Self-similarly expanding regions of phase change yield cavitational instabilities and model deep earthquakes

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ABSTRACT

The dynamical fields that emanate from self-similarly expanding ellipsoidal regions undergoing phase change (change in density, i.e., volume collapse, and change in moduli) under pre-stress, constitute the dynamic generalization of the seminal Eshelby inhomogeneity problem (as an equivalent inclusion problem), and they consist of pressure, shear, and *M* waves emitted by the surface of the expanding ellipsoid and yielding Rayleigh waves. Two fundamental theorems of physics govern the phenomenon, the Cauchy-Kowalewskaya theorem, which based on dimensional analysis and analytic properties alone, dictates that there is zero particle velocity in the interior, and Noether's theorem that governs the configurational forces that drive the expansion: it dictates that in self-similar expansion the boundary takes a shape that extremizes (minimizes for stability) the energy consumed in the process of moving the surface of discontinuity so that it does not become a sink or source of energy (total dynamic *J* integral equals to zero). The expression indicates that the expanding region can be *planar*, thus *breaking the symmetry* of the input and the phenomenon manifests itself as a newly discovered one of a "dynamic collapse/ cavitation instability". The flattened very thin ellipsoid will be oriented in space (directions of axes speeds) so that the energy due to phase change under pre-stress is able to escape out condensed in the core of dislocations of edge and screw-pair type gliding out at maximum speed and minimum loss on the planes where the maximum Peach-Koehler type of configurational force is applied on them and is big enough to overcome the self-stress due to inertia (dyn *J* integral) plus dissipative viscous losses. The radiation patterns are obtained in terms of the equivalent eigenstrain which also contains geometric effects of the planarity through the Dynamic Eshelby Tensor for the penny-shape. The phenomenon can model Deep Focus Earthquakes and models in the literature are evaluated on the basis of the energy to move the boundary of discontinuity. Noether's theorem is valid in anisotropy and nonlinear elasticity, and the phenomenon is independent of scales, valid from the nano to the very large ones, and applicable in general to other dynamic phenomena of stress induced martensitic transformations, shear banding, and amorphization.

I. Introduction

An analysis is presented for the dynamical elastic fields that emanate from self-similarly expanding ellipsoidal regions where the material undergoes phase change (change in density i.e., volume collapse, and change in moduli under pre-stress). Self-similar expansion of the ellipsoidal surface of discontinuity starts from zero dimension with constant axes expansion speeds, with self-similarity grasping the early response of the system (e.g., Barenblatt, 1996). The problem constitutes the dynamic generalization of the Eshelby inclusion (Eshelby, 1957) problem with the "equivalent eigenstrain" obtained through the constant Dynamic Eshelby Tensor (DET) (Ni and Markenscoff , 2016, a, c). The emanated fields consist of *P*, *S*, and *M* waves, and we express the solution in terms of integrals over their slowness surfaces (rather than the unit sphere) in order to directly obtain the wave-front fields ("radiation patterns") at the *P* and *S* fronts and also near the surface of the expanding ellipsoid which is the *M* wave front.

Because of the constant stress Eshelby property in the interior domain for self-similar expansion, constant tractions on the crack surfaces can be cancelled by those of an inclusion, and we thus obtain by a limiting process the elliptically expanding crack of Burridge and Willis (1969) where the M waves yield the Rayleigh fields of elastodynamics. The M waves also give the static Eshelby solution for the ellipsoid (Ni and Markenscoff, 2016c) in the particular limit where time goes to infinity and the axes speeds tend to zero, while their product tends to a constant, the axes lengths. In statics, crack solutions have been obtained as limits of Eshelby inclusions in Mura (1982) and more recently by Markenscoff (2018). For the two respective limits, the Burridge and Willis (1969) self-similarly expanding elliptical crack and the static Eshelby inclusion, the *M* waves may be called "Burridge-Willis-Eshelby Waves". We also refer to the works of Richards, (1973, 1976) who solved the elliptically expanding crack by the Cagniard-de Hoop technique.

Markenscoff and Ni (2010) analyzed a spherical inclusion with dilatational transformation strain expanding spherically in general motion $R = R_0 + l(t)$ by applying the expression given by Willis (1965) in terms of the dynamic Green's function integrated over an expanding region. The *M* waves that arise in the self-similar expansion $R = Vt$ can be shown to be produced from contributions (in the term u_r^{III} in their eqtn (2.20)) of the wavelets emitted (and traveling at the *P/S* wave speeds) by the expanding surface of the phase boundary from the earlier position on the expanding inclusion at time τ_1 to the latest at one at time τ_2 that have the time to reach the field point (*r,t*), and the wavelets emitted at time τ_2 have as degenerate wavefront the expanding surface of the sphere (ellipsoid) and are the *M* waves. The nature of the *M* waves is also seen from the present analysis where the governing equation for the expanding inclusion is made homogeneous by the application of the *M* operator as in Burridge and Willis (1969), so that an additional slowness surface (the expanding ellipsoid) is generated. All the waves, *P*, *S* and *M* together satisfy zero initial and radiation conditions as required for the self-similar problem.

For self-similarly expanding motion of an ellipsoidal surface due to stretch invariance the anisotropic inclusion with uniform eigenstrain is shown here on the basis of dimensional analysis alone and the ellipticity/ analyticity properties by means of the Cauchy-Kowalewskaya theorem (e. g, John, 1978) to have zero particle velocity in the interior domain. This is confirmed in isotropy by the full solution (Ni and Markenscoff, 2016a, eqtn (3.79)), where, in the corresponding expression of the particle velocity, the contributions of pressure P and shear S waves are shown explicitly to be cancelled in the interior domain by the M waves. As a result, no kinetic energy is radiated to the interior, and there is no focusing at the origin (with high stresses and dissipated energy), as it occurs in non-self-similar motion, such as in inclusions with time dependent eigenstrains (Willis, 1965), and in generally accelerating motion of a spherical inclusion in Markenscoff and Ni, 2010. Naturally, when applied to self-similar expansion $R = Vt$ the solution of Markenscoff and Ni (2010) based on integrating the Green's function on an expanding boundary coincides with the one of Ni and Markenscoff (2016, b) based on this self-similar approach for the sphere.

The self-similarly expanding ellipsoidal region of phase change under pressure is applied here to the modeling of deep focus earthquakes (DFEs), thought to be caused by high-pressure phase transformation. For the concept of transformation strain in geophysics we refer to Rice, 1980.

