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A remark on "Study of a Leslie-Gower predator-prey model with prey defense and mutual interference of predators" [Chaos, Solitons & Fractals 120 (2019) 1–16]

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ABSTRACT

In Mishra et al. (2019, [12]) a classical three-species modified Leslie-Gower system is considered, with mutual interference and prey-defense. The existence of a globally attracting invariant set is established for the system, under certain parametric restrictions. We show that the claimed invariant set is *not* globally attracting, and that finite time blow-up can occur for sufficiently large or small initial data. We further show that the invariant set cannot contain any interior equilibrium.

1. Introduction

1.1. Background

In [12] the following three species food chain ODE model is proposed,

$$\frac{du}{dt} = a_0 u - a_1 u^2 - w_1\left(\frac{uv}{iu^2 + d_1}\right), \tag{1}$$

$$\frac{dv}{dt} = -\delta v + w_2\left(\frac{uv}{iu^2 + d_1}\right) - w_3\left(\frac{vp}{v + bp + d_2}\right), \tag{2}$$

$$\frac{dp}{dt} = cp^2 - w_4\frac{p^2}{v + d_3}. \tag{3}$$

This model is based on a modified Leslie-Gower formulation [3], and considers the interactions between a generalist top predator p, depredating on a specialist middle predator v, that in turn is depredating on a prey u, where (u, v, p) are solutions to the above system (1)–(3). Similar models have generated a plethora of past and current research interest [1,2,4–6,10,14–26]. Also see [7,8,11,27].

The novelties in the current model are that the prey u is equipped with defense ability, via a Monod-Haldane functional response for the interaction between prey u and middle predator v. The middle predator v serves as favorite food for top predator p, modeled following a Beddington-DeAngelis functional response. The dynamics of the top predator are modeled via a modified Leslie-Gower formulation [3,9]. For a detailed description of the modeling formulation the reader is referred to Mishra et al. [12].

1.2. Past results

In [12] various theorems on the boundedness of the system (1)–(3) are proved, and the existence of an invariant globally attracting set is established. In particular, we recall the following result from [12],

Theorem 1.1. *If condition* $w_4 > c\eta$ *holds and set*

$$\Lambda = \left\{u(t), v(t), p(t) : u(t) \le \frac{a_0}{a_1}, 0 \le S_1(t) \le \frac{a_0}{a_1} + \frac{a_0^2}{4\delta a_1^2}, 0 \le S_2(t) \le \frac{a_0}{a_1} + \frac{a_0^2}{4\delta a_1^2} + \frac{M}{\delta}\right\}$$

