathcal{D}}}.$$

By the proof of Theorem 1.1, in order to prove Theorem 4.2, we only need to show that

$$\sum_n \lambda(\mathcal{E}_n) < \infty.$$

Like (2.1),

$$\sum_{n=1}^{N} \frac{\varphi(n)\psi(n)}{n|n|_{\mathcal{D}}} = \sum_{n=1}^{N} (\psi(n) - \psi(n+1)) \sum_{m=1}^{n} \frac{\varphi(m)}{m|m|_{\mathcal{D}}} + \psi(N+1) \sum_{m=1}^{N} \frac{\varphi(m)}{m|m|_{\mathcal{D}}}. \tag{4.5}$$

We estimate the inner sums here (denote $d_{k+1} = n_{k+1}/n_k$) by

$$\begin{aligned}
\sum_{m=1}^{n} \frac{\varphi(m)}{m|m|_{\mathcal{D}}} &= \sum_{n_k \le n} \sum_{\substack{m=1 \\ n_k|m,\ n_{k+1}\nmid m}}^{n} \frac{\varphi(m)}{m|m|_{\mathcal{D}}} \\
&= \sum_{n_k \le n} \sum_{\substack{1 \le m \le n/n_k \\ d_{k+1}\nmid m}} \frac{\varphi(n_k m)}{m} \\
&\le \sum_{n_k \le n} \varphi(n_k) \sum_{\substack{1 \le m \le n/n_k \\ d_{k+1}\nmid m}} 1 \\
&\le n \sum_{n_k \le n} \frac{\varphi(n_k)}{n_k}, \\
&= n\mathfrak{M}(n),
\end{aligned}$$

where the first inequality holds by the fact that

$$\varphi(nm) \le m\varphi(n).$$

Therefore, by (4.5) and (4.1),

$$\begin{aligned}
\sum_{n=1}^{N} \lambda(\mathcal{E}_n) &\le \sum_{n=1}^{N} \frac{2\psi_0(n)}{n} \varphi(n) \\
&= 2 \sum_{n=1}^{N} \frac{\varphi(n)\psi(n)}{n|n|_{\mathcal{D}}} \\
&\le C \sum_{n=1}^{N} (\psi(n) - \psi(n+1)) n\mathfrak{M}(n) + C\psi(N+1) N\mathfrak{M}(N)
\end{aligned}$$

$$\leq C\sum_{n=1}^{N}(\psi(n)-\psi(n+1))\sum_{j=1}^{n}\mathfrak{M}(j)+C\psi(N+1)N\mathfrak{M}(N)$$
$$\leq \sum_{n=1}^{N+1}C\psi(n)\mathfrak{M}(n).$$

Combining with assumption (4.4), $\sum_n \lambda(\mathcal{E}_n)<\infty$ follows. □

The remaining part of Theorem 1.5 needs more energy to prove. In the previous two sections, we used the Duffin–Schaeffer theorem to complete the proof. Now, we will apply the following lemma to finish the proof.

Lemma 4.3 [17, Theorem 1.17]. *Let $\psi:\mathbb{N}\to\mathbb{R}$ be a nonnegative function. Suppose that*

$$\sum_{n\in\mathbb{N}:G_n\geq 3}\frac{\log G_n}{n\cdot\log\log G_n}=\infty,$$

where

$$G_n=\sum_{k=2^{2^n}+1}^{2^{2^{n+1}}}\frac{\psi(k)\varphi(k)}{k}.$$

Then, for almost every α, the inequality

$$|n\alpha-p|\leq\psi(n)$$

has infinitely many coprime solutions $(n,p)\in\mathbb{N}\times\mathbb{Z}$.

The next lemma is easy to prove by a Möbius function or follows from Lemma 2.1 ($k=1$) directly.

Lemma 4.4. *For any $d\in\mathbb{N}$,*

$$\sum_{\substack{n=N_1\\ d\nmid n}}^{N_2}\frac{\varphi(n)}{n}\geq\max\Big\{0,\frac{4}{\pi^2}(N_2-N_1)-O(\log N_2)\Big\}\quad\textit{for all}\ \ 0<N_1<N_2.$$

Remark 4.5. The sharp bound $4/\pi^2$ can be achieved when $d=2$.

Theorem 4.6. *Let $\psi:\mathbb{N}\to\mathbb{R}$ be a nonnegative function and $\lim_{n\to\infty}\psi(n)=0$. Define*

$$\mathcal{E}_n(\psi)=\bigcup_{\substack{p=1\\(p,n)=1}}^{n}\Big(\frac{p-\psi(n)}{n},\frac{p+\psi(n)}{n}\Big).$$

Then the following claims are true.

Zero–one law $\lambda(\limsup\mathcal{E}_n(\psi))\in\{0,1\}$ [12].
Subhomogeneity *For any $t\geq 1$, $\lambda(\limsup\mathcal{E}_n(t\psi))\leq t\lambda(\limsup\mathcal{E}_n(\psi))$* [17].

We need another lemma.

