Local well-posedness of the Hall-MHD system in $H^s(\mathbb{R}^n)$ with $s > \frac{n}{2}$

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We establish local well-posedness of the Hall-magneto-hydrodynamics (Hall-MHD) system in the Sobolev space $(H^s(\mathbb{R}^n))^2$ with $s > \frac{n}{2}$, $n \geq 2$. The previously known local well-posedness Sobolev space was $(H^s(\mathbb{R}^n))^2$ with $s > \frac{n}{2} + 1$. Thus the result presented here is an improvement. Moreover, we show that the solution of the Hall-MHD system in the space $(H^s(\mathbb{R}^n))^2$ with $s > \frac{n}{2}$ converges to a solution of the MHD system when the Hall effect coefficient goes to zero.

1 Introduction

Considered here is the incompressible Hall-magneto-hydrodynamics (Hall-MHD) system with fractional magnetic diffusion:

$$\begin{aligned} u_t + u \cdot \nabla u - b \cdot \nabla b + \nabla p = \nu \Delta u, \\ b_t + u \cdot \nabla b - b \cdot \nabla u + \eta \nabla \times ((\nabla \times b) \times b) = -\mu(-\Delta)^\alpha b, \\ \nabla \cdot u = 0, \end{aligned} \tag{1}$$

with $(x, t) \in \mathbb{R}^n \times [0, \infty)$, $n \geq 2$, and initial conditions

$$u(x, 0) = u_0(x), \qquad b(x, 0) = b_0(x), \qquad \nabla \cdot u_0 = \nabla \cdot b_0 = 0. \tag{2}$$

In the system, u is the fluid velocity, scalar function p is the pressure and b is the magnetic field. The constants ν, μ and η denote the kinematic viscosity, the reciprocal of the magnetic Reynolds number and the Hall effect coefficient, respectively. We limit ourselves to the viscous resistive case, $\nu > 0$, $\mu > 0$, and $\alpha > \frac{1}{2}$. The Hall effect parameter η represents ion inertial length scale, see [12], at which ions decouple from electrons. It is natural to say that η is bounded from above. The Hall nonlinearity $\nabla \times ((\nabla \times b) \times b)$ is the only difference between the Hall-MHD and the usual MHD systems. For mathematical study of this model, we refer to [1, 3, 4, 5, 7, 8, 9, 10] and reference therein.

The purpose of this paper is to find the largest possible Sobolev spaces where the Hall-MHD system is locally well-posed. Previously, it was shown in [7] that system (1) with $\alpha = 1$ is locally well-posed in $\left(H^s(\mathbb{R}^3)\right)^2$ with $s > \frac{5}{2}$. Later, in the case of $\frac{1}{2} < \alpha < 1$, local well-posedness was obtained in $(H^s(\mathbb{R}^n))^2$ with $s > \frac{n}{2} + 1$. We aim to improve the aforementioned findings and establish the main result below.

Theorem 1.1 *Let $\nu, \mu > 0$ and $\alpha > \frac{1}{2}$. Assume $(u_0, b_0) \in (H^s(\mathbb{R}^n))^2$ with $s > 2 - 2\alpha + \frac{n}{2}$ and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. There exists a time $T = T(\|u_0\|_{H^s}, \|b_0\|_{H^s}) > 0$ and a unique solution (u, b) of (1) on $[0, T]$ such that*

$$(u, b) \in (C([0, T]; H^s(\mathbb{R}^n)))^2 .$$

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Remark 1.2. Notice that $s > 2 - 2\alpha + \frac{n}{2} = \frac{n}{2}$ for $\alpha = 1$; and $2 - 2\alpha + \frac{n}{2} < \frac{n}{2} + 1$ for $\frac{1}{2} < \alpha < 1$. Thus for the Hall-MHD system (1) with $\alpha = 1$, we obtain the local well-posedness in $(H^s(\mathbb{R}^n))^2$ with $s > \frac{n}{2}$, which is a lager space than $\left(H^{5/2}(\mathbb{R}^3)\right)^2$ for $n = 3$.

Remark 1.3. We shall also show that the solution obtained in Theorem 1.1 converges to a solution of the MHD system when $\eta \to 0$.

The techniques involved are based on the Littlewood-Paley decomposition theory and the frequency-localization approach.

Notation. For the sake of brevity, we denote by: $A \lesssim B$ an estimate of the form $A \leq CB$ with an absolute constant C; $A \sim B$ an estimate of the form $C_1 B \leq A \leq C_2 B$ with absolute constants C_1, C_2; $\|\cdot\|_p$ the norm of space L^p; and $(\cdot,\cdot)$ the L^2-inner product. The notations associated with Littlewood-Paley decomposition theory and related concepts are introduced in Appendix.

2 A priori estimate

The core of the proof of local well-posedness is the a priori estimate satisfied by smooth solutions in H^s with $s > 2-2\alpha+\frac{n}{2}$, which is the content of this section. The local existence of smooth solutions will then follow from certain traditional approximating and limiting process. The uniqueness and continuous dependance on initial data can be also obtained through standard arguments. Thus, we only show the following statement.

Theorem 2.1 *Let $(u_0, b_0) \in (H^s(\mathbb{R}^n))^2$ with $s > 2 - 2\alpha + \frac{n}{2}$ and (u, b) be a smooth solution of (1) starting from the data (u_0, b_0). There exists a time $T = T(\|u_0\|_{H^s}, \|b_0\|_{H^s}) > 0$, such that, for every $t \in [0, T]$ we have*

$$\|u(t)\|_{H^s}^2 + \|b(t)\|_{H^s}^2 \leq C\left(\|u_0\|_{H^s}^2 + \|b_0\|_{H^s}^2\right),$$

where the constant C depends on T, ν, μ, η, $\|u_0\|_{H^s}$, and $\|b_0\|_{H^s}$, and does not blow up as $\eta \to 0$

Proof: The main argument of establishing the a priori estimate relies on identifying the Sobelev norm $\dot{H}^s$ by the Besov norm $\dot{B}^s_{2,2}$, and then combining with the basic energy estimate in L^2. In order to estimate the Besov norm $\dot{B}^s_{2,2}$, we shall encounter several flux terms from the five nonlinear terms in the equations. The most difficult term is the one from the Hall nonlinearity $\nabla \times ((\nabla \times b) \times b)$; thus the major effort will be put on estimating the flux from the Hall term. Besides, to show how the cancellations are exploited and how the optimization is carried out in the estimates of the rest nonlinear terms, we shall focus on the flux from $(b \cdot \nabla)b$ and $(u \cdot \nabla)b$.

