Forward Backward Doubly Stochastic Differential Equations and the Optimal Filtering of Diffusion Processes *

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Abstract

The connection between forward backward doubly stochastic differential equations and the optimal filtering problem is established without using the Zakai's equation. The solutions of forward backward doubly stochastic differential equations are expressed in terms of conditional law of a partially observed Markov diffusion process. It then follows that the adjoint time-inverse forward backward doubly stochastic differential equations governs the evolution of the unnormalized filtering density in the optimal filtering problem.

Keywords. Forward backward doubly stochastic differential equations, optimal filtering problem, Feynman-Kac formula, Itô's formula, adjoint stochastic processes.

1 Introduction

The goal of this work is to study the state of a noise-perturbed dynamical system, U_t, given noisy observation on the dynamics, V_t. This suggests the optimal filtering problem of determining the conditional probability of U_t, given an observed path $\{V_s : 0 \le s \le t\}$. The pioneer work of optimal filtering problems was considered by Kallianpur and Striebel [13] and Zakai [23]. In particular, the Kallianpur-Striebel formula provides a continuous time framework of the optimal filtering that considers the conditional probability density function (PDF) of the state as the solution of a nonlinear stochastic partial differential equation (SPDE); and the approach proposed by Zakai leads to a linear stochastic integro-differential parabolic equation, referred to as the Zakai's equation. Under strong regularity conditions it can be shown that the solution of the Zakai's equation represents an unnormalized conditional density of the state process. Fundamental research of the optimal filtering problem was also conducted by Kalman and Bucy [5, 15], Kushner and Pardoux [16, 18], Shiryaev [21] and Stratonovich [22], among other extensive studies on discrete nonlinear filter solver (see [6, 7, 9, 10, 11]).

The advantage of solving the optimal filter problems with SPDEs such as the Zakai equation is that it provides the "exact" solution for the conditional density of U_t given

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$\{V_s\}_{0\le s\le t}$. However, it has not been considered as an efficient method by the science and engineering community because of its slow convergence and high complexity. Instead of dealing with SPDEs, the unnormalized density function can also be studied through a system of stochastic (ordinary) differential equations (SDEs). Such a system consists of two SDEs, one standard SDE and one backward doubly stochastic differential equation (BDSDE), and is referred to as a system of forward backward doubly stochastic differential equations (FBDSDEs). The FBDSDE system was first studied by Pardoux and Peng in [20], where the equivalence between FBDSDEs and certain parabolic type SPDEs was established. Our recent work [1, 2, 3, 4] indicates that solving optimal filtering problems with FBDSDE systems can be far less costly than that with SPDEs and more accurate than both SPDEs and discrete filter methods such as particle filter methods.

In this paper, we establish a direct link between the optimal filtering problem and a FBDSDE system. First we provide a FBDSDE version of Feynman-Kac formula for the optimal filter problem and obtain the adjoint of this system. To the best of our knowledge, similar results have been obtained before. As a consequence, we show this adjoint, which is a a time-inverse FBDSDE system, provides a solution for the unnormalized condition density of the optimal filter problem.

The rest of this paper is organized as follows. In Section 2 we present the mathematical formulation of the optimal filtering problem and provide a brief introduction of FBDSDEs. In Section 3 we establish the connection between the FBDSDEs and the unnormalized conditional density function. Some closing remarks will be given in Section 4.

2 Preliminaries

In this section, we present the mathematical formulation of the optimal filtering problem and provide a brief introduction of FBDSDEs.

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, and let $T>0$ be fixed throughout the paper. Let $\{W_t\}_{0\le t\le T}$ and $\{B_t\}_{0\le t\le T}$ be two mutually independent standard Brownian motions defined on $(\Omega,\mathcal{F},\mathbb{P})$, with values in $\mathbb{R}^d$ and $\mathbb{R}^l$, respectively. Denote by $\mathcal{N}$ the class of $\mathbb{P}$-null sets of $\mathcal{F}$. For each $t\in[0,T]$ and any process η_t, let

$$\mathcal{F}^{\eta}_{s,t}:=\sigma\{\eta_r-\eta_s:s\le r\le t\}\vee\mathcal{N}$$

be the σ-field generated by $\{\eta_r-\eta_s\}_{s\le r\le t}$ and write $\mathcal{F}^{\eta}_t=\mathcal{F}^{\eta}_{0,t}$.

2.1 The optimal filtering problem

Consider the following stochastic differential system on the probability space $(\Omega,\mathcal{F},\mathbb{P})$

$$\begin{cases}\mathrm{d}U_t=b_t(U_t)\mathrm{d}t+\rho_t\mathrm{d}W_t+\tilde{\rho}_t\mathrm{d}B_t,\\ \mathrm{d}V_t=h(U_t)\mathrm{d}t+\mathrm{d}B_t,\end{cases}\tag{2.1}$$

where $\{U_t\in\mathbb{R}^d:t\ge 0\}$ is the "state process" that describes the state of a dynamical system and $\{V_t\in\mathbb{R}^l:t\ge 0\}$ is the "measurement process" which is the noise perturbed observations of the state U_t. Given an initial state U_0 with probability distribution $p_0(u)$ independent of W_t and B_t, the goal of the optimal filtering problem is to obtain the

best estimate of $\phi(U_t)$ as the conditional expectation with respect to the measurement $\{V_r\}_{0\le s\le t}$, where ϕ is a given test function.

Denote by $\mathcal{F}_t^V := \sigma\{V_r : 0 \le r \le t\}$ the σ-field generated by the measurement process from time 0 to t and denote by $\mathcal{M}_t$ the space of all $\mathcal{F}_t^V$-measurable and square integrable random variables at time t. The optimal filtering problem can be formulated mathematically as to find the conditional expectation

$$\mathbb{E}\left[\phi(U_t)\middle|\mathcal{F}_t^V\right] = \inf\left\{\mathbb{E}\left[|\phi(U_t)-\psi_t|^2\right] : \psi_t \in \mathcal{M}_t\right\}.$$

According to [12, 14], the *optimal* filter is given by

$$\mathbb{E}\left[\phi(U_t)\middle|\mathcal{F}_t^V\right] = \frac{\displaystyle\int_{\mathbb{R}^d}\phi(u)p_t du}{\displaystyle\int_{\mathbb{R}^d} p_t du}, \tag{2.2}$$

where p_t is the *unnormalized filtering density*. (2.2) is the well known Kallianpur–Striebel formula.

Define

$$Q_t^s := \exp\left\{\int_s^t h(U_r)dU_r - \frac{1}{2}\int_s^t |h(U_r)|^2 dr\right\}.$$

When $s=0$ we denote Q_t^0 as Q_t in short. Let $\tilde{\mathbb{P}}$ be the probability measure induced on the space $(\Omega,\mathcal{F})$ such that

$$\left.\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}\right|_{\mathcal{F}_t^V} = Q_t. \tag{2.3}$$

Then according to the Cameron-Martin theorem the probability measures $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are equivalent when the Novikov condition is satisfied [8]. Moreover, it is straightforward to verify that (see [17], Lemma 8.6.2)

$$\mathbb{E}\left[\phi(U_t)\middle|\mathcal{F}_t^V\right] = \frac{\tilde{\mathbb{E}}\left[\phi(U_t)Q_t\middle|\mathcal{F}_t^V\right]}{\tilde{\mathbb{E}}\left[Q_t\middle|\mathcal{F}_t^V\right]}. \tag{2.4}$$

where $\tilde{\mathbb{E}}$ denotes the expectation with respect to $\tilde{\mathbb{P}}$.

2.2 Forward backward doubly stochastic differential equations

For each $t\in[0,T]$, define

$$\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B.$$

Then the collection $\{\mathcal{F}_t : t\in[0,T]\}$ is neither increasing nor decreasing, and thus does not constitute a filtration [20]. For any positive integer $n\in\mathbb{N}$, denote by $\mathcal{M}^2(0,T;\mathbb{R}^n)$ the set of $\mathbb{R}^n$-valued jointly measurable random processes $\{\psi_t : t\in[0,T]\}$ such that ψ_t is $\mathcal{F}_t$ measurable for a.e. $t\in[0,T]$ and satisfies

$$\mathbb{E}\int_0^T |\psi_t|^2 dt < \infty.$$

Similarly, denote by $\mathcal{S}^2([0,T];\mathbb{R}^n)$ the set of continuous $\mathbb{R}^n$-valued random processes $\{\psi_t : t\in[0,T]\}$ such that ψ_t is $\mathcal{F}_t$ measurable for any $t\in[0,T]$ and satisfies

$$\mathbb{E}\sup_{0\le t\le T}|\psi_t|^2 < \infty.$$

We next provide a brief introduction of forward backward doubly stochastic differential equations (FBDSDEs), summarized from [20].

Given $\tau \geq 0$, $x \in \mathbb{R}^d$ and $\varphi \in \mathcal{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$, a system of forward backward doubly stochastic differential equations (FBDSDEs) can be formulated as

$$\begin{aligned} \mathrm{d}X_t &= b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad \tau \leq t \leq T, \\ -\mathrm{d}Y_t &= f(t, X_t, Y_t, Z_t)\mathrm{d}t + g(t, X_t, Y_t, Z_t)\mathrm{d}\overleftarrow{B}_t - Z_t \mathrm{d}W_t, \quad \tau \leq t \leq T, \\ X_\tau &= x, \qquad Y_T = \varphi(X_T), \end{aligned}$$

or, in the integral equation form, for any $t \in [\tau, T]$,

$$X_t = x + \int_\tau^t b(X_s)\mathrm{d}s + \int_\tau^t \sigma(X_s)\mathrm{d}W_s, \tag{2.5}$$

$$Y_t = \varphi(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)\mathrm{d}s + \int_t^T g(s, X_s, Y_s, Z_s)\mathrm{d}\overleftarrow{B}_s - \int_t^T Z_s \mathrm{d}W_s. \tag{2.6}$$

Notice that equation (2.5) is a standard forward SDE with a standard forward Itô integral and equation (2.6) is a backward doubly stochastic differential equation (BDSDE) involving the backward Itô integral $\int \cdot \mathrm{d}\overleftarrow{B}_s$ (see [19] for details on the two types of integrals).

Let the mappings $f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \to \mathbb{R}^k$ and $g : [0,T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \to \mathbb{R}^{k \times l}$ be jointly measurable and for any $(y, z) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}$,

$$f(\cdot, \cdot, y, z) \in \mathcal{M}^2(0, T; \mathbb{R}^k), \qquad g(\cdot, \cdot, y, z) \in \mathcal{M}^2(0, T; \mathbb{R}^{k \times l}).$$

Denote by $|\cdot|$ the Euclidean norm of a vector and by $\|A\| := \sqrt{\mathrm{Tr}(AA^*)}$ the norm of a matrix A. The existence and uniqueness of solutions, moment estimates for the solutions, and the regularity of solutions to Equation (2.6) rely on one or more of the following assumptions.

Assumption 2.1 *f and g satisfy the Lipschitz condition: there exist constants $c > 0$ and $0 < \bar{c} < 1$ such that for any $(t, x) \in [0,T] \times \mathbb{R}^d$, $y_1, y_2 \in \mathbb{R}^k$ and $z_1, z_2 \in \mathbb{R}^{k \times d}$,*

$$\begin{aligned} |f(t, x, y_1, z_1) - f(t, x, y_2, z_2)|^2 &\leq c(|y_1 - y_2|^2 + \|z_1 - z_2\|^2), \\ \|g(t, x, y_1, z_1) - g(t, x, y_2, z_2)\|^2 &\leq c|y_1 - y_2|^2 + \bar{c}\|z_1 - z_2\|^2. \end{aligned}$$

Assumption 2.2 *There exists $c > 0$ such that for all $(t, x, y, z) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$,*

$$gg^*(t, x, y, z) \leq zz^* + c(\|g(t, x, 0, 0)\|^2 + |y|^2)I.$$

Assumption 2.3 *For any $(t, x, y, z) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$ and $\theta \in \mathbb{R}^{k \times d}$*

$$\frac{\partial g}{\partial z}(t, x, y, z)\theta\theta^* \left(\frac{\partial g}{\partial z}(t, x, y, z)\right)^* \leq \theta\theta^*.$$

The following results are due to Pardoux and Peng [20].

Proposition 2.4 *Under Assumption 2.1, the BDSDE* (2.6) *admits a unique solution*

$$(Y, Z) \in \mathcal{S}^2([0,T]; \mathbb{R}^k) \times \mathcal{M}^2(0, T; \mathbb{R}^{k \times d}).$$

Proposition 2.5 *Let Assumptions 2.1 and 2.2 hold, then the solution of the BDSDE* (2.6) *satisfies*

$$\mathbb{E}\sup_{0\le t\le T}|Y_t|^2<\infty.$$

For any positive integer k, denote by $\mathcal{C}^k_{l,b}$ the collection of $\mathcal{C}^k$ functions with bounded partial derivatives of all orders less than or equal to k, and denote by $\mathcal{C}^k_p$ the collection of $\mathcal{C}^k$ functions with partial derivatives of all orders less than or equal to k which grow at most like a polynomial function of x as $x\to\infty$. It is well known that given $b\in\mathcal{C}^3_{l,b}(\mathbb{R}^d,\mathbb{R}^d)$ and $\sigma\in\mathcal{C}^3_{l,b}(\mathbb{R}^d,\mathbb{R}^{d\times d})$, for each $(\tau,x)\in[0,T]\times\mathbb{R}^d$, the SDE (2.5) has a unique strong solution, denoted as $X^{\tau,x}_t$. Consequently denote by $(Y^{\tau,x}_t,Z^{\tau,x}_t)$ the unique solution to the BDSDE

$$Y_t=\varphi(X^{\tau,x}_T)+\int_t^T f(s,X^{\tau,x}_s,Y_s,Z_s)\mathrm{d}s+\int_t^T g(s,X^{\tau,x}_s,Y_s,Z_s)\mathrm{d}\overleftarrow{B}_s-\int_t^T Z_s\mathrm{d}W_s. \quad (2.7)$$

Proposition 2.6 *Let $\varphi\in\mathcal{C}^3_p(\mathbb{R}^d;\mathbb{R}^k)$. Under Assumptions 2.1 – 2.3, the random field $\{Y^{\tau,x}_\tau:\tau\in[0,T],x\in\mathbb{R}^d\}$ admits a continuous version such that for any $\tau\in[0,T]$, $x\mapsto Y^{\tau,x}_\tau$ is of class $\mathcal{C}^2$ a.s..*

The following regularity result can be obtained by using standard techniques of SDEs, FBSDEs and BDSDEs (see Proposition 1 in [3])

Lemma 2.7 *In addition to the Assumption 2.1, assume that $f,g\in\mathcal{C}^1_{l,b}$. Then the solution $(Y^{\tau,x}_t,Z^{\tau,x}_t)$ to the BDSDE* (2.7) *satisfies*

$$\mathbb{E}\left[(Y^{\tau,x}_t-Y^{\tau,x}_\tau)^2\right]\le C(t-\tau),\quad \mathbb{E}\left[(Z^{\tau,x}_t-Z^{\tau,x}_\tau)^2\right]\le C(t-\tau),\quad 0\le\tau\le t\le T,$$

where C is a positive constant independent of τ and t.

Note that with the convention above, the unique solution to the FBDSDE system (2.5) – (2.6) can be written as $(X^{\tau,x}_t,Y^{\tau,x}_t,Z^{\tau,x}_t)$. Denote

$$\nabla X^{\tau,x}_t:=\frac{\partial X^{\tau,x}_t}{\partial x},\qquad \nabla Y^{\tau,x}_t:=\frac{\partial Y^{\tau,x}_t}{\partial x},\qquad \nabla Z^{\tau,x}_t:=\frac{\partial Z^{\tau,x}_t}{\partial x}.$$

Then $(\nabla Y^{\tau,x}_t,\nabla Z^{\tau,x}_t)$ is the unique solution to variational form of the BDSDE (2.6) (see [20])

$$\begin{aligned}\nabla Y^{\tau,x}_t=&\varphi'(X^{\tau,x}_T)\nabla X^{\tau,x}_T+\int_t^T\left(\frac{\partial f}{\partial x}\nabla X^{\tau,x}_s+\frac{\partial f}{\partial Y}\nabla Y^{\tau,x}_s+\frac{\partial f}{\partial Z}\nabla Z^{\tau,x}_s\right)\mathrm{d}s\\&+\int_t^T\left(\frac{\partial g}{\partial x}\nabla X^{\tau,x}_s+\frac{\partial g}{\partial Y}\nabla Y^{\tau,x}_s+\frac{\partial g}{\partial Z}\nabla Z^{\tau,x}_s\right)\mathrm{d}\overleftarrow{B}_s-\int_t^T\nabla Z^{\tau,x}_s\mathrm{d}W_s.\end{aligned}$$

In addition, the random field $\{Z^{\tau,x}_t:t\in[\tau,T],x\in\mathbb{R}^d\}$ has an a.s. continuous version

$$Z^{\tau,x}_t=\nabla Y^{\tau,x}_t(\nabla X^{\tau,x}_t)^{-1}\sigma(X^{\tau,x}_t),\qquad Z^{\tau,x}_\tau=\nabla Y^{\tau,x}_\tau\sigma(x). \quad (2.8)$$

The following Lemma follows directly from Lemma 2.7 and Proposition 2.5.

