Black holes and Bhargava's invariant theory

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Abstract

Attractor black holes in type II string compactifications on $K3\times T^2$ are in correspondence with equivalence classes of binary quadratic forms. The discriminant of the quadratic form governs the black hole entropy, and the count of attractor black holes at a fixed entropy is given by a class number. Here, we show this tantalizing relationship between attractors and arithmetic can be generalized to a rich family, connecting black holes in supergravity and string models with analogous equivalence classes of more general forms under the action of arithmetic groups. Many of the physical theories involved have played an earlier role in the study of "magical" supergravities, while their mathematical counterparts are directly related to geometry-of-numbers examples in the work of Bhargava et al.

1 Introduction

Studies of BPS states provide one of the few windows we have into the structure of non-perturbative field theory and string theory. It is therefore interesting to ask if underlying (perhaps non-manifest) mathematical structures can be uncovered from studies of BPS spectra. One set of hints in this direction appears in the papers [1, 2], where connections between arithmetic and the study of attractor black holes [3] were discussed. A particularly striking observation in these papers relates the numbers of attractor black holes at a fixed entropy in $K3 \times T^2$ compactification of type II strings to class numbers of binary quadratic forms with negative discriminant.

These numbers are defined as follows. Consider quadratic forms

$$ax^2 + bxy + cy^2 \ ,$$

which have integral coefficients and are positive definite, so that in particular the discriminant

$$D = b^2 - 4ac$$

is negative. The arithmetic group $SL(2, \mathbb{Z})$ acts on such quadratic forms by simply acting on (x, y) by the $SL(2, \mathbb{Z})$-transformation. The number $H(D)$ of such $SL(2, \mathbb{Z})$-equivalence classes at a fixed discriminant D defines the (Hurwitz) class number, which in turn may immediately be related to class numbers of imaginary quadratic fields. For example, if D is squarefree, $H(D)$ is simply the class number[1] of $\mathbb{Q}[\sqrt{D}]$.

The low energy effective supergravity describing the $K3 \times T^2$ compactification of type II superstring belongs to the family of $N = 4$ Maxwell-Einstein supergravity theories that describes the coupling of $N = 4$ supergravity to $N = 4$ vector multiplets with the global symmetry group $SO(n, 6) \times SU(1, 1)$ – realizing the specific case $n = 22$. Moore argues in [1, 2] that given a BPS black hole with quantized electric and magnetic charges (q, p) transforming in the $(n+6, 2)$ representation of $SO(n, 6) \times SU(1, 1)$, one may associate the $SL(2, \mathbb{Z})$-equivalence class of binary quadratic forms with

$$a = \frac{1}{2}p^2, \ b = p \cdot q, \ c = \frac{1}{2}q^2 \ .$$

This association arises naturally from the geometry of the attractor $K3 \times T^2$ associated to the black hole with those charges. Then $H(D)$ gives the number of distinct attractor black holes at a fixed value of the discriminant D, which coincides with the quartic invariant of $SO(n, 6) \times SU(1, 1)$ and also governs the supergravity black hole entropy:

$$S = \sqrt{p^2 q^2 - (p \cdot q)^2} \ .$$

This relationship, and its extension to the class groups which endow the equivalence classes at fixed D with a group multiplication law [1, 2], was reviewed recently in [4].

[1] Strictly speaking, the Hurwitz class number also has some fractional contributions to correctly account for automorphisms.

It is natural to wonder whether this connection is an accident of special coincidences about the geometry of $K3$ or if it might presage a more general set of relations between BPS black holes and structures in number theory. Here, we give evidence for the latter viewpoint by generalizing this observation to a wider class of supergravity theories, many of which arose originally in the study of "magical" supergravities in the early 1980s [5, 6]. Many of these theories, in turn, admit embeddings into string theory.[2]

The organization of this note is as follows. In the next section, we describe examples of the physical theories of interest. In each, the classification of attractor points is equivalent to the problem of counting equivalence classes of certain forms modulo the action of an arithmetic group; the black hole entropy is then further controlled by a suitable arithmetic invariant. In several cases, this corresponding algebraic structure also appears in the study of higher composition laws [7], and the class numbers governing attractor degeneracies have been of independent interest in the study of arithmetic statistics.[3] We conclude by discussing some natural directions for future exploration. In an appendix we describe in more detail the connection of the structures we study to Jordan algebras.

