

# Quantum control landscape of bipartite systems

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Received 14 November 2018, revised 28 February 2019

Accepted for publication 7 March 2019

Published 28 March 2019



CrossMark

## Abstract

The control landscape of a quantum system  $A$  interacting with another quantum system  $B$  is studied. Only system  $A$  is accessible through time dependent controls, while system  $B$  is not accessible. The objective is to find controls that implement a desired unitary transformation on  $A$ , regardless of the evolution on  $B$ , at a sufficiently large final time. The freedom in the evolution on  $B$  is used to define an *extended control landscape* on which the critical points are investigated in terms of kinematic and dynamic gradients. A spectral decomposition of the corresponding extended unitary system simplifies the landscape analysis which provides: (i) a sufficient condition on the rank of the dynamic gradient of the extended landscape that guarantees a trap free search for the final time unitary matrix of system  $A$ , and (ii) a detailed decomposition of the components of the overall dynamic gradient matrix. Consequently, if the rank condition is satisfied, a gradient algorithm will find the controls that implements the target unitary on system  $A$ . It is shown that even if the dynamic gradient with respect to the controls alone is not full rank, the additional flexibility due to the parameters that define the extended landscape still can allow for the rank condition of the extended landscape to hold. Moreover, satisfaction of the latter rank condition subsumes any assumptions about controllability, reachability and control resources. Here satisfaction of the rank condition is taken as an assumption. The conditions which ensure that it holds remain an open research question. We lend some numerical support with two common examples for which the rank condition holds.

Keywords: quantum, control, landscape

(Some figures may appear in colour only in the online journal)

## 1. Introduction

Extensive theoretical and experimental evidence supports the relative ease of searching for very good to near optimal controls for quantum systems [1–5]. The ‘ease’ here refers to the search efficiency (e.g. number of iterations), putting aside the overhead of experimental/laboratory/field set up efforts. Similarly, in a computational simulation context, numerical optimization algorithms are typically very successful in finding classical control fields that drive a quantum system towards a desired target. For instance, maximizing the overlap  $F(c(t))$  between a target state (transformation) and the actual state (time evolution operator) of a closed quantum system utilizing time dependent fields  $c(t)$  *almost always* succeeds using a gradient based search [5]. This finding suggests that the underlying *control landscape* [2] defined by the functional  $F(c(t))$  is ‘simple’ in the sense that local optima (traps) rarely occur. Thus, in the absence of traps, a gradient based search typically converges to the global optimum, which, for instance, corresponds to the preparation (implementation) of a target state (gate) with fidelity  $F = 1$ . The scope of the principles underlying this result seem to apply to more than quantum systems, with applicability across a wide range of optimization in the sciences and engineering in many highly complex systems [6].

However, any realistic quantum system interacts with its surrounding environment causing the system dynamics to become perturbed in some fashion. The search for optimal controls, and in particular the study of the control landscape of such open quantum systems, is relatively unexplored. A basic question is whether the interaction with the environment, or with an auxiliary system in general, introduces traps into the control landscape so that a gradient based search would not succeed in finding the global optimum.

In this paper we investigate the control landscape of *bipartite quantum systems*. That is, we study the optimization of a unitary transformation objective for quantum system  $A$  that is subject to time dependent fields and coupled to another quantum system  $B$ , so that the overall combined dynamics is unitary. We derive a sufficient criteria that allows for concluding when a gradient based algorithm will be successful in finding the fields that implement a desired unitary operation on system  $A$ . In particular, using the fidelity measure  $F$  developed in [7] for bipartite systems, we find a sufficient condition for when  $\nabla_c F \neq 0$  holds except at saddles and the global maximum  $F = 1$  (and minimum), i.e. the control landscape is *trap-free*. This result is achieved by noting that, since we are only interested in implementing a unitary operation on  $A$ , maximizing over the parameters describing the final unitary operation on system  $B$ , collected in the vector  $\phi$ , referred to as the extended landscape parameter, yields additional freedom to aid in the optimization process. The sufficient condition is on the rank of the dynamic gradient of the spectral frequencies of an associated extended unitary with respect to both  $c$  and  $\phi$ . Since  $\phi$  becomes a global phase on the target unitary in the absence of  $B$ , our results also hold for closed quantum systems.

In contrast to previous studies of the control landscape of quantum systems, the trap-free condition presented here does not depend on the assumption that the system is controllable, or on any assumption that controls are available that allow for implementing the target unitary transformation. As long as the developed sufficient condition holds, a trap-free search is guaranteed. So as not to mislead the reader, it remains an open challenge to find criteria for when the condition holds. In this work satisfaction of the condition is stated as an assumption, though its general validity is not known at this time. Taking a pragmatic view, the rank condition can be used to test available controls to lend support for their use. In this regard we provide numerical evidence that the condition is satisfied for many simulations of two common bipartite systems; the results are fully consistent with the main assumption and its consequences. The afore stated sufficient condition follows directly by utilizing the spectral

decomposition of the extended landscape unitary matrix, which as a by-product, also yields new detailed expressions for the relevant gradients. These may provide the basis for establishing conditions which underly the presented sufficient condition for a trap-free search.

The work is organized as follows. We begin by introducing the bipartite control system and the fidelity measure, followed by defining the optimization problem. Using the spectral decomposition of unitary operations we first show that the landscape can be expressed as a function of what we refer to as the unitary ‘frequencies’  $\omega(c, \phi)$  that depend on the controls  $c$  and  $\phi$ . An immediate consequence is that the landscape as a function of  $\omega$ , i.e. the so called kinematic landscape, only posses a finite number of saddle points. By defining an *extended control landscape* we will see that if the rank of the gradient of  $\omega$  with respect to  $c$  and  $\phi$ , i.e. the so called dynamic landscape, is full, then the landscape of  $A$  that arises from disregarding the evolution on  $B$  is trap-free. We numerically investigate two common models describing a bipartite system and show that the rank condition holds for the considered systems.

## 2. Bipartite quantum control system

We consider a closed *bipartite* quantum system with parts  $A$  and  $B$  of dimension  $N_A$  and  $N_B$ , respectively. The total system of dimension  $N = N_A N_B$  is described by a time independent (drift) Hamiltonian  $H_0$ , which includes interactions between  $A$  and  $B$ . We further assume that system  $A$  can be influenced by  $M$  time varying control fields  $c_m(t)$ ,  $m = 1, \dots, M$  that are coupled through  $M$  (control) Hamiltonians  $H_m$  in a bilinear way to system  $A$  [8]. The time dependent Hamiltonian describing the total system is then given by,

$$H(t) = H_0 + \sum_{m=1}^M c_m(t)(H_m \otimes I_{N_B}), \quad (1)$$

where  $I_{N_B}$  denotes the identity matrix of dimension  $N_B \times N_B$ . We additionally assume that the controls are piecewise constant over  $L$  uniform intervals  $\delta = T/L$ . The total unitary evolution denoted by  $U(c) \in U(N)$  at time  $t = T$  is then given by a product of  $L$  unitary operations. Specifically,

$$\begin{aligned} U(c) &= U_1 \cdots U_L \\ U_\ell &= e^{-i\delta H_\ell}, \quad \ell = 1, \dots, L, \\ H_\ell &= H_0 + \sum_{m=1}^M c_{\ell m}(H_m \otimes I_{N_B}), \quad \ell = 1, \dots, L \end{aligned} \quad (2)$$

where all the control amplitudes  $c_{\ell m} = c_m(\ell T/L)$  are collected in the vector  $c \in \mathbb{R}^{LM}$ . Note that piecewise constant controls are not crucial for the landscape analysis that follows. Any suitable parameterization of the control fields  $c_m(t)$  will have a similar effect, e.g. via frequencies, amplitude and phases of a Fourier series.