Lemma 2.8 *Assume that $b\in\mathcal{C}^2_{l,b}$, $f\in\mathcal{C}^2_{l,b}$, $g\in\mathcal{C}^2_{l,b}$ and $\varphi\in\mathcal{C}^2_{l,b}$. Then there exists $C>0$ such that*

$$\mathbb{E}[(\nabla Y^{\tau,x}_t-\nabla Y^{\tau,x}_t)^2]\le C(t-\tau),\quad \mathbb{E}[(\nabla Z^{\tau,x}_t-\nabla Z^{\tau,x}_\tau)^2]\le C(t-\tau),\quad 0\le\tau\le t\le T.$$

Moreover,

$$\mathbb{E}\sup_{0\le t\le T}|\nabla Y^{\tau,x}_t|^2<\infty.$$

3 FBDSDEs and Optimal Filtering

In this section, we establish the connection between the optimal filtering problem and a FBDSDE system. In particular, we will first prove a Feynman-Kac formula in the filtering context. Then we present the adjoint relationship between standard FBDSDEs and time-inverse FBDSDEs. In the end we will show that the solution of a time-inverse FBDSDE is the unnormalized filtering density sought in the optimal filtering problem. For simplicity of exposition, we only discuss the one dimensional case with $d=1$ and $l=1$. The same method can also be applied to multi-dimensional cases with more complicated calculations.

3.1 Feynman-Kac type formula for optimal filtering

For $\tau\in[0,T]$ and $x\in\mathbb{R}^d$, consider the following FBDSDE system on the probability space $(\Omega,\mathcal{F},\tilde{\mathbb{P}})$

$$\begin{cases} \mathrm{d}X_t = b_t(X_t)\mathrm{d}t+\sigma_t\mathrm{d}W_t, \quad \tau\le t\le T & \text{(SDE)} \\ -\mathrm{d}Y_t = -Z_t\mathrm{d}W_t+\left(h(X_t)Y_t+\dfrac{\tilde{\rho}_t}{\sigma_t}Z_t\right)\mathrm{d}\overleftarrow{V}_t, \quad \tau\le t\le T & \text{(BDSDE)} \\ X_\tau = x, \qquad Y_T=\phi(X_T), \end{cases} \tag{3.9}$$

where $\sigma_t^2=\rho_t^2+\tilde{\rho}_t^2$, and b, ρ, $\tilde{\rho}$, h are the functions appeared in the optimal filtering problem (2.1). Here W_t is the same Brownian motion as in the nonlinear filtering problem (2.1), while V_t is the measurement process which becomes a standard Brownian motion independent of W_t under the induced probability measure $\tilde{\mathbb{P}}$ defined by (2.3). Then X_t is a $\mathcal{F}_t^W$ adaptive stochastic process and the pair (Y_t,Z_t) is adaptive to $\mathcal{F}_t^W\vee\mathcal{F}_{t,T}^V$. For any single-variable function $F=F(x)$, denote $F':=\frac{dF}{dx}$ and $F''=\frac{d^2F}{dx^2}$.

Lemma 3.1 *Assume that b_t and σ_t are bounded and $h\in\mathcal{C}_b^2(\mathbb{R};\mathbb{R})$. Then for any $0\le s\le t\le T$, there exists a positive constant C independent of s and t such that*

$$\tilde{\mathbb{E}}[(h(X_t)-h(X_s))^2|\mathcal{F}_{t,T}^V]\le C(t-s). \tag{3.10}$$

Proof. The application of Itô's formula to $h(X_t)$ results in

$$h(X_t)=h(X_s)+\int_s^t\left(b_r(X_r)h'(X_r)+\frac{\sigma_r^2}{2}h''(X_r)\right)\mathrm{d}r+\int_s^t\sigma_r h'(X_r)\mathrm{d}W_r,$$

and hence

$$(h(X_t)-h(X_s))^2=\left(\int_s^t\left(b_r(X_r)h'(X_r)+\frac{\sigma_r^2}{2}h''(X_r)\right)\mathrm{d}r+\int_s^t\sigma_r h'(X_r)\mathrm{d}W_r\right)^2. \tag{3.11}$$

Taking expectation $\tilde{\mathbb{E}}$ of the above gives

$$\tilde{\mathbb{E}}\big[(h(X_t)-h(X_s))^2\big]=\tilde{\mathbb{E}}\left[\Big(\int_s^t\big(b_r(X_r)h'(X_r)+\frac{\sigma_r^2}{2}h''(X_r)\big)\mathrm{d}r\Big)^2\right]+\tilde{\mathbb{E}}\left[\int_s^t\big(\sigma_r h'(X_r)\big)^2\mathrm{d}r\right].$$

The inequality (3.10) then follows immediately from the assumptions of the lemma. □

With Proposition 2.5, and Lemmas 2.7 and 3.1, we establish the following Feynman-Kac formula in the optimal filtering context.

Theorem 3.2 *Assume that ϕ is bounded, $b_t,\rho_t,\tilde{\rho}_t\in\mathcal{C}_{l,b}$ and $h\in\mathcal{C}^2_{l,b}(\mathbb{R})$. Then, $\forall\tau\in[0,T]$ and $x\in\mathbb{R}^d$ the following equality holds a.s.*

$$Y^{\tau,x}_\tau=\mathbb{E}^x_\tau[\phi(U_T)Q^\tau_T], \tag{3.12}$$

where $\mathbb{E}^x_\tau[\cdot]:=\tilde{\mathbb{E}}[\cdot|\mathcal{F}^V_{\tau,T},U_\tau=x]$.

Proof. We prove the statement (3.12) for $\tau=0$ only, the general case follows from the $\tau=0$ case trivially. First it is straightforward to verify that under assumptions in Theorem 3.2, all the assumptions of Proposition 2.5, and Lemmas 2.7 and 3.1 are fulfilled. Since $Y^{\tau,x}_\tau$ and $Z^{\tau,x}_\tau$ are functions of x, we write $Y^{\tau,x}_\tau=Y_\tau(x)$ and $Z^{\tau,x}_\tau=Z_\tau(x)$ in the sequel.

Let $0=t_0<t_1<t_2\cdots<t_N=T$ be an equidistant temporal partition with $t_{n+1}-t_n=T/N:=\Delta t$ and define

$$\Delta_n=\mathbb{E}^x_0[Q_{t_{n+1}}Y_{t_{n+1}}(U_{t_{n+1}})-Q_{t_n}Y_{t_n}(U_{t_n})].$$

It follows immediately that

$$\mathbb{E}^x_0[\phi(U_T)Q_T-Y_0(x)]=\sum_{n=0}^{N-1}\Delta_n.$$

Denote $\tilde{\mathbb{P}}_x:=\tilde{\mathbb{P}}(\cdot|U_0=x)$. To prove (3.12) it suffices to verify that

$$\sum_{n=0}^{N-1}\Delta_n\overset{N\to\infty}{\longrightarrow}0\text{ in }\mathcal{L}^1(\Omega,\tilde{\mathbb{P}}_x).$$

For each $n\geq0$, let U_{t_n} be the solution of the state for (2.1) at time step t_n and consider the FBDSDEs system (3.9) on $[t_n,t_{n+1}]$ with initial condition U_{t_n}:

$$\begin{cases}\mathrm{d}\hat{X}_t=b_t(\hat{X}_t)\mathrm{d}t+\sigma_t\mathrm{d}W_t,\\ -\mathrm{d}Y_t=-Z_t\mathrm{d}W_t+\left(h(\hat{X}_t)Y_t+\dfrac{\tilde{\rho}_t}{\sigma_t}Z_t\right)\mathrm{d}\overleftarrow{V}_t,\\ \hat{X}_{t_n}=U_{t_n},\quad Y_{t_{n+1}}=Y_{t_{n+1}}(\hat{X}_{t_{n+1}}).\end{cases} \tag{3.13}$$

From the definition of the state process U_t in (2.1) and the SDE $\hat{X}_t$ in (3.13), we have the relation between $U_{t_{n+1}}$ and $\hat{X}_{t_{n+1}}$:

$$U_{t_{n+1}}=\hat{X}_{t_{n+1}}+\int_{t_n}^{t_{n+1}}\rho_s\mathrm{d}W_s-\int_{t_n}^{t_{n+1}}\sigma_s(\hat{X}_s)\mathrm{d}W_s+\int_{t_n}^{t_{n+1}}\tilde{\rho}_s(\mathrm{d}V_s-h(U_s)\mathrm{d}s)+R^{n+1}_X$$

where

$$R^{n+1}_X=\int_{t_n}^{t_{n+1}}b_s(U_s)ds-\int_{t_n}^{t_{n+1}}b_s(\hat{X}_s)ds.$$

To simplify presentation, for any process ψ_t we write $\hat{\psi}_t:=\psi_t(\hat{X}_t)$ throughout the rest of this proof. Let $\eta_{n+1}=U_{t_{n+1}}-\hat{X}_{t_{n+1}}$ Then from the above we have that

$$\eta_{n+1}=\int_{t_n}^{t_{n+1}}\rho_s\mathrm{d}W_s-\int_{t_n}^{t_{n+1}}\sigma_s\mathrm{d}W_s+\int_{t_n}^{t_{n+1}}\tilde{\rho}_s(\mathrm{d}V_s-h(U_s)\mathrm{d}s)+R^{n+1}_X \tag{3.14}$$

Applying the Taylor expansion to $Y_{t_{n+1}}$ we have that

$$Y_{t_{n+1}}(U_{t_{n+1}}) = \hat{Y}_{t_{n+1}} + \hat{Y}'_{t_{n+1}} \cdot \eta_{n+1} + \frac{1}{2}\hat{Y}''_{t_{n+1}} \cdot (\eta_{n+1})^2 + \xi_{n+1}, \tag{3.15}$$

where ξ_{n+1} is the Taylor remainder such that $\mathbb{E}_0^x[(\xi_{n+1})^2] \le C(\Delta t)^3$. Then for each $n = 0, 1, 2, \cdots, N-1$,

$$\begin{aligned}
\Delta_n &= \mathbb{E}_0^x \left[Q_{t_{n+1}} Y_{t_{n+1}}(U_{t_{n+1}}) - Q_{t_n} \hat{Y}_{t_{n+1}} + Q_{t_n} \hat{Y}_{t_{n+1}} - Q_{t_n} Y_{t_n}(U_{t_n}) \right] \\
&= \underbrace{\mathbb{E}_0^x \left[\left(Q_{t_{n+1}} - Q_{t_n} \right) \hat{Y}_{t_{n+1}} \right]}_{\textbf{(i)}} + \underbrace{\mathbb{E}_0^x \left[Q_{t_n} \left(\hat{Y}_{t_{n+1}} - Y_{t_n}(U_{t_n}) \right) \right]}_{\textbf{(ii)}} \\
&\quad + \underbrace{\mathbb{E}_0^x \left[Q_{t_{n+1}} \left(\hat{Y}'_{t_{n+1}} \eta_{n+1} + \frac{1}{2} \hat{Y}''_{t_{n+1}} \cdot (\eta_{n+1})^2 + \xi_{n+1} \right) \right]}_{\textbf{(iii)}}.
\end{aligned} \tag{3.16}$$

We next estimate terms (**i**), (**ii**) and (**iii**) in (3.16) one by one.

(**i**) Write $h_t = h(U_t)$ and $\hat{h}_t = h(\hat{X}_t)$, and apply Ito's formula to Q_{t_n} we obtain

$$\begin{aligned}
\mathbb{E}_0^x \left[(Q_{t_{n+1}} - Q_{t_n}) \hat{Y}_{t_{n+1}} \right] &= \mathbb{E}_0^x \left[\int_{t_n}^{t_{n+1}} h_s Q_s \mathrm{d}V_s \hat{Y}_{t_{n+1}} \right] \\
&= \mathbb{E}_0^x \left[\int_{t_n}^{t_{n+1}} \hat{h}_s Q_s \mathrm{d}V_s \hat{Y}_{t_{n+1}} \right] + \mathbb{E}_0^x \left[\int_{t_n}^{t_{n+1}} (h_s - \hat{h}_s) Q_s \mathrm{d}V_s \hat{Y}_{t_{n+1}} \right].
\end{aligned} \tag{3.17}$$

Applying Itô formula to function h yields

$$h_s - \hat{h}_s = h'(U_{t_n}) \left(\int_{t_n}^{s} \rho_r \mathrm{d}W_r + \int_{t_n}^{s} \tilde{\rho}_r \mathrm{d}V_r - \int_{t_n}^{s} \sigma_r \mathrm{d}W_r \right) + \mathcal{O}(\Delta t),$$

and consequently with $h'_{t_n} := h'(U_{t_n})$ we have

$$\begin{aligned}
&\mathbb{E}_0^x \left[\int_{t_n}^{t_{n+1}} (h_s - \hat{h}_s) Q_s \mathrm{d}V_s \hat{Y}_{t_{n+1}} \right] \\
=&\mathbb{E}_0^x \left[h'_{t_n} Q_{t_n} Y_{t_n}(U_{t_n}) \int_{t_n}^{t_{n+1}} \mathrm{d}V_s \left(\int_{t_n}^{s} \rho_r \mathrm{d}W_r - \int_{t_n}^{s} \sigma_r \mathrm{d}W_r \right) \right] \\
&+ \mathbb{E}_0^x \left[h'_{t_n} (Q_s - Q_{t_n}) \left(\hat{Y}_{t_{n+1}} - Y_{t_n}(U_{t_n}) \right) \int_{t_n}^{t_{n+1}} \mathrm{d}V_s \left(\int_{t_n}^{s} \rho_r \mathrm{d}W_r - \int_{t_n}^{s} \sigma_r \mathrm{d}W_r \right) \right] \\
&+ \mathbb{E}_0^x \left[h'_{t_n} Q_{t_n} \hat{Y}_{t_{n+1}} \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \tilde{\rho}_r \mathrm{d}V_r \mathrm{d}V_s \right] + \mathcal{O}\left((\Delta t)^{\frac{3}{2}} \right).
\end{aligned} \tag{3.18}$$

First noting that $h'_{t_n} Q_{t_n} Y_{t_n}(U_{t_n}) \int_{t_n}^{t_{n+1}} \mathrm{d}V_s$ is independent of $\int_{t_n}^{s} \rho_r \mathrm{d}W_r - \int_{t_n}^{s} \sigma_r \mathrm{d}W_r$, we have

$$\mathbb{E}_0^x \left[h'_{t_n} Q_{t_n} Y_{t_n}(U_{t_n}) \int_{t_n}^{t_{n+1}} \mathrm{d}V_s \left(\int_{t_n}^{s} \rho_r \mathrm{d}W_r - \int_{t_n}^{s} \sigma_r \mathrm{d}W_r \right) \right] = 0. \tag{3.19}$$

Second, it's straightforward to verify that

$$\mathbb{E}_0^x \left[h'_{t_n} (Q_s - Q_{t_n}) \left(\hat{Y}_{t_{n+1}} - Y_{t_n}(U_{t_n}) \right) \int_{t_n}^{t_{n+1}} \mathrm{d}V_s \left(\int_{t_n}^{s} \rho_r \mathrm{d}W_r - \int_{t_n}^{s} \sigma_r \mathrm{d}W_r \right) \right] \sim \mathcal{O}\left((\Delta t)^{\frac{3}{2}} \right). \tag{3.20}$$