Multiplying the first equation of (1) by $\lambda_q^{2s}\Delta_q^2 u$ and the second one by $\lambda_q^{2s}\Delta_q^2 b$, and taking summation for all $q \geq -1$ gives us

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\sum_{q\geq -1}\left(\lambda_q^{2s}\|u_q\|_2^2 + \lambda_q^{2s}\|b_q\|_2^2\right) \\ &\leq -\nu\sum_{q\geq -1}\lambda_q^{2s+2}\|u_q\|_2^2 - \mu\sum_{q\geq -1}\lambda_q^{2s+2\alpha}\|b_q\|_2^2 + I_1 + I_2 + I_3 + I_4 + \eta I_5, \end{aligned} \tag{3}$$

with

$$I_1 = -\sum_{q\geq -1}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(u\cdot\nabla u)\cdot u_q\,dx, \qquad I_2 = \sum_{q\geq -1}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(b\cdot\nabla b)\cdot u_q\,dx,$$

$$I_3 = -\sum_{q\geq -1}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(u\cdot\nabla b)\cdot b_q\,dx, \qquad I_4 = \sum_{q\geq -1}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(b\cdot\nabla u)\cdot b_q\,dx,$$

$$I_5 = \sum_{q\geq -1}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q((\nabla\times b)\times b)\cdot\nabla\times b_q\,dx.$$

As expected, the estimate of I_1, I_2, I_3, and I_4 are less challenging than that of I_5. On the other hand, due to the similarity of I_1 and I_3, I_2 and I_4, we shall only show the details of handling I_3 and I_2, not I_1 and I_4.

We first decompose I_3 by adapting Bony's paraproduct (18)

$$
\begin{aligned}
I_3 =& -\sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(u_{\leq p-2}\cdot\nabla b_p)\cdot b_q\,dx\\
&-\sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(u_p\cdot\nabla b_{\leq p-2})\cdot b_q\,dx\\
&-\sum_{q\geq -1}\sum_{p\geq q-2}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(u_p\cdot\nabla\tilde{b}_p)\cdot b_q\,dx\\
=&I_{31}+I_{32}+I_{33};
\end{aligned}
$$

and then by commutator (19) to rewrite I_{31}

$$
\begin{aligned}
I_{31} =& -\sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s}\int_{\mathbb{R}^3}[\Delta_q, u_{\leq p-2}\cdot\nabla]b_p\cdot b_q\,dx\\
&-\sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s}\int_{\mathbb{R}^3}(u_{\leq q-2}\cdot\nabla\Delta_q b_p)\cdot b_q\,dx\\
&-\sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s}\int_{\mathbb{R}^3}((u_{\leq p-2}-u_{\leq q-2})\cdot\nabla\Delta_q b_p)\cdot b_q\,dx\\
=&I_{311}+I_{312}+I_{313}.
\end{aligned}
$$

Since $\sum_{|p-q|\leq 2}\Delta_q b_p = b_q$ and $\nabla\cdot u_{\leq q-2}=0$, one can infer $I_{312}=0$.

To estimate I_{311}, we proceed with an optimization strategy. It follows from the commutator estimate in Lemma 4.2, Hölder's inequality, and Bernstein's inequality that

$$
\begin{aligned}
|I_{311}| \leq& \sum_{q\geq -1}\sum_{|p-q|\leq 2}\lambda_q^{2s}\|\nabla u_{\leq p-2}\|_\infty\|b_p\|_2\|b_q\|_2\\
\lesssim& \sum_{q\geq -1}\lambda_q^{2s}\|b_q\|_2^2\sum_{p\leq q}\lambda_p^{1+\frac{n}{2}}\|u_p\|_2\\
\lesssim& \sum_{q\geq -1}\sum_{p\leq q}\lambda_{p-q}^{\delta\alpha}\lambda_p^{1+\frac{n}{2}-\delta-\delta\alpha-s}\left(\lambda_q^{s+\alpha}\|b_q\|_2\right)^\delta\left(\lambda_q^{s}\|b_q\|_2\right)^{2-\delta}\\
&\cdot\left(\lambda_p^{s+1}\|u_p\|_2\right)^\delta\left(\lambda_p^{s}\|u_p\|_2\right)^{1-\delta}\\
\lesssim& \sum_{q\geq -1}\sum_{p\leq q}\lambda_{p-q}^{\delta\alpha}\left(\lambda_q^{s+\alpha}\|b_q\|_2\right)^\delta\left(\lambda_q^{s}\|b_q\|_2\right)^{2-\delta}\left(\lambda_p^{s+1}\|u_p\|_2\right)^\delta\left(\lambda_p^{s}\|u_p\|_2\right)^{1-\delta}
\end{aligned}
$$

for some parameter $0<\delta<1$ satisfying

$$
s\geq 1+\frac{n}{2}-\delta-\delta\alpha. \tag{4}
$$

We continue the estimate of I_{311} by using Young's inequality with parameters satisfying

$$
\begin{aligned}
&\frac{1}{\delta_1}+\frac{1}{\delta_2}+\frac{1}{\delta_3}+\frac{1}{\delta_4}=\delta\alpha,\ \ 0<\delta_1,\delta_2,\delta_3,\delta_4<1\\
&\frac{1}{\theta_1}+\frac{1}{\theta_2}+\frac{1}{\theta_3}+\frac{1}{\theta_4}=1,\ \ \theta_1=\theta_3=\frac{2}{\delta},\ \ 1<\theta_2,\theta_4<\infty.
\end{aligned} \tag{5}
$$