## 3. Fidelity functions

The design objective is to find control parameters  $c \in \mathbb{R}^{LM}$  so that the final time unitary matrix  $U(c)$  is as close as possible to a desired unitary evolution  $W \in U(N_A)$  on system  $A$ . If this is achieved exactly then the final time unitary matrix will factorize, i.e.  $U(c) = W \otimes U_B$  for some  $U_B \in U(N_B)$ . The goal of this paper is to find a condition for when the search for controls achieving this goal is trap-free.

To quantify the search we proceed by introducing a fidelity function for a bipartite quantum system which measures the closeness to a decoupled system with the target unitary  $W$  on system  $A$ . We then formulate the corresponding optimization problem. The introduced fidelity function characterizes the control landscape of a bipartite quantum system so that we can define subsequently what we mean by a trap-free search. We distinguish between a model-based and data-based control design procedure. We show that there is a fidelity function variable that is common to both.

### 3.1. Model-based control design

If we assume that (2) is an accurate model of the bipartite system, the optimization problem to find the controls  $c$  to implement  $W$  on the  $A$ -system can be formulated as minimizing the distance  $D$  between the final-time unitary  $U(c)$  given by (2) and *any* decoupled target system  $W \otimes \Phi$  with  $\Phi \in U(N_B)$ . Specifically, we use the distance measure developed in [7],

$$D(c, \Phi) = \|U(c) - W \otimes \Phi\|_{\text{fro}}^2 \quad (3)$$

where  $\|X\|_{\text{fro}} = \sqrt{\text{Tr} X^\dagger X}$  is the Frobenius norm. Since the control task is to create, at time  $T$ , the unitary matrix  $W$  on system  $A$ , we additionally minimize (3) over  $\Phi$  to obtain a fidelity function  $F(c)$  (given below) that is independent of  $\Phi$ . Note that we are not assuming here that we can create every  $\Phi$ . In fact there may exist only a single  $\Phi$  that allows for creating  $W$ .

As shown in [7], for a given control  $c$ , optimization over  $\Phi$  gives,

$$\min_{\Phi \in U(N_B)} D(c, \Phi) = 2N \left( 1 - (1/N) \max_{\Phi \in U(N_B)} J(c, \Phi) \right) = 2N \left( 1 - \sqrt{F(c)} \right) \quad (4)$$

with,

$$\begin{aligned} J(c, \Phi) &= \text{ReTr}\{(W \otimes \Phi)^\dagger U(c)\} \\ \max_{\Phi \in U(N_B)} J(c, \Phi) &= \|\Gamma(c)\|_{\text{nuc}} \end{aligned} \quad (5)$$

and

$$F(c) = \|\Gamma(c)/N\|_{\text{nuc}}^2, \quad (6)$$

where  $\|X\|_{\text{nuc}} = \text{Tr}\sqrt{X^\dagger X}$  is the nuclear norm (i.e. the sum of the singular values of  $X$ ), and where  $\Gamma(c) = \sum_{a=1}^{N_A} R(c)_{[aa]} \in \mathbb{C}^{N_B \times N_B}$ ,  $R(c) = (W \otimes I_{N_B})^\dagger U(c) \in U(N)$  with  $\{R(c)_{[aa]}\}$ ,  $a = 1, \dots, N_A$  the  $N_A$  block diagonal  $N_B \times N_B$  submatrices of  $R(c)$ .

We refer to (i)  $F(c)$  as a function of  $c \in \mathbb{R}^{LM}$  as the *control landscape*, (ii)  $J(c, \Phi)$  as a function of  $(c, \Phi) \in \mathbb{R}^{LM} \times U(N_B)$  as the *extended control landscape*, and (iii)  $\Phi$  as the *extended landscape unitary matrix*. Both of these functions are bounded:  $F(c) \in [0, 1]$  and  $J(c, \Phi) \in [-N, N]$ . The unitary matrix  $\Phi_{\text{opt}}$  which achieves the above maximum is obtained from the singular value decomposition  $\Gamma(c) = T_{\text{left}} Q T_{\text{right}}^\dagger$  with  $T_{\text{left}}, T_{\text{right}} \in U(N)$  and  $Q = \text{diag}(q_1, \dots, q_{N_B}) \geq 0$ , so that we obtain,

$$\Phi_{\text{opt}}(c) = T_{\text{left}} T_{\text{right}}^\dagger. \quad (7)$$

For our subsequent analysis it is more convenient to parameterize the unitary matrix  $\Phi$  via the generator  $\mathcal{B}(\phi) = \sum_{b=1}^{N_B^2} \phi_b B_b$  by,

$$\Phi(\phi) = \exp\{i\mathcal{B}(\phi)\}, \quad (8)$$

where  $\{B_b\}_{b=1}^{N_B^2}$  is an operator basis for system  $B$ . The real parameters  $\phi_b$  are collected in the vector  $\phi \in \mathbb{R}^{N_B^2}$  referred to as the *extended landscape parameter*. The extended landscape objective is then equivalent to,

$$J(c, \phi) = \text{ReTr}\{(W \otimes \Phi(\phi))^\dagger U(c)\}. \quad (9)$$

In each appropriate context we sometimes use ‘fidelity’ when referring to  $F(c)$ ,  $J(c, \Phi)$ , or  $J(c, \phi)$ . The maxima  $F(c) = 1$  and  $J(c, \phi) = N$  (equivalently  $D(c, \phi) = 0$ ) are obtained iff the target unitary matrix  $W$  on system  $A$  is achieved for the final time unitary matrix, i.e.  $U(c) = W \otimes U_B$  for some unitary matrix  $U_B \in \text{U}(N_B)$ . When this occurs we also have  $\Gamma(c) = N_A U_B$  and  $\Phi_{\text{opt}}(c) = \Phi(\phi_{\text{opt}}(c)) = U_B$ . We emphasize again that even when perfect decoupling is achieved, we do not specify or constrain  $U_B$ , the final-time unitary on system  $B$ .

For a *closed* system, i.e. there is no  $B$ -system present,  $N_B = 1$ ,  $N = N_A$ , and the extended landscape unitary  $\Phi = e^{i\phi}$  is reduced to a global phase on the target unitary matrix  $W$ . Consequently,  $\Gamma(c) = \text{Tr } W^\dagger U(c)$  and  $\max_\phi J(c, e^{i\phi}) = |\text{Tr } W^\dagger U(c)|$  which when squared and normalized gives  $F(c) = |\text{Tr } \{W^\dagger U(c)\}/N_A|^2 \in [0, 1]$ , an often used fidelity measure for closed quantum systems.

### 3.2. Data-based control design

If the model (2) is not known adequately, the control fields can be directly learned in an experiment using data obtained from measurement outcomes [9]. Assuming that the initial state  $\rho_B$  of system  $B$  is uncorrelated with the initial state of system  $A$  we can use quantum process tomography (QPT) to estimate the time evolution of system  $A$ , which is in general given by a quantum channel that is a completely positive trace preserving map [10, 11] represented by a process matrix  $X(c) \in \mathbb{C}^{N_A^2 \times N_A^2}$ . As shown in [7], the *channel fidelity*  $\hat{F}(c)$ , measuring how close the quantum channel is to the target unitary  $W$  on system  $A$ , is given by,

$$\hat{F}(c) = (1/N_A^2) \vec{W}^\dagger X(c) \vec{W} = (1/N_A^2) \text{Tr}\{\Gamma(c) \rho_B \Gamma(c)^\dagger\}, \quad (10)$$

where  $\vec{W} \in \mathbb{C}^{N_A^2}$  is the vectorized version of the target unitary  $W \in \text{U}(N_A)$ . Note that *only*  $X(c)$  is known from the data, whereas  $\Gamma(c)$  and  $\rho_B$  are not. However, this fidelity, like (6) is a direct measure of a norm of  $\Gamma(c)$ : in (6) the norm is the square of the sum of the singular values, whereas in (10) it is a weighted sum of the singular values squared. In either situation these fidelities are in the range  $[0, 1]$  and the maximum of 1 is only achieved when  $W$  is implemented on system  $A$ . Notwithstanding the intermediate step of QPT, because (6) and (10) effectively measure a norm of  $\Gamma(c)$ , their landscape properties are essentially the same, modulo some minor effects on the critical points as discussed in [12].