Putting (3.19) and (3.20) in (3.18), it follows from the regularity condition of $\tilde{\rho}_r$ that

$$\mathbb{E}_0^x\left[\int_{t_n}^{t_{n+1}}(h_s-\hat{h}_s)Q_s\mathrm{d}V_s\hat{Y}_{t_{n+1}}\right]=\mathbb{E}_0^x\left[h_{t_n}^{'}Q_{t_n}\hat{Y}_{t_{n+1}}\tilde{\rho}_{t_n}\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}\mathrm{d}V_r\mathrm{d}V_s\right]+\mathcal{O}\left((\Delta t)^{\frac{3}{2}}\right).$$

Define

$$\nu_n:=h_{t_n}^{'}Q_{t_n}\hat{Y}_{t_{n+1}}\tilde{\rho}_{t_n}\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}\mathrm{d}V_r\mathrm{d}V_s. \tag{3.21}$$

Then by using the facts $\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}\mathrm{d}V_r\mathrm{d}V_s=\frac{1}{2}\big((V_{t_{n+1}}-V_{t_n})^2-\Delta t\big)$ and $h_{t_n}^{'}Q_{t_n}\hat{Y}_{t_{n+1}}\tilde{\rho}_{t_n}$ is independent of $\frac{1}{2}\big((V_{t_{n+1}}-V_{t_n})^2-\Delta t\big)$ we have

$$\sum_{n=0}^{N-1}\nu_n=\sum_{n=0}^{N-1}h_{t_n}^{'}Q_{t_n}\hat{Y}_{t_{n+1}}\tilde{\rho}_{t_n}\cdot\frac{1}{2}\big((V_{t_{n+1}}-V_{t_n})^2-\Delta t\big)\overset{N\to\infty}{\longrightarrow}0\text{ in }\mathcal{L}^1(\Omega,\tilde{\mathbb{P}}_x). \tag{3.22}$$

In summary (3.17) gives the estimate of the term (**i**) in (3.16) as

$$(\mathbf{i})=\mathbb{E}_0^x\left[\int_{t_n}^{t_{n+1}}\hat{h}_sQ_s\mathrm{d}V_s\hat{Y}_{t_{n+1}}\right]+\mathbb{E}_0^x[\nu_n]+\mathcal{O}\left((\Delta)^{\frac{3}{2}}\right), \tag{3.23}$$

with $\sum_{n=0}^{N-1}\mathbb{E}_0^x[\nu_n]\to0$ in $\mathcal{L}^1(\Omega,\tilde{\mathbb{P}}_x)$ as $N\to\infty$.

(**ii**) It follows directly from the FBDSDEs system (3.13) that term (**ii**) in (3.16) satisfies

$$\begin{aligned}(\mathbf{ii})&=\mathbb{E}_0^x\left[Q_{t_n}\int_{t_n}^{t_{n+1}}\hat{Z}_s\mathrm{d}W_s-Q_{t_n}\int_{t_n}^{t_{n+1}}\left(\hat{h}_s\hat{Y}_s+\frac{\tilde{\rho}_s}{\sigma_s}\hat{Z}_s\right)\mathrm{d}\overleftarrow{V}_s\right]\\&=-\mathbb{E}_0^x\left[Q_{t_n}\int_{t_n}^{t_{n+1}}\left(\hat{h}_s\hat{Y}_s+\frac{\tilde{\rho}_s}{\sigma_s}\hat{Z}_s\right)\mathrm{d}\overleftarrow{V}_s\right].\end{aligned} \tag{3.24}$$

(**iii**) By splitting term (**iii**) in (3.16) and using the definition of η_{n+1} in (3.14) we obtain

$$\begin{aligned}(\mathbf{iii})&=\mathbb{E}_0^x\left[Q_{t_{n+1}}\hat{Y}_{t_{n+1}}^{'}\eta_{n+1}\right]+\frac{1}{2}\mathbb{E}_0^x\left[Q_{t_{n+1}}\hat{Y}_{t_{n+1}}^{''}\cdot(\eta_{n+1})^2\right]+\mathbb{E}_0^x\left[Q_{t_{n+1}}\xi_{n+1}\right]\\&=\underbrace{\mathbb{E}_0^x\left[Q_{t_{n+1}}\hat{Y}_{t_{n+1}}^{'}\left(\int_{t_n}^{t_{n+1}}\rho_s\mathrm{d}W_s-\int_{t_n}^{t_{n+1}}\sigma_s\mathrm{d}W_s\right)\right]}_{(\mathbf{iii-1})}+\underbrace{\mathbb{E}_0^x\left[Q_{t_n}\hat{Y}_{t_{n+1}}^{'}\int_{t_n}^{t_{n+1}}\tilde{\rho}_s\mathrm{d}V_s\right]}_{(\mathbf{iii-2})}\\&\quad+\underbrace{\mathbb{E}_0^x\left[(Q_{t_{n+1}}-Q_{t_n})\hat{Y}_{t_{n+1}}^{'}\int_{t_n}^{t_{n+1}}\tilde{\rho}_s\mathrm{d}V_s\right]-\mathbb{E}_0^x\left[Q_{t_{n+1}}\hat{Y}_{t_{n+1}}^{'}\int_{t_n}^{t_{n+1}}\tilde{\rho}_sh_s\mathrm{d}s\right]}_{(\mathbf{iii-3})}\\&\quad+\underbrace{\mathbb{E}_0^x\left[Q_{t_{n+1}}\hat{Y}_{t_{n+1}}^{'}R_X^{n+1}\right]}_{(\mathbf{iii-4})}+\frac{1}{2}\underbrace{\mathbb{E}_0^x\left[Q_{t_{n+1}}\hat{Y}_{t_{n+1}}^{''}\cdot(\eta_{n+1})^2\right]}_{(\mathbf{iii-5})}+\mathbb{E}_0^x\left[Q_{t_{n+1}}\xi_{n+1}\right].\end{aligned} \tag{3.25}$$

We next estimate terms (**iii-1**) – (**iii-5**).

Denote

$$\nabla\hat{X}_t:=\frac{\partial\hat{X}_t^{t_n,x}}{\partial x}|_{x=U_{t_n}},\qquad\nabla\hat{Y}_t:=\frac{\partial Y_t^{t_n,x}}{\partial x}|_{x=U_{t_n}},\qquad\nabla\hat{Z}_t:=\frac{\partial Z_t^{t_n,x}}{\partial x}|_{x=U_{t_n}}.$$

Then term (**iii-1**) can be written as

$$(\mathbf{iii}-\mathbf{1}) = \mathbb{E}_0^x\left[Q_{t_{n+1}}\left(\hat{Y}'_{t_{n+1}}\nabla\hat{X}_{t_{n+1}}\right)\left(\int_{t_n}^{t_{n+1}}\rho_s \mathrm{d}W_s - \int_{t_n}^{t_{n+1}}\sigma_s \mathrm{d}W_s\right)\left(\nabla\hat{X}_{t_{n+1}}\right)^{-1}\right]. \tag{3.26}$$

By using the fact that $|(\nabla\hat{X}_{t_{n+1}})^{-1}| = 1+\mathcal{O}(\Delta t)$ and the following variational equation (see [20])

$$\nabla\hat{Y}_t = \hat{Y}'_{t_{n+1}}\nabla\hat{X}_{t_{n+1}} + \int_t^{t_{n+1}}\left(\hat{h}'_s\hat{Y}_s\nabla\hat{X}_s + \hat{h}_s\nabla\hat{Y}_s + \frac{\tilde{\rho}_s}{\sigma_s}\nabla\hat{Z}_s\right)\mathrm{d}\overleftarrow{V}_s - \int_t^{t_{n+1}}\nabla\hat{Z}_s \mathrm{d}W_s,$$

we deduce that (3.26) becomes

$$\begin{aligned}(\mathbf{iii}-\mathbf{1}) &= \mathbb{E}_0^x\left[Q_{t_{n+1}}\left(\hat{Y}'_{t_{n+1}}\nabla\hat{X}_{t_{n+1}} - \nabla\hat{Y}_{t_n}\right)\left(\int_{t_n}^{t_{n+1}}\rho_s \mathrm{d}W_s - \int_{t_n}^{t_{n+1}}\sigma_s \mathrm{d}W_s\right)\right] \\ &\quad + \mathbb{E}_0^x\left[Q_{t_{n+1}}\nabla\hat{Y}_{t_n}\left(\int_{t_n}^{t_{n+1}}\rho_s \mathrm{d}W_s - \int_{t_n}^{t_{n+1}}\sigma_s \mathrm{d}W_s\right)\right] + \mathcal{O}((\Delta t)^{\frac{3}{2}}) \\ &= \mathbb{E}_0^x\left[\left(Q_{t_n}\int_{t_n}^{t_{n+1}}\nabla\hat{Z}_s \mathrm{d}W_s + \lambda_{t_n}\right)\cdot\left(\int_{t_n}^{t_{n+1}}\rho_s \mathrm{d}W_s - \int_{t_n}^{t_{n+1}}\sigma_s \mathrm{d}W_s\right)\right] + \mathcal{O}((\Delta t)^{\frac{3}{2}}),\end{aligned}$$

where $\lambda_{t_n} = -Q_{t_n}\int_{t_n}^{t_{n+1}}\left(\hat{h}'_s\bar{Y}_s\nabla\hat{X}_s + \hat{h}_s\nabla\hat{Y}_s + \frac{\tilde{\rho}_s}{\sigma_s}\nabla\hat{Z}_s\right)d\overleftarrow{V}_s + Q_{t_{n+1}}\nabla\hat{Y}_{t_n}$ is independent of $\int_{t_n}^{t_{n+1}}\rho_s \mathrm{d}W_s - \int_{t_n}^{t_{n+1}}\sigma_s \mathrm{d}W_s$ and hence gives

$$\mathbb{E}_0^x\left[\lambda_{t_n}\left(\int_{t_n}^{t_{n+1}}\rho_s \mathrm{d}W_s - \int_{t_n}^{t_{n+1}}\sigma_s \mathrm{d}W_s\right)\right] = 0.$$

As a consequence

$$\begin{aligned}(\mathbf{iii}-\mathbf{1}) &= \mathbb{E}_0^x\left[Q_{t_n}\int_{t_n}^{t_{n+1}}\nabla\hat{Z}_s \mathrm{d}W_s\cdot\left(\int_{t_n}^{t_{n+1}}\rho_s \mathrm{d}W_s - \int_{t_n}^{t_{n+1}}\sigma_s \mathrm{d}W_s\right)\right] + \mathcal{O}\left((\Delta t)^{\frac{3}{2}}\right) \\ &= \mathbb{E}_0^x\left[Q_{t_n}\nabla\hat{Z}_{t_{n+1}}\cdot\int_{t_n}^{t_{n+1}}(\rho_s-\sigma_s)\mathrm{d}s\right] + \mathcal{O}\left((\Delta t)^{\frac{3}{2}}\right).\end{aligned} \tag{3.27}$$

Let C represent a generic constant while the context is clear. By the definition of R_X^{n+1}, it is straightforward to verify that

$$(\mathbf{iii}-\mathbf{4}) = \mathbb{E}_0^x\left[Q_{t_{n+1}}\hat{Y}'_{t_{n+1}}R_X^{n+1}\right] \le C(\Delta t)^{\frac{3}{2}}. \tag{3.28}$$

Applying Itô formula to Q_{t_n} in term (**iii-3**) we obtain

$$\begin{aligned}(\mathbf{iii-3}) &= \mathbb{E}_0^x\left[(Q_{t_{n+1}}-Q_{t_n})\hat{Y}'_{t_{n+1}}\int_{t_n}^{t_{n+1}}\tilde{\rho}_s \mathrm{d}V_s\right] - \mathbb{E}_0^x\left[Q_{t_{n+1}}\hat{Y}'_{t_{n+1}}\int_{t_n}^{t_{n+1}}\tilde{\rho}_s h_s \mathrm{d}s\right]\\ &= \mathbb{E}_0^x\left[\int_{t_n}^{t_{n+1}} h_s Q_s \mathrm{d}V_s \int_{t_n}^{t_{n+1}}\tilde{\rho}_s \mathrm{d}V_s \hat{Y}'_{t_{n+1}}\right] - \mathbb{E}_0^x\left[Q_{t_{n+1}}\hat{Y}'_{t_{n+1}}\int_{t_n}^{t_{n+1}}\tilde{\rho}_s h_s \mathrm{d}s\right]\\ &\le \left|\mathbb{E}_0^x\left[\int_{t_n}^{t_{n+1}} h_s (Q_s - Q_{t_{n+1}})\mathrm{d}V_s \int_{t_n}^{t_{n+1}}\tilde{\rho}_s \mathrm{d}V_s \hat{Y}'_{t_{n+1}}\right]\right|\\ &\quad + \left|\mathbb{E}_0^x\left[Q_{t_{n+1}}\hat{Y}'_{t_{n+1}}\left(\int_{t_n}^{t_{n+1}} h_s \mathrm{d}V_s \int_{t_n}^{t_{n+1}}\tilde{\rho}_s \mathrm{d}V_s - \int_{t_n}^{t_{n+1}}\tilde{\rho}_s h_s \mathrm{d}s\right)\right]\right|\\ &\le C(\Delta t)^{\frac{3}{2}}.\end{aligned} \tag{3.29}$$

By using the definition of η_{n+1} in (3.14), we deduce that

$$\begin{aligned}(\mathbf{iii-5}) &= \frac{1}{2}\mathbb{E}_0^x\left[Q_{t_n}\hat{Y}''_{t_{n+1}}\int_{t_n}^{t_{n+1}}\left(\rho_s^2+\tilde{\rho}_s^2+\sigma_s^2-2\rho_s\sigma_s\right)\mathrm{d}s\right] + \mathcal{O}\left((\Delta t)^{\frac{3}{2}}\right)\\ &= \mathbb{E}_0^x\left[Q_{t_n}\hat{Y}''_{t_{n+1}}\int_{t_n}^{t_{n+1}}\left(\sigma_s^2-\rho_s\sigma_s\right)\mathrm{d}s\right] + \mathcal{O}\left((\Delta t)^{\frac{3}{2}}\right).\end{aligned}$$

As a simple corollary of the assertion (2.8), we have $\hat{Y}''_{t_{n+1}}\sigma_{t_{n+1}} = \nabla\hat{Z}_{t_{n+1}} + \mathcal{O}(\Delta t)$ and thus

$$(\mathbf{iii-5}) = \mathbb{E}_0^x\left[Q_{t_n}\nabla\hat{Z}_{t_{n+1}}\int_{t_n}^{t_{n+1}}(\sigma_s-\rho_s)\mathrm{d}s\right] + \mathcal{O}\left((\Delta t)^{\frac{3}{2}}\right). \tag{3.30}$$

It then remains to estimate term (**iii-2**). Notice that due to equations (2.8) and (3.1) we have $\hat{Z}_s/\sigma_s = \nabla\hat{Y}_s(\nabla\hat{X}_s)^{-1}$. Hence for any $s\in[t_n,t_{n+1}]$ it holds

$$\begin{aligned}\hat{Y}'_{t_{n+1}} - \frac{\hat{Z}_s}{\sigma_s} &= \hat{Y}'_{t_{n+1}} - \nabla\hat{Y}_s(\nabla\hat{X}_s)^{-1}\\ &= -\int_s^{t_{n+1}}\left(\hat{h}'_r\hat{Y}_r\nabla\hat{X}_r + \hat{h}_r\nabla\hat{Y}_r + \frac{\tilde{\rho}_r}{\sigma_r}\nabla\hat{Z}_r\right)\mathrm{d}\overleftarrow{V}_r - \int_s^{t_{n+1}}\nabla\hat{Z}_r\mathrm{d}W_r + \mathcal{O}(\Delta t),\end{aligned}$$

and therefore

$$\begin{aligned}-Q_{t_n}\int_{t_n}^{t_{n+1}}\frac{\tilde{\rho}_s}{\sigma_s}\hat{Z}_s\mathrm{d}\overleftarrow{V}_s &= -Q_{t_n}\hat{Y}'_{t_{n+1}}\int_{t_n}^{t_{n+1}}\tilde{\rho}_s\mathrm{d}\overleftarrow{V}_s\\ &\quad - Q_{t_n}\int_{t_n}^{t_{n+1}}\tilde{\rho}_s\int_s^{t_{n+1}}\left(\hat{h}'_r\hat{Y}_r\nabla\hat{X}_r + \hat{h}_r\nabla\hat{Y}_r + \frac{\tilde{\rho}_r}{\sigma_r}\nabla\hat{Z}_r\right)\mathrm{d}\overleftarrow{V}_r\mathrm{d}\overleftarrow{V}_s\\ &\quad - Q_{t_n}\int_{t_n}^{t_{n+1}}\tilde{\rho}_s\int_s^{t_{n+1}}\nabla\hat{Z}_r\mathrm{d}W_r\mathrm{d}\overleftarrow{V}_s + \mathcal{O}\left((\Delta t)^{\frac{3}{2}}\right).\end{aligned}$$