The radiation patterns obtained here for the particle acceleration in terms of the eigenstrains (equivalent to the phase change) can be used to relate seismic signals to information about the source. Nucleation in deep focus earthquakes (DFEs) happens at 400-700 kilometers under the surface of the earth where the mechanism is attributed to "polymorphic phase transitions under high pressures" of over 10GPa (Knopoff and Randall, 1970). Analog experiments (Meade and Jeanloz, 1989) "recorded acoustic emissions and shearing instabilities due to rapid atomic motions across displacive phase transition in Si and Ge at pressures of 70 GPa, well above the brittle-ductile transition in both solids, not by fracturing or cracking of the samples". There is additional literature including works regarding the olivine to spinel transition (Green and Burnley, 1989, Burnley and Green (1989), Green, 2007, Houston and Williams, 1991). Experiments were performed (Schubnel *et al*, 2013) on germanium olivine at 2 to 5 GPa to model in the laboratory the analog of deep focus earthquakes and found that "fault propagation must have been happening instantaneously as phase transition occurs", and there has been a more recent quantitative modeling that is based on static micromechanics (Wang *et al*, 2017). The mechanism " is poorly understood" (Wang et al, 2017) and the phenomenon has remained a deep mystery. Recent publications by Li *et al* (2018) and Romanowicz (2018) attribute the DFEs to anisotropy.

The DFEs are modeled as self-similarly expanding ellipsoidal regions of phase change with the phase change being an equivalent transformation strain obtained by the Dynamic Eshelby Tensor (DET) which depends on the undetermined yet axes speeds, being the variables of the problem. Given the eigenstrain, the strain energy density drop ΔW can be computed, through the DET, inside the self-similarly expanding region of the phase change, which is the input energy rate for the DFE. As for self-similar motion no kinetic energy is radiated inwards the input energy rate goes partly to move the phase boundary (as quantified through the dynamic J integral (Rice, 1968, Markenscoff and Singh, 2015, Markenscoff and Ni (2010), and the rest is radiated outside as strain and kinetic energy which is the seismic energy of the DFEs. A central result of this paper is the determination of the shape that will be assumed by the self-similarly expanding region of phase change.

Following Eshelby (1970) for statics, we define here as "equilibrium" shape of the **self-similarly** expanding phase boundary the one that has invariance of the Hamiltonian (total strain and kinetic energy) under infinitesimal translation of the inhomogeneity position. On the basis of Noether's theorem (1918) it will have to assume the shape that extremizes (*minimizes*, for stability) the energy-rate to move the inhomogeneity so that the *boundary does not become a sink* (or source) *of energy*, which requires that the total dynamic J integral vanishes. The expression gives as a possibility that the normal boundary velocity be equal to zero, so that the assumed shape can be the limit as the ellipsoid becomes flattened to a penny-shape ellipse or flattened elliptical cylinder (two-dimensional band), as in Figure 1(b), (c), respectively. With this flattened (pancake) shape no energy will be lost to move the upper and lower surfaces of the ellipsoid and energy will be lost only to move the perimeter (a line), with the other (symmetric) term in the expression of the J integral determining its orientation in space for minimization. We may note that the dyn J integral expresses the energy spent in an infinitesimal translation of the inhomogeneity position as a necessary and sufficient condition for linear momentum to be preserved in the domain (Gupta and Markenscoff, 2012). Thus, a major result is obtained, which is that the expanding region of phase change can propagate *planarly* for minimization of energy spent in the expansion process for any type of eigenstrain, which constitutes a ***symmetry breaking*** of the volumetric input of volume collapse/phase change, consistent with observations in nature and the laboratory in analog experiments for DFE (2017), and in amorphization experiments under shock loading (e.g., (Zhao et al, 2017), and references within. It has been called a *dynamic collapse instability* by R. Jeanloz and *a dynamic cavitation instability* by J.W. Hutchinson (private communications), *cavitation* for allowing the energy to escape out. For this to happen the pre-stress must be very high, so that the energy in the very thin region can be finite, and not infinitesimal, which explains why the phenomenon happens at the pressures of deep earthquakes.

In summary, it is demonstrated how two fundamental theorems of mathematical physics, the Cauchy-Kowalewskaya and Noether's theorems, taken together result in maximizing the energy rate that is available to radiate out as seismic energy in this model.

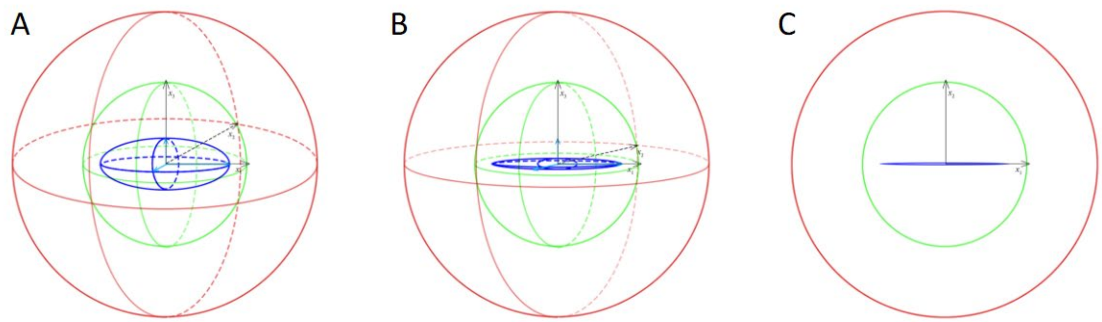

Figure 1: Schematics of the expanding ellipsoidal inclusion, penny-elliptical shape and 2D band, and wave-fronts (*P* in red, *S* in green and *M* in blue)

On the perimeter of the flattened ellipsoid the expansion will be governed by the symmetry-preserving expression of the driving force given by Noether's theorem. The *expression from Noether's theorem gives the physical law that drives the expansion relating the applied load to the phase boundary velocity though the dynamic J integral for an interface*. In the presence of unequal normal stresses, the flattened ellipsoid will orient itself in space on the planes of maximum shear stress where the Peach-Koehler type of configurational force will be maximum so that the self-stress due to inertia can be overcome first, and this determines the orientation where the motion starts and the directions (and magnitudes) of the axes speeds along which the dislocations will glide with maximum velocity and minimum dissipation (Figure 2).