## 4. Trap-free search

Searching over the control parameters  $c \in \mathbb{R}^{LM}$  to maximize the model-based fidelity  $F(c)$  from (6) or the data-based fidelity  $\hat{F}(c)$  from (10) is in general not a convex optimization problem due to the inherent bilinear control structure of the Hamiltonian (1). There are many approaches [13] to determining  $c$  including stochastic algorithms [9] and gradient-based algorithms [14, 15], etc. In almost all cases the search for controls with various objectives is proving to be remarkably successful, both in simulations and in a variety of experiments. As a result, no matter how the search for the controls is conducted, we would like to know under

what conditions the landscape is trap-free, i.e. when there are no sub-optimal local maxima. Specifically, by a *trap-free* search we mean the following:

*The control landscape  $F(c)$  is trap-free if the critical points (where  $\nabla_c F(c) = 0$ ) are either saddles or global extrema.*

Towards reaching this latter assessment, we defined two search landscapes: the control landscape with objective  $F(c)$  from (6), and the extended landscape with objective  $J(c, \phi)$  from (9). We will show that at the optimal parameter  $\phi_{\text{opt}}(c)$  (obtained from (7) via  $\Phi_{\text{opt}}(c) = \exp\{i\mathcal{B}(\phi_{\text{opt}}(c))\}$ ), when the extended landscape  $J(c, \phi_{\text{opt}}(c))$  is trap-free, then so is the control landscape  $F(c)$ .

## 5. Critical points of the control landscape

To see how a trap-free search of the control landscape (6) could arise, we first examine in more detail the character of the critical points of the extended landscape fidelity (9).

### 5.1. Spectral decomposition

The extended landscape  $J(c, \phi)$  from (9) depends on a unitary matrix denoted by  $U_{\text{ext}}(c, \phi) \in \text{U}(N)$  which can be decomposed via,

$$U_{\text{ext}}(c, \phi) = (W \otimes \Phi(\phi))^{\dagger} U(c) = V(c, \phi) e^{i\Omega(c, \phi)} V(c, \phi)^{\dagger}, \quad (11)$$

where  $\Omega(c, \phi) = \text{diag}(\omega_1(c, \phi), \dots, \omega_N(c, \phi))$ , the unitary matrix satisfies  $V(c, \phi) \in \text{U}(N)$ , and spectral frequencies  $\{\omega_n(c, \phi)\}$  are conveniently collected in the vector  $\omega(c, \phi) \in \mathbb{R}^N$ . The fidelity then simply becomes,

$$J(c, \phi) = \sum_{n=1}^N \cos(\omega_n(c, \phi)) \quad (12)$$

and the gradient of  $J(c, \phi)$  takes the form

$$\nabla_{c, \phi} J(c, \phi) = G_{c, \phi}(c, \phi) g(\omega(c, \phi)) \in \mathbb{R}^{LM+N_B^2}, \quad (13)$$

where, following the terminology in [2], the matrix

$$G_{c, \phi}(c, \phi) = \nabla_{c, \phi} \omega(c, \phi) = \begin{bmatrix} G_c(c, \phi) \\ G_{\phi}(c, \phi) \end{bmatrix} \in \mathbb{R}^{(LM+N_B^2) \times N} \quad (14)$$

is referred to as the *dynamic gradient* and the vector

$$g(\omega) = -(\sin(\omega_1(c, \phi)), \dots, \sin(\omega_N(c, \phi)))^T \in \mathbb{R}^N \quad (15)$$

as the *kinematic gradient*. Detailed expressions for the constituent dynamic gradient matrices  $G_c(c, \phi) = \nabla_c \omega(c, \phi) \in \mathbb{R}^{LM \times N}$  and  $G_{\phi}(c, \phi) = \nabla_{\phi} \omega(c, \phi) \in \mathbb{R}^{N_B^2 \times N}$  are presented in appendix A along with some of their properties.

### 5.2. Kinematic critical points

In terms of the vector of spectral frequencies  $\omega \in \mathbb{R}^N$  we define the *kinematic fidelity* in (12) by  $J(\omega) = \sum_n \cos \omega_n$ , and the associated *kinematic gradient* by  $\nabla_{\omega} J(\omega) = g(\omega)$ , so that the *kinematic Hessian* is given by  $\nabla_{\omega}^2 J(\omega) = -\text{diag}(\cos(\omega_1(c, \phi)), \dots, \cos(\omega_N(c, \phi)))$ . A

kinematic critical point at which  $g(\omega) = 0$  is obtained iff  $\sin(\omega_n(c, \phi)) = 0$  for all  $n$  (equivalently  $\cos(\omega_n(c, \phi)) = \pm 1$ ). Using standard trigonometry (appendix B) yields the following picture. Trivially the maximum and minimum of  $J$  is given by  $J(\omega) = \pm N$  with a corresponding Hessian  $\nabla_\omega^2 J(\omega) = \mp I_N$ , respectively. The interior is defined through  $J(\omega) = (2p - N)$  and  $\nabla_\omega^2 J(\omega) = \text{diag}(-I_p, I_{N-p})$  for  $1 \leq p \leq N - 1$ . Thus, the kinematic critical points associated at the top and bottom of the landscape where  $J(\omega) = \pm N$  are global extrema because the Hessians there are, respectively, negative and positive definite. In the interior of the landscape where  $J(\omega) \in (-N, N)$  there are no traps, only saddles; there the Hessians are all indefinite with both positive and negative eigenvalues of unit magnitude. Moreover, these critical points produce exactly  $N - 1$  critical values at which  $J(\omega) \in \{2 - N, \dots, N - 2\}$ . However, there may be an infinite number of frequencies and/or control combinations which produce the  $N - 1$  saddle point objective values. Landing on one of these saddles is highly unlikely with the gradient algorithm, so these are usually of no practical importance; however their presence can influence the efficiency of climbing the landscape.

## 6. Rank conditions for a trap-free search

To arrive at our main result first note that from (5) and (6) together with (8) where  $\Phi_{\text{opt}}(c) = \Phi(\phi_{\text{opt}}(c))$ , the extended landscape objective at  $\phi_{\text{opt}}(c)$  becomes,

$$J(c, \phi_{\text{opt}}(c)) = \|\Gamma(c)\|_{\text{nuc}} = N\sqrt{F(c)}, \quad (16)$$

and the gradient with respect to the controls  $c$  then reads,

$$\nabla_c J(c, \phi_{\text{opt}}(c)) = \left( N / \sqrt{F(c)} \right) \nabla_c F(c). \quad (17)$$

This result shows that at the optimal  $\phi_{\text{opt}}(c)$  the extended landscape fidelity is positive, and additionally (see appendix C) we also find that the following two landscape properties must hold:

$$\sum_{n=1}^N \sin \omega_n(c, \phi_{\text{opt}}(c)) = 0, \quad (18)$$

$$\nabla_\phi J(c, \phi_{\text{opt}}(c)) = G_\phi(c, \phi_{\text{opt}}(c))g(\omega(c, \phi_{\text{opt}}(c))) = 0.$$

The second property above follows by definition of  $\phi_{\text{opt}}(c)$ : the extended landscape objective is maximized by  $\max_\phi J(c, \phi) = J(c, \phi_{\text{opt}}(c))$ . Consequently, at  $\phi_{\text{opt}}(c)$ , the gradient of the extended landscape objective with respect to the extended landscape parameters is zero. Together with the definitions of the gradient matrices in (13), the equivalent expression for the gradient in (16) is,

$$\nabla_c J(c, \phi_{\text{opt}}(c)) = G_c(c, \phi_{\text{opt}}(c))g(\omega(c, \phi_{\text{opt}}(c))), \quad (19)$$

so that with (14) and (17), (18) we arrive at,

$$\nabla_{c,\phi} J(c, \phi)|_{\phi=\phi_{\text{opt}}(c)} = \begin{bmatrix} G_c(c, \phi) \\ G_\phi(c, \phi) \end{bmatrix} g(c, \phi) \Big|_{\phi=\phi_{\text{opt}}(c)} = \begin{bmatrix} \left( N / \sqrt{F(c)} \right) \nabla_c F(c) \\ \mathbf{0} \end{bmatrix}. \quad (20)$$

Since the extended landscape gradient at the optimal landscape parameter as expressed above is a product of the dynamic and the kinematic gradient (13)–(15), and since the kinematic gradient  $g(\omega)$  is zero *only* at the global extrema  $J = \pm N$  or at saddles, thereby forming a *sufficient condition* to ensure that there are no traps (local extrema) in the control landscape



interior  $F(c) \in (0, 1)$  is reflected in the following rank condition for the dynamic gradient  $G_{c,\phi}(c, \phi_{\text{opt}}(c))$ .