Since W and V are two independent Brownian motions,

$$\begin{aligned}&\mathbb{E}_0^x\left[Q_{t_n}\int_{t_n}^{t_{n+1}}\tilde{\rho}_s\int_s^{t_{n+1}}\nabla\hat{Z}_r\mathrm{d}W_r\mathrm{d}\overleftarrow{V}_s\right]\\ &= \mathbb{E}_0^x\left[Q_{t_n}\int_{t_n}^{t_{n+1}}\tilde{\rho}_s\int_s^{t_{n+1}}(\nabla\hat{Z}_r-\nabla\hat{Z}_{t_n})\mathrm{d}W_r\mathrm{d}\overleftarrow{V}_s\right] \le C(\Delta t)^{\frac{3}{2}}.\end{aligned} \tag{3.31}$$

As a result,

$$
\begin{aligned}
(\mathbf{iii}-\mathbf{2}) &= \mathbb{E}_0^x\left[Q_{t_n}\int_{t_n}^{t_{n+1}}\frac{\tilde{\rho}_s}{\sigma_s}\hat{Z}_s\mathrm{d}\overleftarrow{V}_s\right] \\
&\quad -\mathbb{E}_0^x\left[Q_{t_n}\int_{t_n}^{t_{n+1}}\tilde{\rho}_s\int_s^{t_{n+1}}\left(\hat{h}_r'\hat{Y}_r\nabla\hat{X}_r+\hat{h}_r\nabla\hat{Y}_r+\frac{\tilde{\rho}_r}{\sigma_r}\nabla\hat{Z}_r\right)\mathrm{d}\overleftarrow{V}_r\mathrm{d}\overleftarrow{V}_s\right]+\mathcal{O}\left((\Delta t)^{\frac{3}{2}}\right) \\
&= \mathbb{E}_0^x\left[Q_{t_n}\int_{t_n}^{t_{n+1}}\frac{\tilde{\rho}_s}{\sigma_s}\hat{Z}_s\mathrm{d}\overleftarrow{V}_s\right]-\mathbb{E}_0^x\left[\tilde{\lambda}_{t_n}\int_{t_n}^{t_{n+1}}\int_s^{t_{n+1}}\mathrm{d}\overleftarrow{V}_r\mathrm{d}\overleftarrow{V}_s\right]+\mathcal{O}\left((\Delta t)^{\frac{3}{2}}\right),
\end{aligned}
\tag{3.32}
$$

where $\tilde{\lambda}_{t_n}=Q_{t_n}\tilde{\rho}_{t_{n+1}}\left(\hat{h}_{t_n}'\hat{Y}_{t_{n+1}}\nabla\hat{X}_{t_n}+\hat{h}_{t_n}\nabla\hat{Y}_{t_{n+1}}+\frac{\tilde{\rho}_{t_{n+1}}}{\sigma_{t_{n+1}}}\nabla\hat{Z}_{t_{n+1}}\right)$ is independent of $\int_{t_n}^{t_{n+1}}\int_s^{t_{n+1}}d\overleftarrow{V}_r d\overleftarrow{V}_s=\frac{1}{2}\big((V_{t_{n+1}}-V_{t_n})^2-\Delta t\big)$. By an argument similar to (3.22), we obtain

$$
\sum_{n=0}^{N-1}\mathbb{E}_0^x\left[\tilde{\lambda}_{t_n}\int_{t_n}^{t_{n+1}}\int_s^{t_{n+1}}\mathrm{d}\overleftarrow{V}_r\mathrm{d}\overleftarrow{V}_s\right]\overset{N\to\infty}{\longrightarrow}0 \text{ in } \mathcal{L}^1(\Omega,\tilde{\mathbb{P}}_x). \tag{3.33}
$$

Collecting estimates (3.27), (3.28), (3.29), (3.30) and (3.32) into (3.25); then inserting (3.25), (3.23) and (3.24) into (3.16) we finally obtain

$$
\begin{aligned}
\Delta_n &= \mathbb{E}_0^x\left[\int_{t_n}^{t_{n+1}}\hat{h}_sQ_s\mathrm{d}V_s\hat{Y}_{t_{n+1}}-Q_{t_n}\int_{t_n}^{t_{n+1}}\hat{h}_s\hat{Y}_s\mathrm{d}\overleftarrow{V}_s+\nu_n\right]+\mathcal{O}\left((\Delta t)^{\frac{3}{2}}\right) \\
&= \mathbb{E}_0^x[\alpha_n]+\mathbb{E}_0^x[\beta_n]+\mathbb{E}_0^x[\gamma_n]+\mathbb{E}_0^x[\nu_n]+\mathcal{O}\left((\Delta t)^{\frac{3}{2}}\right),
\end{aligned}
\tag{3.34}
$$

where $\{\nu_n\}$ is defined as in (3.21) satisfying (3.22), and

$$
\begin{aligned}
\alpha_n &:= \int_{t_n}^{t_{n+1}}\left(Q_s\hat{h}_s-Q_{t_n}\hat{h}_{t_n}\right)\hat{Y}_{t_{n+1}}\mathrm{d}V_s, \\
\beta_n &:= \int_{t_n}^{t_{n+1}}Q_{t_n}\left(\hat{h}_{t_{n+1}}\hat{Y}_{t_{n+1}}-\hat{h}_s\hat{Y}_s\right)\mathrm{d}\overleftarrow{V}_s, \\
\gamma_n &:= Q_{t_n}\hat{Y}_{t_{n+1}}\left(\hat{h}_{t_n}-\hat{h}_{t_{n+1}}\right)\cdot(V_{t_{n+1}}-V_{t_n}).
\end{aligned}
$$

The last steps are to show that $\sum_{n=0}^{N-1}\mathbb{E}_0^x[\alpha_n]\to 0$, $\sum_{n=0}^{N-1}\mathbb{E}_0^x[\beta_n]\to 0$, and $\sum_{n=0}^{N-1}\mathbb{E}_0^x[\gamma_n]\to 0$ in $\mathcal{L}^1(\Omega,\tilde{\mathbb{P}}_x)$ as $N\to\infty$.

First write $\alpha_n=\alpha_n^{(1)}+\alpha_n^{(2)}+\alpha_n^{(3)}$ with

$$
\begin{aligned}
\alpha_n^{(1)} &:= \int_{t_n}^{t_{n+1}}(Q_s-Q_{t_n})\hat{h}_s\mathrm{d}V_s\cdot\hat{Y}_{t_{n+1}}, \\
\alpha_n^{(2)} &:= \int_{t_n}^{t_{n+1}}Q_{t_n}(\hat{h}_s-\hat{h}_{t_n})\mathrm{d}V_s\cdot\hat{Y}_{t_n}, \\
\alpha_n^{(3)} &:= \int_{t_n}^{t_{n+1}}Q_{t_n}(\hat{h}_s-\hat{h}_{t_n})\mathrm{d}V_s\cdot\left(\hat{Y}_{t_{n+1}}-\hat{Y}_{t_n}\right).
\end{aligned}
$$

Denote by $\tilde{\mathbb{E}}_x$ the expectation with respect to $\tilde{\mathbb{P}}_x$, where $\tilde{\mathbb{P}}_x:=\tilde{\mathbb{P}}(\cdot|U_0=x)$ is the induced probability measure. Notice that $\hat{Y}_{t_n}=Y_{t_n}(U_{t_n})$ due to $\hat{X}_{t_n}=U_{t_n}$ given in (3.13), and

that h_t is a bounded function, we apply Itô's formula to $(Q_s - Q_{t_n})$ in $\alpha_n^{(1)}$ to get

$$\begin{aligned}
\tilde{\mathbb{E}}_x\left[\left|\mathbb{E}_0^x\big[\sum_{n=0}^{N-1}\alpha_n^{(1)}\big]\right|\right] &= \tilde{\mathbb{E}}_x\left[\left|\mathbb{E}_0^x\big[\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}\hat{h}_r Q_r \mathrm{d}V_r \hat{h}_s \mathrm{d}V_s \cdot \hat{Y}_{t_{n+1}}\big]\right|\right] \\
&\le \tilde{\mathbb{E}}_x\left[\left|\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}\hat{h}_r (Q_r - Q_{t_n}) \mathrm{d}V_r \hat{h}_s \mathrm{d}V_s \cdot \hat{Y}_{t_{n+1}}\right|\right] \\
&\quad + \tilde{\mathbb{E}}_x\left[\left|\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}\hat{h}_r Q_{t_n} \mathrm{d}V_r \hat{h}_s \mathrm{d}V_s \cdot \bar{Y}_{t_{n+1}}\right|\right] \\
&\le C\sum_{n=0}^{N-1}(\Delta t)^{\frac{3}{2}} + C\tilde{\mathbb{E}}_x\left[\left|\sum_{n=0}^{N-1} Q_{t_n}\bar{Y}_{t_{n+1}} \cdot \frac{1}{2}\big((V_{t_{n+1}} - V_{t_n})^2 - \Delta t\big)\right|\right].
\end{aligned}$$

and from the fact that

$$\sum_{n=0}^{N-1} Q_{t_n}\bar{Y}_{t_{n+1}} \cdot \frac{1}{2}\big((V_{t_{n+1}} - V_{t_n})^2 - \Delta t\big) \to 0 \text{ in } \mathcal{L}^1(\Omega, \tilde{P}_x)$$

we have

$$\mathbb{E}_0^x\big[\sum_{n=0}^{N-1}\alpha_n^{(1)}\big] \to 0 \text{ in } \mathcal{L}^1(\Omega, \tilde{P}_x). \tag{3.35}$$

For $\alpha_n^{(2)}$, we apply Itô's formula to h_s to get

$$\begin{aligned}
\mathbb{E}_0^x[\alpha_n^{(2)}] &= \mathbb{E}_0^x\big[\int_{t_n}^{t_{n+1}} Q_{t_n} \cdot \left(\int_{t_n}^{s}[\hat{b}_r \cdot \hat{h}_r' + \frac{(\sigma_r)^2}{2} \cdot \hat{h}_r'']dr + \int_{t_n}^{s}\sigma_r \cdot \hat{h}_r' dW_r\right) \cdot \hat{Y}_{t_n} dV_s\big] \\
&= \mathbb{E}_0^x\big[\int_{t_n}^{t_{n+1}} Q_{t_n}\hat{Y}_{t_n} \cdot \int_{t_n}^{s}[\hat{b}_r \cdot \hat{h}_r' + \frac{(\sigma_r)^2}{2}\hat{h}_r'']dr dV_s\big].
\end{aligned}$$

Since b_t, σ_t, h' and h'' are all bounded, we have

$$\tilde{E}_x[|\mathbb{E}_0^x[\alpha_n^{(2)}]|] \le C\Delta t^{3/2}.$$

Moreover, it follows from Hölder's inequality, Lemma 2.7 and Lemma 3.1 that

$$\begin{aligned}
\tilde{E}_x[|\mathbb{E}_0^x[\alpha_n^{(3)}]|] &= \tilde{E}_x\Big[\big|\mathbb{E}_0^x\big[\int_{t_n}^{t_{n+1}} Q_{t_n}(\hat{h}_s - \hat{h}_{t_n})dV_s \cdot \big(\hat{Y}_{t_{n+1}} - \hat{Y}_{t_n}\big)\big]\big|\Big] \\
&\le \Big(\tilde{E}_x\big[\big(\int_{t_n}^{t_{n+1}} Q_{t_n}(\hat{h}_s - \hat{h}_{t_n})dV_s\big)^2\big]\Big)^{\frac{1}{2}} \cdot \Big(\tilde{E}_x\big[\big(\hat{Y}_{t_{n+1}} - \hat{Y}_{t_n}\big)^2\big]\Big)^{\frac{1}{2}} \\
&\le C\Delta t^{3/2}.
\end{aligned}$$

Hence,

$$= \sum_{n=0}^{N-1}\mathbb{E}_0^x[\alpha_n^{(2)}] \to 0 \text{ in } \mathcal{L}^1(\Omega, \tilde{P}_x), \tag{3.36}$$

and

$$\sum_{n=0}^{N-1}\mathbb{E}_0^x[\alpha_n^{(3)}] \to 0 \text{ in } \mathcal{L}^1(\Omega, \tilde{P}_x). \tag{3.37}$$

Then, from (3.35), (3.36) and (3.37), we get

$$\sum_{n=0}^{N-1}\mathbb{E}_0^x[\alpha_n]\to 0 \text{ in } \mathcal{L}^1(\Omega,\tilde{P}_x). \tag{3.38}$$

For the term β_n in (3.34), we have

$$\begin{aligned}\beta_n=&\int_{t_n}^{t_{n+1}} Q_{t_n}\Big((\hat{h}_{t_{n+1}}-\hat{h}_s)\hat{Y}_{t_{n+1}}+\hat{h}_s(\hat{Y}_{t_{n+1}}-\hat{Y}_s)\Big)d\overleftarrow{V}_s\\ =&\int_{t_n}^{t_{n+1}} Q_{t_n}(\hat{h}_{t_{n+1}}-\hat{h}_s)\hat{Y}_{t_n}d\overleftarrow{V}_s+\int_{t_n}^{t_{n+1}} Q_{t_n}\hat{h}_s(\hat{Y}_{t_{n+1}}-\hat{Y}_s)d\overleftarrow{V}_s\\ &+\int_{t_n}^{t_{n+1}} Q_{t_n}(\hat{h}_{t_{n+1}}-\hat{h}_s)d\overleftarrow{V}_s\cdot\big(\hat{Y}_{t_{n+1}}-\hat{Y}_{t_n}\big)\\ =&\beta_n^{(1)}+\beta_n^{(2)}+\beta_n^{(3)}\end{aligned}$$

with

$$\beta_n^{(1)}=\int_{t_n}^{t_{n+1}} Q_{t_n}(\hat{h}_{t_{n+1}}-\hat{h}_s)\hat{Y}_{t_n}d\overleftarrow{V}_s,$$

$$\beta_n^{(2)}=\int_{t_n}^{t_{n+1}} Q_{t_n}\hat{h}_s(\hat{Y}_{t_{n+1}}-\hat{Y}_s)d\overleftarrow{V}_s$$

and

$$\beta_n^{(3)}=\int_{t_n}^{t_{n+1}} Q_{t_n}(\hat{h}_{t_{n+1}}-\hat{h}_s)d\overleftarrow{V}_s\cdot\big(\hat{Y}_{t_{n+1}}-\hat{Y}_{t_n}\big).$$

Following the similar the approaches to $\alpha_n^{(2)}$ and $\alpha_n^{(3)}$, we have

$$\tilde{E}_x[|\mathbb{E}_0^x[\beta_n^{(1)}]|]\le C\Delta t^{3/2}$$

and

$$\tilde{E}_x[|\mathbb{E}_0^x[\beta_n^{(3)}]|]\le C\Delta t^{3/2}.$$

Hence,

$$\sum_{n=0}^{N-1}\mathbb{E}_0^x[\beta_n^{(1)}]\to 0 \text{ in } \mathcal{L}^1(\Omega,\tilde{P}_x), \tag{3.39}$$

and

$$\sum_{n=0}^{N-1}\mathbb{E}_0^x[\beta_n^{(3)}]\to 0 \text{ in } \mathcal{L}^1(\Omega,\tilde{P}_x). \tag{3.40}$$