2 Physical theories and mathematical structures

In this section, we describe in telegraphic terms some of the supergravity theories (often with string embedding) which figure in the sequel. They give rise to mathematical structures which can be related to the work of Bhargava, as discussed in a different language in e.g. the work of Krutelevich [9]; see particularly Table 1 there. We should stress that Krutelevich uses the language of Jordan algebras and related Freudenthal triple systems which underlie the symmetry groups and geometries of all extended Maxwell-Einstein supergravity theories with symmetric target spaces in d$= 5, 4, 3$, in particular

[2] More precisely, we are only concerned with the geometry of the vector multiplet moduli space here. So realizations of these theories with extra purely neutral hypermultiplets will constitute UV completions of our picture. In what follows, whether we know an explicit string embedding or not, we will assume that the supergravity duality group is broken to a corresponding arithmetic group in the full theory. This is certainly what happens when a known string embedding exists.

[3] For an accessible introduction to this subject, see e.g. [8].

the magical supergravity theories [5], as well as $N > 4$ simple supergravity theories. For a review and references on the subject we refer to [10].

2.1 Example one

Consider pure 5d $N = 2$ Poincare supergravity, or more generally a 5d $N = 2$ theory with no vector multiplets (but perhaps coupled to hypermultiplets), a theory with eight supercharges. Upon compactification on a circle, one obtains 4d $N = 2$ supergravity with a single vector multiplet (coming from Kaluza-Klein reduction of the graviton on the circle, plus its superpartners) plus, perhaps, neutral hypers.[4] The cubic form that defines this theory is simply $\mathcal{N} = X^3$ which leads to the prepotential

$$\mathcal{F} = X^3/X_0$$

in four dimensions.

The black holes in this theory are characterized by two electric charges (under the matter vector multiplet and the graviphoton), and the corresponding magnetic charges. They transform in the spin $s = 3/2$ representation of an $SL(2, \mathbb{R})$ duality group of the 4d supergravity. We should note that the attractor flows for BPS black holes in general homogeneous supergravity theories, including this theory, were studied in [11, 12]. The attractor flows for non-supersymmetric black holes in this theory were studied in detail in [13].

In a non-perturbative completion of this theory, one expects $SL(2, \mathbb{R})$ to be demoted to $SL(2, \mathbb{Z})$. There is a U-duality invariant formula for the black hole entropy in such a theory, given in e.g. [14]. We find it to be in beautiful correspondence with the following construction.

Consider a binary cubic form

$$F(x, y) = ax^3 + bx^2y + cxy^2 + dy^3 \ . \tag{1}$$

It has a discriminant given by

$$D = 18abcd + b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 \ . \tag{2}$$

By a simple change of variables,

$$a = -\xi_0/3, \quad b = \xi_1/3, \quad c = -\eta_1/3, \quad d = \eta_0/3$$

[4] We should note that there is another d=4 $N = 2$ Maxwell-Einstein supergravity with one vector multiplet which does not have an uplift to five dimensions. Its quartic invariant is given by the square of a quadratic form.

it takes the form

$$D = -\tilde{I}_4/3$$

where

$$\tilde{I}_4 = (\xi_0)^2(\eta_0)^2 + 4(\xi_1)^3(\eta_0) + 2\xi_0\eta_0\xi_1\eta_1 - 1/3(\xi_1)^2(\eta_1)^2 - 4/27(\xi_0)(\eta_1)^3$$

is the $SL(2,\mathbb{R})$ invariant quartic form corresponding to eqn. (3.40) of [14]. This in turn governs the entropy of 4d charged black holes in our supergravity theory.

There is a natural action of $SL(2,\mathbb{Z})$ on (x, y) which induces an action of $SL(2,\mathbb{Z})$ on the set of binary cubic forms. Defining equivalence classes as we did in the case of binary quadratic forms, we obtain class numbers for cubic forms $h_3(D)$. These were studied by Davenport in 1951 [15]. These count the inequivalent attractor black holes (at fixed entropy) in the physical theory we are describing. The promotion of these class numbers to orders of class groups is described in [7].

