PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
https://doi.org/10.1090/proc/14665
Article electronically published on July 8, 2019

WEIGHTED ESTIMATES FOR ONE-SIDED MARTINGALE TRANSFORMS

WEI CHEN, RUI HAN, AND MICHAEL T. LACEY

(Communicated by Svitlana Mayboroda)

ABSTRACT. Let $Tf = \sum_I \varepsilon_I \langle f, h_{I^+}\rangle h_{I^-}$. Here, $|\varepsilon_I| = 1$, and h_J is the Haar function defined on dyadic interval J. We show that, for instance,

$$\|T\|_{L^2(w)\to L^2(w)} \lesssim [w]_{A_2^+}.$$

Above, we use the one-sided A_2 characteristic for the weight w. This is an instance of a one-sided A_2 conjecture. Our proof of this fact is difficult, as the very quick known proofs of the A_2 theorem do not seem to apply in the one-sided setting.

1. One-sided martingale transform

We prove new sharp one-sided weighted inequalities for certain kinds of one-sided martingale transforms. One-sided weights are variants of the usual A_p weights on the real line, for which the following supremum is finite:

$$[w]_{A_p^+} = \sup_I \frac{w(I^-)}{|I^-|}\Big[\frac{\sigma(I^+)}{|I^+|}\Big]^{p-1},$$

where w is a non-negative function, locally integrable, and $\sigma = w^{-\frac{1}{p-1}}$ is also locally integrable. The set $I = (a, a+2\delta)$ is an interval, with left and right halves, $I^- = (a, a+\delta)$ and $I^+ = (a+\delta, a+2\delta)$, respectively.

For the endpoint cases of $p = 1, \infty$, we use the one-sided maximal function

$$M_+ f = \sup_I \frac{\mathbf{1}_{I^-}}{|I^+|}\int_{I^+} |f|\,dx.$$

Define M_- similarly. We set

$$[w]_{A_1^+} = \Big\|\frac{M_- w}{w}\Big\|_\infty, \qquad [w]_{A_\infty^+} = \sup_I w(I)^{-1}\int_I M_-(w\mathbf{1}_I)\,dx.$$

The definitions of A_p^- follow similarly, and it is important to note the duality between these expressions. In particular, $[w]_{A_p^+} = [\sigma]_{A_{p'}^-}^{p-1}$.

Received by the editors November 5, 2018, and, in revised form, April 12, 2019.

2010 *Mathematics Subject Classification.* Primary 42B25.

The first author was supported by the National Natural Science Foundation of China (11771379), the Natural Science Foundation of Jiangsu Province (BK20161326), and the Jiangsu Government Scholarship for Overseas Studies (JS-2017-228).

The research of the second author was supported in part by a grant from the US National Science Foundation, DMS-1800689.

The research of the third author was supported in part by a grant from the US National Science Foundation, DMS-1600693, and the Australian Research Council, ARC DP160100153.

The qualitative aspects of the A_p^+ theory closely match those of the usual A_p theory. The study of sharp constants for the A_p theory has been under very rapid development. But some of the corresponding results for A_p^+ seem much harder to establish, as we will see, both in the main theorem below and in the concluding section of this paper. The maximal function estimates were established by Sawyer [18], and we have this result, matching known results for the usual maximal function.

Theorem 1.1 ([14, Thms. 1.5 and 1.9])**.** *These inequalities hold:*

$$\|M_+\|_{L^p(w)\to L^{p,\infty}(w)} \lesssim [w]_{A_p^+}^{1/p}, \qquad 1<p<\infty, \tag{1.1}$$

$$\|M_+\|_{L^p(w)\to L^p(w)} \lesssim ([w]_{A_p^+}[\sigma]_{A_\infty^-})^{1/p}, \qquad 1<p<\infty. \tag{1.2}$$

We remark that we have

$$[\sigma]_{A_\infty^-} \le [\sigma]_{A_{p'}^-} = [w]_{A_p^+}^{p'-1}.$$

The main contribution of this paper is to establish some sharp A_p^+ estimates for (maximal truncations of) one-sided martingale transforms, which are defined by

$$Tf = \sum_{\text{dyadic } I} \varepsilon_I \langle f, h_{I^+}\rangle h_{I^-},$$

$$T_\sharp f = \sup_{\delta>0} \Big| \sum_{\substack{\text{dyadic } I\\ |I|>\delta}} \varepsilon_I \langle f, h_{I^+}\rangle h_{I^-} \Big|,$$

where ε_I takes values in $\{-1,1\}$ and h is the Haar function defined by

$$h_I(x) = \frac{\mathbf{1}_{I^+}(x) - \mathbf{1}_{I^-}(x)}{\sqrt{|I|}}.$$

Our main result is the sharp weak L^p inequality, for all $1<p<\infty$, and maximal truncations. We also prove sharp inequality on L^2, but *not* for the maximal truncations.