It then follows that

$$
\begin{aligned}
|I_{311}| \le & \frac{\mu}{16}\sum_{q\ge -1}\sum_{p\le q}\lambda_{p-q}^{\delta_1\theta_1}\lambda_q^{2s+2\alpha}\|b_q\|_2^2 + C_{\nu,\mu}\sum_{q\ge -1}\sum_{p\le q}\lambda_{p-q}^{\delta_2\theta_2}\left(\lambda_q^s\|b_q\|_2\right)^{(2-\delta)\theta_2} \\
& + \frac{\nu}{16}\sum_{q\ge -1}\sum_{p\le q}\lambda_{p-q}^{\delta_3\theta_3}\lambda_p^{2s+2}\|u_p\|_2^2 + C_{\nu,\mu}\sum_{q\ge -1}\sum_{p\le q}\lambda_{p-q}^{\delta_4\theta_4}\left(\lambda_q^s\|u_p\|_2\right)^{(1-\delta)\theta_4} \\
\le & \frac{\mu}{16}\sum_{q\ge -1}\lambda_q^{2s+2\alpha}\|b_q\|_2^2 + \frac{\nu}{16}\sum_{q\ge -1}\lambda_q^{2s+2}\|u_q\|_2^2 \\
& + C_{\nu,\mu}\sum_{q\ge -1}\left(\lambda_q^s\|b_q\|_2\right)^{(2-\delta)\theta_2} + C_{\nu,\mu}\sum_{q\ge -1}\left(\lambda_q^s\|u_q\|_2\right)^{(1-\delta)\theta_4},
\end{aligned}
$$

with various constants $C_{\nu,\mu}$ that depend on ν,μ and tend to infinity as $\nu,\mu\to 0$. We pause to analyze the parameters. In view of (4) and (5), we obtain that

$$
s \ge \frac{n}{2}-\alpha+\left(\frac{1}{\theta_2}+\frac{1}{\theta_4}\right)(1+\alpha) \ge \frac{n}{2}-\alpha+\epsilon \tag{6}
$$

provided θ_2 and θ_4 are large enough.

Other terms in I_3 are simpler and can be estimated in an analogous way; thus the details are omitted. As a conclusion, we have for s satisfying (6)

$$
\begin{aligned}
|I_3| \le & \frac{\mu}{8}\sum_{q\ge -1}\lambda_q^{2s+2\alpha}\|b_q\|_2^2 + \frac{\nu}{8}\sum_{q\ge -1}\lambda_q^{2s+2}\|u_q\|_2^2 \\
& + C_{\nu,\mu}\left(\sum_{q\ge -1}\lambda_q^{2s}\|b_q\|_2^2\right)^{\gamma_1} + C_{\nu,\mu}\left(\sum_{q\ge -1}\lambda_q^{2s}\|u_q\|_2^2\right)^{\gamma_2},
\end{aligned} \tag{7}
$$

with certain constants $\gamma_1,\gamma_2>1$.

Adapting the same decomposition strategy of using Bony's paraproduct and commutator, we deconstruct I_2 and I_4 as follows

$$
\begin{aligned}
I_2 = & \sum_{q\ge -1}\sum_{|q-p|\le 2}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(b_{\le p-2}\cdot\nabla b_p)\cdot u_q\,dx \\
& + \sum_{q\ge -1}\sum_{|q-p|\le 2}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(b_p\cdot\nabla b_{\le p-2})\cdot u_q\,dx \\
& + \sum_{q\ge -1}\sum_{p\ge q-2}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(b_p\cdot\nabla\tilde{b}_p)\cdot u_q\,dx \\
= & I_{21}+I_{22}+I_{23},
\end{aligned}
$$

with

$$
\begin{aligned}
I_{21} = & \sum_{q\ge -1}\sum_{|q-p|\le 2}\lambda_q^{2s}\int_{\mathbb{R}^3}[\Delta_q, b_{\le p-2}\cdot\nabla]b_p\cdot u_q\,dx \\
& + \sum_{q\ge -1}\sum_{|q-p|\le 2}\lambda_q^{2s}\int_{\mathbb{R}^3}(b_{\le q-2}\cdot\nabla\Delta_q b_p)\cdot u_q\,dx \\
& + \sum_{q\ge -1}\sum_{|q-p|\le 2}\lambda_q^{2s}\int_{\mathbb{R}^3}((b_{\le p-2}-b_{\le q-2})\cdot\nabla\Delta_q b_p)\cdot u_q\,dx \\
= & I_{211}+I_{212}+I_{213};
\end{aligned}
$$

and

$$
\begin{aligned}
I_4 = & \sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(b_{\leq p-2}\cdot\nabla u_p)\cdot b_q\,dx \\
& + \sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(b_p\cdot\nabla u_{\leq p-2})\cdot b_q\,dx \\
& + \sum_{q\geq -1}\sum_{p\geq q-2}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(\tilde{b}_p\cdot\nabla u_p)\cdot b_q\,dx \\
= & I_{41}+I_{42}+I_{43},
\end{aligned}
$$

with

$$
\begin{aligned}
I_{41} = & \sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s}\int_{\mathbb{R}^3}[\Delta_q, b_{\leq p-2}\cdot\nabla]u_p\cdot b_q\,dx \\
& + \sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s}\int_{\mathbb{R}^3}(b_{\leq q-2}\cdot\nabla\Delta_q u_p)\cdot b_q\,dx \\
& + \sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s}\int_{\mathbb{R}^3}((b_{\leq p-2}-b_{\leq q-2})\cdot\nabla\Delta_q u_p)\cdot b_q\,dx \\
= & I_{411}+I_{412}+I_{413}.
\end{aligned}
$$

We claim that $I_{212}+I_{412}=0$. Indeed, we have

$$
\begin{aligned}
I_{212}+I_{412} = & \sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s}\int_{\mathbb{R}^3}(b_{\leq q-2}\cdot\nabla\Delta_q b_p)\cdot u_q\,dx \\
& + \sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s}\int_{\mathbb{R}^3}(b_{\leq q-2}\cdot\nabla\Delta_q u_p)\cdot b_q\,dx \\
= & \sum_{q\geq -1}\lambda_q^{2s}\int_{\mathbb{R}^3}(b_{\leq q-2}\cdot\nabla b_q)\cdot u_q\,dx + \sum_{q\geq -1}\lambda_q^{2s}\int_{\mathbb{R}^3}(b_{\leq q-2}\cdot\nabla u_q)\cdot b_q\,dx \\
= & 0.
\end{aligned}
$$

The fact $\sum_{|p-q|\leq 2}\Delta_q b_p = b_q$ and $\sum_{|p-q|\leq 2}\Delta_q u_p = u_q$ justifies the second equality above.