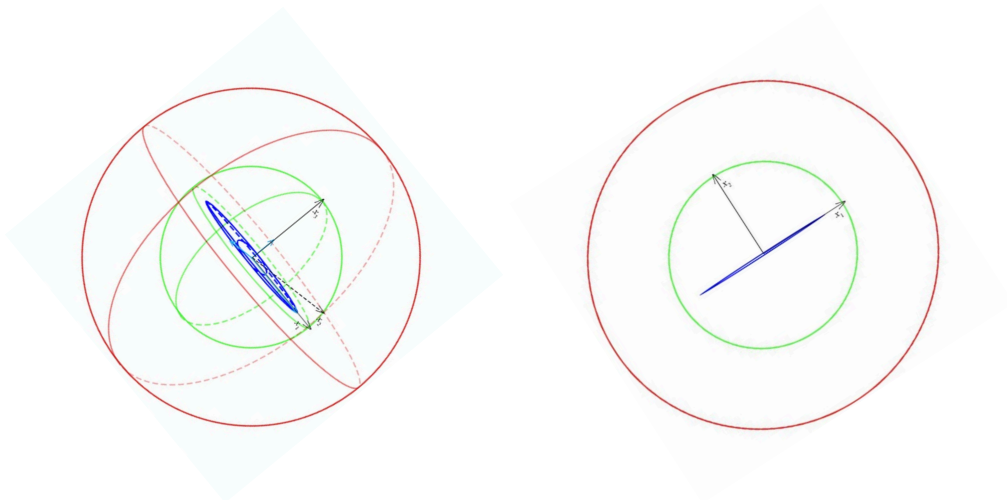

Figure 2: Orientation of the flattened ellipsoid/elliptical cylinder in space with dislocations emitted on directions of maximum Peach-Koehler force applied on them: edge (double couple) or screw-pair (linear vector dipole).

The radiation patterns for the particle acceleration at the wave-front are obtained and they relate it to the equivalent eigenstrain and the expansion speeds explicitly (the equivalent eigenstrain is also a function of the expansion speeds implicitly). There are six components of the eigenstrain, the three of them being the components of a Burgers vector and producing a dislocation loop in the limit of the flattened ellipsoid (double couple), while the in-plane components $\varepsilon_{11}^{*}, \varepsilon_{22}^{*}, \varepsilon_{12}^{*}$ are different in nature, e.g, ε_{11}^{*} in the flattened limit gives screw –pair (attributed to Weertman by Knopoff and Randall, 1970 . Volume collapse propagating planarly will produce eigenstrains $\varepsilon_{11}^{*}, \varepsilon_{22}^{*}$ (and ε_{33}^{*} obtained through the Dynamic Eshelby Tensor for the penny-shape ellipsoid where the geometric effect gives a Poisson's ratio dependence to the eigenstrains which may appear as anisotropy in the moment tensor. The self-similarly expanding planar circular disk with in-plane transformation strain may be considered a new dynamic singularity. All eigenstrains contribute to the shear wave-front except for the purely volumetric one with equal longitudinal eigenstrains which yields only *P* waves in the radiation patterns.

The concept of the energy consumed to move the boundary of phase change has not been considered before in the earthquake literature, and so the planar mode was not discovered. Based on the energy needed to move the boundary some proposed models for DFEs are excluded: Model B of Knopoff and Randall (1970) of drop in bulk modulus under pressure and spherical expansion is excluded as it is shown that it does not have enough energy to move an expanding boundary while maintaining the spherical symmetry of the input and it will have to break it expanding planarly. Essentially, in the Eshelby approach the equivalent eigenstrain due to drop in moduli under pressure is proportional to the pressure. If you increase the pressure, the eigenstrain increases, so the pressure increase does not give you enough driving force to move the boundary because the eigenstrain induced by pressure also has increased proportionally. However, if you have change in density, this is an equivalent eigenstrain which in magnitude is *independent* of the pressure. So more pressure will provide more energy to move the phase boundary that has eigenstrain independent of the pressure. (This does not include the deformation due to the pressure itself). So, volume collapse under high enough pressure has enough energy to expand and may occur with bulk modulus drop at the same time (and provide the difference in the energies needed for the drop in bulk modulus as discussed). Volume collapse may also not happen spherically, as in Randall (1964), but may preferably happen planarly minimizing the energy to expand the boundary, an issue to be further investigated.

While the phenomenon of expanding regions of phase change under pressure as manifested in DFEs occurs at large scales, it is valid independently of scale, from the nano to the very large ones, and is applicable to other phenomena of phase induced martensitic transformations under pressure (Escobar, et al 2000), amorphization (Zhang, et al, 2017), dynamic shear banding, etc. Noether's theorem is valid in anisotropy and nonlinear elasticity.

II. The waves of phase change from a self-similarly expanding ellipsoid

(a) *P*, *S*, and *M* waves

We consider a self-similarly expanding inclusion with uniform transformation strain (which will be the "equivalent eigenstrain of the phase change" problem) in the interior domain

$$\varepsilon^*_{lm}(\vec{x},t) = \varepsilon^*_{lm} H(t-(s_r^2 x_r^2)^{1/2}), \tag{1}$$

where x_i is the position vector, t the time, $1/s_1, 1/s_2, 1/s_3$, are the axes speeds of the expanding ellipsoid in the argument of the Heaviside step function $H(.)$.

The governing equations expressing conservation of linear momentum in elastodynamics , where C_{ijkl} are the elastic stiffness tensor and ρ is the density are (Ni and Markenscoff, 2016a, c)

$$\rho\frac{\partial^2 u_j}{\partial t^2} - C_{jklm}\frac{\partial^2 u_l}{\partial x_k \partial x_m} = -C_{jklm}\varepsilon^*_{lm}\frac{\partial}{\partial x_k}H(t-(s_r^2 x_r^2)^{1/2}) \tag{2}$$

with initial conditions

$$\vec{u}(\vec{x},0) = 0 \text{ and } \frac{\partial \vec{u}}{\partial t}(\vec{x},0) = 0 \tag{3}$$

and where vanishing radiation conditions at infinity are applied. In terms of the Navier operator of elastodynamics

$$L_{jl}(\frac{\partial}{\partial t},\nabla) = \rho\frac{\partial^2}{\partial t^2}\delta_{jl} - C_{jklm}\frac{\partial^2}{\partial x_k \partial x_m} \tag{4}$$

with δ_{ij} denoting the Kroenecker delta, the system (2) is written as

$$L_{jl}(\frac{\partial}{\partial t},\nabla)u_l = -K_j(\nabla)H(t^2 - s_r^2 x_r^2) \quad \text{with} \quad K_j(\nabla) = C_{jklm}\varepsilon_{lm}^{*}\frac{\partial}{\partial x_k} \tag{5}$$

The system (2) was solved by the Radon transform (e.g., Willis, 1973, Wang and Achenbach, 1994) in Ni and Markenscoff (2016, a). Here we follow the method of Burridge and Willis (1969) for the elliptically expanding crack. We introduce the operator

$$M\{\frac{\partial}{\partial t},\nabla\} = \frac{\partial^2}{\partial t^2} - \frac{1}{s_r^2}\frac{\partial^2}{\partial x_r^2} \text{ which satisfies the identity } M^2\{\frac{\partial}{\partial t},\nabla\}H(t^2 - s_r^2 x_r^2) = \frac{8\pi}{s_1 s_2 s_3}\delta(t)\delta(x)$$

so that, with the application of Duhamel's principle the governing system of equations becomes the homogeneous system of equations

$$M^2\{\frac{\partial}{\partial t},\nabla\}L\{\frac{\partial}{\partial t},\nabla\}u = 0 \qquad t > 0 \tag{6}$$

with inhomogeneous initial condition

$\partial^k u_i(0)/\partial t^k = 0$ for $k = 1,2,3,4$ and $\partial^5 u_i(0)/\partial t^5 = -8\pi/(\rho s_1 s_2 s_3)K_i\delta(x)$
(7)