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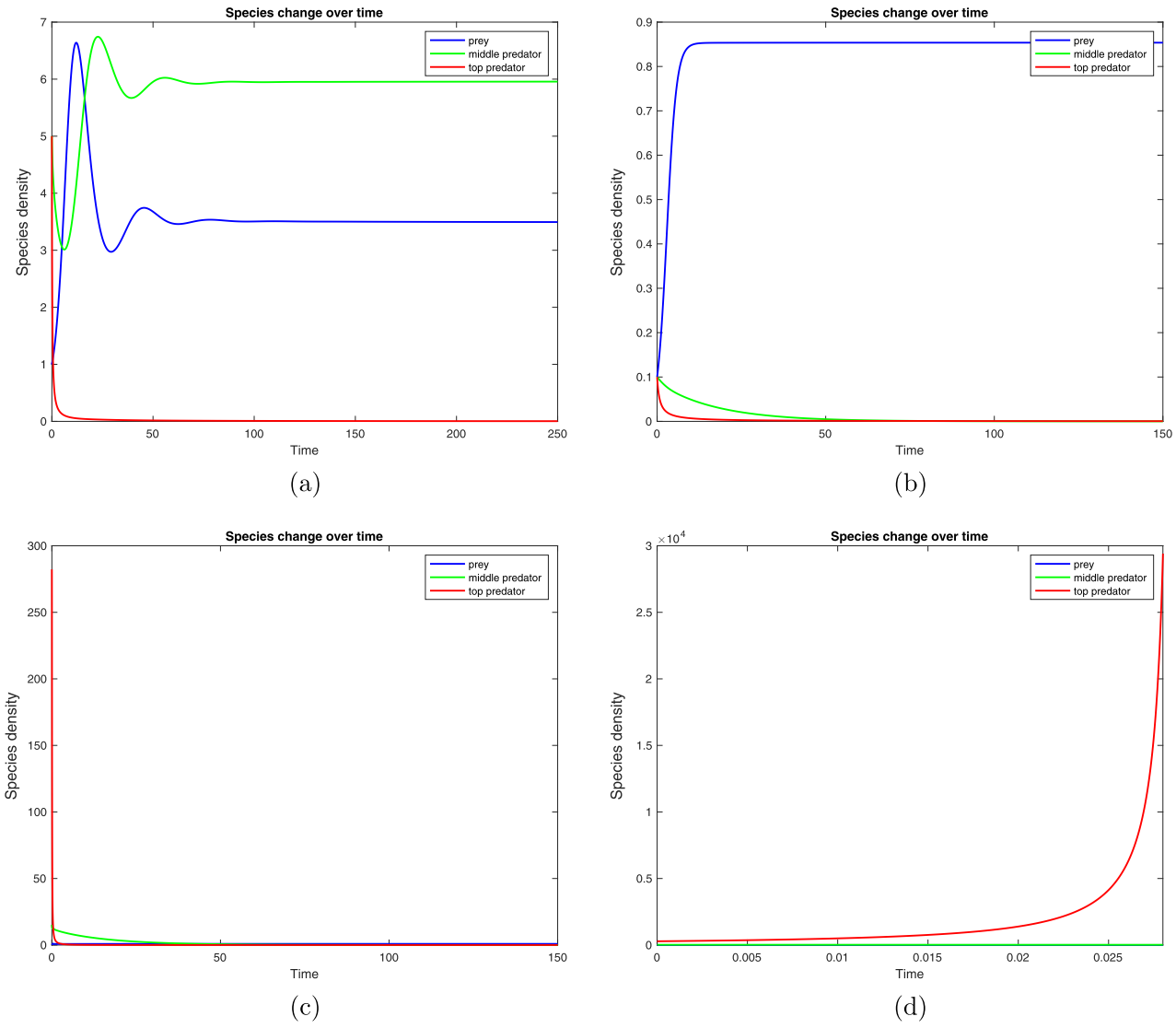

Fig. 1. (a) Simulation with parameters in (16) starting from the IC $= [1, 5, 5] \in \Lambda$. We see a $[u^*, v^*, 0]$ equilibrium. (b) Simulation with parameters in (18) starting from the IC $= [0.1, 0.1, 0.1] \in \Lambda$. We see a $[u^*, 0, 0]$ equilibrium. (c) Simulation with parameters in (18) starting from the IC $= [0.85, 15.18, 282.3] \in \Lambda$. Again we see a $[u^*, 0, 0]$ equilibrium (d) Simulation with parameters in (18) starting from the IC $= [0.853658536585366, 18, 282.3] \notin \Lambda$, blows up at time $t \approx 0.028$.

is positively invariant set and attracts all solutions initiating in the interior of positive octant, then all the positive solutions of the system (1)–(3) are uniformly bounded, where $\eta = \frac{a_0 w_1}{a_1 w_2} + \frac{a_0^2 w_2}{4\delta a_1^2 w_1} + d_3$ *and* $M = \frac{1}{4(w_4 - c\eta)}$. *Moreover, the system (1)–(3), is dissipative in* $\mathbb{R}^3_+$.

We note that $S_1(t) = u(t) + \frac{w_1}{w_2} v(t)$, $S_2(t) = u(t) + \frac{w_1}{w_2} v(t) + \alpha p(t)$ and $\alpha = \frac{1}{\eta a_1^2}$.