The rest terms in I_2+I_4 are relatively simple. We only choose one representative term, I_{211}, to carry out the details of estimating. Applying Hölder's inequality and Bernstein's inequality leads to

$$
\begin{aligned}
|I_{211}| \leq & \sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s}\|\nabla b_{\leq p-2}\|_\infty\|b_p\|_2\|u_q\|_2 \\
\lesssim & \sum_{q\geq -1}\lambda_q^{2s}\|b_q\|_2\|u_q\|_2\sum_{p\leq q}\lambda_p^{1+\frac{n}{2}}\|b_p\|_2 \\
= & \sum_{q\geq -1}\sum_{p\leq q}\lambda_{p-q}^{\delta_1\alpha+\delta_2}\lambda_p^{1+\frac{n}{2}-\delta_1\alpha-\delta_3\alpha-\delta_2-s}\left(\lambda_q^{s+\alpha}\|b_q\|_2\right)^{\delta_1}\left(\lambda_q^{s}\|b_q\|_2\right)^{1-\delta_1} \\
& \cdot\left(\lambda_q^{s+1}\|u_q\|_2\right)^{\delta_2}\left(\lambda_q^{s}\|u_q\|_2\right)^{1-\delta_2}\left(\lambda_p^{s+\alpha}\|b_p\|_2\right)^{\delta_3}\left(\lambda_p^{s}\|b_p\|_2\right)^{1-\delta_3} \\
\leq & C\sum_{q\geq -1}\sum_{p\leq q}\lambda_{p-q}^{\delta_1\alpha+\delta_2}\left(\lambda_q^{s+\alpha}\|b_q\|_2\right)^{\delta_1}\left(\lambda_q^{s}\|b_q\|_2\right)^{1-\delta_1}\cdot\left(\lambda_q^{s+1}\|u_q\|_2\right)^{\delta_2} \\
& \left(\lambda_q^{s}\|u_q\|_2\right)^{1-\delta_2}\left(\lambda_p^{s+\alpha}\|b_p\|_2\right)^{\delta_3}\left(\lambda_p^{s}\|b_p\|_2\right)^{1-\delta_3}
\end{aligned}
$$

for parameters $0<\delta_1,\delta_2,\delta_3<1$, $\delta_2=(2-\delta_1-\delta_2)\alpha$, and

$$
s\geq 1+\frac{n}{2}-\delta_1\alpha-\delta_3\alpha-\delta_2. \tag{8}
$$

Adapting Young's inequality with parameters ζ_i, $1 \leq i \leq 6$, such that

$$\begin{aligned} \zeta_1+\zeta_2+\zeta_3+\zeta_4+\zeta_5+\zeta_6=\delta_1\alpha+\delta_2,\ \ \zeta_1,...,\zeta_6>0 \\ \frac{1}{\theta_1}+\frac{1}{\theta_2}+\frac{1}{\theta_3}+\frac{1}{\theta_4}+\frac{1}{\theta_5}+\frac{1}{\theta_6}=1, \\ \theta_1=\frac{2}{\delta_1},\ \theta_3=\frac{2}{\delta_2},\ \theta_5=\frac{2}{\delta_3},\ 1<\theta_2,\theta_4,\theta_6<\infty \end{aligned} \tag{9}$$

we have

$$\begin{aligned} |I_{211}| \leq & \frac{\mu}{8}\sum_{q\geq -1}\lambda_q^{2s+2\alpha}\|b_q\|_2^2+\frac{\nu}{16}\sum_{q\geq -1}\lambda_q^{2s+2}\|u_q\|_2^2+C_{\nu,\mu}\sum_{q\geq -1}\left(\lambda_q^s\|b_q\|_2\right)^{(1-\delta_1)\theta_2} \\ & + C_{\nu,\mu}\sum_{q\geq -1}\left(\lambda_q^s\|b_q\|_2\right)^{(1-\delta_3)\theta_6}+C_{\nu,\mu}\sum_{q\geq -1}\left(\lambda_q^s\|u_q\|_2\right)^{(1-\delta_2)\theta_4}. \end{aligned}$$

Again, the parameter constraints (8) and (9) imply that

$$\begin{aligned} s \geq & 1+\frac{n}{2}-2\alpha+(\alpha-1)\delta_2+2\alpha\left(\frac{1}{\theta_2}+\frac{1}{\theta_4}+\frac{1}{\theta_6}\right) \\ = & 1+\frac{n}{2}-2\alpha+(\alpha-1)\delta_2+\epsilon \end{aligned}$$

for large enough θ_2, θ_4, and θ_6. Notice that $s \geq \frac{n}{2}-1+\epsilon$ for $\alpha = 1$. In general for δ_2 close enough to 1, we have

$$s \geq \frac{n}{2}-\alpha+\epsilon. \tag{10}$$

To conclude, we expect to have for s satisfying (10)

$$\begin{aligned} |I_2| \leq & \frac{\mu}{8}\sum_{q\geq -1}\lambda_q^{2s+2\alpha}\|b_q\|_2^2+\frac{\nu}{16}\sum_{q\geq -1}\lambda_q^{2s+2}\|u_q\|_2^2 \\ & + C_{\nu,\mu}\left(\sum_{q\geq -1}\lambda_q^{2s}\|b_q\|_2^2\right)^{\gamma_1}+C_{\nu,\mu}\left(\sum_{q\geq -1}\lambda_q^{2s}\|u_q\|_2^2\right)^{\gamma_2}, \end{aligned} \tag{11}$$

for some constants γ_1, γ_2.