The solution of the system (6) under the initial conditions (7) consists of contributions from the poles of the differential operators L and M, which give waves emanated from the three slowness surfaces, S^P, S^S, S^M (the first two corresponding to the poles of L , and the third one to those of M), which, for an isotropic material with $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, where λ and μ denote the Lame' constants, are:

$$S^P:\ |\xi|^2 = \frac{\rho}{\lambda + 2\mu} = \frac{1}{a^2},\ S^S:\ |\xi|^2 = \frac{\rho}{\mu} = \frac{1}{b^2},\ S^M:\ \frac{\xi_1^2}{s_1^2} + \frac{\xi_2^2}{s_3^2} + \frac{\xi_3^2}{s_3^2} = 1 \tag{8}$$

and where a and b denote the pressure and shear wave speeds. The solution of (6) with (7) is obtained following the procedure in Burridge and Willis (1969) in terms of integrals over the slowness surfaces, rather than over the unit sphere as in Ni and Markenscoff, 2016a, c), since in this formulation we can readily obtain the wave front behavior for the radiation patterns for the *P* and *S* waves and also for the M wave front (surface of the ellipsoid). By asymptotically expanding around the *M* wave front we obtain the crack limit and Rayleigh waves as we will show below. The solution for the displacement is

$$u_i(\mathbf{x},t) = \frac{1}{\pi s_1 s_2 s_3}\sum\int_{S^L}\frac{N_{ik}(1,\xi)K_k(\xi)}{\partial D/\partial\gamma(1,\xi)M^2(1,\xi)}(t + \xi\cdot\mathbf{x})\operatorname{sgn}(t + \xi\cdot\mathbf{x})\frac{dS}{|\nabla\gamma^L(\xi)|}$$
$$-\frac{1}{4\pi s_1 s_2 s_3}\int_{S^M}\{\rho L_{ik}^{-2}(1,\xi)K_k(\xi)[(t + \xi\cdot x)\operatorname{sgn}(t + \xi\cdot x) - (t - \xi\cdot x)\operatorname{sgn}(t - \xi\cdot x)]$$

$$+L^{-1}_{lk}(1,\xi)K_k(\xi)(\xi\cdot x)H(t-|\xi\cdot x|)\}\frac{dS}{|\nabla\gamma^M(\xi)|} \tag{9}$$

with, for isotropy,

$$K_k(\vec{\xi}) = C_{jklm}\varepsilon^*_{lm}\xi_j = \lambda\varepsilon^*_{mm}\delta_{jk}\xi_j + \mu(\varepsilon^*_{jk}+\varepsilon^*_{kj})\xi_j$$

and where $N_{ij}(\gamma,\xi_i)$ and $D(\gamma,\xi_i)$ are defined by $L^{-1}{}_{ij}(\gamma,\xi_i) = \dfrac{N_{ij}(\gamma,\xi_i)}{D}$
with $N_{lj}(\gamma,\xi) = \rho[(\gamma^2 - a^2|\xi|^2)\delta_{jl} + (a^2-b^2)\xi_j\xi_l]$ and $D(\gamma,\xi) = \rho^2(\gamma^2 - b^2|\xi|^2)(\gamma^2 - a^2|\xi|^2)$

$$\gamma^M(\vec{\xi}) = \pm(\xi_k^2/s_k^2)^{1/2},\quad |\nabla\gamma(\vec{\xi})| = (\xi_k^2/s_k^4)^{1/2} \tag{10}$$

The solution (9) satisfies the Hadamard jump conditions across a moving surface of discontinuity, with normal boundary velocity $\dot{l}$, which are

$$[[\frac{\partial u_l}{\partial t}]] = -\dot{l}[[\frac{\partial u_l}{\partial x_m}]]n_m \text{ and } [[\sigma_{ij}n_j]] = -\rho\dot{l}[[\frac{\partial u_i}{\partial t}]] \tag{11}$$

with the brackets denoting the jumps across the moving interface.

For the expanding ellipsoid the normal $\vec{n}$ at a point (x_1,x_2,x_3) on the boundary is $\vec{n} = (s_1^2x_1, s_2^2x_2, s_3^2x_3)/(s_k^4x_k^2)^{1/2}$, the velocity vector is $(x_1/t, x_2/t, x_3/t)$, and it can be written as, $((1/s_1)\cos\theta\sin\phi, (1/s_2)\sin\theta\sin\phi, (1/s_3)\cos\phi)$ --where the angles θ and ϕ are those of spherical coordinates--, and the normal velocity $\dot{l}$ is its projection on the normal vector which is $\dot{l} = 1/(s_k^4(x_k/t)^2)^{1/2}$.

It may be remarked that the solution obtained for a spherical inclusion in self-similar expansion (Ni and Markenscoff, 2016b) giving the *P*, *S* and *M* waves coincides with the one of Markenscoff and Ni (2010) obtained by the Green's function method for a spherically expanding inclusion with general acceleration when applied to self-similar expansion.

Radiation patterns

We next obtain the radiation patterns for the particle acceleration at the *P* and *S* wave fronts following Lighthill, 1960, and Burridge, 1967. Singularities occur when the plane $t+\vec{\xi}\cdot\vec{x}=0$ is tangent to the slowness surfaces S^L, S^M at the point $\vec{\xi}_0$. The particle acceleration is

$$\frac{\partial^2 u_i(\vec{x},t)}{\partial t^2} = \frac{1}{\pi s_1 s_2 s_3}\int_{S^L}\frac{N_{ik}(1,\vec{\xi})C_{jklm}\varepsilon^*_{lm}\xi_j\delta(t+\vec{x}\cdot\vec{\xi})}{\partial D/\partial\gamma(1,\vec{\xi})M^2(1,\vec{\xi})}\frac{dS}{|\vec{\nabla}\gamma^L(\vec{\xi})|}$$

$$+\frac{1}{4\pi s_1 s_2 s_3}\int_{S^M}\left\{-2\rho L_{ik}^{-2}(1,\vec{\xi})C_{jklm}\varepsilon_{lm}^{*}\xi_j\delta(t+\vec{x}\cdot\vec{\xi})\right.$$

$$\left.+L_{ik}^{-1}(1,\vec{\xi})C_{jklm}\varepsilon_{lm}^{*}\xi_j\left[\delta(t+\vec{\xi}\cdot\vec{x})+t\delta'(t+\vec{\xi}\cdot\vec{x})\right]\right\}\frac{dS}{\left|\vec{\nabla}\gamma^M(\vec{\xi})\right|}$$