Theorem 1.2. *The following inequalities are uniform over all one-sided martingale transforms:*

$$\|T_\sharp\|_{L^p(w)\to L^{p,\infty}(w)} \lesssim [w]_{A_p^+}^{\frac{1}{p}} [w]_{A_\infty^+}^{\frac{1}{p'}}, \qquad 1<p<\infty,$$

$$\|T\|_{L^2(w)\to L^2(w)} \lesssim [w]_{A_2^+}^{\frac12} \max\big\{[\sigma]_{A_\infty^-}, [w]_{A_\infty^+}\big\}^{\frac12}. \tag{1.3}$$

This is a very modest assertion, compared to the advanced state of the A_p theory. We will address what we think is true, and some of the unexpected complications we encountered, in the concluding section of this paper.

The theory of A_p^+ weights was started by Sawyer [18]. He showed the natural analog of Muckenhoupt's famous theorem: The A_p^+ condition is characterized by those positive a.e. weights w for which M_+ is bounded on $L^p(w)$. Amir, Foranzi, and Martín-Reyes [1] established several natural analogs of the A_p^+ theory for singular integrals. The recent paper of Chill and Król [5] gives an elegant development of the A_p^+ theory, especially with an eye towards applications in parabolic PDEs. While there is a sizable literature on A_p^+ weights, even in the singular integral setting, the results we could find were qualitative in nature. As far as we are aware, the

question of sharp constants has not been addressed before for singular integral-like objects in the A_p^+ setting.

Concerning the proof, it is noteworthy that the proof of the maximal function estimates (1.2) is relatively simple. A variant of the argument used by Buckley [3] can be used. The weak-type result can be combined with a sharp reverse Hölder estimate [14, Thm. 1.8]. Marcinkiewicz interpolation finishes the proof.

No such argument can work for the martingale transforms, since the L^∞ endpoint estimate is too large. Indeed, most of the strategies that are so successful for the usual A_p weights do not seem to generalize to the one-sided setting, a subject we return to in §5. We adapt methods from [11, 13] in order to complete the proof. Namely, in §3, we establish a general version of a distributional inequality, which is the key to the characteristic of the A_p^+ condition. We also make some initial remarks on a one-sided two weight theory in §2. These two elements are combined in §4 via a corona construction.

2. Background

It is a useful remark that the one-weight inequality $\|Tf\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}$ is equivalent to the two-weight inequality

$$\|T(\sigma f)\|_{L^p(w)} \lesssim \|f\|_{L^p(\sigma)}, \qquad \sigma = w^{1-p'}. \tag{2.1}$$

The latter inequality is a two-weight inequality and is convenient as it dualizes correctly; namely, the inequality above is equivalent to $\|T^*(w\phi)\|_{L^{p'}(\sigma)} \lesssim \|\phi\|_{L^{p'}(w)}$.

There is a characterization of the two-weight inequality (2.1) in the case of $p = 2$, due to Nazarov, Treil, and Volberg [15]. It completely solves the question for dyadic operators. For the sake of clarity, we recall it in the context of our one-sided martingale transforms.

Theorem 2.1 ([15])**.** *Let (w, σ) be a pair of weights, and let T be a one-sided martingale transform. We have $\|T(\sigma\cdot)\|_{L^2(\sigma)\to L^2(w)} \lesssim \mathfrak{T}$, where $\mathfrak{T}$ is the best constant in the inequalities below, uniformly over intervals I:*

$$\begin{aligned} \|\mathbf{1}_I T(\sigma \mathbf{1}_I)\|_{L^2(w)} &\le \mathfrak{T}\sigma(I)^{1/2}, \\ \|\mathbf{1}_I T^*(w \mathbf{1}_I)\|_{L^2(\sigma)} &\le \mathfrak{T} w(I)^{1/2}. \end{aligned} \tag{2.2}$$

The condition (2.2) is referred to as a *testing condition.* It is a fundamental reduction in complexity in proving the L^2 bound.

A corresponding theorem in the L^p setting is necessarily more complicated. A true characterization has not yet been found; see [19]. Indeed, the main result of this last paper is powerful but is stated with an assumption about "quadratic A_p condition" [19, (3.2)], which would have to be adapted to the one-sided case.

We do not need a full characterization, and if one restricts attention to dyadic Calderón-Zygmund operators, very sharp conditions can be given. We will use the following two-weight inequality for maximal truncations of one-sided martingale transforms that essentially follows from [11, Thm. 4.3].