Now we are left to estimate I_5. By Bony's paraproduct and commutator (21), the routine decomposition procedure yields

$$\begin{aligned} I_5 = & \sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(b_{\leq p-2}\times(\nabla\times b_p))\cdot\nabla\times b_q\,dx \\ & + \sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(b_p\times(\nabla\times b_{\leq p-2}))\cdot\nabla\times b_q\,dx \\ & + \sum_{q\geq -1}\sum_{p\geq q-2}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(b_p\times(\nabla\times\tilde{b}_p))\cdot\nabla\times b_q\,dx \\ = & I_{51}+I_{52}+I_{53}; \end{aligned}$$

with

$$\begin{aligned} I_{51} = & \sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s}\int_{\mathbb{R}^3}[\Delta_q, b_{\leq p-2}\times\nabla\times]b_p\cdot\nabla\times b_q\,dx \\ & + \sum_{q\geq -1}\lambda_q^{2s}\int_{\mathbb{R}^3}b_{\leq q-2}\times(\nabla\times b_q)\cdot\nabla\times b_q\,dx \\ & + \sum_{q\geq -1}\sum_{|p-q|\leq 2}\lambda_q^{2s}\int_{\mathbb{R}^3}(b_{\leq p-2}-b_{\leq q-2})\times(\nabla\times(b_p)_q)\cdot\nabla\times b_q\,dx \\ = & I_{511}+I_{512}+I_{513}. \end{aligned}$$

The cross product property implies immediately that $I_{512}=0$. We deduce from the commutator estimate in Lemma 4.3 that

$$\begin{aligned}
|I_{511}| \lesssim & \sum_{q\geq -1}\sum_{|p-q|\leq 2}\lambda_q^{2s+1}\|\nabla b_{\leq p-2}\|_\infty\|b_p\|_2\|b_q\|_2\\
\lesssim & \sum_{q\geq -1}\lambda_q^{2s+1}\|b_q\|_2^2\sum_{p\leq q}\lambda_p^{1+\frac{n}{2}}\|b_p\|_2\\
= & \sum_{q\geq -1}\sum_{p\leq q}\lambda_{p-q}^{\delta_1\alpha-1}\lambda_p^{2+\frac{n}{2}-\delta_1\alpha-\delta_2\alpha-s}\left(\lambda_q^{s+\alpha}\|b_q\|_2\right)^{\delta_1}\left(\lambda_q^{s}\|b_q\|_2\right)^{2-\delta_1}\\
& \cdot\left(\lambda_p^{s+\alpha}\|b_p\|_2\right)^{\delta_2}\left(\lambda_p^{s}\|b_p\|_2\right)^{1-\delta_2}\\
\leq & C\sum_{q\geq -1}\sum_{p\leq q}\lambda_{p-q}^{\delta_1\alpha-1}\left(\lambda_q^{s+\alpha}\|b_q\|_2\right)^{\delta_1}\left(\lambda_q^{s}\|b_q\|_2\right)^{2-\delta_1}\left(\lambda_p^{s+\alpha}\|b_p\|_2\right)^{\delta_2}\left(\lambda_p^{s}\|b_p\|_2\right)^{1-\delta_2}
\end{aligned}$$

for parameters satisfying $\frac{1}{\alpha}<\delta_1<2$, $0<\delta_2<1$, and

$$s\geq 2+\frac{n}{2}-\delta_1\alpha-\delta_2\alpha. \tag{12}$$

By Young's inequality we have for the parameters

$$\begin{gathered}
\zeta_1+\zeta_2+\zeta_3+\zeta_4=\delta_1\alpha-1,\;\;\zeta_1,...,\zeta_4>0\\
\frac{1}{\theta_1}+\frac{1}{\theta_2}+\frac{1}{\theta_3}+\frac{1}{\theta_4}=1,\;\;\theta_1=\frac{2}{\delta_1},\;\;\theta_3=\frac{2}{\delta_2},\;\;1<\theta_2,\theta_4<\infty,
\end{gathered} \tag{13}$$

such that

$$\begin{aligned}
|I_{511}| \leq & \frac{\mu}{16\eta}\sum_{q\geq -1}\sum_{p\leq q}\lambda_{p-q}^{\zeta_1\theta_1}\lambda_q^{2s+2\alpha}\|b_q\|_2^2+C_\mu\eta\sum_{q\geq -1}\sum_{p\leq q}\lambda_{p-q}^{\zeta_2\theta_2}\left(\lambda_q^{s}\|b_q\|_2\right)^{(2-\delta_1)\theta_2}\\
& +\frac{\mu}{16\eta}\sum_{q\geq -1}\sum_{p\leq q}\lambda_{p-q}^{\zeta_3\theta_3}\lambda_p^{2s+2\alpha}\|b_p\|_2^2+C_\mu\eta\sum_{q\geq -1}\sum_{p\leq q}\lambda_{p-q}^{\zeta_4\theta_4}\left(\lambda_p^{s}\|b_p\|_2\right)^{(1-\delta_2)\theta_4}\\
\leq & \frac{\mu}{8\eta}\sum_{q\geq -1}\lambda_q^{2s+2\alpha}\|b_q\|_2^2+C_\mu\eta\sum_{q\geq -1}\left(\lambda_q^{s}\|b_q\|_2\right)^{(2-\delta_1)\theta_2}\\
& +C_\mu\eta\sum_{q\geq -1}\left(\lambda_p^{s}\|b_p\|_2\right)^{(1-\delta_2)\theta_4}.
\end{aligned}$$

Regarding the parameters, (12) and (13) imply that

$$s\geq\frac{n}{2}+2-2\alpha+2\alpha\left(\frac{1}{\theta_2}+\frac{1}{\theta_4}\right)\geq\frac{n}{2}+2-2\alpha+\epsilon \tag{14}$$

for large enough θ_2 and θ_4.

By Hölder's inequality,

$$
\begin{aligned}
|I_{513}| \leq & \sum_{q\geq -1}\sum_{|p-q|\leq 2}\lambda_q^{2s}\int_{\mathbb{R}^3}\left|(b_{\leq p-2}-b_{\leq q-2})\times(\nabla\times(b_p)_q)\cdot\nabla\times b_q\right|\,dx\\
\lesssim & \sum_{q\geq -1}\sum_{|p-q|\leq 2}\lambda_q^{2s}\|\nabla b_q\|_\infty\|b_{\leq p-2}-b_{\leq q-2}\|_2\|\nabla b_p\|_2\\
\lesssim & \sum_{q\geq -1}\lambda_q^{2s+2+\frac{n}{2}}\|b_q\|_2^3\\
=& C\sum_{q\geq -1}\lambda_q^{2+\frac{n}{2}-\delta\alpha-s}\left(\lambda_q^{s+\alpha}\|b_q\|_2\right)^{\delta}\left(\lambda_q^{s}\|b_q\|_2\right)^{3-\delta}\\
\leq & C\sum_{q\geq -1}\left(\lambda_q^{s+\alpha}\|b_q\|_2\right)^{\delta}\left(\lambda_q^{s}\|b_q\|_2\right)^{3-\delta}\\
\leq & \frac{\mu}{16\eta}\sum_{q\geq -1}\lambda_q^{2s+2\alpha}\|b_q\|_2^2+C_\mu\eta\sum_{q\geq -1}\left(\lambda_q^{s}\|b_q\|_2\right)^{\frac{2(3-\delta)}{2-\delta}}
\end{aligned}
$$

for $0<\delta<2$ and $s\geq 2+\frac{n}{2}-\delta\alpha>2+\frac{n}{2}-2\alpha$.