Thus the authors in [12] claim,

- Λ is positively invariant.
- Λ is attracting. That is all solutions initiating in $\mathbb{R}^3_+$ of (1)–(3) are uniformly bounded and eventually enter Λ.
- System (1)–(3) is dissipative.

In the current manuscript we show,

- Λ is *not* attracting. In particular for various initial data in $\mathbb{R}^3_+$, finite time blow-up occurs. Thus system (1)–(3) is *not* dissipative. These are shown via Theorems 2.2 and 2.3 and Fig. 1.
- Λ is invariant only for boundary equilibrium. That is Λ as constructed, cannot contain a feasible interior equilibrium (u^*, v^*, p^*). This is shown via Theorem 2.1.

2. The invariant set and finite time blow-up

2.1. No interior equilibrium in Λ

Theorem 2.1. *Consider the three species food chain model (1)–(3), and* Λ *as defined in Theorem 1.1. For* $\forall (u_0, v_0, p_0) \in \Lambda$, *p goes extinct eventually, and the only dynamics are boundary equilibrium* $[u^*, 0, 0]$, $[u^*, v^*, 0]$ *or a closed loop in the* $v - p$ *plane.*

Proof. From the form of Λ we see that we require,

$$\frac{w_1}{w_2} v_0 \leq S_1(t) \leq \frac{a_0}{a_1} + \frac{a_0^2}{4\delta a_1^2}. \tag{4}$$

Standard estimates (see [12]) show

$$\lim_{t\to\infty}\max(v(t)) \le \frac{w_2a_0}{w_1a_1} + \frac{w_2(a_0)^2}{4w_1\delta(a_1)^2}. \tag{5}$$

Note, the upper bound for v is as above, and not $\frac{w_1a_0}{w_2a_1} + \frac{w_2(a_0)^2}{4w_1\delta(a_1)^2}$, as claimed in [12]. The bound in [12] is the result of a simple algebraic error, as is apparent from equation (3.2) in [12].

Furthermore, the parametric restriction as required for Λ is

$$c < \frac{w_4}{\eta} = \frac{w_4}{\frac{w_2a_0}{w_1a_1} + \frac{w_2(a_0)^2}{4w_1\delta(a_1)^2} + d_3}. \tag{6}$$

This implies,

$$v \le \max(v(t)) \le \frac{w_2a_0}{w_1a_1} + \frac{w_2(a_0)^2}{4w_1\delta(a_1)^2} < \frac{w_4}{c} - d_3 \tag{7}$$

while

$$v_0 \le \frac{w_2a_0}{w_1a_1} + \frac{w_2(a_0)^2}{4w_1\delta(a_1)^2} < \frac{w_4}{c} - d_3. \tag{8}$$

Thus, via (7)-(8) the coefficient of p^2 in (3), $\left(c - \frac{w_4}{v+d_3}\right)$ is enforced to be negative initially and subsequently for all time, and so the only dynamics possible for p, via the restrictions of Theorem 1.1, is that $p \to 0$ eventually. This reduces (1)–(3) to a 2-species subsystem in u, v only. Standard theory [13] yields that the only dynamics are $[u^*, 0, 0]$, $[u^*, v^*, 0]$ or a closed loop in the $v - p$ plane. □

2.2. Finite time blow-up

We state the following theorem,

Theorem 2.2. *Consider the three species food chain model (1)–(3), and any set of parameters under which there exists a feasible interior equilibrium. Even if $c < \frac{w_4}{\eta}$, p will blow-up in finite time for any data large or small as long as the data meets the following sufficiency requirement*

$$\frac{(p_0)^{\frac{1}{\beta}}}{(v_0)} > \left(\frac{\beta\left[\frac{w_2a_1}{a_1d_1} - \delta\right]}{c(v_{min})^\beta}\right)^{\frac{1}{\beta}}.$$