**Rank condition for trap-free search.** Assume that the dynamic gradient of the extended landscape,  $G_{c,\phi}(c, \phi)$  evaluated at the optimal extended landscape parameter satisfies,

$$\begin{aligned} \text{rank } G_{c,\phi}(c, \phi_{\text{opt}}(c)) &= N, & U_{\text{ext}}(c, \phi_{\text{opt}}(c)) &\in \text{U}(N) \\ &\geq N-1, & U_{\text{ext}}(c, \phi_{\text{opt}}(c)) &\in \text{SU}(N) \end{aligned} \quad (21)$$

with  $U_{\text{ext}}(c, \phi)$  from (11). Under this condition, the control landscape  $F(c)$  is trap-free, meaning that  $\nabla_c F(c) = 0$  either at a saddle where  $F(c) \in (0, 1)$  or at the global extrema, including the global maximum  $F(c) = 1$ .

We emphasize again that the rank condition (21) is a sufficient condition for a trap-free search. We have not established the conditions which guaranty that (21) holds. Moreover, it is a strong condition in the sense that it does subsume any prior assumptions about controllability, reachability, and control resources.

To emphasize this point, if, in fact, (21) holds, then a gradient algorithm *will* reach the top of the control landscape; *a fortiori*, a final-time decoupling control exists.

Interestingly, if (21) holds, then in the landscape interior  $F(c) \in (0, 1)$ , except at saddles, the fidelity gradient satisfies  $\nabla_c F(c) \neq 0$  even if the rank of the dynamic control gradient  $G_c(c, \phi_{\text{opt}}(c))$  is not full. Recall that the matrix  $G_c$  describes the gradient of the unitary frequencies  $\omega(c, \phi)$  with respect to  $c$  with  $\phi$  fixed, where the controls only act on system  $A$ . Intuitively we expect that when the total  $AB$  system cannot be *fully* controlled by acting on  $A$  alone, the frequencies cannot arbitrarily be varied as a function of  $c$ . Furthermore, we expect that then  $G_c$  is not full rank. However, the rank condition (21) can hold by virtue of the apparent *additional flexibility* offered by  $G_\phi(c, \phi_{\text{opt}}(c))$ , the dynamic gradient with respect to the extended landscape parameter. Clearly the  $N_B^2$  extended landscape parameters can offset rank deficiencies due to the  $LM$  control parameters alone.

At the top we have:  $F(c) = 1$ , the final time unitary  $U(c) = W \otimes U_B$ , the optimal extended landscape variable  $\Phi_{\text{opt}}(c) = U_B$ , and the extended landscape unitary  $U_{\text{ext}}(c, \phi_{\text{opt}}(c)) = I_N$ . This means that as a search converges to the top of the landscape,  $U_{\text{ext}}(c, \phi_{\text{opt}}(c)) \rightarrow I_N$  (clearly in  $\text{SU}(N)$ ), and so the sum of the spectral frequencies converges to zero, and hence, the rank condition converges to **rank**  $G_{c,\phi}(c, \phi_{\text{opt}}(c)) \geq N-1$ .

An interesting special case of  $U_{\text{ext}} \in \text{SU}(N)$  is when the spectral frequencies are symmetric about zero and otherwise unequal. When this occurs the rank condition for a trap-free search drops to  $N/2$ . To see how this comes about, assume that the spectral frequencies are ordered so that  $\omega = [-\bar{\omega}, \bar{\omega}]$  with  $\bar{\omega}_1 > \bar{\omega}_2 > \dots > \bar{\omega}_{N/2} > 0$ . It follows that the dynamic gradient has the form  $G(c, \phi) = [-\bar{G}(c, \phi) \ \bar{G}(c, \phi)]$  with  $\bar{G}(c, \phi) \in \mathbb{R}^{(LM+N_B^2) \times N/2}$ . Hence, under these condition the control landscape is trap-free if **rank**  $G_{c,\phi}(c, \phi_{\text{opt}}(c)) \geq N/2$ . We show an example of this reduced rank condition in figure 2 of a single spin coupled to a random bath with the identity target.

Before presenting numerical support for the main result, it is informative to examine (21) for a (non-bipartite) closed-system. Since there is no  $B$  system we have  $N_B = 1$ ,  $N = N_A$ , and  $\Phi(\phi)$  reduces to a global phase on the target unitary matrix, i.e. the extended landscape unitary matrix becomes  $U_{\text{ext}}(c, \phi) = e^{i\phi} W^\dagger U(c)$ . As shown in appendix E the dynamic gradient simplifies to,

$$G_{c,\phi} = \begin{bmatrix} G_c \\ \mathbf{1} \end{bmatrix} \in \mathbb{R}^{(LM+1) \times N} \quad (22)$$



from which is obtained,

$$\mathbf{rank} G_{c,\phi} = \min\{\mathbf{rank} G_c + 1, N\}. \quad (23)$$

Thus, if for a closed quantum system  $\mathbf{rank} G_c \geq N - 1$  is satisfied, a gradient algorithm will find the controls that implement a target unitary transformation up to global phase. We emphasize here again that no statement is made about the validity of this assumption on the rank of  $G_c$ . However, assuming that sufficient control resources are available, it has recently been shown that almost all closed quantum systems are trap free [5]. That is, if one picks the drift and the control Hamiltonian at random, and furthermore, there are no control field constraints, the control landscape of almost all (except a null set) closed systems is trap free. We therefore expect that when sufficient control resources are available, the condition  $\mathbf{rank} G_c \geq N - 1$  holds for almost all closed quantum systems, though a rigorous proof connecting the results in [5] and the analysis presented above for closed systems is left for future studies.

The single extra degree of freedom due to the extended landscape parameter  $\phi$  makes it clear that  $G_c$  can have a reduced rank. For example, if  $U_{\text{ext}} \in \text{SU}(N)$ , the landscape is trap-free even if the rank of  $G_c$  is as low as  $N - 2$ , provided that  $N - 2 \leq LM$ . The numerical results to follow suggest that this flexibility is more pronounced for bipartite system.

## 7. Numerical examples

In order to find the controls that implement a desired target unitary on system  $A$  we use a gradient algorithm starting with an initial control  $c^0 \in \mathbb{R}^{LM}$  which we choose throughout this section to be the zero vector, and iterate for  $i = 1, 2, \dots$ , according to,

$$c^i = c^{i-1} + \gamma_i \nabla_c F(c^{i-1}), \quad (24)$$

where the gradient is obtained numerically. The step-size,  $\gamma^i$ , is increased, and the control update accepted, whenever  $F(c^i) > F(c^{i-1})$ . Otherwise the control update is not accepted, the step-size is decreased and (24) is repeated with the previous control. The algorithm is halted when  $F(c^i)$  is insufficiently increasing. In order to see whether the rank condition holds, the singular values of  $G_{c,\phi}$  and  $G_c$  at  $\phi_{\text{opt}}$  are calculated during the iteration.