From the BDSDE in (3.9), Lemma 2.7, Lemma 2.5 and estimate (3.31), we get

$$\begin{aligned}\sum_{n=0}^{N-1}\mathbb{E}_0^x[\beta_n^{(2)}]=&\sum_{n=0}^{N-1}\mathbb{E}_0^x\Big[\int_{t_n}^{t_{n+1}} Q_{t_n}\hat{h}_s\Big(\int_s^{t_{n+1}}\hat{Z}_r dW_r-\int_s^{t_{n+1}}\big(\hat{h}_r\hat{Y}_r+\frac{\tilde{\rho}_r}{\sigma_r}\hat{Z}_r\big)d\overleftarrow{V}_r\Big)d\overleftarrow{V}_s\Big]\\ =&\sum_{n=0}^{N-1}\mathbb{E}_0^x\Big[\int_{t_n}^{t_{n+1}} Q_{t_n}\hat{h}_s\big(-\int_s^{t_{n+1}}\big(\hat{h}_r\hat{Y}_{t_{n+1}}+\frac{\tilde{\rho}_r}{\sigma_r}\hat{Z}_{t_{n+1}}\big)d\overleftarrow{V}_r\big)d\overleftarrow{V}_s\Big]+O((\Delta t)^{\frac{3}{2}})\\ &+\sum_{n=0}^{N-1}\mathbb{E}_0^x\Big[\int_{t_n}^{t_{n+1}} Q_{t_n}\hat{h}_s\big(\int_s^{t_{n+1}}\hat{h}_r(\hat{Y}_{t_{n+1}}-\hat{Y}_r)+\frac{\tilde{\rho}_r}{\sigma_r}(\hat{Z}_{t_{n+1}}-\hat{Z}_r)d\overleftarrow{V}_r\big)d\overleftarrow{V}_s\Big]\end{aligned}$$

Next, we take conditional expectation $\tilde{E}_x$ to the absolute value of the above equation. Since

$$\tilde{E}_x\big[(\hat{Y}_{t_{n+1}}-\hat{Y}_r)^2\big]\le C(\Delta t)^3,\ \tilde{E}_x\big[(\hat{Z}_{t_{n+1}}-\hat{Z}_r)^2\big]\le C(\Delta t)^3$$

and

$$\begin{aligned}&\tilde{E}_x\Big[\big|\mathbb{E}_0^x\big[\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}Q_{t_n}\hat{h}_s\big(-\int_s^{t_{n+1}}\hat{h}_r\hat{Y}_{t_{n+1}}d\overleftarrow{V}_r\big)d\overleftarrow{V}_s\big]\big|\Big]\\ \le& C\tilde{E}_x\Big[\big|\sum_{n=0}^{N-1}Q_{t_n}\hat{Y}_{t_{n+1}}\cdot\frac{1}{2}\big((V_{t_{n+1}}-V_{t_n})^2-\Delta t\big)\big|\Big],\end{aligned}$$

it follows from the fact

$$\sum_{n=0}^{N-1}Q_{t_n}\hat{Y}_{t_{n+1}}\cdot\frac{1}{2}\big((V_{t_{n+1}}-V_{t_n})^2-\Delta t\big)\to 0 \text{ in } \mathcal{L}^1(\Omega,\tilde{P}_x)$$

that

$$\mathbb{E}_0^x[\sum_{n=0}^{N-1}\beta_n^{(2)}]\to 0 \text{ in } \mathcal{L}^1(\Omega,\tilde{P}_x). \tag{3.41}$$

Hence,

$$\mathbb{E}_0^x[\sum_{n=0}^{N-1}\beta_n]\to 0 \text{ in } L^1(\Omega,\tilde{P}_x). \tag{3.42}$$

For γ_n, applying Itô formula to h_t, it's easy to verify that

$$|\mathbb{E}_0^x[\hat{h}_{t_{n+1}}-\hat{h}_{t_n}]|\le C\Delta t. \tag{3.43}$$

Since $\hat{h}_{t_n}-\hat{h}_{t_{n+1}}$ is independent from $Q_{t_n}\hat{Y}_{t_n}(V_{t_{n+1}}-V_{t_n})$,

$$\begin{aligned}\mathbb{E}_0^x[\gamma_n]=&\ \mathbb{E}_0^x\big[Q_{t_n}\hat{Y}_{t_n}\big(\hat{h}_{t_n}-\hat{h}_{t_{n+1}}\big)\cdot(V_{t_{n+1}}-V_{t_n})\big]\\ &+\mathbb{E}_0^x\big[Q_{t_n}\big(\hat{Y}_{t_{n+1}}-\hat{Y}_{t_n}\big)\cdot\big(\hat{h}_{t_n}-\hat{h}_{t_{n+1}}\big)\cdot(V_{t_{n+1}}-V_{t_n})\big]\\ =&\ \mathbb{E}_0^x\big[\hat{h}_{t_n}-\hat{h}_{t_{n+1}}\big]\cdot\mathbb{E}_0^x\big[Q_{t_n}\hat{Y}_{t_n}\cdot(V_{t_{n+1}}-V_{t_n})\big]\\ &+\mathbb{E}_0^x\big[Q_{t_n}\big(\hat{Y}_{t_{n+1}}-\hat{Y}_{t_n}\big)\cdot\big(\hat{h}_{t_n}-\hat{h}_{t_{n+1}}\big)\cdot(V_{t_{n+1}}-V_{t_n})\big].\end{aligned}$$

Then, from estimate (3.43), lemma 2.7, lemma 3.1, we get

$$\begin{aligned}\tilde{E}_x\big[|\mathbb{E}_0^x[\gamma_n]|\big]\le&\ C\Delta t\cdot\tilde{E}_x\big[\big|Q_{t_n}\hat{Y}_{t_n}\cdot(V_{t_{n+1}}-V_{t_n})\big|\big]\\ &+\tilde{E}_x\big[|Q_{t_n}\big(\hat{Y}_{t_{n+1}}-\hat{Y}_{t_n}\big)\cdot\big(\hat{h}_{t_n}-\hat{h}_{t_{n+1}}\big)\cdot(V_{t_{n+1}}-V_{t_n})|\big]\\ \le&C(\Delta t)^{\frac{3}{2}}\end{aligned} \tag{3.44}$$

and therefore

$$\mathbb{E}_0^x[\sum_{n=0}^{N-1}\gamma_n]\to \text{ in } L^1(\Omega,\tilde{P}_x). \tag{3.45}$$

Finally with convergence results in (3.38), (3.42) and (3.45), we have

$$\sum_{n=0}^{N-1}\Delta_n\to 0 \text{ in } L^1(\Omega,\tilde{P}_x)$$

as required. □

3.2 Adjoint FBDSDEs

In this subsection, we consider the following FBDSDEs system, in which the "forward SDE" (2.5) goes backward and the "Backward SDE" (2.6) goes forward

$$\begin{cases} \mathrm{d}\overleftarrow{X}_t = b_t(\overleftarrow{X}_t)\mathrm{d}t - \sigma_t \mathrm{d}\overleftarrow{W}_t, \quad 0 \le t \le \tau & \text{(SDE)} \\ \mathrm{d}\overrightarrow{Y}_t = -b_t'(\overleftarrow{X}_t)\overrightarrow{Y}_t \mathrm{d}t - \overrightarrow{Z}_t^{T,x}\mathrm{d}\overleftarrow{W}_t + \left(h(\overleftarrow{X}_t)\overrightarrow{Y}_t - \dfrac{\tilde{\rho}_t}{\sigma_t}\overrightarrow{Z}_t\right)\mathrm{d}V_t, \quad 0 \le t \le \tau & \text{(BDSDE)} \\ \overleftarrow{X}_\tau = x, \qquad \overrightarrow{Y}_0 = p_0(\overleftarrow{X}_0), \end{cases} \tag{3.46}$$

where $0 \le \tau \le T$, $\int_t^T \cdot \mathrm{d}\overleftarrow{W}_s$ is a backward Itô Integral and $\int_t^T \cdot \mathrm{d}V_s$ is a standard forward Itô integral. Write the solution to (3.46) as $(\overleftarrow{X}_t^{T,x}, \overrightarrow{Y}_t^{T,x}, \overrightarrow{Z}_t^{T,x})$. Then by inverting the time index in the standard FBDSDEs system, $\overleftarrow{X}_t^{T,x}$ is a $\mathcal{F}_{t,T}^W$ adaptive stochastic process and the solution $\left(\overrightarrow{Y}_t^{T,x}, \overrightarrow{Z}_t^{T,x}\right)$ of the BDSDE in (3.46) is adaptive to $\mathcal{F}_{t,T}^W \vee \mathcal{F}_t^V$. Similar to the notation used in Section 2.1, we denote $\overrightarrow{Y}_t(x) := \overrightarrow{Y}_t^{t,x}$ and $\overrightarrow{Z}_t(x) := \overrightarrow{Z}_t^{t,x}$. In addition, for any non-negative integer m and function $\eta_t(x)$ we write $\eta_t^{(m)} := \frac{\partial^m \eta}{\partial x^m}$.

We need the following regularity properties for b and σ.

Assumption 3.3 *For $0 \le s \le t \le T$, functions b and σ satisfy*

$$|b_t(x) - b_s(x)| + |b_t'(x) - b_s'(x)| \le C|t-s|, \qquad |\sigma_t - \sigma_s| \le C|t-s|,$$

where C is a given positive constant independent of b, σ, s and t.

Lemma 3.4 can be proved by using repeatedly the variational form of BDSDEs [20].

Lemma 3.4 *Assume that $b \in \mathcal{C}_{l,b}^4$, $\phi \in \mathcal{C}_{l,b}^3$, $h \in \mathcal{C}_{l,b}^3$ and every derivative of b, ϕ and h has bounded support in $\mathbb{R}$. Then for each $m_1 = 0,1,2$ and $m_2 = 0,1,2,3$, $Y_t^{(m_1)}$, $\overrightarrow{Y}_t^{(m_2)}$ have bounded support and satisfy*

$$\int_{\mathbb{R}} \tilde{\mathbb{E}}\left[\sup_{0\le t\le T}\left|Y_t^{(m_1)}\right|^2\right]\mathrm{d}x < \infty \text{ and } \int_{\mathbb{R}} \tilde{\mathbb{E}}\left[\sup_{0\le t\le T}\left|\overrightarrow{Y}_t^{(m_2)}\right|^2\right]\mathrm{d}x < \infty. \tag{3.47}$$

Denote by $\langle \cdot, \cdot \rangle$ the standard inner product in $\mathcal{L}^2$. The following theorem shows that $\overrightarrow{Y}_t$ is the adjoint stochastic process of Y_t defined in the FBDSDEs system (3.9).

Theorem 3.5 *Assume that, in addition to Assumption 3.3holds, σ is uniformly bounded, $b \in \mathcal{C}_{l,b}^4$, $\phi \in \mathcal{C}_{l,b}^3$, $h \in \mathcal{C}_{l,b}^3$ and each derivative of b, ϕ and h has bounded support in $\mathbb{R}$. Then the process $R_t := \left\langle Y_t, \overrightarrow{Y}_t \right\rangle$, $t \in [0,T]$ is a constant for almost all trajectories.*

Proof. According to [20], R_t has a.s. continuous paths, it suffices to show that $\forall s,t \in [0,T]$, $R_s = R_t$ a.s..

For $0 \le s \le t \le T$ let $s = t_0 < t_1 < \cdots < t_N = t$ be a temporal partition with uniform stepsize $t_{n+1} - t_n = \frac{t-s}{N} = \Delta t$. For simplification of notations, we denote

$$\Delta V_{t_n} := V_{t_{n+1}} - V_{t_n}, \qquad Y_n := Y_{t_n}, \qquad Z_n := Z_{t_n}, \qquad \overrightarrow{Y}_n := \overrightarrow{Y}_{t_n}, \qquad \overrightarrow{Z}_n := \overrightarrow{Z}_{t_n}.$$

By Corollary 2.2 in [20], we have

$$\begin{aligned}
&Y_n(x)=Y_{t_n}^{t_n,x}, \quad \overrightarrow{Y}_n(x) \quad =\overrightarrow{Y}_{t_n}^{t_n,x}, \quad Y_{n+1}(X_{t_{n+1}}^{t_n,x})=Y_{t_{n+1}}^{t_n,x}, \quad \overrightarrow{Y}_n(\overleftarrow{X}_{t_n}^{t_{n+1},x}) \quad =\overrightarrow{Y}_{t_n}^{t_{n+1},x},\\
&Z_n(x)=Z_{t_n}^{t_n,x}, \quad \overrightarrow{Z}_n(x) \quad =\overrightarrow{Z}_{t_n}^{t_n,x}, \quad Z_{n+1}(X_{t_{n+1}}^{t_n,x})=Z_{t_{n+1}}^{t_n,x}, \quad \overrightarrow{Z}_n(\overleftarrow{X}_{t_n}^{t_{n+1},x}) \quad =\overrightarrow{Z}_{t_n}^{t_{n+1},x}.
\end{aligned}$$

Denote conditional expectations

$$\mathbb{E}[\cdot]:=\tilde{\mathbb{E}}[\cdot|\mathcal{F}_T^V], \quad \mathbb{E}_x^n[\cdot]:=\tilde{\mathbb{E}}[\cdot|\mathcal{F}_T^V,X_{t_n}=x], \quad \overleftarrow{\mathbb{E}}_x^n[\cdot]:=\tilde{\mathbb{E}}[\cdot|\mathcal{F}_T^V,\overleftarrow{X}_{t_n}=x].$$

It then follows from the definitions of $\mathbb{E}_x^n$ and $\overleftarrow{\mathbb{E}}_x^n$ that

$$\mathbb{E}_x^n[Y_n]=Y_n(x), \qquad \overleftarrow{\mathbb{E}}_x^n[\overrightarrow{Y}_n]=\overrightarrow{Y}_n(x).$$

Without loss of generality suppose that $\Delta t<s\wedge(T-t)$ and define

$$Y_N=\frac{1}{\Delta t}\int_t^{t+\Delta t}Y_r\mathrm{d}r, \qquad \overrightarrow{Y}_0=\frac{1}{\Delta t}\int_{s-\Delta t}^{s}\overrightarrow{Y}_r\mathrm{d}r.$$

For $n=0,1,\ldots,N-1$, taking the conditional expectations $\mathbb{E}_x^n$ and $\mathbb{E}_x^{\overleftarrow{n+1}}$ of temporal discretized approximations of the BDSDEs in (3.9) and (3.46), respectively, we have that (see [3])

$$\mathbb{E}_x^n[Y_n]=\mathbb{E}_x^n[Y_{n+1}]+\mathbb{E}_x^n[h_{n+1}Y_{n+1}]\Delta V_{t_n}+\mathbb{E}_x^n\left[\frac{\tilde{\rho}_{t_{n+1}}}{\sigma_{t_{n+1}}}Z_{n+1}\right]\Delta V_{t_n}, \quad (3.48)$$

$$\begin{aligned}
\overleftarrow{\mathbb{E}}_x^{n+1}[\overrightarrow{Y}_{n+1}]=&\overleftarrow{\mathbb{E}}_x^{n+1}[\overrightarrow{Y}_n]+\overleftarrow{\mathbb{E}}_x^n\left[-\overleftarrow{b}'_n\overrightarrow{Y}_n\right]\Delta t\\
&+\overleftarrow{\mathbb{E}}_x^{n+1}\left[\overleftarrow{h}_n\overrightarrow{Y}_n\right]\Delta V_{t_n}-\overleftarrow{\mathbb{E}}_x^{n+1}\left[\frac{\tilde{\rho}_{t_n}}{\sigma_{t_n}}\overrightarrow{Z}_n\right]\Delta V_{t_n}, \qquad (3.49)
\end{aligned}$$

where

$$h_{n+1}:=h(X_{t_{n+1}}), \qquad \overleftarrow{b}'_n:=b'_{t_n}(\overleftarrow{X}_{t_n}), \qquad \overleftarrow{h}_n:=h(\overleftarrow{X}_{t_n}).$$

By the definition of expectations $\mathbb{E}_x^n$ and $\mathbb{E}_x^{\overleftarrow{n+1}}$,

$$\mathbb{E}_x^n[h_{n+1}]=\mathbb{E}\left[h(X_{t_{n+1}}^{t_n,x})\right], \quad \overleftarrow{\mathbb{E}}_x^{n+1}\left[\overleftarrow{b}'_n\right]=\mathbb{E}\left[b'_n(\overleftarrow{X}_{t_n}^{t_{n+1},x})\right], \quad \overleftarrow{\mathbb{E}}_x^{n+1}\left[\overleftarrow{h}_n\right]=\mathbb{E}\left[h(\overleftarrow{X}_{t_n}^{t_{n+1},x})\right].$$