(12)

Based on the above expression for the particle acceleration, the jump at the wave front $\vec{\xi}_0$ is obtained following Lighthill (1960) and Burridge (1967), as

$$\left[\ddot{u}_i\left(t,\vec{x}\right)\right]=\frac{-2f\left(\vec{\xi}_0\right)}{s_1 s_2 s_3 K^{1/2}\left(\vec{\xi}_0\right)}\frac{H\left(t+\vec{\xi}_0\cdot\vec{x}\right)}{|\vec{x}|} \tag{13}$$

where $K(\vec{\xi}_0)$ is the Gaussian curvature of $\boldsymbol{S^L}$ at $\vec{\xi}_0$, and where

$$f\left(\vec{\xi}_0\right)=\frac{N_{ik}\left(1,\vec{\xi}_0\right)K_k\left(\vec{\xi}_0\right)}{\partial D/\partial\gamma\left(1,\vec{\xi}_0\right)M^2\left(1,\vec{\xi}_0\right)}\frac{1}{\left|\vec{\nabla}\gamma^L\left(\vec{\xi}_0\right)\right|} \tag{14}$$

For the $\boldsymbol{P}$ wave-front $\boldsymbol{K^{1/2}=a}$:

$$\left|\vec{\xi}\right|^2=1/a^2\ ,\ \gamma^L\left(\vec{\xi}\right)=a\left|\vec{\xi}\right|,\ \boldsymbol{M^2=\left\{1-\frac{1}{a^2}\left(\frac{\hat{x}_1^2}{s_1^2}+\frac{\hat{x}_2^2}{s_2^2}+\frac{\hat{x}_3^2}{s_3^2}\right)\right\}^2},\ \boldsymbol{\frac{\partial D}{\partial\gamma}=2\rho^2\frac{\lambda+\mu}{\lambda+2\mu}=\frac{2\rho}{a^2}(\lambda+\mu)},$$

$$\boldsymbol{\frac{N_{ik}(1,\xi)}{\partial D/\partial\gamma(1,\xi)M^2(1,\xi)}=\frac{a^4\hat{\xi}_i\hat{\xi}_k}{2\rho\left(a^2-p^2\right)^2}}\ ,\ \boldsymbol{p^2=\frac{\hat{\xi}_k^2}{s_k^2}}\ ,\quad\left|\vec{\nabla}\gamma^L\left(\vec{\xi}\right)\right|=a \tag{15}$$

Setting $\vec{\xi}_0=-\frac{\hat{x}}{a}$, where $\hat{\boldsymbol{x}}$ is a unit vector at the $\boldsymbol{P}$ wave-front, the radiation pattern for acceleration emitted by an expanding three-dimensional ellipsoidal inclusion with eigenstrain $\boldsymbol{\varepsilon_{ij}^{*}}$, at the $\boldsymbol{P}$ front we have the radiation pattern

$$\frac{\partial^2 u_i}{\partial t^2}=\frac{\left(\lambda\delta_{jk}\varepsilon_{mm}^{*}+\mu\varepsilon_{jk}^{*}+\mu\varepsilon_{kj}^{*}\right)\hat{x}_i\hat{x}_k\hat{x}_j}{\pi\rho a^3 s_1 s_2 s_3\left\{1-\frac{1}{a^2}\left(\frac{\hat{x}_1^2}{s_1^2}+\frac{\hat{x}_2^2}{s_2^2}+\frac{\hat{x}_3^2}{s_3^2}\right)\right\}^2}\frac{H\left(t-\frac{|\vec{x}|}{a}\right)}{|\vec{x}|} \tag{16}$$

and, analogously, at the $\boldsymbol{S}$ wave-front we have

$$\frac{N_{lk}(1,\xi)}{\partial D/\partial\gamma(1,\xi)M^2(1,\xi)}=\frac{b^4\left[\delta_{lk}-\hat{\xi}_l\hat{\xi}_k\right]}{2\rho\left(b^2-p^2\right)^2} \tag{17}$$

and we obtain the radiation pattern at the shear wave-front

$$\frac{\partial^2 u_l}{\partial t^2}=\frac{\left[\delta_{lk}-\hat{x}_l\hat{x}_k\right]\left(\lambda\delta_{jk}\varepsilon^*_{nm}+\mu\varepsilon^*_{jk}+\mu\varepsilon^*_{kj}\right)\hat{x}_j}{\pi\rho b^3 s_1 s_2 s_3\left\{1-\frac{1}{b^2}\left(\frac{\hat{x}_1^2}{s_1^2}+\frac{\hat{x}_2^2}{s_2^2}+\frac{\hat{x}_3^2}{s_3^2}\right)\right\}^2}\frac{H\left(t-\frac{|\vec{x}|}{b}\right)}{|\vec{x}|} \tag{18}$$

In the limit, as the third axis speed goes to zero, if we set $\lim_{1/s_3\to 0} 1/s_3\varepsilon^*_{3s}=b_s$, we retrieve the radiation patterns of Burridge and Willis (1969) (which has a typo). The above radiation patterns depend on the "equivalent" eigenstrain, which is a function of the input parameters of pre-stress and phase change and the expansion speeds through the DET, in addition to the explicit dependence of (18) on the expansion speeds $1/s_i$. The expansion speeds themselves are functions of the phase change and pressure determined in the sequel from the minimization of the energy to move the boundary. It is observed that in the limit of the flattened ellipsoid in equation (18) shear eigenstrains produce dislocations (edge dislocation a double couple), while the longitudinal ones produce a screw-pair (attributed to Weertman in Randall and Knopoff (1970). The contributions in the shear wave front in (18) will only vanish if the three longitudinal eigenstrains are equal, but that is not the case in the flattened ellipsoid, where the third one is necessarily unequal, so there will always be both P and S wave radiation due to planarity.