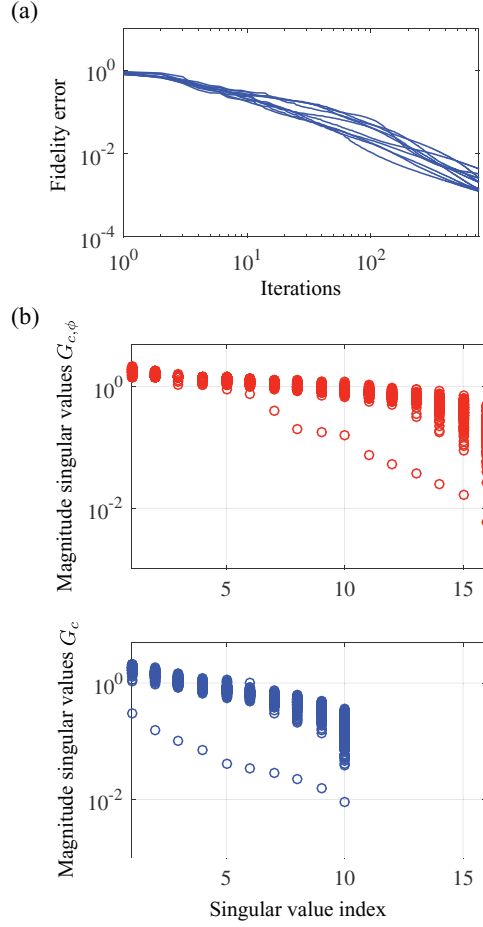
### 7.1. Central spin system

We begin by investigating a single spin ( $N_A = 2$ ) that interacts through a Heisenberg type interaction with  $q_B$  environmental spins ( $N_B = 2^{q_B}$ ) and a single control  $c(t)$  is applied in the  $z$  direction on the system spin. The total Hamiltonian describing the control system reads

$$H(t) = \sigma_y \otimes I_{N_B} + \sum_{q=1}^{q_B} a_q \sum_{s=x,y,z} \sigma_s \otimes \sigma_s^{(q)} + c(t) (\sigma_z \otimes I_{N_B}), \quad (25)$$

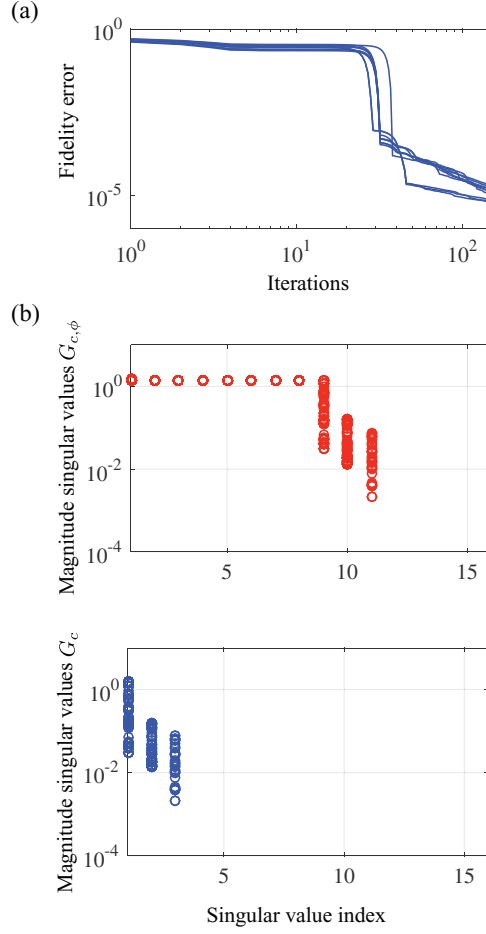
where  $\sigma_s$ ,  $s \in \{x, y, z\}$  are the Pauli spin operators. The controllability aspects of this model, also called the *central spin model*, were studied in [16]. It was shown that for all values of the coupling constants  $a_q$ , and independent of the number  $q_B$  of environmental spins, the system spin (i.e. the system  $A$ ) is fully controllable. Moreover, if all the coupling constants are different from each other, the total system (i.e. the system spin + environmental spins) becomes fully controllable.

In figure 1 we studied the implementation of some randomly chosen unitary transformation  $W \in \text{U}(2)$  on the system spin for a total evolution time of  $T = 20$  with  $L = 100$  piecewise constant controls and the coupling constants  $a_q$  in (25) all chosen to be equal. We remark here that,



**Figure 1.** Numerical investigation of the convergence of the gradient algorithm (24) for the central spin system (25) with  $q_B = 3$  and equal coupling constants  $a_q$ . (a) Fidelity error  $1 - F$  as a function of the iteration step for 10 randomly chosen unitary transformations as a target on the system spin, for a fixed evolution time  $T = 20$  and  $L = 100$  piecewise constant controls. (b) Singular values (circles) of the gradient  $G_{c,\phi}$  and the gradient  $G_c$  at  $\phi_{\text{opt}}$  during the iteration. The vertical spread of the values correspond to singular values during the iteration process for the different targets. The number of singular values of  $G_{c,\phi}$  is seen to be fully consistent with the assumption (21), i.e. **rank**  $G_{c,\phi} = N = 16$  even though **rank**  $G_c = 10$ .

even though the control system (25) evolves on  $\text{SU}(N)$ , the ability to additionally maximize over  $\phi$  allows for implementing  $W$ . Figure 1(a) shows the fidelity error  $1 - F$  as a function of the iteration step  $i$ , whereas each curve corresponds to a randomly chosen target unitary. The results suggest that independently of the target transformation on the system spin, the gradient algorithm finds the controls that implement the target unitary (up to some error which continues to decrease when the iterations stopped), even though the total system is not fully controllable. We remark here that the same behavior was found for additional numerical simulations using different  $q_B$  and  $a_q$ , which suggest that traps rarely occur in the control landscape for the central spin model. We now turn to whether this behavior is in agreement with the sufficient condition (21) for a trap free search. In figure 1(b) we show the singular values (circles) of



**Figure 2.** Numerical investigation of the convergence of the gradient algorithm for the random bath model (26) where  $B_z$  with  $\|B_z\| = 1$  is randomly chosen Hermitian  $8 \times 8$  matrix. (a) Fidelity error as a function of the iteration step for 10 randomly chosen  $B_z$ 's, for a fixed evolution time  $T = 1$  and  $L = 4$  number of piecewise constant controls. (b) Singular values (circles) of the gradient  $G_{c,\phi}$  and the gradient  $G_c$  at  $\phi_{\text{opt}}$  during the iteration. The vertical spread of the values correspond to singular values during the iteration process for the different targets. The number of singular values of  $G_{c,\phi}$  is seen to be fully consistent with the assumption (21), i.e. **rank**  $G_{c,\phi} = 11 \geq N/2 = 8$  even though **rank**  $G_c = 3$ .

$G_{c,\phi}$  and  $G_c$  at  $\phi_{\text{opt}}$  determined by (7) during the iterative process. We see that the singular values displayed are consistent with the corresponding rank condition expressed in (21), i.e. for  $N = 16$ , **rank**  $G_{c,\phi} = 16$  despite **rank**  $G_c = 10$ . As we have previously remarked, it is unlikely that the total  $AB$  system can be *fully* controlled by acting on  $A$  alone; the frequencies cannot arbitrarily be varied as a function of  $c$ .