Multiplying (3.49) by $\mathbb{E}_x^{\overleftarrow{n}}[\overrightarrow{Y}_n]$ and (3.49) by $\mathbb{E}_x^{n+1}[Y_{n+1}]$, then taking integral with respect to $\mathrm{d}x$, we obtain

$$\begin{aligned}
\left\langle\mathbb{E}_x^n[Y_n],\overleftarrow{\mathbb{E}}_x^n[\overrightarrow{Y}_n]\right\rangle=&\left\langle\mathbb{E}_x^n[Y_{n+1}],\overleftarrow{\mathbb{E}}_x^n[\overrightarrow{Y}_n]\right\rangle+\left\langle\mathbb{E}_x^n[h_{n+1}Y_{n+1}],\overleftarrow{\mathbb{E}}_x^n[\overrightarrow{Y}_n]\right\rangle\Delta V_{t_n}\\
&+\left\langle\mathbb{E}_x^n\left[\frac{\tilde{\rho}_{t_{n+1}}}{\sigma_{t_{n+1}}}Z_{n+1}\right],\overleftarrow{\mathbb{E}}_x^n[\overrightarrow{Y}_n]\right\rangle\Delta V_{t_n}
\end{aligned} \quad (3.50)$$

and

$$\begin{aligned}
&\left\langle\overleftarrow{\mathbb{E}}_x^{n+1}[\overrightarrow{Y}_{n+1}],\mathbb{E}_x^{n+1}[Y_{n+1}]\right\rangle\\
=&\left\langle\overleftarrow{\mathbb{E}}_x^{n+1}[\overrightarrow{Y}_n],\mathbb{E}_x^{n+1}[Y_{n+1}]\right\rangle+\left\langle\overleftarrow{\mathbb{E}}_x^{n+1}\left[-\overleftarrow{b}'_n\overrightarrow{Y}_n\right],\mathbb{E}_x^{n+1}[Y_{n+1}]\right\rangle\Delta t\\
&+\left\langle\overleftarrow{\mathbb{E}}_x^{n+1}\left[\overleftarrow{h}_n\overrightarrow{Y}_n\right],\mathbb{E}_x^{n+1}[Y_{n+1}]\right\rangle\Delta V_{t_n}-\left\langle\overleftarrow{\mathbb{E}}_x^{n+1}\left[\frac{\tilde{\rho}_{t_n}}{\sigma_{t_n}}\overrightarrow{Z}_n\right],\mathbb{E}_x^{n+1}[Y_{n+1}]\right\rangle\Delta V_{t_n}.
\end{aligned} \quad (3.51)$$

Subtraction of (3.51) from (3.50) results in

$$\begin{aligned}
&\Big\langle \mathbb{E}_x^n[Y_n], \overleftarrow{\mathbb{E}}_x^n[\overrightarrow{Y}_n]\Big\rangle - \Big\langle \overleftarrow{\mathbb{E}}_x^{n+1}[\overrightarrow{Y}_{n+1}], \mathbb{E}_x^{n+1}[Y_{n+1}]\Big\rangle \\
&= \underbrace{\Big\langle \mathbb{E}_x^n[Y_{n+1}], \overleftarrow{\mathbb{E}}_x^n[\overrightarrow{Y}_n] - \overleftarrow{\mathbb{E}}_x^{n+1}[\overrightarrow{Y}_n]\Big\rangle + \Big\langle \overleftarrow{\mathbb{E}}_x^{n+1}[\overrightarrow{Y}_n], \mathbb{E}_x^n[Y_{n+1}] - \mathbb{E}_x^{n+1}[Y_{n+1}]\Big\rangle}_{(\mathbf{iv})} \\
&\quad + \underbrace{\Big\langle \mathbb{E}_x^n[h_{n+1}Y_{n+1}], \overleftarrow{\mathbb{E}}_x^n[\overrightarrow{Y}_n]\Big\rangle \Delta V_{t_n} - \Big\langle \overleftarrow{\mathbb{E}}_x^{n+1}\Big[\overleftarrow{h}_n \overrightarrow{Y}_n\Big], \mathbb{E}_x^{n+1}[Y_{n+1}]\Big\rangle \Delta V_{t_n}}_{(\mathbf{v})} \\
&\quad + \underbrace{\Big\langle \mathbb{E}_x^n\Big[\frac{\tilde{\rho}_{t_{n+1}}}{\sigma_{t_{n+1}}} Z_{n+1}\Big], \overleftarrow{\mathbb{E}}_x^n[\overrightarrow{Y}_n]\Big\rangle \Delta V_{t_n} + \Big\langle \overleftarrow{\mathbb{E}}_x^{n+1}\Big[\frac{\tilde{\rho}_{t_n}}{\sigma_{t_n}} \overrightarrow{Z}_n\Big], \mathbb{E}_x^{n+1}[Y_{n+1}]\Big\rangle \Delta V_{t_n}}_{(\mathbf{vi})} \\
&\quad - \Big\langle \overleftarrow{\mathbb{E}}_x^{n+1}\Big[-\overleftarrow{b}'_n \overrightarrow{Y}_n\Big], \mathbb{E}_x^{n+1}[Y_{n+1}]\Big\rangle \Delta t.
\end{aligned} \tag{3.52}$$

In what follows, we prove that by taking the sum of equation (3.52) from $n=0$ to $n=N-1$, the right hand side of the resulting equation converges to 0 as $\Delta t \to 0$. To this end we estimate terms (**iv**), (**v**) and (**vi**) one by one.

(**iv**) By the definitions $\mathbb{E}_x^{\overleftarrow{n}}$ and $\mathbb{E}_x^n$, we have

$$\begin{aligned}
\overleftarrow{\mathbb{E}}_x^n[\overrightarrow{Y}_n] - \overleftarrow{\mathbb{E}}_x^{n+1}[\overrightarrow{Y}_n] &= \mathbb{E}\Big[\overrightarrow{Y}_n(x) - \overrightarrow{Y}_n(\overleftarrow{X}_{t_n}^{t_{n+1},x})\Big], \\
\mathbb{E}_x^n[Y_{n+1}] - \mathbb{E}_x^{n+1}[Y_{n+1}] &= \mathbb{E}\Big[Y_{n+1}(X_{t_{n+1}}^{t_n,x}) - Y_{n+1}(x)\Big].
\end{aligned}$$

It follows from Itô's formula that

$$\begin{aligned}
\overrightarrow{Y}_n(\overleftarrow{X}_{t_n}^{t_{n+1},x}) &= \overrightarrow{Y}_n(x) + \int_{t_n}^{t_{n+1}} \big(-b_s(\overleftarrow{X}_s^{t_{n+1},x})\overrightarrow{Y}'_n(\overleftarrow{X}_s^{t_{n+1},x}) + \frac{(\sigma_s)^2}{2}\overrightarrow{Y}''_n(\overleftarrow{X}_s^{t_{n+1},x})\big)\mathrm{d}s \\
&\quad + \int_{t_n}^{t_{n+1}} \sigma_s \overrightarrow{Y}'_n(\overleftarrow{X}_s^{t_{n+1},x})\mathrm{d}\overleftarrow{W}_s, \qquad (3.53) \\
Y_{n+1}(X_{t_{n+1}}^{t_n,x}) &= Y_{n+1}(x) + \int_{t_n}^{t_{n+1}} \big(b_s(X_s^{t_n,x})Y'_{n+1}(X_s^{t_n,x}) + \frac{(\sigma_s)^2}{2}Y''_{n+1}(X_s^{t_n,x})\big)\mathrm{d}s \\
&\quad + \int_{t_n}^{t_{n+1}} \sigma_s Y'_{n+1}(X_s^{t_n,x})\mathrm{d}W_s. \qquad (3.54)
\end{aligned}$$

Taking conditional expectation $\mathbb{E}$ to Equations (3.53) and (3.54), we obtain

$$\begin{aligned}
\overleftarrow{\mathbb{E}}_x^n[\overrightarrow{Y}_n] - \overleftarrow{\mathbb{E}}_x^{n+1}[\overrightarrow{Y}_n] &= -\mathbb{E}\left[\int_{t_n}^{t_{n+1}} \big(-b_s(\overleftarrow{X}_s^{t_{n+1},x})\overrightarrow{Y}'_n(\overleftarrow{X}_s^{t_{n+1},x}) + \frac{(\sigma_s)^2}{2}\overrightarrow{Y}''_n(\overleftarrow{X}_s^{t_{n+1},x})\big)\mathrm{d}s\right] \\
&= -\mathbb{E}\left[-b_n(\overleftarrow{X}_{t_n}^{t_{n+1},x})\overrightarrow{Y}'_n(\overleftarrow{X}_{t_n}^{t_{n+1},x}) + \frac{(\sigma_{t_n})^2}{2}\overrightarrow{Y}''_n(\overleftarrow{X}_{t_n}^{t_{n+1},x})\right]\cdot\Delta t + \overrightarrow{R}_n, \\
\mathbb{E}_x^n[Y_{n+1}] - \mathbb{E}_x^{n+1}[Y_{n+1}] &= \mathbb{E}\left[\int_{t_n}^{t_{n+1}} \big(b_s(X_s^{t_n,x})Y'_{n+1}(X_s^{t_n,x}) + \frac{(\sigma_s)^2}{2}Y''_{n+1}(X_s^{t_n,x})\big)\mathrm{d}s\right] \\
&= \mathbb{E}\left[b_n(X_{t_n}^{t_n,x})Y'_{n+1}(X_{t_n}^{t_n,x}) + \frac{(\sigma_{t_n})^2}{2}Y''_{n+1}(X_{t_n}^{t_n,x})\right]\cdot\Delta t + R_n,
\end{aligned}$$

where

$$\begin{aligned}\overrightarrow{R}_n :=& -\mathbb{E}\left[\int_{t_n}^{t_{n+1}} \big(-b_s(\overleftarrow{X}_s^{t_{n+1},x})\overrightarrow{Y}'_n(\overleftarrow{X}_s^{t_{n+1},x})+\frac{(\sigma_s)^2}{2}\overrightarrow{Y}''_n(\overleftarrow{X}_s^{t_{n+1},x})\big)\mathrm{d}s\right]\\ &+\mathbb{E}\left[-b_n(\overleftarrow{X}_{t_n}^{t_{n+1},x})\overrightarrow{Y}'_n(\overleftarrow{X}_{t_n}^{t_{n+1},x})+\frac{(\sigma_{t_n})^2}{2}\overrightarrow{Y}''_n(\overleftarrow{X}_{t_n}^{t_{n+1},x})\right]\cdot\Delta t,\\ R_n :=& \mathbb{E}\left[\int_{t_n}^{t_{n+1}} \big(b_s(X_s^{t_n,x})Y'_{n+1}(X_s^{t_n,x})+\frac{(\sigma_s)^2}{2}Y''_{n+1}(X_s^{t_n,x})\big)\mathrm{d}s\right]\\ &-\mathbb{E}\big[b_n(X_{t_n}^{t_n,x})Y'_{n+1}(X_{t_n}^{t_n,x})+\frac{(\sigma_{t_n})^2}{2}Y''_{n+1}(X_{t_n}^{t_n,x})\big]\cdot\Delta t.\end{aligned}$$

As a consequence

$$\begin{aligned}&\Big\langle \mathbb{E}_x^n[Y_{n+1}], \overleftarrow{\mathbb{E}}_x^n[\overrightarrow{Y}_n]-\overleftarrow{\mathbb{E}}_x^{n+1}[\overrightarrow{Y}_n]\Big\rangle\\ =& -\int_{\mathbb{R}}\mathbb{E}\Big[Y_{n+1}(X_{t_{n+1}}^{t_n,x})\Big]\Big(\mathbb{E}\big[-b_n(\overleftarrow{X}_{t_n}^{t_{n+1},x})\overrightarrow{Y}'_n(\overleftarrow{X}_{t_n}^{t_{n+1},x})+\frac{(\sigma_{t_n})^2}{2}\overrightarrow{Y}''_n(\overleftarrow{X}_{t_n}^{t_{n+1},x})\big]\cdot\Delta t-\overrightarrow{R}_n\Big)\mathrm{d}x.\end{aligned} \tag{3.55}$$

Similarly

$$\begin{aligned}&\Big\langle \overleftarrow{\mathbb{E}}_x^{n+1}[\overrightarrow{Y}_n], \mathbb{E}_x^n[Y_{n+1}]-\mathbb{E}_x^{n+1}[Y_{n+1}]\Big\rangle\\ =& \int_{\mathbb{R}}\mathbb{E}\Big[\overrightarrow{Y}_n(\overleftarrow{X}_{t_n}^{t_{n+1},x})\Big]\Big(\mathbb{E}\big[b_n(X_{t_n}^{t_n,x})Y'_{n+1}(X_{t_n}^{t_n,x})+\frac{(\sigma_{t_n})^2}{2}Y''_{n+1}(X_{t_n}^{t_n,x})\big]\cdot\Delta t+R_n\Big)\mathrm{d}x.\end{aligned} \tag{3.56}$$

Adding (3.55) to (3.56) we have that

$$\begin{aligned}(\mathbf{iv})=&\Big(\underbrace{-\int_{\mathbb{R}}\mathbb{E}\Big[Y_{n+1}(X_{t_{n+1}}^{t_n,x})\Big]\mathbb{E}\big[-b_n(\overleftarrow{X}_{t_n}^{t_{n+1},x})\overrightarrow{Y}'_n(\overleftarrow{X}_{t_n}^{t_{n+1},x})\big]\mathrm{d}x}_{(\mathbf{iv-1})}\\ &+\underbrace{\int_{\mathbb{R}}\mathbb{E}\Big[\overrightarrow{Y}_n(\overleftarrow{X}_{t_n}^{t_{n+1},x})\Big]\mathbb{E}\big[b_n(X_{t_n}^{t_n,x})Y'_{n+1}(X_{t_n}^{t_n,x})\big]\mathrm{d}x}_{(\mathbf{iv-2})}\Big)\cdot\Delta t\\ &+\Big(\underbrace{-\int_{\mathbb{R}}\mathbb{E}\Big[Y_{n+1}(X_{t_{n+1}}^{t_n,x})\Big]\mathbb{E}\big[\frac{(\sigma_{t_n})^2}{2}\overrightarrow{Y}''_n(\overleftarrow{X}_{t_n}^{t_{n+1},x})\big]\mathrm{d}x}_{(\mathbf{iv-3})}\\ &+\underbrace{\int_{\mathbb{R}}\mathbb{E}\Big[\overrightarrow{Y}_n(\overleftarrow{X}_{t_n}^{t_{n+1},x})\Big]\mathbb{E}\big[\frac{(\sigma_{t_n})^2}{2}Y''_{n+1}(X_{t_n}^{t_n,x})\big]\mathrm{d}x}_{(\mathbf{iv-4})}\Big)\cdot\Delta t+R_n^x,\end{aligned} \tag{3.57}$$

where

$$R_n^x=\int_{\mathbb{R}}\mathbb{E}[Y_{n+1}(X_{t_{n+1}}^{t_n,x})]\overrightarrow{R}_n dx+\int_{\mathbb{R}}\mathbb{E}[\overrightarrow{Y}_n(\overleftarrow{X}_{t_n}^{t_{n+1},x})]R_n\mathrm{d}x.$$