The geometry of numbers of this example proceeds by an intriguing quadratic map from binary cubic forms to binary quadratic forms originally investigated by Eisenstein [16]. We refer to [17] for a review of Eisenstein's work and further references on the subject, which we explain here to indicate the special point structure as well as the fundamental importance of the quadratic transformation of the charge lattice.

Indeed, given a binary cubic form $F(x, y)$ as in equation (1) with discriminant D one can associate with it a closely-related binary quadratic form $Q_F(x, y)$ essentially given by the Hessian of F, so that the association is naturally $SL(2,\mathbb{Z})$-equivariant. To be more explicit, it is convenient to restrict our cubic form slightly and instead take

$$F(x, y) = ax^3 + 3bx^2y + 3cxy^2 + dy^3,$$

with a, b, c, d still all integers. Then the associated quadratic

$$Q_F(x, y) = Ax^2 + Bxy + Cy^2 \tag{3}$$

is given by

$$A = b^2 - ac, \quad B = bc - ad, \quad C = c^2 - bd \tag{4}$$

and the discriminant D_Q of the quadratic form is related to the discriminant D of the cubic form as

$$D_Q = B^2 - 4AC = -\frac{D}{27}. \tag{5}$$

As mentioned, the mapping from $F(x,y)$ into $Q_F(x,y)$ commutes with the $SL(2,\mathbb{Z})$ action, so we have a map from equivalence classes of integral binary cubic forms at a given discriminant to those of binary quadratic forms at (roughly) the same discriminant. It is obvious interest to understand if this map is one-to-one and to characterize its image. We mention here only the simplest case, of $D_Q \equiv 0 \pmod 4$ and $D_Q/4$ squarefree, in which case the map is indeed injective but the image is striking: one obtains exactly those points of order 3 in the class group $\mathrm{Cl}(\mathbb{Q}(\sqrt{D_Q}))$.

Of particular interest is the formula for the growth of the class numbers $h_3(D)$ with D. Davenport proves that

$$\sum_{N=1}^{D} h_3(N) = \frac{\pi^2}{108} D + \mathcal{O}(D^{\frac{15}{16}}).$$

It would be interesting to reproduce this formula directly from the perspective of supergravity counts of black hole attractors.

We do not know of a string construction of the pure supergravity theory with 8 supercharges in five dimensions, although its possible existence is discussed (as a "fantasy island") in [18].

2.2 Example two

In a similar spirit, one can consider a theory of 4d $N=2$ supergravity coupled to two vector multiplets that descends from $N=2$ supergravity coupled to one vector multiplet in $d=5$. The cubic norm that defines this theory is $\mathcal{N} = X^2 Y$ which leads to the prepotential $X^2 Y/X_0$ in four dimensions. The scalar manifold of the $4d$ supergravity is the symmetric space

$$\mathcal{M}_4 = \frac{SO(2,2)}{U(1)\times U(1)}$$

The discrete U-duality group is $SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$ under which the six charges (including electric and magnetic graviphoton charge) transform in the $(3,2)$ representation.

The mathematical structure here is that of **pairs** of binary quadratic forms, which are exchanged by one of the $SL(2,\mathbb{Z})$ symmetries. The other acts on both (simultaneously) as described in the natural action on binary quadratic forms above. A higher composition on pairs of binary quadratic forms is described in [7].

Given such a pair

$$ax^2 + bxy + cy^2$$
$$Ax^2 + Bxy + Cy^2$$

there are three natural discriminants one can define; those of each of the quadratic forms $\Delta_{1,2}$, and the codiscriminant

$$\Delta_c = bB - 2aC - 2cA \ .$$

The problem of counting equivalence classes of such forms (up to action of the arithmetic group) for triples of these quantities has been discussed by Morales [19]. This is a more refined count than is natural in supergravity, where the entropy would be determined by the single duality invariant

$$\Delta_c^2 - 4\Delta_1\Delta_2 \ .$$

Hence, (sums of) the class numbers of Morales govern the attractor counts of black holes in this supergravity theory.

2.3 Example three

We can consider the famous STU model with three vector multiplets coupled to 4d $N = 2$ supergravity. This theory descends from the 5d Maxwell-Einstein supergravity with two vector multiplets defined by the cubic norm XYZ that was first studied in [20]. A nice discussion of BPS black holes in this model can be found in [21]. A promotion of this model to string theory is described in [22] (where it is basically $N = 2$ Example D) and in [23].[5] The prepotential in d$= 4$ is XYZ/X_0 and is generally denoted as

$$F = STU$$

in the gauge $X_0 = 1$. The model has an $SL(2,\mathbb{Z})^3$ duality symmetry.[6] The 8 electric and magnetic charges transform in the $(2, 2, 2)$ representation of the duality group.