## 7.2. Single qubit: random bath with identity gate

In the last subsection we investigated the central spin model for which the system  $A$  spin is fully controllable with a single control field. However, as mentioned before, the rank condition

(21) for a trap free search has no evident direct relation to whether the system  $A$  is fully controllable or not. Now we consider a system  $A$  which is not fully controllable with a single control field. That is, we consider a spin interacting with a random bath of dimension  $N_B = 8$  described by the total Hamiltonian

$$H(t) = c(t)\sigma_x \otimes I_{N_B} + \sigma_z \otimes B_z, \quad (26)$$

where  $B_z$  with  $\|B_z\| = 1$  is a randomly chosen Hermitian matrix. We remark here that such types of Hamiltonians are typically used to model pure dephasing of a single spin interacting with a spin-bath. For instance, in nitrogen vacancy centers dephasing of the electron spin caused by an interaction with surrounding nuclear spins can be modeled by an interaction Hamiltonian of the form  $H = \sigma_z \otimes \sum_q a_q \sigma_z^{(q)}$ . Even though the system spin is not fully controllable with a single control field  $c(t)$ , which can easily be seen from the underlying dynamical Lie algebra [8, 17], at any time  $T$  the identity operation can be implemented on the system spin, for instance, by using delta function like controls that invert the sign of the interaction part. Even though the identity operation on  $A$  is a very specific choice for a target operation, such a target operation is of practical importance in the context of dynamical decoupling, i.e. suppressing the interactions with system  $B$ .

In figure 2 we show results for the implementation of the identity operation on the system spin with a total evolution time  $T = 1$  and  $L = 4$  piecewise constant controls with  $B_z$  a randomly chosen  $N_B \times N_B$  Hermitian matrix normalized to  $\|B_z\| = 1$  and with  $N_B = 8$ . Figure 2(a) shows the fidelity error  $1 - F$  as a function of the iteration step  $i$ , where each curve corresponds to a randomly chosen  $B_z$ . The curves show that for all choices of  $B_z$  the gradient algorithm finds the controls that implement the identity. In figure 2(b) we show the singular values (circles) of  $G_{c,\phi}$  and  $G_c$  at  $\phi_{\text{opt}}$  during the iteration process. Because we chose the target to be the identity and the control system (26) is expressed through Pauli operators, the spectral frequencies  $\omega$  are symmetric around zero. As previously noted, when this occurs the rank condition for a trap-free search drops to  $N/2 = 8$ . From figure 1(b) we see that this condition is satisfied, i.e. with  $N = 16$ , **rank**  $G_{c,\phi} = 11 > 8$  despite **rank**  $G_c = 3$ .

## 8. Summary and outlook

We have investigated the quantum control landscape of a bipartite quantum system where only one part,  $A$ , is accessible through time dependent controls; part  $B$  is not. The control objective was to implement a desired unitary operation on part  $A$  of the bipartite system, regardless of which unitary transformation is implemented on part  $B$ . By defining an extended control landscape, and along with the controls, additionally optimizing over the corresponding extended landscape parameters, we found that if a sufficient condition is satisfied for the extended landscape to be trap-free, then it implies that the original landscape is also trap-free. Specifically, since the kinematic gradient only possess saddles in the interior of the landscape, if the rank of the dynamic gradient at the optimal extended landscape parameters is full, in the particular sense specified, a gradient search on the original landscape will converge to the global maximum ( $F = 1$ ). In addition, it is not necessary for the control gradient  $G_c$  to have full rank. As we have noted, the developed rank condition on the extended landscape can hold by virtue of the apparent flexibility offered by maximizing over the parameters that define the extended landscape. We remark here that this additional freedom in the optimization task does not lead to an increase of the set of unitary transformations reachable on system  $A$ . The characterization of the type of interactions with system  $B$  that can enhance the control properties of  $A$ , perhaps even turning system  $A$  into a fully controllable system, is left for future

studies. We have provided two common supporting numerical examples for which the rank condition holds. Although we have restricted the presentation to a bipartite  $AB$ -system with *both* control and target on the  $A$ -system, other system combinations are clearly possible. For example, if the controlled system space consisted of target and ancilla qubits, the latter to help facilitate error reduction, then the extended landscape unitary  $\Phi$  would have the dimension of the ancilla-bath system.

As we have remarked in various places throughout the text, the rank condition (21), presented as an assumption, is sufficient upon satisfaction to ensure a successful gradient search to the top of the landscape. Though it can be used informally to test various control sequences and/or search algorithms, establishing properties of specific physical implementations, i.e. Hamiltonians, for which (21) is valid is an open research issue. In contrast to the closed system case, we do not expect an almost ‘always statement’ [5] to hold for bipartite control systems. If the rank condition would hold for almost all bipartite systems with controls applied on system  $A$  only, this would imply that every unitary transformation can almost always be perfectly implemented on  $A$  (full controllability), regardless of the form of the control Hamiltonians as well as the interactions with system  $B$ . Even though we do not expect that an almost always statement exists for bipartite systems, the detailed expressions of the dynamic gradients given in the appendix may provide a path forward for identifying physical conditions under which the developed rank condition holds. However, regarding controllability of system  $A$  in the presence of system  $B$ , it is interesting to note that for a slightly different, but allied scenario in which system  $A$  is controlled while the goal is to implement a target unitary operation on system  $B$ , recently the possibility of such *indirect* control of system  $B$  has been fully characterized in terms of the underlying dynamical Lie algebra and the initial state of system  $A$  [18–20]. It would be interesting to study such open system control landscapes using the methodology developed here.

## Acknowledgments

The authors thank Benjamin Russell for many helpful discussions and comments during the drafting of this work. CA was supported by the NSF grant CHE-1763198 and HR was supported by the DOE grant DE-FG02-02ER15344.

## Appendix A. Dynamic gradients

Detailed expressions for the dynamic gradients in (13) are herein derived. We first present the expressions followed by the derivation and mention some properties along the way.

### A.1. Expressions of dynamic gradients

Dropping the  $(c, \phi)$ -dependencies in (13) for ease of reading, the dynamic gradients can be expressed as follows:

$$\begin{aligned} G_{c,\phi} &= \begin{bmatrix} G_c \\ G_\phi \end{bmatrix} \in \mathbb{R}^{(LM+N_B^2) \times N} \\ G_c &= \nabla_c \omega = -(I_L \otimes \mathcal{H}) \mathcal{C} \mathcal{V}_A \in \mathbb{R}^{LM \times N} \\ G_\phi &= \nabla_\phi \omega = -\mathcal{P} \mathcal{V}_B \in \mathbb{R}^{N_B^2 \times N}. \end{aligned} \tag{A.1}$$

The matrices in  $G_c$  are:

$$\begin{aligned}
 \mathcal{H} &= \begin{bmatrix} \overrightarrow{H_1 \otimes I_B}^\dagger \\ \vdots \\ \overrightarrow{H_M \otimes I_B}^\dagger \end{bmatrix} \in \mathbb{C}^{M \times N^2} \\
 \mathcal{C} &= \text{diag}\{\mathcal{C}_1, \dots, \mathcal{C}_L\} \in \mathbb{C}^{LN^2 \times LN^2} \\
 \mathcal{C}_\ell &= \int_0^\delta \mathbf{e}^{i\tau H_\ell^T} \otimes \mathbf{e}^{-i\tau H_\ell} d\tau \in \mathbb{C}^{N^2 \times N^2} \\
 \mathcal{V}_A &= \begin{bmatrix} \mathcal{V}_{A,1} \\ \vdots \\ \mathcal{V}_{A,L} \end{bmatrix} \in \mathbb{C}^{LN^2 \times N} \\
 \mathcal{V}_{A,\ell} &= [v_{\ell 1}^* \otimes v_{\ell 1} \cdots v_{\ell N}^* \otimes v_{\ell N}] \in \mathbb{C}^{N^2 \times N} \\
 [v_{\ell 1} \cdots v_{\ell N}] &= U_{\ell+1} \cdots U_L V \in \text{U}(N^2).
 \end{aligned} \tag{A.2}$$

Note that  $\mathcal{V}_{A,\ell}^\dagger \mathcal{V}_{A,\ell} = I_N$ , hence  $\mathcal{V}_A^\dagger \mathcal{V}_A = LI_N$  and thus both  $\mathcal{V}_{A,\ell}$  and  $\mathcal{V}_A$  have rank  $N$ .