Theorem 2.2. *Let w, σ be two weights, and let T be a one-sided martingale transform. We have the weak-type bound*

$$\|T_\sharp(\sigma f)\|_{L^{p,\infty}(w)} \lesssim (\mathfrak{M}_p^+ + \mathfrak{T}_p)\|f\|_{L^p(\sigma)},$$

where the two constants on the right are the best constants in the following inequalities, holding for all functions f and intervals I:

$$\|M_+(\sigma f)\|_{L^{p,\infty}(w)} \le \mathfrak{M}_p^+ \|f\|_{L^p(\sigma)}, \tag{2.3}$$

$$\int_I T_\sharp(\sigma \mathbf{1}_I f)\, dw \le \mathfrak{T}_p \|f\|_{L^p(\sigma)} w(I)^{1/p'}. \tag{2.4}$$

The theorem as formulated above is not a direct corollary, since the cited [11, Thm. 4.3] has the usual maximal function appearing in (2.3). The proof however extends immediately to the version above. We will comment more on this in the final section of the paper.

The condition (2.4) does not look at all like the corresponding testing condition (2.2). Let us explain here why they are similar. The maximal truncation operator is not linear. But its boundedness is equivalent to the boundedness of a family of linear operators given as follows. For any measurable $\delta : \mathbb{R} \to (0, \infty)$, let us set

$$T_\delta f(x) = \sum_{I\,:\,|I|>\delta(x)} \langle f, h_{I^+}\rangle h_{I^-}(x)$$

to be a linearization of the maximal truncations. It is easy to see that $T_\sharp$ is bounded if and only if the family of linear operators T_δ satisfies a norm bound independent of the choice of measurable δ.

Using linearity, note that the integral in (2.4) can be written as

$$\langle T_\delta(f \mathbf{1}_I \sigma), \varphi_I \cdot w\rangle \le \mathfrak{T}_p \|f\|_{L^p(\sigma)} w(I)^{1/p'}.$$

This inequality should hold uniformly over all choices of truncation δ, and $|\varphi_I| = \mathbf{1}_I$. Therefore, the condition (2.4) is the same as

$$\|\mathbf{1}_I T_\delta^*(w\varphi_I)\|_{L^{p'}(\sigma)} \le \mathfrak{T}_p w(I)^{1/p'}. \tag{2.5}$$

This is the form of (2.4) that we will use. It is a testing condition.

The appearance of T_δ^* is not so familiar. Crucially, these operators satisfy a weak L^1 inequality. We state here and remark that the proof is not easy.

Proposition 2.3 ([11, Thm. 9.3])**.** *T_δ^* maps L^1 to weak L^1 uniformly over the choice of the one-sided martingale transform and the choice of the linearization, δ.*

Finally, we need a lemma which states that a weak-type distributional estimate is sufficient for a John-Nirenberg-type estimate.

Lemma 2.4 ([8, Lemma 5.5], [11, Lemma 10.2])**.** *Let $\{\phi_I : I \in \mathcal{D}\}$ be a collection of functions indexed by dyadic intervals which are supported on I and constant on the grandchildren of I. Suppose that there is a constant C so that for any dyadic interval I_0 and any collection $\mathcal{E}$ of dyadic subintervals $I \subset I_0$, there holds*

$$\Big|\Big\{\Big|\sum_{I\in\mathcal{E}} \phi_I\Big| > C\Big\}\Big| < \tfrac{1}{2}|I_0|. \tag{2.6}$$

Then, we have

$$\Big|\Big\{\Big|\sum_{I\in\mathcal{E}} \phi_I\Big| > (C+1)\lambda\Big\}\Big| < 2^{(1-\lambda)/2}|I_0|, \qquad \lambda > 1.$$

3. The distributional lemma

The key component is a one-sided variant of a distributional lemma discovered in [13]. Fix an interval I_0, and for an integer $a \in \mathbb{Z}$, let $\mathcal{K}_a$ be a set of those dyadic intervals $I \subseteq I_0^-$ for which we have

$$2^a < \langle \sigma \rangle_{I^+}^{p-1} \langle w \rangle_{I^-} \le 2^{a+1}, \tag{3.1}$$

$$\langle w \rangle_{I^-} \le 2 \langle w \rangle_{I_0^-}. \tag{3.2}$$

Lemma 3.1. *For an absolute constant $c > 0$ and constant $C_p > 0$ that depends (only) on p, we have*

$$\sigma\Big(\big| T^*_{\delta, \mathcal{K}_a}(w \varphi_{I_0}) \big| > C_p \lambda \langle w \rangle_{I_0^-} \Big) \lesssim \begin{cases} \lambda^{-\frac{2p'}{p+1}} \cdot \sigma(I_0^+) & \text{for } 0 < \lambda < 1, \\ e^{-c\lambda} \cdot \sigma(I_0^+) & \text{for } \lambda \ge 1. \end{cases} \tag{3.3}$$

Above, for any collection $\mathcal{K}$,

$$(T^*_{\delta, \mathcal{K}}(f))(x) := \sum_{I \in \mathcal{K}} \varepsilon_I \langle f, h_{I^-} \mathbf{1}_{|I| > \delta(x)} \rangle h_{I^+}(x). \tag{3.4}$$

The function φ_I satisfies $|\varphi_I| = \mathbf{1}_I$.