[5] It is also easy to construct Calabi-Yau threefolds with $h^{2,1} = 3$ and a complex structure moduli space governed by a prepotential of this form; we thank J. Bryan for discussions of this point.

[6] The $SL(2,\mathbb{Z})$ factors are broken slightly to congruence subgroups in the known string embeddings, which would cause minor modification to the discussion below.

The algebraic structure involved here is

$$V_2 \otimes V_2 \otimes V_2$$

with V_2 the two dimensional representation of $SL(2,\mathbb{Z})$. Explicit actions of the duality group on electric and magnetic charges can be found in [21]. In a suitable basis of the electric and magnetic charges, the duality invariant entropy is again given by

$$S = \sqrt{p^2 q^2 - (p \cdot q)^2} \ .$$

Again, class numbers are determined by the numbers of elements in $V_2^{\otimes 3}$ modulo the action of $SL(2,\mathbb{Z})^3$, and give the numbers of distinct attractor points at different values of the entropy.

2.4 Example four

There is a similar story relating counts of BPS black holes to binary quartic forms. Consider the family of forms

$$f(x,y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 \ .$$

Again, there is a natural $SL(2,\mathbb{Z})$ symmetry that acts on (x,y) and induces an action on the space of forms. There are now two invariants:

$$I(f) = 12ae - 3bd + c^2$$

$$J(f) = 72ace + 9bcd - 27ad^2 - 27eb^2 - 2c^3 \ .$$

The discriminant of f is

$$\Delta(f) = \frac{1}{27}\left(4I(f)^3 - J(f)^2\right) \ .$$

The discriminant $\Delta(f)$ has the form of the entropy of the 5d uplift of an extremal black hole of $N = 2$ Maxwell-Einstein supergravity with one vector multiplet in d=4, except for the fact that it involves an extra parameter in addition to four charges (two electric and two magnetic) with respect to the 4d vector fields. This is easily established by setting the integer e in the binary cubic form equal to zero. We then find that the discriminant takes the form

$$\Delta(f)|_{e=0} = d^2 D$$

where D is the discriminant of the binary cubic form given above. The quartic invariant I_4 that defines the entropy of charged black holes given in equation (3.40) of [5] can be written in a more symmetrical way between the electric and magnetic charges, via the identification

$$\xi_0 = -q_0 = -9a, \quad \eta_0 = p_0 = d/3, \quad \xi_1 = q_1/3 = b/3, \quad \eta_1 = 3p_1 = c$$

Then the quartic invariant takes the form

$$I_4 = \tilde{I}_4/9 = -D/3 = 4(p_1)^3 q_0 + 4(q_1)^3 p_0 + (p_0)^2(q_0)^2 - 6p_0 q_0 p_1 q_1 - 3(q_1)^2(p_1)^2$$

The quadratic and cubic invariants $I(f)$ and $J(f)$ associated with the binary quartic form then take the following form in terms of these variables and the extra parameter e

$$I(f) = -4/3eq_0 - 9p_0q_1 + 9(p_1)^2$$

$$J(f) = 3\left(-9e(q_1)^2 - 9q_0(p_0)^2 + 8eq_0p_1 + 27p_1p_0q_1 - 18(p_1)^3\right)$$

Now for $e = 0$ the discriminant Δ describes the duality invariant form governing the entropy of a 5d uplift of an extremal black hole of a 4d, $N = 2$ Maxwell-Einstein supergravity with one vector multiplet, since

$$\Delta|_{e=0} = -27(p_0)^2 I_4 \ . \tag{6}$$

This follows from the general result that the entropy of a spinning charged black hole (or ring) that is an uplift of 4d extremal black hole with charges p^0, p^i, q_0, q_i has the form [24, 25, 26]

$$S_{5d} = (2\pi)\sqrt{(N_3(Q_i) - J^2)} \ .$$

where

$$Q_i = p^0 q_i + \frac{1}{2}C_{ijk}p^j p^k$$

$$N_3(Q_i) = \frac{1}{6}C^{ijk}Q_iQ_jQ_k$$

$$J = \frac{1}{2}\left(p^0(p^0q_0 + p^iq_i) + \frac{1}{3}C_{ijk}p^ip^jp^k\right) \ .$$

Then the entropy of the corresponding 4d BPS black hole is given by

$$S_{4d} = \frac{2\pi}{|p_0|}\sqrt{N_3(Q_i) - J^2},$$

where J is the 5d angular momentum.