Likewise the matrices in  $G_\phi$  are:

$$\begin{aligned}
 \mathcal{P} &= \begin{bmatrix} \overrightarrow{I_S \otimes P_1}^\dagger \\ \vdots \\ \overrightarrow{I_S \otimes P_{N_B^2}}^\dagger \end{bmatrix} \in \mathbb{C}^{N_B^2 \times N^2} \\
 P_b &= \int_{\tau=0}^1 \mathbf{e}^{i\tau \mathcal{B}(\phi)} B_b \mathbf{e}^{-i\tau \mathcal{B}(\phi)} d\tau \in \mathbb{C}^{N_B \times N_B} \\
 \mathcal{V}_B &= [v_1^* \otimes v_1 \cdots v_N^* \otimes v_N] \in \mathbb{C}^{N^2 \times N}
 \end{aligned} \tag{A.3}$$

with  $v_n \in \mathbb{C}^N$  a column of  $V$  in (11). Thus  $\mathcal{V}_B^\dagger \mathcal{V}_B = I_N$  from which it follows that **rank**  $\mathcal{V}_B = N$ . In addition,

$$\begin{aligned}
 \vec{P}_b &= \mathcal{K} \vec{B}_b \\
 \mathcal{K} &= \int_0^1 \mathbf{e}^{-i\tau \mathcal{B}(\phi)^T} \otimes \mathbf{e}^{i\tau \mathcal{B}(\phi)} d\tau \in \mathbb{C}^{N_B^2 \times N_B^2}.
 \end{aligned} \tag{A.4}$$

Conditions derived in [6] show that  $\mathcal{C}$  is almost always invertible and similarly for  $\mathcal{K}$ .

### A.2. Derivation of gradient expressions

The variables  $(c, \phi)$  are to be selected to maximize the extended landscape objective function,

$$J(c, \phi) = \text{ReTr}(W \otimes \Phi(\phi))^\dagger U(c). \tag{A.5}$$

The gradient of  $U(c)$  in (2) with respect to each element  $c_{\ell m}$  of  $c$  is (for ease of reading we drop the  $(c, \phi)$ -dependence),

$$\begin{aligned}
 \nabla_{c_{\ell m}} U &= -iU_1 \cdots U_\ell Q_{\ell m} U_{\ell+1} \cdots U_L \\
 Q_{\ell m} &= \int_0^\delta \mathbf{e}^{itH_\ell} (H_m \otimes I_B) \mathbf{e}^{-itH_\ell} dt.
 \end{aligned} \tag{A.6}$$

Using the unitary decomposition (11) write  $(W \otimes \Phi)^\dagger U = V e^{i\Omega} V^\dagger$  together with (A.5) gives,

$$\begin{aligned} J &= \text{ReTr } e^{i\Omega} = \sum_{n=1}^N \cos \omega_n \\ \nabla_{c_{\ell m}} J &= \text{ImTr } (W \otimes \Phi)^\dagger U_1 \cdots U_\ell Q_{\ell k} U_{\ell+1} \cdots U_L \\ &= \text{ImTr } V_\ell^\dagger Q_{\ell m} V_\ell e^{i\Omega} \\ V_\ell &= \begin{cases} U_{\ell+1} \cdots U_L V, & \ell = 1, \dots, L-1 \\ V, & \ell = L \end{cases}. \end{aligned} \quad (\text{A.7})$$

The last lines follow from  $(W \otimes \Phi)^\dagger = V e^{i\Omega} V^\dagger U^\dagger$ . Since  $V_\ell^\dagger Q_{\ell m} V_\ell$  is Hermitian and  $\Omega$  is real and diagonal, it follows that the diagonal elements  $(V_\ell^\dagger Q_{\ell m} V_\ell)_{nn}, n = 1, \dots, N$  are also real. Hence,

$$\nabla_{c_{\ell m}} J = \sum_{n=1, \dots, N} \left( V_\ell^\dagger Q_{\ell m} V_\ell \right)_{nn} \sin \omega_n. \quad (\text{A.8})$$

Comparing the above expression for the gradient with  $\nabla_{c_{\ell m}} J = \nabla_{c_{\ell m}} \sum_n \cos \omega_n = -\sum_n (\nabla_{c_{\ell m}} \omega_n) \sin \omega_n$  gives the elements of the gradient matrix  $G_c \in \mathbb{R}^{LM \times N}$  as,

$$(G_c)_{\ell m, n} = - \left( V_\ell^\dagger Q_{\ell m} V_\ell \right)_{nn} = -v_{\ell n}^\dagger Q_{\ell m} v_{\ell n} \quad (\text{A.9})$$

where  $v_{\ell n} \in \mathbb{C}^N$  are columns of the unitary  $V_\ell$  in (A.7), that is,  $V_\ell = [v_{\ell 1} \cdots v_{\ell N}]$ . Using repeated applications of the ‘vec’ operator  $(\vec{\cdot})$ , which stacks the columns of a matrix into a vector, we arrive at the expression for  $G_c$  is (A.1).

The expression for  $G_\phi$  (A.1) is obtained in a similar way. First from (8) to get  $\nabla_{\phi_b} \Phi(\phi) = \nabla_{\phi_b} e^{iB(\phi)} = i\mathcal{P}_b$  with  $\mathcal{P}_b$  from (A.3) and then application of (11) to (A.5) returns

$$(G_\phi)_{b, n} = -v_n^\dagger (I_A \otimes \mathcal{P}_b) v_n, \quad b = 1, \dots, N_B^2, \quad n = 1, \dots, N \quad (\text{A.10})$$

with  $V = [v_1 \cdots v_N] \in U(N)$  from (11). The rest of the expressions in (A.3) follow from ‘vec’ operation. In addition, using the fact that  $\sum_{b=1}^{N_B^2} \vec{B}_b \vec{B}_b^\dagger = I_{N_B^2}$  together with the invertibility of  $\mathcal{K}$ , which implies that  $\mathcal{K}\mathcal{K}^\dagger \geq \kappa I_{N_B^2}, \kappa > 0$ , we get,

$$G_\phi^T G_\phi \geq \kappa \sum_{b=1}^{N_B^2} z_b z_b^\dagger, \quad z_b = \begin{bmatrix} v_1^\dagger (I_S \otimes B_b) v_1 \\ \vdots \\ v_N^\dagger (I_S \otimes B_b) v_N \end{bmatrix} \in \mathbb{R}^N. \quad (\text{A.11})$$

Though seemingly tractable, establishing the rank of  $G_\phi$  from this formulation is currently an open problem.

## Appendix B. Kinematic critical points

The kinematic objective, gradient, and Hessian are,

$$\begin{aligned} J(\omega) &= \sum_{n=1}^N \cos \omega_n \in [-N, N] \\ \nabla_\omega J(\omega) &= -\sin \omega \in \mathbb{R}^N \\ \nabla_\omega^2 J(\omega) &= -\text{diag}(\cos \omega) \in \mathbb{R}^{N \times N}. \end{aligned} \quad (\text{B.1})$$

The kinematic critical points are where  $\sin \omega_n = 0, \cos \omega_n \in \{+1, -1\}, n = 1, \dots, N$ . Let  $p$  denote the number of times  $\cos \omega_n = 1$ . Reordering the frequencies gives,



$$\cos \omega_n = \begin{cases} +1 & n = 1, \dots, p \\ -1 & n = p+1, \dots, N \end{cases} \quad (\text{B.2})$$

At  $p = \{0, N\}$ ,  $J(\omega) = \{-N, +N\}$ , the extrema objective values, and for  $p = 1, \dots, N-1$ ,  $J(\omega) = 2p - N \in (-N, +N)$ , the objective values in the interior of the landscape. The kinematic critical points, denoted by  $\omega \in \mathbb{R}^N$ , and the associated *kinematic critical values*,  $J(\omega)$ , can be obtained from standard trigonometry producing the summary of their properties in section 5.2.