The point of this inequality is that it holds in the two-weight setting, provided the A_p^+ product is approximately constant. Moreover, one should note that the set on the left is contained in I_0^-, and its σ-measure is controlled by $\sigma(I_0^+)$.

Proof. The collection $\mathcal{K}_a$ is further divided into collections $\mathcal{K}_a^b$, for $b \in \mathbb{N}$, where $I \in \mathcal{K}_a^b$ if $I \in \mathcal{K}_a$, and in addition

$$2^{-b} \langle w \rangle_{I_0^-} < \langle w \rangle_{I^-} \le 2^{1-b} \langle w \rangle_{I_0^-}. \tag{3.5}$$

By (3.2), this is a decomposition of $\mathcal{K}_a$.

We show that for constants $c, C > 0$, and $\lambda > 0$,

$$\sigma\Big(\big| T^*_{\delta, \mathcal{K}_a^b}(w \varphi_{I_0}) \big| > C \lambda 2^{1-b} \langle w \rangle_{I_0^-} \Big) \lesssim 2^{b/(p-1) - c\lambda} \sigma(I_0^+). \tag{3.6}$$

This is a strong condition, and another easy subadditivity argument completes the proof of (3.3).

Indeed, for $0 < \lambda < 1$, since $\sum_{b=0}^\infty 2^{-b(1-\frac{p+1}{2p})} = 1/(1 - 2^{\frac{1-p}{2p}}) =: C_p < \infty$, we have

$$\begin{aligned} \sigma\Big(|T^*_{\delta, \mathcal{K}_a}(w \varphi_{I_0})| > 2CC_p \lambda \langle w \rangle_{I_0^-} \Big) &\le \sum_{b=0}^\infty \sigma\Big(T^*_{\delta, \mathcal{K}_a^b}(w \varphi_{I_0}) > C \lambda 2^{1-b(1-\frac{p+1}{2p})} \langle w \rangle_{I_0^-} \Big) \\ &\lesssim \sigma(I_0^+) \sum_{b=0}^\infty 2^{b/(p-1) - c\lambda 2^{\frac{b(p+1)}{2p}}} \lesssim \lambda^{-\frac{2p'}{p+1}} \cdot \sigma(I_0^+). \end{aligned}$$

For $\lambda \ge 1$, since $\sum_{b=0}^\infty 2^{-b/2} < 10$, we have

$$\begin{aligned} \sigma\Big(|T^*_{\delta, \mathcal{K}_a}(w \varphi_{I_0})| > 20C \lambda \langle w \rangle_{I_0^-} \Big) &\le \sum_{b=0}^\infty \sigma\Big(T^*_{\delta, \mathcal{K}_a^b}(w \varphi_{I_0}) > C \lambda 2^{1-b/2} \langle w \rangle_{I_0^-} \Big) \\ &\lesssim e^{-c\lambda} \cdot \sigma(I_0^+) \sum_{b=0}^\infty 2^{b/(p-1) - c2^{b/2}} \lesssim e^{-c\lambda} \cdot \sigma(I_0^+). \end{aligned}$$

This proves (3.3).

In general, we know little more than the local integrability condition on $w\varphi_{I_0}$. But we have this consequence of the weak-type inequality Proposition 2.3: There is a constant C so that for any subinterval $J \subset I_0$ and any collection $\mathcal{E} \subset \mathcal{K}_a^b$ of subintervals of J, we have

$$\bigl|\bigl\{|T^*_{\delta,\mathcal{E}}(w\varphi_J)| > C2^{1-b}\langle w\rangle_{I_0^-}\bigr\}\bigr| \le \tfrac{1}{2}|J|. \tag{3.7}$$

This follows from the dyadic structure and condition (3.5), which controls the local L^1 norms of $w\varphi_{I_0}$.

To apply the John-Nirenberg estimate in Lemma 2.4, we check that all its sufficient conditions are satisfied. The inequality (3.7) is the hypothesis (2.6). The other hypotheses of Lemma 2.4 are easy to check. It follows that for $\lambda > 0$,

$$\Bigl|\Bigl\{x : |T^*_{\delta,\mathcal{K}_a^b}(x)(w\varphi_{I_0})| > C\lambda 2^{1-b}\langle w\rangle_{I_0^-}\Bigr\}\Bigr| \lesssim 2^{-c\lambda}|I_0|. \tag{3.8}$$

Note that for $0 < \lambda < 1$, this estimate is trivial.