(b) Self-similarity and dimensional analysis: zero particle velocity in the interior domain

From the field solution for the displacement equation (10) the particle velocity is obtained in the form of integrals over the slowness surfaces as

$$\begin{aligned}\frac{\partial u_l}{\partial t}(\mathbf{x},t)=&\frac{-1}{\pi s_1 s_2 s_3}\sum\int_{S^L}\frac{N_{lk}(1,\xi)C_{jkpq}\varepsilon^*_{pq}\xi_j}{\partial D/\partial\gamma(1,\xi)M^2(1,\xi)}\operatorname{sgn}(t-\xi\cdot\mathbf{x})\frac{dS}{|\nabla\gamma^L(\xi)|}\\&+\frac{1}{2\pi s_1 s_2 s_3}\int_{S^M}\left\{\rho L^{-2}_{lk}(1,\xi)C_{jklm}\varepsilon^*_{lm}\xi_j\operatorname{sgn}(t-\xi\cdot x)\right.\\&\left.-L^{-1}_{lk}(1,\xi)C_{jklm}\varepsilon^*_{lm}\xi_j(\xi\cdot x)\delta(t-|\xi\cdot x|)\right\}\frac{dS}{|\nabla\gamma^M(\xi)|}\end{aligned} \tag{19}$$

The critical property of the self-similarly expanding inclusion is that the contributions from the slowness surfaces pertaining to the pressure and shear waves are cancelled by those of the M surface in the interior domain, as the corresponding terms (integrated though over the unit sphere) can be identified in the expression given in Ni and Markenscoff (2016, a, eqtn 3.79). It must be noted here that for self-similar expanding motion these contributions to the particle velocity have constant values in the interior domain (and are not singular at the origin).

The governing system of p.d.e.'s (2) for general anisotropy with initial conditions (2) is stretch-invariant (Ni and Markenscoff, 2016a) so that the number of independent variables can be reduced by one, and it allows for a self-similar solution in terms of the variable $\vec{z} = \vec{x}/t$ and the function $\phi(\vec{z},t) = \frac{\vec{u}(\vec{x},t)}{t}$. In the new variables, the system is easily shown (Ni and Markenscoff, 2016a) to be elliptic in the region $|z| < \upsilon_0^{\min}$ (in the interior of the velocity surfaces for an anisotropic solid corresponding to the operator L), in which region the pertinent determinant does not vanish, while in the exterior the system is hyperbolic.

We then consider the fundamental properties of the system (2) with initial conditions (3). In the interior domain of the expanding anisotropic inclusion the system is an *elliptic* system of partial differential equations with analytic coefficients. The Cauchy-Kowalewskaya theorem (e.g. John, 1978) will provide the solution fully determined in the region of analyticity from the initial conditions (3) (on the displacement and particle velocity). Since these are zero, then the solution will be zero everywhere in the region of analyticity, which is the interior domain of the inclusion. Thus, the vanishing of the particle velocity in the interior domain is dictated by dimensional analysis alone and analytic arguments (answering a question asked by J.R. Rice in 2015 at the Broberg meeting, "what follows from dimensional analysis alone? ") so that the M waves have to be emitted to cancel the pressure and shear ones to fulfill this requirement. Note that for subsonic motion the inclusion lies within the region of ellipticity of the system of p.d.e's and this is the reason that the property is valid only for subsonically expanding inclusions.

(c) The phase change as an equivalent eigenstrain of the expanding ellipsoid with phase change under pressure

For isotropic materials the displacement gradient can be explicitly calculated and the Eshelby property of constant constrained strain proven explicitly. From eqtn (10) we obtain the displacement gradient as integrals over the slowness surfaces,

$$\frac{\partial u_i(x,t)}{\partial x_m} = \frac{1}{\pi s_1 s_2 s_3}\sum\int_{S^L}\frac{\xi_m N_{ik}(1,\xi)K_k(\xi)}{\partial D/\partial r(1,\xi)M^2(1,\xi)}\mathrm{sgn}(t+\xi\cdot x)\frac{dS}{|\nabla\gamma^L(\xi)|}$$
$$-\frac{1}{4\pi s_1 s_2 s_3}\int_{S^M}\{2\rho\xi_M L_{ik}^{-2}(1,\xi)K_k(\xi)H(t-|\xi\cdot x|)$$
$$+\xi_M L_{ik}^{-1}(1,\xi)K_k(\xi)\left[H(t-|\xi\cdot x|-t\delta(t-|\xi\cdot x|)\right]\}\frac{dS}{|\nabla\gamma^M(\xi)|} \quad (20)$$

In the interior domain, in equation (20), by the Schwarz inequality, the step and signum functions are 1, the delta function zero, and thus, the displacement gradient does not depend on the position $\vec{x}$ (Eshelby property) (Ni and Markenscoff, 2016a).

From (20) the Dynamic Eshelby Tensor (DET) $S^{dyn}_{ijkl}(1/s_i)$, defined by the relation of the strain to the eigenstrain $\varepsilon_{ij} = S^{dyn}_{ijkl}\varepsilon^*_{kl}$, is constant in space and time, depending on the axes speeds $1/s_i$ and was given for isotropic materials by Ni and Markenscoff, 2016c, in terms of integrals over the unit sphere. The constant DET allows to solve the problem of a self-similarly expanding ellipsoidal inhomogeneity with "phase change" . Due to change in moduli to C^*_{ijkl} as the inhomogeneity expands in a matrix with C_{ijkl} *under a uniform applied strain* ε^0_{ij} *at infinity (prestress)*, the "phase change" is an equivalent "eigenstrain" ε^*_{ij} . If there is a change in density, i.e., a volume collapse, with a corresponding trace of the strain $\varepsilon_{kk}{}^{vc}$, then the volume collapse is equivalent to an eigenstrain ε^{*vc}_{kk} such that

$$\Delta V / V = \varepsilon^{vc}_{kk} = (1/3) S^{dyn}_{mmij}(1/s_r)\varepsilon^{*vc}_{ij}, \tag{21}$$

so that ε^{*vc}_{kk} is dependent on the axes speeds. In the presence of both modulus drop and volume collapse, we will have $\varepsilon^{**}_{ij} = \varepsilon^*_{ij} + \varepsilon^{*vc}_{ij}$ as in Eshelby (1957), and equation

$$C^*_{ijkl}(\varepsilon^0_{ij} + S^{dyn}_{klmn}(1/s_r)\varepsilon^{**}_{mn} - \varepsilon^{*vc}_{kl}) = C_{ijkl}(\varepsilon^0_{ij} + S^{dyn}_{klmn}(1/s_r)\varepsilon^{**}_{kl} - \varepsilon^{**}_{kl}) \tag{22}$$

expresses the point-wise Hooke's law (Eshelby, 1957). Eqtn (22) can be separated into a volumetric and a deviatoric part (Eshelby, 1957). The components of equivalent eigenstrain ε^{*vc}_{ij} due to volume collapse are determined through the DET reflecting the geometry (such as expanding penny-shape).