We continue to I_{52} and decompose it by adapting commutator (22),

$$
\begin{aligned}
I_{52}=&\sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(\nabla\times b_{\leq p-2}\times b_p)\cdot\nabla\times b_q\,dx\\
=&\sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s}\int_{\mathbb{R}^3}[\Delta_q,\nabla\times b_{\leq p-2}\times]b_p\cdot\nabla\times b_q\,dx\\
&+\sum_{q\geq -1}\lambda_q^{2s}\int_{\mathbb{R}^3}\nabla\times b_{\leq q-2}\times b_q\cdot\nabla\times b_q\,dx\\
&+\sum_{q\geq -1}\sum_{|p-q|\leq 2}\lambda_q^{2s}\int_{\mathbb{R}^3}\nabla\times(b_{\leq p-2}-b_{\leq q-2})\times(b_p)_q\cdot\nabla\times b_q\,dx\\
=&I_{521}+I_{522}+I_{523}.
\end{aligned}
$$

We will only show the estimate of I_{522}, since I_{521} enjoys the same estimate as I_{511} due to the commutator estimate in Lemma 4.4 and I_{523} can be estimated as I_{513}. Integration by parts, identity (20) along with the fact that $\nabla\cdot b_q=0$ infers

$$
\begin{aligned}
I_{522}=&\sum_{q\geq -1}\lambda_q^{2s}\int_{\mathbb{R}^3}\nabla\times(\nabla\times b_{\leq q-2}\times b_q)\cdot b_q\,dx\\
=&\sum_{q\geq -1}\lambda_q^{2s}\int_{\mathbb{R}^3}[(b_q\cdot\nabla)\nabla\times b_{\leq q-2}-(\nabla\cdot\nabla\times b_{\leq q-2})b_q]\cdot b_q\,dx\\
&-\sum_{q\geq -1}\lambda_q^{2s}\int_{\mathbb{R}^3}(\nabla\times b_{\leq q-2}\cdot\nabla)b_q\cdot b_q\,dx.
\end{aligned}
$$

Since $\nabla\cdot(\nabla\times b_{\leq q-2})=0$, it is obvious the last integral vanishes. Thus we have

$$
\begin{aligned}
|I_{522}|\leq&\sum_{q\geq -1}\lambda_q^{2s}\int_{\mathbb{R}^3}\left|[(b_q\cdot\nabla)\nabla\times b_{\leq q-2}-(\nabla\cdot\nabla\times b_{\leq q-2})b_q]\cdot b_q\right|\,dx\\
\lesssim&\sum_{q\geq -1}\lambda_q^{2s}\|\nabla^2 b_{\leq q-2}\|_\infty\|b_q\|_2^2\\
\lesssim&\sum_{q\geq -1}\lambda_q^{2s}\|b_q\|_2^2\sum_{p\leq q}\lambda_p^{2+\frac{n}{2}}\|b_p\|_2
\end{aligned}
$$

which share the same estimate of I_{511}.

The last term I_{53} is treated as

$$
\begin{aligned}
|I_{53}| &\le \sum_{q\ge -1}\sum_{p\ge q-2}\lambda_q^{2s}\int_{\mathbb{R}^3}|\Delta_q(b_p\times\nabla\times\tilde b_p)\cdot\nabla\times b_q|\,dx\\
&\lesssim \sum_{q\ge -1}\lambda_q^{2s}\|\nabla b_q\|_\infty\sum_{p\ge q-3}\|b_p\|_2\|\nabla\tilde b_p\|_2\\
&\lesssim \sum_{q\ge -1}\lambda_q^{2s+1+\frac{n}{2}}\|b_q\|_2\sum_{p\ge q-3}\lambda_p\|b_p\|_2^2\\
&\lesssim \sum_{p\ge -1}\lambda_p\|b_p\|_2^2\sum_{q\le p+3}\lambda_q^{2s+1+\frac{n}{2}}\|b_q\|_2
\end{aligned}
$$

which turns out to be similar as I_{511} again. Summarizing the analysis above, we obtain

$$
|I_5|\le\frac{\mu}{4\eta}\sum_{q\ge -1}\lambda_q^{2s+2\alpha}\|b_q\|_2^2+C_\mu\eta\left(\sum_{q\ge -1}\lambda_q^{2s}\|b_q\|_2^2\right)^{\gamma_1}+C_\mu\eta\left(\sum_{q\ge -1}\lambda_q^{2s}\|b_q\|_2^2\right)^{\gamma_2} \tag{15}
$$

for some $\gamma_1,\gamma_2>1$. Putting together of (3), (7), (11), and (15), there exist constants C_ν, C_μ, and $C_{\nu,\mu}$ such that

$$
\begin{aligned}
&\frac{d}{dt}\left(\|u\|_{\dot H^s}^2+\|b\|_{\dot H^s}^2\right)+\nu\sum_{q\ge -1}\lambda_q^{2s+2}\|u_q\|_2^2+\mu\sum_{q\ge -1}\lambda_q^{2s+2\alpha}\|b_q\|_2^2\\
\le&C_\nu\left(\sum_{q\ge -1}\lambda_q^{2s}\|u_q\|_2^2\right)^{\gamma_1}+C_\nu\left(\sum_{q\ge -1}\lambda_q^{2s}\|u_q\|_2^2\right)^{\gamma_2}\\
&+C_\mu\eta^2\left(\sum_{q\ge -1}\lambda_q^{2s}\|b_q\|_2^2\right)^{\gamma_1}+C_\mu\eta^2\left(\sum_{q\ge -1}\lambda_q^{2s}\|b_q\|_2^2\right)^{\gamma_2}\\
\le&C_{\nu,\mu}(1+\eta^2)\left(\|u\|_{\dot H^s}^2+\|b\|_{\dot H^s}^2\right)^{\gamma_1}+C_{\nu,\mu}(1+\eta^2)\left(\|u\|_{\dot H^s}^2+\|b\|_{\dot H^s}^2\right)^{\gamma_2}
\end{aligned} \tag{16}
$$

Notice that $\gamma_1,\gamma_2>1$ and hence the energy inequality (16) is in the type of Riccati. It follows that, there exists a time $T>0$ which depends on ν,μ,η and $\|u_0\|_{H^s},\|b_0\|_{H^s}$ such that

$$
\|u(t)\|_{H^s}^2+\|b(t)\|_{H^s}^2\le C(\nu,\mu,\eta,T,\|u_0\|_{H^s},\|b_0\|_{H^s})\left(\|u_0\|_{H^s}^2+\|b_0\|_{H^s}^2\right)
$$

for $0\le t<T$, and a constant C depending on ν,μ,η,T and $\|u_0\|_{H^s},\|b_0\|_{H^s}$. We note that the constant C does not blow up as $\eta\to 0$.