Our task is to convert this Lebesgue measure estimate into σ-measure. By combining conditions (3.1) and (3.5), we have for any $I \in \mathcal{K}_a^b$,

$$\begin{aligned}\langle\sigma\rangle_{I^+}^{p-1} &\sim 2^a\langle w\rangle_{I^-}^{-1}\\ &\sim 2^{a+b}\langle w\rangle_{I_0^-}^{-1} \sim 2^b\langle\sigma\rangle_{I_0^+}^{p-1}.\end{aligned}$$

That is, $\sigma(I^+) \sim 2^{b/(p-1)}\rho|I|$, with constant ρ *independent of* $I \in \mathcal{K}_a^b$.

The set on the left in (3.8) is a union of children of intervals I^- for $I \in \mathcal{K}_a^b$. Therefore, (3.6) follows, and the lemma is proved. □

4. Proof of Theorem 1.2

There are two assertions: the weak L^p bound and the L^2 bound. Both are proved by appealing to the two-weight theorems, Theorems 2.2 and 2.1. By inspection, it suffices to prove the weak L^p bound for the maximal truncations. This inequality specialized to the case of $p = 2$, and taking duality into account, gives the full L^2 result.

From Theorem 2.2, we need to estimate the quantities $\mathfrak{M}_p^+$ and $\mathfrak{T}_p$ defined in (2.3) and (2.4). The first $\mathfrak{M}_p^+$ is the weak-type norm for the maximal function M_+. That is given in (1.1) and is smaller than what is claimed. So we turn to the second constant, $\mathfrak{T}_p$, which is the testing constant for the maximal truncations of one-sided martingale transforms. Namely, the task is to show that uniformly over all dyadic intervals I_0, the inequality (2.5) holds. More exactly, we need to see that

$$\|\mathbf{1}_{I_0}T^*_\delta(w\varphi_{I_0})\|_{L^{p'}(\sigma)} \le [w]_{A_p^+}^{1/p}\,[w]_{A_\infty^+}^{1/p'}\cdot w(I_0)^{1/p'}. \tag{4.1}$$

The martingale transform is a sum over all dyadic intervals. We can restrict the sum to those dyadic intervals I that intersect I_0. Those I that strictly contain I_0 can also be dismissed, since they contribute nothing to the left side of (2.5). The case of $I = I_0$ is trivial. Thus, the difficult case is $I \subsetneq I_0$, and we restrict our attention to this case by assuming that the multiplier coefficients $\epsilon_I = 0$ unless $I \subsetneq I_0$.

Our principal tools are the distribution lemma, Lemma 3.1, and a corona-type decomposition. The latter is needed to get to a point that we can apply Lemma 3.1.

For an integer a with $2^a < [w]_{A_p^+}$, let

$$\mathcal{K}_a = \{I \subsetneq I_0 \ : \ 2^a < \langle \sigma \rangle_{I^+}^{p-1} \langle w \rangle_{I^-} \leq 2^{a+1}\}.$$

Assuming that $\epsilon_I = 0$ unless $I \in \mathcal{K}_a$, we will show that

$$\int_{I_0} T^*_{\delta, \mathcal{K}_a} (w \varphi_{I_0})^{p'} \, d\sigma \lesssim 2^{a(p'-1)} [w]_{A_\infty^+} w(I_0). \tag{4.2}$$

Above, we are using the notation of (3.4). Summing over a will prove (4.1).

Definition 4.1 (The corona)**.** Let

$$C_{a,1} := \{I \in \mathcal{K}_a : I^- \text{ is maximal w.r.t. inclusion}\}.$$

For $I \in \mathcal{C}_{a,1}$, let $\mathcal{C}_{a,2}(I)$ be the "bad" children of I, defined as

$$\mathcal{C}_{a,2}(I) := \{J \in \mathcal{K}_a : J^- \subset I^-, \langle w \rangle_{J_-} > 2 \langle w \rangle_{I^-}, J^- \text{ is maximal}\},$$

and let

$$\mathcal{C}_{a,2} = \bigcup_{I \in \mathcal{C}_{a,1}} \mathcal{C}_{a,2}(I).$$

We also define $\mathcal{C}_{a,t}$ for $t \geq 3$ inductively. Let

$$\mathcal{C}_a = \bigcup_{t=1}^{\infty} \mathcal{C}_{a,t}.$$

We refine the corona decomposition.

Definition 4.2. For $J \in \mathcal{K}_a$, let J^s stand for the minimal stopping interval $S \in \mathcal{C}_a$ with $J^- \subseteq S^-$.