Proof. WLOG, we consider the case $b = 0$, if $b > 0$, and there are predator interference effects, then the propensity to blow-up is increased, see Theorem 3.3 [26]. We consider the following function,

$$\Phi(t) = \frac{p^\alpha}{v^\beta},\ \alpha > 0, \beta > 0.$$

Standard estimates upon taking $\alpha = 1$,

$$\frac{d}{dt}\Phi = \left[c + \left(\frac{\beta w_3}{v+d_2} - \frac{w_4}{v+d_3}\right)\right]v^\beta\Phi^2 + \beta\left[\delta - \frac{w_2u}{iu^2+d_1}\right]\Phi.$$

Suppose that $d_2 > d_3$, we choose $\beta > \frac{w_4}{w_3}\frac{D_2}{D_3}$, as this yields

$$\beta > \frac{w_4}{w_3}\frac{d_2}{d_3} > \frac{w_4}{w_3}\left(\frac{v+d_2}{v+d_3}\right).$$

On the other hand if $d_3 > d_2$, we choose $\beta > \frac{w_4}{w_3}\frac{d_3}{d_2}$, as this yields

$$\beta > \frac{w_4}{w_3}\frac{d_3}{d_2} > \frac{w_4}{w_3}\left(\frac{v+d_2}{v+d_3}\right).$$

In either case, an appropriate choice of β yields,

$$\left(\frac{\beta w_3}{v+d_2} - \frac{w_4}{v+d_3}\right) > 0.$$

Note from feasibility $\delta < \frac{w_2u^*}{i(u^*)^2+d_1}$, see [12]. Thus, since u is bounded by $\frac{a_0}{a_1}$, standard estimates yield,

$$\delta < \frac{w_2u^*}{i(u^*)^2+d_1} < \frac{w_2u^*}{d_1} < \frac{w_2a_0}{a_1d_1}.$$

Thus by choosing β appropriately we have

$$\frac{d}{dt}\Phi \ge cv^\beta\Phi^2 + \beta\left[\delta - \frac{w_2u}{iu^2+d_1}\right]\Phi > cv^\beta\Phi^2 - \beta\left(\frac{w_2a_0}{a_1d_1} - \delta\right)\Phi.$$

This yields

$$\frac{d}{dt}\Phi > c(v_{min})^\beta\Phi^2 - \beta\left(\frac{w_2a_0}{a_1d_1} - \delta\right)\Phi.$$

Thus we have finite time blow-up for any data large or small, as long as,

$$\Phi(0) > \frac{\beta\left(\frac{w_2a_0}{a_1d_1} - \delta\right)}{c(v_{min})^\beta}.$$

□

2.3. Alternate proof for finite time blow-up

Theorem 2.3. *Consider the three species food chain model (1)–(3), and any $0 < \delta_1 < c$. Even if $c < \frac{w_4}{\eta}$, p will blow-up in finite time for any data large or small as long as the data meets the following sufficiency requirement,*

$$p_0\ln\left(\frac{v_0}{\frac{w_4}{c-\delta_1} - d_3}\right) > \frac{\delta + \frac{w_3}{b}}{\delta_1},\ v_0 > \frac{w_4}{c-\delta_1} - d_3.$$

Proof. We consider (1)–(3). Note that via simple comparison

$$\frac{dv}{dt} = -\delta v + w_2\left(\frac{uv}{iu^2+d_1}\right) - w_3\left(\frac{vp}{v+bp+d_2}\right) \ge -\delta v - w_3\left(\frac{v}{b}\right). \tag{9}$$

Thus

$$v \ge v_0e^{-\left(\delta + \frac{w_3}{b}\right)t}. \tag{10}$$

Now consider $c < \frac{w_4}{\eta}$, then $c < \frac{w_4}{d_3}$. In order for p to blow-up in finite time it suffices for

$$c - \frac{w_4}{v+d_3} > \delta_1 > 0 \tag{11}$$

for $t \in \left[0, \frac{1}{\delta_1p_0}\right]$. This follows via simple comparison with,

$$\frac{dp}{dt} = \delta_1p^2,\ p(0) = p_0. \tag{12}$$

In order for (11) to hold we use (10) to yield,

$$v \ge v_0e^{-\left(\delta + \frac{w_3}{b}\right)t} > \frac{w_4}{c-\delta_1} - d_3 > \frac{w_4}{c} - d_3 > 0, \tag{13}$$

which implies

$$\frac{1}{\delta + \frac{w_3}{b}}\ln\left(\frac{v_0}{\frac{w_4}{c-\delta_1} - d_3}\right) > t. \tag{14}$$

Now if $t > \frac{1}{\delta_1p_0}$, blow-up is ensured.