LEMMA 4.7. *Let $\mathcal{D} = \{n_k\}$ be a pseudo-absolute-value sequence. Then*

$$\sum_{n_k \le n} n_k \log \frac{n}{n_k} \le Cn \tag{4.6}$$

and

$$\sum_{2^{2^N} \le n_k \le 2^{2^{N+1}}} \frac{1}{\log n_k} = O(1). \tag{4.7}$$

PROOF. Since $\{n_k\}$ is a pseudo-absolute-value sequence, there exists at most one n_k such that $2^j \le n_k < 2^{j+1}$. Thus,

$$\begin{aligned}\sum_{n_k \le n} n_k \log \frac{n}{n_k} &\le \sum_{j=0}^{\log_2 n} \sum_{2^j \le n_k < 2^{j+1}} n_k \log \frac{n}{n_k}\\ &\le \sum_{j=0}^{\log_2 n} 2^{j+1} \log \frac{n}{2^j}\\ &\le Cn.\end{aligned}$$

This proves (4.6).

Similarly,

$$\begin{aligned}\sum_{n=2^{2^N}+1}^{2^{2^{N+1}}} \frac{1}{\log n_k} &\le \sum_{j=2^N}^{2^{N+1}} \sum_{2^j \le n_k < 2^{j+1}} \frac{1}{\log n_k}\\ &\le O(1) \sum_{j=2^N}^{2^{N+1}} \frac{1}{j}\\ &= O(1).\end{aligned}$$

We have finished the proof. □

After the preparations, we can prove the case $\epsilon = 0$ of Theorem 1.5.

THEOREM 4.8. *Let $\mathcal{D} = \{n_k\}$ be a pseudo-absolute-value sequence and $\mathfrak{M}(n)$ be given by (1.11). Then, for almost every $\alpha \in \mathbb{R}$,*

$$\liminf_{n\to\infty} n\mathfrak{M}(n)(\log n)|n|_{\mathcal{D}}\|n\alpha\|' = 0.$$

PROOF. Without loss of generality, assume that $\alpha \in [0, 1)$. Let

$$\psi_0(n) = \frac{1}{|n|_{\mathcal{D}} n\mathfrak{M}(n)(\log n)}$$

and

$$\psi(n) = \frac{1}{n\mathfrak{M}(n)(\log n)}.$$

It suffices to show that there exists some $c > 0$ such that

$$G_N = \sum_{n=2^{2^N}+1}^{2^{2^{N+1}}} \frac{\psi_0(n)\varphi(n)}{n} > c \tag{4.8}$$

for $N \in \mathbb{N}$. Indeed, if (4.8) holds, then, for any $\varepsilon > 0$, there exists some $C > 0$ such that

$$\sum_{n=2^{2^N}+1}^{2^{2^{N+1}}} \frac{C\varepsilon\psi_0(n)\varphi(n)}{n} \geq 3 \quad \text{for all } N.$$

Applying Lemma 4.3 (letting $\psi = C\varepsilon\psi_0$),

$$\lambda(\limsup \mathcal{E}_n(C\varepsilon\psi_0)) = 1. \tag{4.9}$$

Applying Theorem 4.6 (subhomogeneity) to (4.9),

$$\lambda(\limsup \mathcal{E}_n(\varepsilon\psi_0)) \geq \frac{1}{C}.$$

By the zero–one law of Theorem 4.6,

$$\lambda(\limsup \mathcal{E}_n(\varepsilon\psi_0)) = 1.$$

Thus, for any $\varepsilon > 0$, we have that, for almost every α, the inequality

$$|n\alpha - p| \leq \varepsilon\psi_0(n)$$

has infinitely many coprime solutions $(n, p) \in \mathbb{N} \times \mathbb{Z}$. This implies that for almost every $\alpha \in \mathbb{R}$,

$$\liminf_{n\to\infty} n\mathfrak{M}(n)(\log n)|n|_{\mathcal{D}}\|n\alpha\|' = 0.$$

Now we focus on the proof of (4.8).

As usual,

$$\sum_{n=2^{2^N}+1}^{2^{2^{N+1}}} \frac{\varphi(n)\psi(n)}{n|n|_{\mathcal{D}}}$$
$$= \sum_{n=2^{2^N}+1}^{2^{2^{N+1}}} (\psi(n) - \psi(n+1)) \sum_{j=2^{2^N}+1}^{n} \frac{\varphi(j)}{j|j|_{\mathcal{D}}} + \psi(2^{2^{N+1}} + 1) \sum_{j=2^{2^N}+1}^{2^{2^{N+1}}} \frac{\varphi(j)}{j|j|_{\mathcal{D}}}. \tag{4.10}$$