Again by using the Itô formula we obtain

$$\begin{aligned}
\overrightarrow{Y}'_n(\overleftarrow{X}^{t_{n+1},x}_{t_n}) &= \overrightarrow{Y}'_n(x)+\int_{t_n}^{t_{n+1}}\big(-b_s(\overleftarrow{X}^{t_{n+1},x}_s)\overrightarrow{Y}''_n(\overleftarrow{X}^{t_{n+1},x}_s)+\frac{(\sigma_s)^2}{2}\overrightarrow{Y}^{(3)}_n(\overleftarrow{X}^{t_{n+1},x}_s)\big)\mathrm{d}s\\
&\quad+\int_{t_n}^{t_{n+1}}\sigma_s\overrightarrow{Y}''_n(\overleftarrow{X}^{t_{n+1},x}_s)\mathrm{d}\overleftarrow{W}_s,\\
\overrightarrow{Y}''_n(\overleftarrow{X}^{t_{n+1},x}_{t_n}) &= \overrightarrow{Y}''_n(x)+\int_{t_n}^{t_{n+1}}\big(-b_s(\overleftarrow{X}^{t_{n+1},x}_s)\overrightarrow{Y}^{(3)}_n(\overleftarrow{X}^{t_{n+1},x}_s)+\frac{(\sigma_s)^2}{2}\overrightarrow{Y}^{(4)}_n(\overleftarrow{X}^{t_{n+1},x}_s)\big)\mathrm{d}s\\
&\quad+\int_{t_n}^{t_{n+1}}\sigma_s\overrightarrow{Y}^{(3)}_n(\overleftarrow{X}^{t_{n+1},x}_s)\mathrm{d}\overleftarrow{W}_s,\\
b_n(\overleftarrow{X}^{t_{n+1},x}_{t_n}) &= b_n(x)+\int_{t_n}^{t_{n+1}}-b_s(\overleftarrow{X}^{t_{n+1},x}_s)b'_n(\overleftarrow{X}^{t_{n+1},x}_s)+\frac{(\sigma_s)^2}{2}b''_n(\overleftarrow{X}^{t_{n+1},x}_s)\big)\mathrm{d}s\\
&\quad+\int_{t_n}^{t_{n+1}}\sigma_s b'_n(\overleftarrow{X}^{t_{n+1},x}_s)\mathrm{d}\overleftarrow{W}_s.
\end{aligned}$$

Hence, the term $\mathbb{E}\big[b_n(\overleftarrow{X}^{t_{n+1},x}_{t_n})\overrightarrow{Y}'_n(\overleftarrow{X}^{t_{n+1},x}_{t_n})\big]$ on the right hand side of (3.57) can be written as $\mathbb{E}\big[b_n(\overleftarrow{X}^{t_{n+1},x}_{t_n})\overrightarrow{Y}'_n(\overleftarrow{X}^{t_{n+1},x}_{t_n})\big]=b_n(x)\overrightarrow{Y}'_n(x)+P_n(x)$ with

$$\begin{aligned}
P_n(x)=\mathbb{E}\Big[&b_n(x)\cdot\int_{t_n}^{t_{n+1}}\big(-b_s(\overleftarrow{X}^{t_{n+1},x}_s)\overrightarrow{Y}''_n(\overleftarrow{X}^{t_{n+1},x}_s)+\frac{(\sigma_s)^2}{2}\overrightarrow{Y}^{(3)}_n(\overleftarrow{X}^{t_{n+1},x}_s)\big)\mathrm{d}s\\
&+\overrightarrow{Y}'_n(x)\cdot\int_{t_n}^{t_{n+1}}\big(-b_s(\overleftarrow{X}^{t_{n+1},x}_s)b'_n(\overleftarrow{X}^{t_{n+1},x}_s)+\frac{(\sigma_s)^2}{2}b''_n(\overleftarrow{X}^{t_{n+1},x}_s)\big)\mathrm{d}s\\
&+\int_{t_n}^{t_{n+1}}\big(-b_s(\overleftarrow{X}^{t_{n+1},x}_s)b'_n(\overleftarrow{X}^{t_{n+1},x}_s)+\frac{(\sigma_s)^2}{2}b''_n(\overleftarrow{X}^{t_{n+1},x}_s)\big)\mathrm{d}s\\
&\cdot\int_{t_n}^{t_{n+1}}\big(-b_s(\overleftarrow{X}^{t_{n+1},x}_s)\overrightarrow{Y}''_n(\overleftarrow{X}^{t_{n+1},x}_s)+\frac{(\sigma_s)^2}{2}\overrightarrow{Y}^{(3)}_n(\overleftarrow{X}^{t_{n+1},x}_s)\big)\mathrm{d}s\\
&+\int_{t_n}^{t_{n+1}}(\sigma_s)^2b'_n(\overleftarrow{X}^{t_{n+1},x}_s)\overrightarrow{Y}''_n(\overleftarrow{X}^{t_{n+1},x}_s)\mathrm{d}s\Big].
\end{aligned}$$

As a result the terms on the right hand side of (3.57) can be rewritten as

$$(\mathbf{iv}-\mathbf{1}) = \int_{\mathbb{R}}\big(Y_{n+1}(x)\cdot b_n(x)\overrightarrow{Y}'_n(x)\big)\mathrm{d}x+H^1_n, \tag{3.58}$$

$$(\mathbf{iv}-\mathbf{2}) = \int_{\mathbb{R}}\big(\overrightarrow{Y}_n(x)b_n(x)Y'_{n+1}(x)\big)\mathrm{d}x+H^2_n, \tag{3.59}$$

$$(\mathbf{iv}-\mathbf{3}) = -\int_{\mathbb{R}}\frac{(\sigma_{t_n})^2}{2}Y_{n+1}(x)\overrightarrow{Y}''_n(x)\mathrm{d}x+H^3_n, \tag{3.60}$$

$$(\mathbf{iv}-\mathbf{4}) = \int_{\mathbb{R}}\frac{(\sigma_{t_n})^2}{2}\overrightarrow{Y}_n(x)Y''_{n+1}(x)\mathrm{d}x+H^4_n, \tag{3.61}$$

where

$$
\begin{aligned}
H_n^1 =& \int_{\mathbb{R}} \{ \Big(Y_{n+1}(x) + \mathbb{E}[\int_{t_n}^{t_{n+1}} \big(b_s(X_s^{t_n,x}) Y'_{n+1}(X_s^{t_n,x}) + \frac{(\sigma_s)^2}{2} Y''_{n+1}(X_s^{t_n,x}) \big) ds \big)] \Big) \cdot P_n(x) \\
&+ \mathbb{E}[\int_{t_n}^{t_{n+1}} \big(b_s(X_s^{t_n,x}) Y'_{n+1}(X_s^{t_n,x}) + \frac{(\sigma_s)^2}{2} Y''_{n+1}(X_s^{t_n,x}) \big) ds \big)] \cdot b_n(x) \overrightarrow{Y}'_n(x) \} \mathrm{d}x, \\
H_n^2 =& \int_{\mathbb{R}} \{ \mathbb{E}[\int_{t_n}^{t_{n+1}} \big(- b_s(\overleftarrow{X}_s^{t_{n+1},x}) \overrightarrow{Y}'_n(\overleftarrow{X}_s^{t_{n+1},x}) + \frac{(\sigma_s)^2}{2} \overrightarrow{Y}''_n(\overleftarrow{X}_s^{t_{n+1},x}) \big) ds] \cdot b_n(x) Y'_{n+1}(x) \} \mathrm{d}x, \\
H_n^3 =& - \int_{\mathbb{R}} \{ Y_{n+1}(x) \cdot \frac{(\sigma_{t_n})^2}{2} \mathbb{E}[\int_{t_n}^{t_{n+1}} \big(- b_s(\overleftarrow{X}_s^{t_{n+1},x}) \overrightarrow{Y}_n^{(3)}(\overleftarrow{X}_s^{t_{n+1},x}) + \frac{(\sigma_s)^2}{2} \overrightarrow{Y}_n^{(4)}(\overleftarrow{X}_s^{t_{n+1},x}) \big) \mathrm{d}s] \\
&+ \mathbb{E}[\int_{t_n}^{t_{n+1}} \big(b_s(X_s^{t_n,x}) Y'_{n+1}(X_s^{t_n,x}) + \frac{(\sigma_s)^2}{2} Y''_{n+1}(X_s^{t_n,x}) \big) \mathrm{d}s)] \cdot \frac{(\sigma_{t_n})^2}{2} \overrightarrow{Y}''_n(x) \\
&+ \mathbb{E}[\int_{t_n}^{t_{n+1}} \big(b_s(X_s^{t_n,x}) Y'_{n+1}(X_s^{t_n,x}) + \frac{(\sigma_s)^2}{2} Y''_{n+1}(X_s^{t_n,x}) \big) \mathrm{d}s)] \\
&\cdot \frac{(\sigma_{t_n})^2}{2} \mathbb{E}[\int_{t_n}^{t_{n+1}} \big(- b_s(\overleftarrow{X}_s^{t_{n+1},x}) \overrightarrow{Y}_n^{(3)}(\overleftarrow{X}_s^{t_{n+1},x}) + \frac{(\sigma_s)^2}{2} \overrightarrow{Y}_n^{(4)}(\overleftarrow{X}_s^{t_{n+1},x}) \big) \mathrm{d}s] \} \mathrm{d}x, \\
H_n^4 =& \int_{\mathbb{R}} \{ \mathbb{E}[\int_{t_n}^{t_{n+1}} \big(- b_s(\overleftarrow{X}_s^{t_{n+1},x}) \overrightarrow{Y}'_n(\overleftarrow{X}_s^{t_{n+1},x}) + \frac{(\sigma_s)^2}{2} \overrightarrow{Y}''_n(\overleftarrow{X}_s^{t_{n+1},x}) \big)] \mathrm{d}s \cdot \frac{(\sigma_{t_n})^2}{2} Y''_{n+1}(x) \} \mathrm{d}x.
\end{aligned}
$$

Integrating by parts, we obtain

$$
\begin{aligned}
\int_{\mathbb{R}} \big(Y_{n+1}(x) \cdot b_n(x) \overrightarrow{Y}'_n(x) \big) \mathrm{d}x &= - \int_{\mathbb{R}} Y'_{n+1}(x) b_n(x) \overrightarrow{Y}_n(x) \mathrm{d}x \\
&\quad - \int_{\mathbb{R}} Y_{n+1}(x) b'_n(x) \overrightarrow{Y}_n(x) \mathrm{d}x, \qquad (3.62) \\
- \int_{\mathbb{R}} \frac{(\sigma_{t_n})^2}{2} Y_{n+1}(x) \overrightarrow{Y}''_n(x) \mathrm{d}x &= \int_{\mathbb{R}} \frac{(\sigma_{t_n})^2}{2} Y'_{n+1}(x) \overrightarrow{Y}'_n(x) \mathrm{d}x, \qquad (3.63) \\
\int_{\mathbb{R}} \frac{(\sigma_{t_n})^2}{2} \overrightarrow{Y}_n(x) Y''_{n+1}(x) \mathrm{d}x &= - \int_{\mathbb{R}} \frac{(\sigma_{t_n})^2}{2} \overrightarrow{Y}'_n(x) Y'_{n+1}(x) \mathrm{d}x. \qquad (3.64)
\end{aligned}
$$

Adding (3.58) to (3.59) and applying (3.62), the sum of the first two terms on the right hand side of (3.57) becomes

$$
\begin{aligned}
&- \int_{\mathbb{R}} \mathbb{E}[Y_{n+1}(X_{t_{n+1}}^{t_n,x})] \mathbb{E}\big[- b_n(\overleftarrow{X}_{t_n}^{t_{n+1},x}) \overrightarrow{Y}'_n(\overleftarrow{X}_{t_n}^{t_{n+1},x}) \big] dx \\
&\quad + \int_{\mathbb{R}} \mathbb{E}[\overrightarrow{Y}_n(\overleftarrow{X}_{t_n}^{t_{n+1},x})] \mathbb{E}\big[b_n(X_{t_n}^{t_n,x}) Y'_{n+1}(X_{t_n}^{t_n,x}) \big] dx \qquad (3.65) \\
&= - \int_{\mathbb{R}} Y_{n+1}(x) b'_n(x) \overrightarrow{Y}_n(x) dx + H_n^1 + H_n^2.
\end{aligned}
$$

Similarly, adding (3.60) to (3.61) and applying (3.64), (3.64) yields

$$
\begin{aligned}
&- \int_{\mathbb{R}} \mathbb{E}[Y_{n+1}(\tilde{X}_{t_{n+1}}^{t_n,x})] \mathbb{E}\big[\frac{(\sigma_{t_n})^2}{2} \overrightarrow{Y}''_n(\overleftarrow{X}_{t_n}^{t_{n+1},x}) \big] dx + \int_{\mathbb{R}} \mathbb{E}[\overrightarrow{Y}_n(\overleftarrow{X}_{t_n}^{t_{n+1},x})] \mathbb{E}\big[\frac{(\sigma_{t_n})^2}{2} Y''_{n+1}(\tilde{X}_{t_n}^{t_n,x}) \big] dx \\
&= H_n^3 + H_n^4, \qquad (3.66)
\end{aligned}
$$

which is the sum of the last two terms on the right hand side of (3.57).

For first two terms on the right hand side of equation (3.52), we insert (3.65) and (3.66) into (3.57) to obtain the following equation

$$\begin{aligned}&\Big\langle \mathbb{E}_x^n[Y_{n+1}], \overleftarrow{\mathbb{E}}_x^n[\overrightarrow{Y}_n] - \overleftarrow{\mathbb{E}}_x^{n+1}[\overrightarrow{Y}_n]\Big\rangle + \Big\langle \overleftarrow{\mathbb{E}}_x^{n+1}[\overrightarrow{Y}_n], \mathbb{E}_x^n[Y_{n+1}] - \mathbb{E}_x^{n+1}[Y_{n+1}]\Big\rangle \\ &= \big(-\int_{\mathbb{R}} Y_{n+1}(x) b_n'(x) \overrightarrow{Y}_n(x) dx\big)\Delta t + (H_n^1 + H_n^2 + H_n^3 + H_n^4)\Delta t + R_n^x.\end{aligned} \tag{3.67}$$

Next, we consider the term

$$\Big\langle \mathbb{E}_x^n[h_{n+1}Y_{n+1}], \overleftarrow{\mathbb{E}}_x^n[\overrightarrow{Y}_n]\Big\rangle \Delta V_{t_n} - \Big\langle \overleftarrow{\mathbb{E}}_x^{n+1}[\overleftarrow{h}_n \overrightarrow{Y}_n], \mathbb{E}_x^{n+1}[Y_{n+1}]\Big\rangle \Delta V_{t_n}$$

in (3.52). From the definition of $\mathbb{E}_x^n$ and $\overleftarrow{\mathbb{E}}_x^{n+1}$, one has

$$\begin{aligned}&\Big\langle \mathbb{E}_x^n[h_{n+1}Y_{n+1}], \overleftarrow{\mathbb{E}}_x^n[\overrightarrow{Y}_n]\Big\rangle \Delta V_{t_n} - \Big\langle \overleftarrow{\mathbb{E}}_x^{n+1}[\overleftarrow{h}_n \overrightarrow{Y}_n], \mathbb{E}_x^{n+1}[Y_{n+1}]\Big\rangle \Delta V_{t_n} \\ &= \int_{\mathbb{R}} \mathbb{E}[h(X_{t_{n+1}}^{t_n,x}) Y_{n+1}(X_{t_{n+1}}^{t_n,x}) \overrightarrow{Y}_n(x)] - \mathbb{E}[h(\overleftarrow{X}_{t_n}^{t_{n+1},x}) \overrightarrow{Y}_n(\overleftarrow{X}_{t_n}^{t_{n+1},x}) Y_{n+1}(x)] dx \Delta V_{t_n}.\end{aligned} \tag{3.68}$$

We apply Itô formula to h on time interval $[t_n, t_{n+1}]$ to get

$$\begin{aligned}h(X_{t_{n+1}}^{t_n,x}) =& h(x) + \int_{t_n}^{t_{n+1}} b_s(X_s^{t_n,x}) h'(X_s^{t_n,x}) + \frac{(\sigma_s)^2}{2} h''(X_s^{t_n,x})\big) ds \\ &+ \int_{t_n}^{t_{n+1}} \sigma_s h'(X_s^{t_n,x}) dW_s,\end{aligned}$$

and

$$\begin{aligned}h(\overleftarrow{X}_{t_n}^{t_{n+1},x}) =& h(x) + \int_{t_n}^{t_{n+1}} -b_s(\overleftarrow{X}_s^{t_{n+1},x}) h'(\overleftarrow{X}_s^{t_{n+1},x}) + \frac{(\sigma_s)^2}{2} h''(\overleftarrow{X}_s^{t_{n+1},x})\big) ds \\ &+ \int_{t_n}^{t_{n+1}} \sigma_s h'(\overleftarrow{X}_s^{t_{n+1},x}) d\overleftarrow{W}_s.\end{aligned}$$