Equations (22) are also applicable for interior anisotropy C^*_{ijkl} (see Eshelby, 1961, for statics) but not exterior anisotropy (when the DET is not known) in a matrix of isotropic material with Lame constants (λ, μ) when they are written as

$$C^*_{ijkl}(\varepsilon_{kl} + \varepsilon^0_{kl}) = \lambda(\varepsilon_{kk} - \varepsilon^*_{kk} + \varepsilon^0_{kk})\delta_{ij} + 2\mu(\varepsilon_{ij} - \varepsilon^*_{ij} + \varepsilon^0_{ij}) \tag{23}$$

Anisotropicity has been considered central in DFEs in recent literature (Li *et al* (2018) and Romanowicz (2018)). We may remark that the DET for the sphere Ni and Markenscoff (2016,b) is smaller than the static one for subsonic motion, and, if this is also true for the ellipsoid, the system (22) would be always invertible for ε^*_{ij}. The equivalent eigenstrain ε^{**}_{ij} depends through (22) on the axes speeds, which will be determined below in terms of the input parameters as a minimization problem. This ultimately will relate the radiation patterns to the equivalent eigenstrain that will contain the effects of phase change *and* the geometry of the expanding region, which through the DET, will have Poisson's ratio effect. We note that the full field quantities are the superposition of the uniform pre-stress field and those of the equations (1) with (2), as in the static Eshelby (1957).

(d) A limit yields the Burridge and Willis (1969) elliptically expanding crack and Rayleigh waves

An important limiting property of the expanding ellipsoidal inclusion with eigenstrain arises from the fact that the Burridge and Willis expanding elliptic crack solution can be retrieved, as anticipated by them, and, hence, the natural name to give to these waves is "Burridge-Willis-Eshelby Waves". The crack solution is obtained in the limit of an ellipsoidal inclusion as the speed of the third axes tends to zero, i.e. $1/s_3 \to 0$, and the eigenstrain ε^*_{31} tends to infinity, in such a way that

$$\lim_{1/s_3 \to 0}\left(1/s_3 \, \varepsilon^*_{3i} \right) = \lim_{1/s_3 \to 0}\left(1/s_3 \frac{\partial u^*_i}{\partial x_3} \right) = const = \Delta \dot{u}^*_i \tag{24}$$

tends to a constant, the *rate of displacement discontinuity* $\Delta \dot{u}^*_i$, with crack opening displacement

$$\Delta u_i(x_1, x_2, t) = \Delta \dot{u}^*_i (t^2 - s_1^2 x_1^2 - s_2^2 x_2^2)^{\frac{1}{2}} H(t^2 - s_1^2 x_1^2 - s_2^2 x_2^2) \tag{25}$$

The constant interior stresses (Eshelby property in dynamics) of the expanding inclusion with eigenstrain ε^*_{ij} can cancel constant applied tractions on the crack faces (since in the Hadamard condition (11) the R.H.S is zero across the crack faces), i.e.

$$\sigma^0_{3i} + C_{3imn}\left(S^{dyn}_{mnkl} \varepsilon^*_{kl} - \varepsilon^*_{mn} \right) = 0 \tag{26}$$

which determines the corresponding eigenstrain as to cancel the applied tractions. The idea of obtaining a crack solution from an inclusion in statics is initially due to Eshelby (1957,1961) and treated in Mura,1982, chapter 5 and is due to the constant stress Eshelby property in the interior domain. Recently a full analysis for the Griffith cracks as limits of Eshelby inclusions was presented in Markenscoff (2018).

The elliptically expanding crack surface is actually the slowness surface S^M that is a degenerate wave-front with the curvature becoming infinite at the elliptical tip of the crack. The near tip field is obtained by the asymptotic expansion near the *M* wave-front according to Burridge and Willis (1969) and we will follow them to obtain the traction on the plane $x_3 = 0$ in front of the expanding flattened elliptical inclusion in the limit as it becomes a crack. The slowness surface S^M is now a cylinder parallel to the ξ_3 axis, $\xi_3 \to \pm\infty$. The near tip field is obtained following Burridge and Willis (1969) for the asymptotic evaluation of the *M* slowness surface contribution in (20). The traction vector $\vec{t}^{(3)}$ on the plane in front of the elliptical edge with the normal boundary velocity $\frac{1}{l^2} = s_1^2 \cos^2\theta + s_2^2 \sin^2\theta$, is obtained in the limit as $1/s_3 \to 0$ and $x_3 \to 0$, from

$$t_i^{(3)} = C_{i3lm}\frac{\partial u_l}{\partial x_m} =$$

$$= \lim_{1/s_3\to 0}\left(\frac{1}{\pi s_1 s_2}\int_{S^L} C_{i3lm}\xi_m N_{lk}(1,\vec{\xi})C_{kn3q}\xi_n\frac{\varepsilon_{3q}^*}{s_3}\operatorname{sgn}(1+\vec{\xi}\cdot\frac{\vec{x}}{t})\frac{dS}{\left|\nabla\gamma^L(\vec{\xi})\right|}\right.$$

$$-\frac{1}{4\pi s_1 s_2}\int_{S^M}\{2\rho C_{i3lm}\xi_m L_{lk}^{-2}C_{kn3q}\xi_n\frac{\varepsilon_{3q}^*}{s_3}H(t-\left|\vec{\xi}\cdot\vec{x}\right|)+\left[C_{i3lm}\xi_m L_{lk}^{-1}(1,\vec{\xi})C_{kn3q}\xi_n\frac{\varepsilon_{3q}^*}{s_3}\right.$$

$$\left.+C_{i3k3}(\xi_1=0,\xi_2=0,\xi_3=1)\frac{\varepsilon_{3k}^*}{s_3}\right]\left[H(t-\left|\vec{\xi}\cdot\vec{x}\right|)-t\delta(t-\left|\vec{\xi}\cdot\vec{x}\right|)\right]\}\frac{dS}{\left|\nabla\gamma^L(\vec{\xi})\right|} \qquad (27)$$

In (27) the leading contribution is from the *M* slowness surface, and the leading term of the integral over S^M (the axis parallel to ξ_3) near the inclusion (crack) edge is evaluated at $x_1 = \frac{(1+\varepsilon)t}{s_1}$,$x_2 = 0$, $x_3 = 0$, as $\varepsilon \to 0$ from the integral

$$I \approx \lim_{1/s_3\to 0}\frac{1}{4\pi}\frac{1}{(2\varepsilon)^{1/2}}\int_{-\infty}^{\infty}\left[\{C_{i3lm}\xi_m L_{lk}^{-1}(1,\vec{\xi})C_{kn3q}\xi_n + C_{i3q3}\}\frac{\varepsilon_{3q}^*}{s_3}\right]d\xi_3 + O(1) \qquad (28)$$

with

$$\xi_1 = s_1^2 x_1/t, \xi_2 = s_2^2 x_2/t$$

Following Burridge and Willis (1969) the traction vector at a distance of order ε from the axes tips of the expanding inclusion (crack)
for prevailing plane strain, at $x_1 = \frac{(1+\varepsilon)t}{s_1}$, $x_2 = x_3 = 0$, along the $-x_1$ axis, has stress intensity factors