□

3 Convergence of the Hall-MHD to the MHD system

In this section, we show that solutions (u^η,b^η,p^η) of (1) with $\alpha=1$ in $H^{\frac{n}{2}}$ converges to a solution (u,b,p) of the MHD system, as $\eta\to 0$. Namely, we prove

Theorem 3.1 *Let (u^η,b^η,p^η) be a solution to (1) with $\alpha=1$ obtained in Theorem 1.1 associated with initial data (u_0,b_0). Let (u,b,p) be a solution to (1) with $\eta=0$ and $\alpha=1$ under the same initial data. Then we have*

$$
\lim_{\eta\to 0}(\|u^\eta-u\|_2+\|b^\eta-b\|_2)=0.
$$

Proof: Take the difference $U=u^\eta-u$, $B=b^\eta-b$ and $\pi=p^\eta-p$, which satisfy the equations:

$$
\begin{aligned}
U_t+u\cdot\nabla U-b\cdot\nabla B+U\cdot\nabla u^\eta-B\cdot\nabla b^\eta+\nabla\pi=\nu\Delta U,\\
B_t+u\cdot\nabla B-b\cdot\nabla U+U\cdot\nabla b^\eta-B\cdot\nabla u^\eta-\eta\nabla\times((\nabla\times b^\eta)\times b^\eta)=\mu\Delta B,\\
\nabla\cdot U=0,\quad\nabla\cdot B=0.
\end{aligned} \tag{17}
$$

Multiplying the first equation by U and the second by B, we obtain (formally)

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|U\|_2^2+\nu\|\nabla U\|_2^2\\
=&\int_{\mathbb{R}^3} b\cdot\nabla B\cdot U\,dx-\int_{\mathbb{R}^3} U\cdot\nabla u^\eta\cdot U\,dx+\int_{\mathbb{R}^3} B\cdot\nabla b^\eta\cdot U\,dx,\\
&\frac{1}{2}\frac{d}{dt}\|B\|_2^2+\mu\|\nabla B\|_2^2\\
=&\int_{\mathbb{R}^3} b\cdot\nabla U\cdot B\,dx-\int_{\mathbb{R}^3} U\cdot\nabla b^\eta\cdot B\,dx+\int_{\mathbb{R}^3} B\cdot\nabla u^\eta\cdot B\,dx\\
&+\eta\int_{\mathbb{R}^3}\nabla\times((\nabla\times b^\eta)\times b^\eta)\cdot B\,dx.
\end{aligned}$$

Adding the two yields, provided that (u^η, b^η, p^η) and (u, b, p) are regular enough,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|U\|_2^2+\|B\|_2^2\right)+\nu\|\nabla U\|_2^2+\mu\|\nabla B\|_2^2\\
=&-\int_{\mathbb{R}^3} U\cdot\nabla u^\eta\cdot U\,dx+\int_{\mathbb{R}^3} B\cdot\nabla b^\eta\cdot U\,dx-\int_{\mathbb{R}^3} U\cdot\nabla b^\eta\cdot B\,dx\\
&+\int_{\mathbb{R}^3} B\cdot\nabla u^\eta\cdot B\,dx+\eta\int_{\mathbb{R}^3}\nabla\times((\nabla\times b^\eta)\times b^\eta)\cdot B\,dx\\
\equiv& I_1+I_2+I_3+I_4+\eta I_5.
\end{aligned}$$

It is straight forward to notice that

$$|I_1+I_2+I_3+I_4|\le C\left(\|\nabla u^\eta\|_\infty+\|\nabla b^\eta\|_\infty\right)\left(\|U\|_2^2+\|B\|_2^2\right);$$

and also

$$\begin{aligned}
|I_1+I_2+I_3+I_4|\le& C(\nu^{-1}+\mu^{-1})\left(\|u^\eta\|_\infty+\|b^\eta\|_\infty\right)\left(\|U\|_2^2+\|B\|_2^2\right)\\
&+\frac{1}{4}\nu\|\nabla U\|_2^2+\frac{1}{4}\mu\|\nabla B\|_2^2.
\end{aligned}$$

We estimate I_5 as

$$\begin{aligned}
\eta|I_5|=&\left|\eta\int_{\mathbb{R}^3}((\nabla\times b^\eta)\times b^\eta)\cdot\nabla\times B\,dx\right|\\
\le& C\eta\|\nabla b^\eta\|_\infty\|b^\eta\|_2\|\nabla B\|_2\\
\le& C\eta^2\mu^{-1}\|\nabla b^\eta\|_\infty^2\|b^\eta\|_2^2+\frac{1}{4}\mu\|\nabla B\|_2^2
\end{aligned}$$

or as

$$\begin{aligned}
\eta|I_5|=&\left|\eta\int_{\mathbb{R}^3}((\nabla\times b^\eta)\times b^\eta)\cdot\nabla\times B\,dx\right|\\
\le& C\eta\|b^\eta\|_\infty\|\nabla b^\eta\|_2\|\nabla B\|_2\\
\le& C\eta^2\mu^{-1}\|b^\eta\|_\infty^2\|\nabla b^\eta\|_2^2+\frac{1}{4}\mu\|\nabla B\|_2^2
\end{aligned}$$

Combining the above estimates leads to, for $s>\frac{n}{2}$

$$\frac{d}{dt}\left(\|U\|_2^2+\|B\|_2^2\right)\le C\left(\|U\|_2^2+\|B\|_2^2\right)+C\eta^2\mu^{-1}\|\nabla b^\eta\|_2^2,$$

from which Grönwall's inequality implies that

$$\|U(t)\|_2^2+\|B(t)\|_2^2\le C\eta^2\mu^{-1}\|\nabla b^\eta\|_2^2+(\|U(0)\|_2^2+\|B(0)\|_2^2+C\eta^2\mu^{-1}\|\nabla b^\eta\|_2^2)e^{Ct}.$$

Following estimates in Section 2, one can see that $\|\nabla b^\eta\|_2$ has an upper bound as $\eta \to 0$. We also note that $U(0) = B(0) = 0$. Thus

$$\lim_{\eta\to 0}(\|U(t)\|_2^2 + \|B(t)\|_2^2) = 0,$$

and the convergence rate is $\mathcal{O}(\eta^2)$.