Thus if we choose the initial data s.t.,

$$\frac{1}{\delta+\frac{w_3}{b}}\ln\left(\frac{v_0}{\frac{w_4}{c-\delta_1}-d_3}\right)>t>\frac{1}{\delta_1 p_0},$$

we have the following sufficient condition for finite time blow-up of p,

$$p_0\ln\left(\frac{v_0}{\frac{w_4}{c-\delta_1}-d_3}\right)>\frac{\delta+\frac{w_3}{b}}{\delta_1}. \tag{15}$$

□

Remark 1. Theorem 3 in [12] is also incorrect. Therein, via equations (3.16), (3.17) and (3.19) the state variables u, v are required to be bounded. In order to prove a global result, one cannot bound the state variables in the first place - rather one needs to prove these bounds, from the enforced parametric restrictions. In particular, (3.19) in [12] enforces $v\leq\frac{w_4}{c}-d_3$. As we have pointed out in Theorem 2.1, this trivially causes p to decay to 0. This would not occur if $v_0>>1$, which would cause p to blow-up in finite time. Thus there is no global asymptotic stability for (1)–(3).

3. Numerical simulations

We consider the following parameter set,

$$\begin{cases}a_0=0.55, a_1=0.04, d_1=8.7, d_2=3, d_3=4, c=0.05, i=0.001,\\ b=0.025, w_1=0.6, w_2=0.25, w_3=0.7, w_4=11.7, \delta=0.1.\end{cases} \tag{16}$$

We choose the following initial condition [1, 5, 5]. It can easily be seen via Theorem 1.1, this IC lies in the claimed invariant set Λ. We check this next,

$$\begin{aligned}&w_4=11.7>11.6970=c\eta, u(t)=1\leq\frac{a_0}{a_1}=13.75,\\ &0\leq S_1(t)=13\leq 486.4063=\frac{a_0}{a_1}+\frac{a_0^2}{4\delta a_1^2},\\ &0\leq S_2(t)=26.3581\leq 1321.2=\frac{a_0}{a_1}+\frac{a_0^2}{4\delta a_1^2}+\frac{M}{\delta}.\end{aligned} \tag{17}$$

The above calculations show that, the IC [1, 5, 5] is in Λ.

We consider another parameter set,

$$\begin{cases}a_0=0.7, a_1=0.82, d_1=10, d_2=10, d_3=1.3, c=1.2, i=0.1,\\ b=0.024, w_1=0.06, w_2=0.5, w_3=0.2, w_4=20, \delta=0.1,\end{cases} \tag{18}$$

with different ICs [0.1, 0.1, 0.1], [0.85, 15.18, 282.3] and [0.853658536585366, 18, 282.3].

Simple calculations as in (17) show that the ICs [0.1, 0.1, 0.1] and [0.85, 15.18, 282.3] are in Λ but [0.853658536585366, 18, 282.3] is *not*.

4. Conclusion

In the current manuscript we show finite time blow-up in (1)–(3). Thus the claimed invariant globally attracting set for (1)–(3), Λ, as constructed in [12], is not globally attracting. It is invariant, only to contain boundary equilibrium or closed loops in the $v-p$. We caution against trying to prove global boundedness results in systems, where the top predator is modeled using the modified Leslie-Gower scheme. A much more promising research direction for such systems is the restriction of initial data that will lead to globally existing solutions, or damping mechanisms that would yield the same, such as [27]. Furthermore, the construction of an invariant set that could contain feasible interior equilibrium solutions, remains an open challenge for the type of system considered here, or three species models where the top predator is modeled via the modified Leslie-Gower scheme in general, in both the ODE or PDE cases.

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