The collections below form a partition of $\mathcal{K}_a$:

$$\mathcal{K}_a(S) := \{J \in \mathcal{K}_a : J^s = S\}, \quad S \in \mathcal{C}_a.$$

We are now at the core of the argument. Set $\tau_S = T^*_{\delta, \mathcal{K}_a(S)}(w \varphi_{I_0})$ and

$$X_{S,n} = \mathbf{1}_{\{2^{n-1} \langle w \rangle_{S^-} < |\tau_S| \leq 2^n \langle w \rangle_{S^-}\}} \tau_S, \quad n \in \mathbb{Z}.$$

Then, estimate

$$\left\| \mathbf{1}_{I_0} T^*_{\delta, \mathcal{K}_a} (w \varphi_{I_0}) \right\|_{L^{p'}(\sigma)} = \Big\| \sum_{S \in \mathcal{C}_a} \tau_S \Big\|_{L^{p'}(\sigma)} \tag{4.3}$$

$$\leq \sum_{n=-\infty}^{\infty} \Big\| \sum_{S \in \mathcal{C}_a} X_{S,n} \Big\|_{L^{p'}(\sigma)} \tag{4.4}$$

$$\lesssim \sum_{n=-\infty}^{\infty} \Big[\sum_{S \in \mathcal{C}_a} \|X_{S,n}\|_{L^{p'}(\sigma)}^{p'} \Big]^{1/p'}. \tag{4.5}$$

The last inequality follows from the construction of the corona: the values $\langle w \rangle_{S^-}$ form a geometric sequence of reals.

Concerning the norms in (4.5), the case of $n \geq n_p$ and $n < n_p$ are different. Here n_p is the least positive integer such that $2^{n-1} \geq C_p$. For $n \geq n_p$, we have,

again by the construction of the corona and the distribution estimate (3.3) with $\lambda = 2^{n-1}/C_p \geq 1$,

$$\begin{aligned}\sum_{S\in\mathcal{C}_a} \|X_{S,n}\|_{L^{p'}(\sigma)}^{p'} &\lesssim 2^{np'} e^{-c2^{n-1}} \sum_{S\in\mathcal{C}_a} \langle w\rangle_{S^-}^{p'} \sigma(S^+) \\ &\lesssim 2^{np'} e^{-c2^{n-1}} 2^{a(p'-1)} \sum_{S\in\mathcal{C}_a} \langle w\rangle_{S^-} |S^+| \\ &\lesssim 2^{np'} e^{-c2^{n-1}} 2^{a(p'-1)} [w]_{A_\infty^+} w(I_0).\end{aligned}$$

This estimate decreases rapidly in n, so it is strong enough to imply (4.2).

The case of $n < n_p$ is very similar. We use the distribution estimate (3.3) with $\lambda = 2^{n-1}/C_p < 1$:

$$\begin{aligned}\sum_{S\in\mathcal{C}_a} \|X_{S,n}\|_{L^{p'}(\sigma)}^{p'} &\lesssim 2^{np'} 2^{-n\frac{2p'}{p+1}} \sum_{S\in\mathcal{C}_a} \langle w\rangle_{S^-}^{p'} \sigma(S^+) \\ &\lesssim 2^{np/(p+1)} 2^{a(p'-1)} \sum_{S\in\mathcal{C}_a} \langle w\rangle_{S^-} |S^+| \\ &\lesssim 2^{np/(p+1)} 2^{a(p'-1)} [w]_{A_\infty^+} w(I_0).\end{aligned}$$

Since $n \leq n_p$, this is again strong enough to complete the proof of (4.2). And the proof is finished. □

5. Complements

(1) We note that the main theorem includes a weak-type inequality for the maximal truncations, but a strong-type inequality *without truncations*. Our proof verifies the correct testing conditions for maximal truncations. Then, one needs a corresponding two-weight inequality, reducing the norm inequality for maximal truncations to testing inequalities. There is a model theorem for Haar shifts [11, Thm. 4.7]. We have not checked that the (delicate and involved) proof of that theorem continues to hold in the one-sided setting.

(2) The inequalities in Theorem 1.2 are sharp in the A_p^+ characteristic. It is easy to build single-scale examples, which we leave to the reader. Here is a soft proof. Let $\alpha > 0$ be the smallest index in the inequality

$$\|T\|_{L^2(w)\to L^2(w)} \lesssim [w]_{A_2^+}^{\alpha},$$

valid for the particular one-sided martingale transform

$$Tf = \sum_I \langle f, h_{I^+}\rangle (h_{I^{--}} + h_{I^{-+}}),$$

where $I^{-\pm}$ denote the dyadic children of I^-. As Petermichl [16, 17] has shown us, we can average translates of T to recover the Hilbert transform H. We then learn that

$$\|T\|_{L^2(w)\to L^2(w)} \lesssim \sup_x [w(\cdot - x)]_{A_2^+}^{\alpha} \lesssim [w]_{A_2}^{\alpha}.$$

As a consequence, $\alpha \geq 1$.

(3) It would be very natural to seek a proof of our main theorem using ideas related to sparse bounds. The latter subject was started in [12] and has been very successful. But we could not find such a bound in the current setting. That is

why we returned to an earlier proof [13] of the A_2 bound, one that was explored in [8, 9, 11].