□

4 Appendix

4.1 Littlewood-Paley decomposition

Our analysis is built on the Littlewood-Paley decomposition theory.Basic languages and concepts are introduced briefly below.

We choose a nonnegative radial function $\chi \in C_0^\infty(\mathbb{R}^n)$ satisfying

$$\chi(\xi) = \begin{cases} 1, & \text{for } |\xi| \le \frac{3}{4} \\ 0, & \text{for } |\xi| \ge 1. \end{cases}$$

Denote $\lambda_q = 2^q$ for integers q. A sequence of cut-off functions are defined,

$$\varphi(\xi) = \chi(\frac{\xi}{2}) - \chi(\xi), \;\; \varphi_q(\xi) = \begin{cases} \varphi(\lambda_q^{-1}\xi) & \text{for } q \ge 0, \\ \chi(\xi) & \text{for } q = -1. \end{cases}$$

For a tempered distribution vector field u we define the Littlewood-Paley projection

$$\begin{aligned} &h = \mathcal{F}^{-1}\varphi, \qquad \tilde{h} = \mathcal{F}^{-1}\chi, \\ &u_q := \Delta_q u = \mathcal{F}^{-1}(\varphi(\lambda_q^{-1}\xi)\mathcal{F}u) = \lambda_q^n \int h(\lambda_q y)u(x-y)dy, \qquad \text{for } q \ge 0, \\ &u_{-1} = \mathcal{F}^{-1}(\chi(\xi)\mathcal{F}u) = \int \tilde{h}(y)u(x-y)dy, \end{aligned}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and inverse Fourier transform, respectively. Due to the Littlewood-Paley theory, the identity

$$u = \sum_{q=-1}^{\infty} u_q$$

holds in the sense of distribution, which is the fundamental idea of shell decomposition. We also denote the various summation terms simply by

$$u_{\le Q} = \sum_{q=-1}^{Q} u_q, \qquad u_{(Q,N]} = \sum_{p=Q+1}^{N} u_p, \qquad \tilde{u}_q = \sum_{|p-q|\le 1} u_p.$$

We can adapt the norm of Sobolev space $\dot{H}^s$ as

$$\|u\|_{\dot{H}^s} \sim \left(\sum_{q=-1}^{\infty} \lambda_q^{2s} \|u_q\|_2^2 \right)^{1/2}, \quad s \in \mathbb{R}.$$

Bernstein's inequality satisfied by the dyadic blocks u_q is introduced below.

Lemma 4.1 *Let n be the space dimension and $r \ge s \ge 1$. Then for all tempered distributions u, we have*

$$\|u_q\|_r \lesssim \lambda_q^{n(\frac{1}{s}-\frac{1}{r})} \|u_q\|_s.$$

4.2 Bony's paraproduct and commutators

We adapt the following version of Bony's paraproduct

$$\begin{aligned}\Delta_q(u\cdot\nabla v) = &\sum_{|q-p|\le 2}\Delta_q(u_{\le p-2}\cdot\nabla v_p) + \sum_{|q-p|\le 2}\Delta_q(u_p\cdot\nabla v_{\le p-2})\\ &+\sum_{p\ge q-2}\Delta_q(\tilde{u}_p\cdot\nabla v_p),\end{aligned} \tag{18}$$

which is used through the paper to decompose the nonlinear terms. We introduce a commutator as

$$[\Delta_q, u_{\le p-2}\cdot\nabla]v_p = \Delta_q(u_{\le p-2}\cdot\nabla v_p) - u_{\le p-2}\cdot\nabla\Delta_q v_p. \tag{19}$$

Lemma 4.2 *The following estimate holds, for any* $1 < r < \infty$

$$\|[\Delta_q, u_{\le p-2}\cdot\nabla]v_p\|_r \lesssim \|\nabla u_{\le p-2}\|_\infty\|v_p\|_r.$$

To treat the Hall term, we recall a fundamental identity for vector valued functions F and G,

$$\nabla\times(F\times G) = [(G\cdot\nabla)F - (\nabla\cdot F)G] - [(F\cdot\nabla)G - (\nabla\cdot G)F]. \tag{20}$$

In addition, two more commutators are defined

$$[\Delta_q, F\times\nabla\times]G = \Delta_q(F\times(\nabla\times G)) - F\times(\nabla\times G_q), \tag{21}$$

$$[\Delta_q, (\nabla\times F)\times]G = \Delta_q((\nabla\times F)\times G) - (\nabla\times F)\times G_q. \tag{22}$$

They satisfy the estimates below.

Lemma 4.3 *Assume* $\nabla\cdot F = 0$ *and* F, G *vanish at large* $|x|\in\mathbb{R}^3$. *For any* $1\le r\le\infty$, *we have*

$$\|[\Delta_q, F\times\nabla\times]G\|_r \lesssim \|\nabla F\|_\infty\|G\|_r;$$

$$\|[\Delta_q, (\nabla\times F)\times]G\|_r \lesssim \|\nabla F\|_\infty\|G\|_r.$$

Lemma 4.4 *Assume the vector valued functions* F, G *and* H *vanish at large* $|x|\in\mathbb{R}^3$. *For any* $1\le r_1, r_2\le\infty$ *with* $\frac{1}{r_1}+\frac{1}{r_2}=1$, *we have*

$$\left|\int_{\mathbb{R}^3}[\Delta_q, (\nabla\times F)\times]G\cdot\nabla\times H\,dx\right| \lesssim \|\nabla^2 F\|_\infty\|G\|_{r_1}\|H\|_{r_2}.$$

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The style of the following references should be used in all documents.

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