### Appendix C. Extended landscape gradient

To establish (18), note first that from (24) for a given control  $c$ , at the optimal extended landscape variable  $\Phi_{\text{opt}}(c)$ , equivalently  $\Phi(\phi_{\text{opt}}(c))$ , and from (11) and (7),

$$\begin{aligned} \text{Tr}(W \otimes \Phi(\phi_{\text{opt}}(c))) U(c) &= \sum_{n=1}^N \cos \omega_n(c, \phi_{\text{opt}}(c)) \\ &\quad + i \sin \omega_n(c, \phi_{\text{opt}}(c)) = \text{Tr} Q \end{aligned} \quad (\text{C.1})$$

where  $Q$  is the diagonal matrix of singular values of  $\Gamma$ . Since the singular values are real, it follows that the first result in (18) holds, namely,

$$\sum_{n=1}^N \sin \omega_n(c, \phi_{\text{opt}}(c)) = 0. \quad (\text{C.2})$$

The second result in (18) is established by using the expressions in appendix A, i.e. the gradient of the objective with respect to an element of the extended landscape parameter vector  $\phi$  is,

$$\begin{aligned} \nabla_b J &= \text{Re Tr} \nabla_b \Phi^\dagger \Gamma = \text{Re Tr} (i \Phi P_b)^\dagger \Gamma \\ &= \text{Im Tr} P_b \Phi^\dagger \Gamma = \text{Im Tr} (T_{\text{right}}^\dagger P_b T_{\text{right}}) Q \end{aligned} \quad (\text{C.3})$$

where the last expression arises by evaluating the gradient at the optimal extended landscape variable (7). Since  $T_{\text{right}}^\dagger P_b T_{\text{right}}$  is Hermitian and  $Q$  is the diagonal matrix of singular values of  $\Gamma$ , then the trace is a real number. It follows that  $\nabla_b J = 0$  at the optimal extended landscape, and hence we get the second result in (18).

### Appendix D. Rank condition for $\text{SU}(N)$

From (21), if  $U_{\text{ext}} \in \text{SU}(N)$  then  $\det U_{\text{ext}} = 1$ . From the decomposition (11),  $\det U_{\text{ext}} = \exp\{i \sum_n \omega_n\} = 1$  iff  $\sum_n \omega_n = 2\pi k$  for any integer  $k$ , so a degree of freedom is lost from  $N$  to  $N-1$ . To account for this in the dynamic gradient set  $\omega = (\bar{\omega}, \omega_N) \in \mathbb{R}^N$  with  $\bar{\omega} = (\omega_1, \dots, \omega_{N-1}) \in \mathbb{R}^{N-1}$  and  $\omega_N = 2\pi k - \sum_{n=1}^{N-1} \omega_n = 2\pi k - \mathbb{1}_{N-1}^T \bar{\omega}$ . The dynamic gradient is then,

$$\begin{aligned} G_{c,\phi} &= \bar{G}_{c,\phi} [I_{N-1}, -\mathbb{1}_{N-1}] \in \mathbb{R}^{LM+N_B^2 \times N} \\ \bar{G}_{c,\phi} &= \nabla_{c,\phi} \bar{\omega} \in \mathbb{R}^{LM+N_B^2 \times N-1}. \end{aligned} \quad (\text{D.1})$$

It follows that  $\text{rank } \bar{G}_{c,\phi} = N-1$  implies that  $\text{rank } G_{c,\phi} = N-1$  which establishes (21) for  $U_{\text{ext}} \in \text{SU}(N)$ .

## Appendix E. Closed-system

For a closed-system the extended landscape variable reduces to a scalar,  $\Phi = e^{i\phi}$ , equivalently,  $N_B = 1$  and  $N = N_A$ . The extended landscape objective and gradient are then,

$$\begin{aligned} J(c, \phi) &= \text{Re } \text{Tr } U_{\text{ext}}(c, \phi) \\ \nabla_\phi J(c, \phi) &= \text{Re}(i) \text{Tr } U_{\text{ext}}(c, \phi) \\ U_{\text{ext}}(c, \phi) &= e^{i\phi} W^\dagger U(c) \in \text{U}(N). \end{aligned} \quad (\text{E.1})$$

Using the spectral decomposition set  $U_{\text{ext}} = V e^{i\Omega} V^\dagger$ ,  $V \in \text{U}(N)$ ,  $\Omega = \text{diag}(\omega)$ , then,

$$\nabla_\phi J(c, \phi) = \text{Re}(i) \sum_{n=1}^N e^{i\omega_n} = - \sum_n \sin \omega_n = \mathbb{1}_N^T g(\omega) \quad (\text{E.2})$$

where  $g(\omega) = -\sin \omega \in \mathbb{R}^N$ . At the optimal extended landscape  $\phi_{\text{opt}}(c) = (\text{Tr } W^\dagger U(c))^* / |\text{Tr } W^\dagger U(c)|$ , the gradient matrix then becomes (dropping the  $c, \phi_{\text{opt}}(c)$  dependence for ease of reading),

$$G_{c,\phi} = \begin{bmatrix} G_c \\ \mathbb{1}_N^T \end{bmatrix} \in \mathbb{R}^{LM+1 \times N}. \quad (\text{E.3})$$

If **rank**  $G_c = r$ , then  $G_c$  has the singular value decomposition:  $G_c = U_c \begin{bmatrix} S_r & 0 \\ 0 & 0 \end{bmatrix} V_c^T$  with  $U_c \in \text{U}(LM)$ ,  $V_c \in \text{U}(N)$ ,  $S_r = \text{diag}(s_1, \dots, s_r)$  and  $s_1 = \|G_c\| \geq s_2 \geq \dots \geq s_r > 0$ . Since **rank**  $G_{c,\phi} = \text{rank } G_{c,\phi}^T G_{c,\phi}$ , then,

$$\begin{aligned} G_{c,\phi}^T G_{c,\phi} &= \begin{bmatrix} S_r^2 + v_1 v_1^T & v_1 v_2^T \\ v_2 v_1^T & v_2 v_2^T \end{bmatrix} \\ v &= V_c^T \mathbb{1}_N = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^N \end{aligned} \quad (\text{E.4})$$

with  $v_1 \in \mathbb{R}^r$ ,  $v_2 \in \mathbb{R}^{N-r}$  and  $v^T v = v_1^T v_1 + v_2^T v_2 = N$ . Because  $S_r$  is invertible, we can apply the *rank additivity lemma* associated with the *Schur Complement* for an Hermitian matrix, i.e.

$$\begin{aligned} \text{for } X = \begin{bmatrix} A & B \\ B^\dagger & C \end{bmatrix} = X^\dagger, \text{ if } A \text{ is nonsingular then,} \\ \text{rank } X = \text{rank } A + \text{rank } (C - B^\dagger A^{-1} B). \end{aligned} \quad (\text{E.5})$$

Applying this to (E.4) with  $A = S_r^2 + v_1 v_1^T$ ,  $B = v_1 v_2^T$ , and  $C = v_2 v_2^T$  gives,

$$\begin{aligned} \text{rank } G_{c,\phi}^T G_{c,\phi} &= \text{rank } (S_r^2 + v_1 v_1^T) \\ &\quad + \text{rank } v_2 (1 - v_1^T (S_r^2 + v_1 v_1^T)^{-1} v_1) v_2^T \\ &= r + \text{rank } v_2 v_2^T \\ &= r + 1 \end{aligned} \quad (\text{E.6})$$

provided that  $v_1^T (S_r^2 + v_1 v_1^T)^{-1} v_1 < 1$  which is always the case. This establishes (23).

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