(4) The source of many of our difficulties seems to be linked to this point. A key element of the recent developments in the A_p theory is the following universal fact: For any weight μ on $\mathbb{R}$, the maximal function

$$M_\mu f = \sup_{I\,:\,\mu(I)>0} \mathbf{1}_I \mu(I)^{-1} \int_I f \, d\mu$$

is weakly bounded on $L^1(\mu)$. This does not seem to be true for the plus versions of this maximal function.

(5) The absence of this "universal maximal function" reminds us of a similar obstruction in the multiparameter setting. Recent results of Barron and Pipher [2] have shown that in fact sparse bounds do not hold in that setting. Would some version of these arguments hold in the one-sided case?

(6) We pose the question: What is the simplest natural condition that one can place on a weight μ so that the maximal operator

$$M_\mu^+ f = \sup_{I\,:\,\mu(I^+)>0} \mathbf{1}_{I^-} \mu(I^+)^{-1} \int_{I^+} f \, d\mu$$

is weakly bounded on $L^1(\mu)$? Or on some $L^p(\mu)$ for $1 \le p < \infty$? Is it $\mu \in A_\infty^+$?

(7) Our main theorem admits a straightforward extension to paraproducts, that is, operators of the form

$$Tf = \sum_I \tau_I \langle f \rangle_{I^+} h_{I^-},$$

where τ_I is a Carleson sequence. It likewise admits an extension to the setting where the martingale transforms are replaced with Haar shifts with complexity, in the sense of [9, 10]. In the latter case, with complexity, one wants bounds that are polynomial in complexity. There are versions of our two-weight theorems, Theorems 2.1 and 2.2, that account for complexity. In the interest of clarity, we have not pursued these points here.

(8) One of the ways that the classical A_p theory and the A_p^+ are similar is in the area of extrapolation. A robust theory holds in both places. A quantified version of the extrapolation of A_p^+ is described in Carro, Lorente, and Martín-Reyes [4]. The extrapolation of strong-type norms in the A_p^+ setting is described in [6]. That discussion is not quantitative, but there should not be a major obstacle to devising such a theory. With it, one could deduce new strong-type inequalities from our inequality, (1.3).

(9) Given the role of two-weight inequalities Theorems 2.1 and 2.2 in our argument, it might be reasonable to investigate these theorems in the setting of one-sided operators. We comment that [11, Thm. 4.7] has a general strong-type two-weight inequality. But, as for Theorem 2.2, the discussion is not geared towards one-sided operators. Moreover, the proof of that theorem is not so easy, and the sufficient conditions are substantially more involved. So, we felt that appealing to the one-sided version of that result was not in the spirit of this paper.

(10) The paper of Vuorinen [19] aims for a characterization of the strong-type inequality for general dyadic operators. But, note that the main theorem of this paper begins with an assumption of a "quadratic two-weight $\mathcal{A}_p$" condition, specified in section 3 of that paper. This condition is not of a one-sided nature. So again,

the theorem might be attractive to apply in the one-sided setting, but it is not completely straightforward to do so.

(11) The overarching conjecture here concerns *one-sided Calderón-Zygmund operators* and a one-sided version of the main results of [7, 9]. These are operators T, bounded on L^2, that have a kernel representation

$$\langle Tf, g\rangle = \int K(x,y) f(y) g(x)\, dx\, dy,$$

where $K(x,y)$ satisfies the standard size and smoothness conditions

$$|K(x,y)| \lesssim |x-y|^{-1}, \qquad x \neq y,$$
$$|\nabla^{\alpha} K(x,y)| \lesssim |x-y|^{-1-\alpha},$$

where $0 < \alpha \leq 1$ is fixed. But, one imposes the one-sided condition: $K(x,y) = 0$ if $x < y$.

Conjecture 5.1. *Let T be a one-sided Calderón-Zygmund operator. Then, there holds*

$$\|T\|_{L^p(w)\to L^p(w)} \lesssim [w]_{A_p^+} \max\{[\sigma]_{A_\infty^-}^{1/p}, [w]_{A_\infty^+}^{1/p'}\} \lesssim [w]_{A_p}^{\max\{1,(p-1)^{-1}\}}.$$

(12) One of the beautiful aspects of proofs of the A_2 theorem in [9] is the *Hytönen Representation Theorem*, which gives a representation of a Calderón-Zygmund operator as a rapidly convergent in complexity sum of dyadic shifts. We could not prove the analogous result in the one-sided setting. *Does a one-sided Hytönen Representation hold?* We see no reason why it should not hold but could not find a proof.

Acknowledgment

We are grateful to the referee for offering several improvements to the presentation of our results.

References

[1] H. Aimar, L. Forzani, and F. J. Martín-Reyes, *On weighted inequalities for singular integrals*, Proc. Amer. Math. Soc. **125** (1997), no. 7, 2057–2064. MR1376747

[2] A. Barron and J. Pipher, *Sparse domination for bi-parameter operators using square functions*, arXiv:1709.05009, 2017.