Therefore the invariant $J(f)|_{e=0}$ describes the 5d angular momentum of the spinning black hole in our example. We should note that the general black hole solution of pure $N = 2$ supergravity in five dimensions involving six parameters, namely the four electromagnetic charges, mass and angular momentum was given in [27] to which we refer for references on earlier work on the subject. However the general solution has not been written in terms of an invariant corresponding to the discriminant Δ with non-vanishing parameter e. On the basis of the above analysis we predict that there exists a five parameter family of extremal black ring solutions of pure $N = 2$ 5d supergravity whose entropy is described by the invariants $J(f), I(f)$ and $\Delta(f)$.

As was shown in [28] for fixed invariants $I(f)$ and $J(f)$ there exists a single orbit of $SL(2,\mathbb{R})$ in the space of quartic forms $F(x,y)$ if $\Delta(f) < 0$ and the orbit lies in the subspace of quartic forms with one pair of complex roots. For fixed invariants $I(f)$ and $J(f)$ with positive discriminant $\Delta(f) > 0$ one finds three different orbits of $SL(2,\mathbb{R})$. Two of these orbits lie in the subspace where the quartic form admits two pairs of complex roots while the third orbit lies in the subspace with real roots only.

Aspects of the asymptotics of class numbers $h(I,J)$ of binary quartic forms with given values of I, J are determined by Bhargava and Shankar in [28]. Theorem 1.6 of that paper tells us the following. If we define

$$H(I,J) = \max|I^3|, |J^2/4| \ ,$$

then

$$\sum_{H(I,J)<X} h(I,J) \sim \zeta(2)X^{5/6} + \mathcal{O}(X^{3/4+\epsilon}) \ .$$

Again, it would be nice to present a physics proof.

2.5 A magical example

We close with a brief mention of a slightly richer example. It is the complex magical $N = 2$ Maxwell-Einstein supergravity with 9 vector multiplets, spanning the moduli space[5, 6]

$$\mathcal{M}_{\text{vector}} = SU(3,3;\mathbb{Z})\backslash SU(3,3)/(SU(3)\times SU(3)\times U(1)) \ .$$

The moduli space is 9 (complex) dimensional, and the prepotential is given by

$$\mathcal{F} \sim i\frac{\det(X)}{X^0} ~,$$

where X is a three-by-three Hermitian matrix. It descends from the complex magical supergravity in $5d$ with the scalar manifold

$$\mathcal{M}_5 = SL(3,\mathbb{C})/SU(3).$$

This theory is discussed in detail in [29].

This vector multiplet moduli space arises as item 5 in Table 1 of [9]. The charges transform in the 20 dimensional irreducible representation of the arithmetic group, and an appropriate highest weight module $V(\omega_3)$ is identified in [9] and serves as the analogue of the binary forms in our earlier examples.

This model has a known string theory embedding, as a $\mathbb{Z}_3$ orbifold of T^6 compactification.

3 Discussion

In this note, we outlined a connection between the class numbers characterizing forms with fixed values of various arithmetic invariants and BPS black holes in supergravity and string theory. While we provided a few examples in §2, it should be clear that similar considerations extend to many further constructions. Extension of our results to other supergravity theories, in particular those that have stringy extensions, and possible extensions along the lines of example four above will be left to future investigations.

In the prototype case involving binary quadratic forms and BPS black holes on $K3 \times T^2$, it has been observed that the generating function

$$\sum_D H(D)\; q^D$$

(with suitable constant term) provides the holomorphic piece of a weight 3/2 mock modular form [30]. It would be very interesting to relate the black hole counting problems above to modular objects in a similar manner. In fact it seems very likely that the count of attractor black holes in our Example 2 is governed by a degree 2 weight 2 Siegel form studied by Kohnen [31].