Direct computation yields

$$\begin{aligned}
\sum_{j=2^{2^N}+1}^{n} \frac{\varphi(j)}{j|j|_{\mathcal{D}}} &= \sum_{k:1\le n_k\le n} \sum_{\substack{j=2^{2^N}+1\\ n_k|j,\ n_{k+1}\nmid j}}^{n} \frac{\varphi(j)}{j|j|_{\mathcal{D}}} \\
&= \sum_{n_k\le n} \sum_{\substack{(2^{2^N}+1/n_k)\le j\le(n/n_k)\\ d_{k+1}\nmid j}} \frac{\varphi(n_k j)}{j} \\
&\ge \sum_{n_k\le n} \varphi(n_k) \sum_{\substack{(2^{2^N}+1/n_k)\le j\le(n/n_k)\\ d_{k+1}\nmid j}} \frac{\varphi(j)}{j} \\
&\ge \frac{4}{\pi^2} \sum_{n_k\le n} \varphi(n_k) \max\Big\{0, \frac{n-2^{2^N}}{n_k} - O\Big(\log\Big(\frac{n}{n_k}\Big)\Big)\Big\} \\
&\ge \frac{4}{\pi^2} \sum_{n_k\le n} \frac{\varphi(n_k)}{n_k}\Big((n-2^{2^N}) - O\Big(n_k \log\frac{n}{n_k}\Big)\Big) \\
&\ge \frac{4}{\pi^2}\mathfrak{M}(n)(n-2^{2^N}) - \sum_{n_k\le n} O\Big(n_k\log\frac{n}{n_k}\Big), \qquad (4.11)
\end{aligned}$$

where the second inequality holds by Lemma 4.4.

By the definition of $\psi(n)$, for $n \ne n_k$,

$$\psi(n) - \psi(n+1) = \frac{O(1)}{n^2\mathfrak{M}(n)\log n} \qquad (4.12)$$

and

$$\psi(n_k) - \psi(n_k+1) = \frac{O(1)}{n_k\mathfrak{M}^2(n_k)\log n_k}. \qquad (4.13)$$

By (4.6), (4.12) and (4.13),

$$\begin{aligned}
&\sum_{n=2^{2^N}+1}^{2^{2^{N+1}}} (\psi(n)-\psi(n+1)) \sum_{n_k\le n} n_k \log\frac{n}{n_k} + \psi(2^{2^{N+1}}+1) \sum_{n_k\le 2^{2^{N+1}}} n_k \log\frac{2^{2^{N+1}}}{n_k} \\
&\le \sum_{n=2^{2^N}+1}^{2^{2^{N+1}}} \frac{O(1)}{n\mathfrak{M}(n)\log n} + \sum_{2^{2^N}\le n_k\le 2^{2^{N+1}}} \frac{O(1)}{\mathfrak{M}^2(n_k)\log n_k} + \frac{O(1)}{\mathfrak{M}(2^{2^{N+1}})} \\
&\le \frac{O(1)}{\mathfrak{M}^2(2^{2^N})} + \frac{O(1)}{\mathfrak{M}(2^{2^{N+1}})} + \frac{O(1)}{\mathfrak{M}(2^{2^N})} \sum_{n=2^{2^N}+1}^{2^{2^{N+1}}} \frac{1}{n\log n} \\
&= \frac{O(1)}{\mathfrak{M}(2^{2^N})}, \qquad (4.14)
\end{aligned}$$

where the second inequality holds by (4.7) and the third inequality holds because of $(a = 2^{2^N}$ and $b = 2^{2^{N+1}})$

$$\sum_a^b \frac{1}{n\log n} \asymp \int_a^b \frac{dx}{x\log x} = \log\log b - \log\log a \quad \text{for any } b > a > 1. \tag{4.15}$$

Putting (4.11) and (4.14) into (4.10),

$$\begin{aligned}
&\sum_{n=2^{2^N}+1}^{2^{2^{N+1}}} \frac{\varphi(n)\psi(n)}{n|n|_{\mathcal{D}}} \\
&\quad \geq \sum_{n=2^{2^N}+1}^{2^{2^{N+1}}} c\left(\frac{1}{n\log n\mathfrak{M}(n)} - \frac{1}{(n+1)\log(n+1)\mathfrak{M}(n+1)}\right)\mathfrak{M}(n)(n-2^{2^N}) \\
&\quad\quad - \frac{O(1)}{\mathfrak{M}(2^{2^N})} \\
&\quad \geq \sum_{n=2^{(2^N+4)}}^{2^{2^{N+1}}} \frac{c}{2}\left(\frac{1}{n\log n\mathfrak{M}(n)} - \frac{1}{(n+1)\log(n+1)\mathfrak{M}(n+1)}\right)n\mathfrak{M}(n) - \frac{O(1)}{\mathfrak{M}(2^{2^N})} \\
&\quad \geq c\sum_{n=2^{(2^N+4)}}^{2^{2^{N+1}}} \frac{1}{n\log n} - \frac{O(1)}{\mathfrak{M}(2^{2^N})}.
\end{aligned}$$

Using (4.15) again,

$$\sum_{n=2^{(2^N+4)}}^{2^{2^{N+1}}} \frac{1}{n\log n} \asymp 1.$$

This yields that, for some $c > 0$,

$$G_N \geq c.$$

We have finished the proof. □

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