[3] Stephen M. Buckley, *Estimates for operator norms on weighted spaces and reverse Jensen inequalities*, Trans. Amer. Math. Soc. **340** (1993), no. 1, 253–272. MR1124164

[4] María J. Carro, María Lorente, and Francisco J. Martín-Reyes, *A counting problem in ergodic theory and extrapolation for one-sided weights*, J. Anal. Math. **134** (2018), no. 1, 237–254. MR3771482

[5] Ralph Chill and Sebastian Król, *Weighted inequalities for singular integral operators on the half-line*, Studia Math. **243** (2018), no. 2, 171–206. MR3811173

[6] David V. Cruz-Uribe, José Maria Martell, and Carlos Pérez, *Weights, extrapolation and the theory of Rubio de Francia*, Operator Theory: Advances and Applications, vol. 215, Birkhäuser/Springer Basel AG, Basel, 2011. MR2797562

[7] Tuomas Hytönen and Carlos Pérez, *Sharp weighted bounds involving A_∞*, Anal. PDE **6** (2013), no. 4, 777–818, DOI 10.2140/apde.2013.6.777. MR3092729

[8] Tuomas Hytönen, Carlos Pérez, Sergei Treil, and Alexander Volberg, *Sharp weighted estimates for dyadic shifts and the A_2 conjecture*, J. Reine Angew. Math. **687** (2014), 43–86, DOI 10.1515/crelle-2012-0047. MR3176607

[9] Tuomas P. Hytönen, *The sharp weighted bound for general Calderón-Zygmund operators*, Ann. of Math. (2) **175** (2012), no. 3, 1473–1506. MR2912709

[10] Tuomas P. Hytönen, *The A_2 theorem: remarks and complements*, Harmonic analysis and partial differential equations, Contemp. Math., vol. 612, Amer. Math. Soc., Providence, RI, 2014, pp. 91–106. MR3204859

[11] Tuomas P. Hytönen, Michael T. Lacey, Henri Martikainen, Tuomas Orponen, Maria Carmen Reguera, Eric T. Sawyer, and Ignacio Uriarte-Tuero, *Weak and strong type estimates for maximal truncations of Calderón-Zygmund operators on A_p weighted spaces*, J. Anal. Math. **118** (2012), no. 1, 177–220, DOI 10.1007/s11854-012-0033-3. MR2993026

[12] Michael T. Lacey, *An elementary proof of the A_2 bound*, Israel J. Math. **217** (2017), no. 1, 181–195, DOI 10.1007/s11856-017-1442-x. MR3625108

[13] Michael T. Lacey, Stefanie Petermichl, and Maria Carmen Reguera, *Sharp A_2 inequality for Haar shift operators*, Math. Ann. **348** (2010), no. 1, 127–141, DOI 10.1007/s00208-009-0473-y. MR2657437

[14] Francisco J. Martín-Reyes and Alberto de la Torre, *Sharp weighted bounds for one-sided maximal operators*, Collect. Math. **66** (2015), no. 2, 161–174, DOI 10.1007/s13348-015-0132-4. MR3338703

[15] F. Nazarov, S. Treil, and A. Volberg, *Two weight inequalities for individual Haar multipliers and other well localized operators*, Math. Res. Lett. **15** (2008), no. 3, 583–597, DOI 10.4310/MRL.2008.v15.n3.a16. MR2407233

[16] Stefanie Petermichl, *Dyadic shifts and a logarithmic estimate for Hankel operators with matrix symbol* (English, with English and French summaries), C. R. Acad. Sci. Paris Sér. I Math. **330** (2000), no. 6, 455–460, DOI 10.1016/S0764-4442(00)00162-2. MR1756958

[17] S. Petermichl, *The sharp bound for the Hilbert transform on weighted Lebesgue spaces in terms of the classical A_p characteristic*, Amer. J. Math. **129** (2007), no. 5, 1355–1375, DOI 10.1353/ajm.2007.0036. MR2354322

[18] E. Sawyer, *Weighted inequalities for the one-sided Hardy-Littlewood maximal functions*, Trans. Amer. Math. Soc. **297** (1986), no. 1, 53–61, DOI 10.2307/2000455. MR849466

[19] Emil Vuorinen, *Two-weight L^p-inequalities for dyadic shifts and the dyadic square function*, Studia Math. **237** (2017), no. 1, 25–56, DOI 10.4064/sm8288-9-2016. MR3612889

School of Mathematical Sciences, Yangzhou University, Yangzhou 225002, People's Republic of China; School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332

Email address: weichen@yzu.edu.cn

School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332
Email address: rui.han@math.gatech.edu

School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332
Email address: lacey@math.gatech.edu