This should follow from the relationship of the general philosophy of Kudla-Millson theory in arithmetic geometry to physics described in [32].

The results here suggest many other natural questions:

• Can we find proofs of the asymptotic results governing class numbers (some of which were presented above) using physical arguments?

• The theories most closely tied to these arithmetic variants seem to be $N \geq 2$ supersymmetric theories which enjoy a certain finiteness property (no instanton corrections to the prepotential). Can we demonstrate a connection of all such $N = 2$ theories to the theory of arithmetic invariants? Can we turn this around and propose a classification of such theories in terms of invariant theory?

• Most ambitiously, one would like to find a version of this story which holds in quantum geometry, e.g. for counts of BPS black holes in the mirror quintic. This should involve a suitable generalization of the various classical number theoretic notions that played a role here. Progress on this front would be most interesting.

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Appendix

The examples we considered in this paper correspond to the $N = 2$ Maxwell-Einstein supergravity theories with zero, one, two and eight vector multiplets in d=5 and their 4d counterparts. Five dimensional $N=2$ Maxwell-Einstein supergravities with symmetric target spaces G/H,such that G is a symmetry of the Lagrangian, are in one-to-one correspondence with the Euclidean Jordan algebras of degree three [5, 6]. The C-tensor C_{IJK} appearing in the coupling

$$C_{IJK} F^I \wedge F^J \wedge A^K$$

in these theories determines their Lagrangian uniquely and is given by the cubic norm of the underlying Jordan algebra J.

The Jordan algebras of degree three admit 3 idempotents e_1, e_2 and e_3 with the identity element given by

$$\mathbb{I} = e_1 + e_2 + e_3$$

which corresponds to the bare graviphoton in the corresponding supergravity theory. Hence the pure $N=2$ supergravity is described by the norm

$$\mathcal{N}_3(X\mathbb{I}) = X^3 \ .$$

The example 2 we considered above corresponds to the $N=2$ Maxwell-Einstein supergravity described by the cubic norm

$$\mathcal{N}_3(Xe_1 + Ye_2 + Ye_3) = XY^2$$

and the STU model in five dimensions is defined by

$$\mathcal{N}_3(Xe_1 + Ye_2 + Ze_3) = XYZ$$

as found in [20]. The complex magical supergravity in 5d is defined by the Jordan algebra $J_3^{\mathbb{C}}$ of three-by-three complex Hermitian matrices and the norm of an element J is given by its determinant

$$\mathcal{N}_3(J) = det(J) \ .$$

By the action of the automorphism group H of the underlying Jordan algebra every element of J can be brought to the form $(Xe_1 + Ye_2 + Ze_3)$ of the STU model. For the complex magical supergravity the automorphism group of $J_3^{\mathbb{C}}$ is $SU(3)$. This shows clearly that there exist natural extensions of the invariants we studied to more general classes of invariants related to the arithmetic subgroups of the global symmetry groups of the $N=2$ supergravity theories. For $N=4$ Maxwell-Einstein supergravities in 5d, the underlying Jordan algebras of degree three are non-Euclidean. Similarly the symmetries of maximal $N = 8$ supergravity are given by the symmetries of the non-Euclidean exceptional Jordan algebra of Hermitian 3×3 matrices over the split octonions [33]. Under dimensional reduction to four dimensions the correspondence between Jordan algebras of degree three and 5d supergravities goes over to a correspondence between Freudenthal triple systems $\mathcal{F}(J)$ associated with the Jordan algebras J of degree three and 4d supergravities [33, 34, 10].

Orbits of extremal black hole solutions of supergravity theories under the action their continuous U-duality groups G have been studied extensively beginning with the work of [33]. For those theories that have stringy extensions the relevant orbits are with respect to the discrete U-duality groups which are typically the maximal arithmetic subgroups $G(\mathbb{Z})$ of G [35, 36]. Motivated by the works of [33, 35] Krutelevich considered the integral version of Freudenthal's construction of exceptional groups and studied their connection to higher composition laws of Bhargava [9]. There is a one-to-one or two-to-one correspondence between known supergravity theories with symmetric target spaces and the rows of the Table 1 in [9]. The first three examples we gave above correspond to the first three rows of the Table 1 of [9], and our fifth (magical) example is the fifth entry in his Table. However our fourth example does not have a corresponding entry in Krutelevich's table.

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