Thus

$$\begin{aligned}&\mathbb{E}[h(X_{t_{n+1}}^{t_n,x}) Y_{n+1}(X_{t_{n+1}}^{t_n,x}) \overrightarrow{Y}_n(x)] \\ &= h(x) Y_{n+1}(x) \overrightarrow{Y}_n(x) + \overrightarrow{Y}_n(x) \cdot \mathbb{E}\big[\big(h(X_{t_{n+1}}^{t_n,x}) - h(x)\big) Y_{n+1}(x) \\ &\quad + h(x)\big(Y_{n+1}(X_{t_{n+1}}^{t_n,x}) - Y_{n+1}(x)\big) + \big(h(X_{t_{n+1}}^{t_n,x}) - h(x)\big)\big(Y_{n+1}(X_{t_{n+1}}^{t_n,x}) - Y_{n+1}(x)\big)\big]\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}[h(\overleftarrow{X}_{t_n}^{t_{n+1},x}) \overrightarrow{Y}_n(\overleftarrow{X}_{t_n}^{t_{n+1},x}) Y_{n+1}(x)] \\ &= h(x) \overrightarrow{Y}_n(x) Y_{n+1}(x) + Y_{n+1}(x) \cdot \mathbb{E}\big[\big(h(\overleftarrow{X}_{t_n}^{t_{n+1},x}) - h(x)\big) \overrightarrow{Y}_n(x) \\ &\quad + h(x)\big(\overrightarrow{Y}_n(\overleftarrow{X}_{t_n}^{t_{n+1},x}) - \overrightarrow{Y}_n(x)\big) + \big(h(\overleftarrow{X}_{t_n}^{t_{n+1},x}) - h(x)\big)\big(\overrightarrow{Y}_n(\overleftarrow{X}_{t_n}^{t_{n+1},x}) - \overrightarrow{Y}_n(x)\big)\big].\end{aligned}$$

With the above equations, (3.68) becomes

$$\Big\langle \mathbb{E}_x^n[h_{n+1}Y_{n+1}], \overleftarrow{\mathbb{E}}_x^n[\overrightarrow{Y}_n]\Big\rangle \Delta V_{t_n} - \Big\langle \overleftarrow{\mathbb{E}}_x^{n+1}[\overleftarrow{h}_n \overrightarrow{Y}_n], \mathbb{E}_x^{n+1}[Y_{n+1}]\Big\rangle \Delta V_{t_n} = G_n^1 \Delta V_{t_n}, \tag{3.69}$$

where

$$
\begin{aligned}
G_n^1 = \int_{\mathbb{R}} \{ & \overrightarrow{Y}_n(x)\cdot\mathbb{E}\big[\big(h(X_{t_{n+1}}^{t_n,x})-h(x)\big)Y_{n+1}(x) \\
& +h(x)\big(Y_{n+1}(X_{t_{n+1}}^{t_n,x})-Y_{n+1}(x)\big)+\big(h(X_{t_{n+1}}^{t_n,x})-h(x)\big)\big(Y_{n+1}(X_{t_{n+1}}^{t_n,x})-Y_{n+1}(x)\big)\big] \\
& -Y_{n+1}(x)\cdot\mathbb{E}\big[\big(h(\overleftarrow{X}_{t_n}^{t_{n+1},x})-h(x)\big)\overrightarrow{Y}_n(x) \\
& +h(x)\big(\overrightarrow{Y}_n(\overleftarrow{X}_{t_n}^{t_{n+1},x})-\overrightarrow{Y}_n(x)\big)+\big(h(\overleftarrow{X}_{t_n}^{t_{n+1},x})-h(x)\big)\big(\overrightarrow{Y}_n(\overleftarrow{X}_{t_n}^{t_{n+1},x})-\overrightarrow{Y}_n(x)\big)\big]\}dx.
\end{aligned}
$$

Finally, we consider the term

$$
\Big\langle \mathbb{E}_x^n[\frac{\tilde{\rho}_{t_{n+1}}}{\sigma_{t_{n+1}}}Z_{n+1}], \overleftarrow{\mathbb{E}}_x^n[\overrightarrow{Y}_n]\Big\rangle \Delta V_{t_n} + \Big\langle \overleftarrow{\mathbb{E}}_x^{n+1}[\frac{\tilde{\rho}_{t_n}}{\sigma_{t_n}}\overrightarrow{Z}_n], \mathbb{E}_x^{n+1}[Y_{n+1}]\Big\rangle \Delta V_{t_n}
$$

on the right hand side of equation (3.52). From the relation between Z_t and $\frac{\partial Y_t}{\partial x}$ given in (2.8), we know that

$$
Z_{n+1}(X_{t_{n+1}}^{t_n,x}) = \frac{\partial Y_{n+1}(X_{t_{n+1}}^{t_n,x})}{\partial x}(\nabla X_{t_{n+1}}^{t_n,x})^{-1}\sigma_{t_{n+1}}
$$

and

$$
\overrightarrow{Z}_n(\overleftarrow{X}_{t_n}^{t_{n+1},x}) = \frac{\partial \overrightarrow{Y}_n(\overleftarrow{X}_{t_n}^{t_{n+1},x})}{\partial x}(\nabla \overleftarrow{X}_{t_n}^{t_{n+1},x})^{-1}\sigma_{t_n}.
$$

Therefore we have

$$
\begin{aligned}
\Big\langle \mathbb{E}_x^n[\frac{\tilde{\rho}_{t_{n+1}}}{\sigma_{t_{n+1}}}Z_{n+1}], \overleftarrow{\mathbb{E}}_x^n[\overrightarrow{Y}_n]\Big\rangle &= \mathbb{E}\Big[\int_{\mathbb{R}} \frac{\tilde{\rho}_{t_{n+1}}}{\sigma_{t_{n+1}}}Z_{n+1}(X_{t_{n+1}}^{t_n,x})\cdot\overrightarrow{Y}_n(x)dx\Big] \\
&= \mathbb{E}\Big[\int_{\mathbb{R}} \tilde{\rho}_{t_{n+1}}\frac{\partial Y_{n+1}(X_{t_{n+1}}^{t_n,x})}{\partial x}\cdot\overrightarrow{Y}_n(x)\cdot(\nabla X_{t_{n+1}}^{t_n,x})^{-1}dx\Big]
\end{aligned} \tag{3.70}
$$

and

$$
\begin{aligned}
\Big\langle \overleftarrow{\mathbb{E}}_x^{n+1}[\frac{\tilde{\rho}_{t_n}}{\sigma_{t_n}}\overrightarrow{Z}_n], \mathbb{E}_x^{n+1}[Y_{n+1}]\Big\rangle &= \mathbb{E}\Big[\int_{\mathbb{R}} \frac{\tilde{\rho}_{t_n}}{\sigma_{t_n}}\overrightarrow{Z}_n(\overleftarrow{X}_{t_n}^{t_{n+1},x})\cdot Y_{n+1}(x)dx\Big] \\
&= \mathbb{E}\Big[\int_{\mathbb{R}} \tilde{\rho}_{t_n}\frac{\partial \overrightarrow{Y}_n(\overleftarrow{X}_{t_n}^{t_{n+1},x})}{\partial x}\cdot Y_{n+1}(x)\cdot(\nabla \overleftarrow{X}_{t_n}^{t_{n+1},x})^{-1}dx\Big],
\end{aligned} \tag{3.71}
$$

Adding (3.70) and (3.71) together, we obtain

$$
\begin{aligned}
& \Big\langle \mathbb{E}_x^n[\frac{\tilde{\rho}_{t_{n+1}}}{\sigma_{t_{n+1}}}Z_{n+1}], \overleftarrow{\mathbb{E}}_x^n[\overrightarrow{Y}_n]\Big\rangle + \Big\langle \overleftarrow{\mathbb{E}}_x^{n+1}[\frac{\tilde{\rho}_{t_n}}{\sigma_{t_n}}\overrightarrow{Z}_n], \mathbb{E}_x^{n+1}[Y_{n+1}]\Big\rangle \\
&= \mathbb{E}\Big[\int_{\mathbb{R}} \tilde{\rho}_{t_{n+1}}\frac{\partial Y_{n+1}}{\partial x}(x)\cdot\overrightarrow{Y}_n(x)dx + \int_{\mathbb{R}} \tilde{\rho}_{t_{n+1}}\frac{\partial \overrightarrow{Y}_n}{\partial x}(x)\cdot Y_{n+1}(x)dx\Big] + G_n^2,
\end{aligned} \tag{3.72}
$$

where

$$
\begin{aligned}
G_n^2 = & \mathbb{E}\Big[\int_{\mathbb{R}} \tilde{\rho}_{t_{n+1}}\frac{\partial Y_{n+1}}{\partial x}(X_{t_{n+1}}^{t_n,x})\cdot\overrightarrow{Y}_n(x)\cdot(\nabla X_{t_{n+1}}^{t_n,x})^{-1}dx - \int_{\mathbb{R}} \tilde{\rho}_{t_{n+1}}\frac{\partial Y_{n+1}}{\partial x}(x)\cdot\overrightarrow{Y}_{t_n}(x)dx\Big] \\
& +\mathbb{E}\Big[\int_{\mathbb{R}} \tilde{\rho}_{t_n}\frac{\partial \overrightarrow{Y}_n}{\partial x}(\overleftarrow{X}_{t_n}^{t_{n+1},x}))\cdot Y_{n+1}(x)\cdot(\nabla \overleftarrow{X}_{t_n}^{t_{n+1},x})^{-1}dx - \int_{\mathbb{R}} \tilde{\rho}_{t_{n+1}}\frac{\partial \overrightarrow{Y}_n}{\partial x}(x)\cdot Y_{n+1}(x)dx\Big].
\end{aligned}
$$

Integrating by parts gives,

$$\int_{\mathbb{R}} \tilde{\rho}_{t_{n+1}} \frac{\partial Y_{n+1}}{\partial x}(x) \cdot \overrightarrow{Y}_n(x) dx = -\int_{\mathbb{R}} \tilde{\rho}_{t_{n+1}} \frac{\partial \overrightarrow{Y}_n}{\partial x}(x) \cdot Y_{n+1}(x) dx.$$

Therefore

$$G_n^2 = \left\langle \mathbb{E}_x^n[\frac{\tilde{\rho}_{t_{n+1}}}{\sigma_{t_{n+1}}} Z_{n+1}], \mathbb{E}_x^{\overleftarrow{n}}[\overrightarrow{Y}_n] \right\rangle \Delta V_{t_n} + \left\langle \mathbb{E}_x^{\overleftarrow{n+1}}[\frac{\tilde{\rho}_{t_n}}{\sigma_{t_n}} \overrightarrow{Z}_n], \mathbb{E}_x^{n+1}[Y_{n+1}] \right\rangle. \tag{3.73}$$

From (3.67), (3.69) and (3.73), equation (3.52) becomes

$$\begin{aligned}
&\left\langle \mathbb{E}_x^n[Y_n], \overleftarrow{\mathbb{E}}_x^n[\overrightarrow{Y}_n] \right\rangle - \left\langle \mathbb{E}_x^{n+1}[Y_{n+1}], \overleftarrow{\mathbb{E}}_x^{n+1}[\overrightarrow{Y}_{n+1}] \right\rangle \\
=&\big(- \int_{\mathbb{R}} Y_{n+1}(x) b_n'(x) \overrightarrow{Y}_n(x) dx \big) \Delta t + (H_n^1 + H_n^2 + H_n^3 + H_n^4) \Delta t + R_n^x \\
&+ \int_{\mathbb{R}} \mathbb{E}[b_n'(\overleftarrow{X}_{t_n}^{t_{n+1},x}) \overrightarrow{Y}_n(\overleftarrow{X}_{t_n}^{t_{n+1},x})] Y_{n+1}(x) dx \Delta t + G_n \Delta V_{t_n} \\
=&H_n \Delta t + R_n^x + G_n \Delta V_{t_n} + F_n \Delta t,
\end{aligned} \tag{3.74}$$

where

$$H_n = H_n^1 + H_n^2 + H_n^3 + H_n^4, \quad G_n = G_n^1 + G_n^2$$

and

$$F_n = \int_{\mathbb{R}} Y_{n+1}(x) \Big(\mathbb{E}\big[b_n'(\overleftarrow{X}_{t_n}^{t_{n+1},x}) \overrightarrow{Y}_n(\overleftarrow{X}_{t_n}^{t_{n+1},x}) \big] - b_n'(x) \overrightarrow{Y}_n(x) \Big) dx.$$

Next, we sum (3.74) from $n=0$ to $n=N-1$ to get

$$\begin{aligned}
&\left\langle \mathbb{E}_x^0[Y_0], \overleftarrow{\mathbb{E}}_x^0[\overrightarrow{Y}_0] \right\rangle - \left\langle \overleftarrow{\mathbb{E}}_x^N[\overrightarrow{Y}_N], \mathbb{E}_x^N[Y_N] \right\rangle \\
=&\sum_{n=0}^{N-1} (H_n \Delta t + R_n^x + G_n \Delta V_{t_n} + F_n \Delta t).
\end{aligned} \tag{3.75}$$

From definitions of H_n , R_n^x, G_n and F_n, it's easy to verify that $E[(H_n)^2] \le C(\Delta t)^2$, $E[(R_n^x)^2] \le C(\Delta t)^4$,$E[(G_n)^2] \le C(\Delta t)^2$ and $E[(F_n)^2] \le C(\Delta t)$. Therefore,

$$\lim_{\Delta t \to 0} \sum_{n=0}^{N-1} (H_n \Delta t + R_n^x + G_n \Delta V_{t_n} + F_n \Delta t) = 0, \ a.s..$$

Also, since $\lim_{\Delta t \to 0} Y_0 = Y_s$ and $\lim_{\Delta t \to 0} Y_N = Y_t$, we have

$$\left\langle Y_s, \overrightarrow{Y}_s \right\rangle = \left\langle Y_t, \overrightarrow{Y}_t \right\rangle$$

as required. □

Now are ready to state the main result in this paper. It is a direct consequence of Theorems 3.2 and 3.5.

Theorem 3.6 *Assume that the ssumptions in Theorem 3.2 and Theorem 3.5 hold. Then*

$$\left\langle \overrightarrow{Y}_T, \phi \right\rangle = \tilde{E}\left[\phi(U_T) Q_T \middle| \mathcal{F}_T^V \right], \qquad \forall \phi \in \mathcal{L}^\infty(\mathbb{R}^d).$$

Proof. Applying Theorem 3.5, one has

$$\left\langle \overrightarrow{Y}_T, Y_T \right\rangle = \left\langle \overrightarrow{Y}_0, Y_0 \right\rangle.$$

Since $Y_T = \phi$ as given in (3.9), $\overrightarrow{Y}_0 = p_0$ as given in (3.46) and $Y_0 = \tilde{E}_x\left[\phi(S_T)Q_T \middle| \mathcal{F}_T^V\right]$ as proved in Theorem 3.2, we have

$$\left\langle \overrightarrow{Y}_T, \phi \right\rangle = \int_{\mathbb{R}} p_0(x)\tilde{E}_x[\phi(U_T)Q_T \middle| \mathcal{F}_T^V]dx.$$

Let φ be any bounded $\mathcal{F}_T^V$ measurable random variable,

$$\tilde{E}_x[\left\langle \overrightarrow{Y}_T, \phi \right\rangle \varphi] = \int_{\mathbb{R}} p_0(x)\tilde{E}_x[\phi(U_T)Q_T\varphi]dx.$$

It then follows from the fact that $\tilde{P}_x(\cdot|\mathcal{F}_T^V) = \tilde{P}(\cdot|\mathcal{F}_T^V)$, and definition of $\tilde{P}$

$$\tilde{E}[\left\langle \overrightarrow{Y}_T, \phi \right\rangle \varphi] = \tilde{E}[\phi(U_T)Q_T\varphi],$$

as required in the theorem. □

Remark. From (2.4), we can see that

$$E\left[\phi(U_T)\middle|\mathcal{F}_T^V\right] = \frac{\left\langle \overrightarrow{Y}_T, Y_T \right\rangle}{\tilde{\mathbb{E}}\left[Q_t\middle|\mathcal{F}_t^V\right]}$$

Thus the solution $\overrightarrow{Y}_T$ of the FBDSDE (3.46) indeed provides an unnormalized solution for the optimal filter problem.

4 Closing Remarks

In this paper, we derived a Feymann-Kac type BDSDE formula for optimal filter problems and its adjoint. Then we show that the adjoint provides a unnormalized solution for the optimal filter problem (BSDE filter). As our preliminary work has shown, the BSDE filter has the potential to solve the optimal filter problem with more accuracy and less complexity than traditional filter methods.

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