$$K_I = \sqrt{2\varepsilon}\sigma_{33} \sim \Delta\dot{u}_3^*\frac{\mu^2 R(s_1)}{2\rho(s_1^2-1/a^2)^{1/2}},\ K_{II} = \sqrt{2\varepsilon}\sigma_{31} \sim \Delta\dot{u}_1^*\frac{\mu^2 R(s_1)}{2\rho(s_1^2-1/b^2)^{1/2}} \qquad (29)$$

where

$$R(s_1) = 4s_1^2\left(s_1^2-1/a^2\right)^{1/2}\left(s_1^2-1/b^2\right)^{1/2} - \left(2s_1^2-1/b^2\right)^2 \qquad (30)$$

is the Rayleigh wave function, which vanishes as the speed $1/s_1$ of the $-x_1$ axis approaches the Rayleigh wave speed c_R. If, at this end, conditions of anti-plane strain prevail due to $\Delta\dot{u}_2^*$, we have asymptotically

$$K_{III} = \sqrt{2\varepsilon}\sigma_{32} \sim \Delta u_2^* \frac{(-\mu)(s_1^2 - 1/b^2)^{1/2}}{2} \quad (31)$$

which vanishes as the boundary propagates at the shear-wave-speed *b*.

III. The shape of the self-similarly expanding ellipsoidal inclusion and a dynamic cavitational instability

The criterion for the assumed self-similar shape is based on the extension of the static "equilibrium shape" of a region of martensitic transformation due to Eshelby, 1970, to self-similar dynamics, where the geometrical shape of the ellipsoidal boundary (with constant axes speeds) is independent of time (with the surface area scaling as t^2). Eshelby, 1970, reasoned on a basis of translational invariance in a thought experiment to reach the conclusion that we must have $\delta E^{tot} = 0$ for *the static **equilibrium position of phase boundaries***. Indeed, the shape is the one that, when assumed,

$$\Pi^H(u_i, u_{i,j}) = \int_\Omega W(x_i, u_i, u_{i,j}) dV \quad (32)$$

the Hamiltonian energy of an elastic body with strain energy density *W* *under total loading will be invariant under any family of infinitesimal translations* *(parallel displacement) of the inhomogeneity position* $x_k^* = x_k + \varepsilon a_k$, $u_i^* = u_i$, (Figure 3).

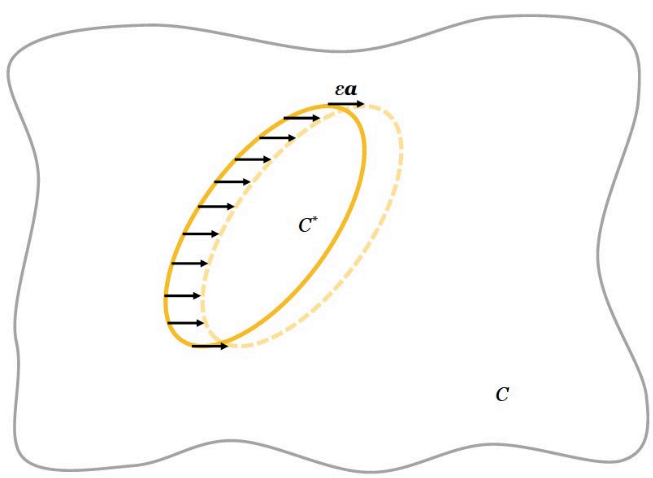

Figure 3: Translation of the inhomogeneity position

Then, according to Noether's theorem (i.e., Gelfand and Fomin p. 177) the shape will be an extremal surface of $u_i(x_i)$ on which the total variation of the total energy of the body under an arbitrary translation $\delta\xi_l = \varepsilon a_l$ of the inhomogeneity position should be equal to zero, i.e.,

$$\delta E^{tot} = \delta \Pi^{H} = -\int_{S} \delta\xi_l([[W]]\delta_{lj} - <\sigma_{ij}>[[u_{i,l}]])dS_j = -\delta\xi_l J_l^{dyn} =$$

$$-\int_{S} \delta\xi_k([[W]] - <\sigma_{ij}>[[u_{i,j}]])dS_k = 0 \qquad (33)$$

as in Eshelby,1970. The double brackets [[.]] denote jumps across the phase boundary and <.> denotes the average of the values across the two sides of the interface (surface of discontinuity). Eshelby (1970) further argued that, since $\delta E^{tot} = 0$ must be zero for any arbitrary $\delta\xi_k$, the integrand in (33) must vanish, i.e., the "effective normal force" (Eshelby, 1970)

$$[[W]] - <\sigma_{ij}>[[u_{i,j}]] = 0 \qquad (34)$$

must vanish point-wise, "*which determines the shape that the boundary will take under a given applied loading in phase equilibrium* ***when the stresses due both to self-forces and applied loading are included in (34)*** of Eshelby (1970*)"*.

We extend the above static definition to elastodynamics with the Hamiltonian functional containing the strain energy density *W* and the kinetic energy density *T*,

$$\Pi(u_i) = \int_{t_1}^{t_2} \int_{\Omega} \left\{ W(x_i, u_{i,j}) + T(\dot{u}_i) \right\} dV\, dt$$

and, for the self-similarly expanding phase boundary, *we postulate that the shape that the boundary will assume, and which is independent of time for self-similar expansion (just scaling), must be such that the rate of the Hamiltonian,* ***will remain invariant for any family of infinitesimal translations*** $x_k^* = x_k + \varepsilon a_k, u_k^* = u_k$ ***of the inhomogeneity position***. Consequently, by Noether's theorem, $u_i(x_k,t)$ will be an **extremal surface** both of the rate of the Lagrangean and also of the Hamiltonian *(since the variation of the Hamiltonian is minus to the variation of the Lagrangean for a purely mechanical system*)(e.g., Stolz, 2003, Markenscoff and Singh, 2015). For the **extremization** of the rate of the Lagrangean the dynamic J_k^{dyn} integral under total loading must vanish, and since the variation of the Hamiltonian is minus the variation of the Lagrangean, the variation of the rate of the Hamiltonian must also vanish.

We give to the inhomogeneity position a translation *per unit time* such that $\varepsilon\dot{a}_k = \varepsilon\upsilon_k$ where υ_k is the defect velocity. With the near-field asymptotic behavior of the dynamic field of the defect satisfying $u_{j,k}\upsilon_k = \dot{u}_j$, we obtain from J^{dyn} the expression for the energy release rate for a moving surface of discontinuity (instantaneous rate of energy flow through S^d, same as for a crack (without the brackets in Freund, 1972) as

$$\delta \dot{E}^{tot} / \varepsilon = \dot{a}_k J_k^{dyn} = \upsilon_k \int_{S^d} \{[[W+T]]n_k - [[\sigma_{ij} u_{j,k}]]n_i\} dS = \int_{s^d} \{[[W+T]]\upsilon_n + [[\sigma_{ij}\dot{u}_j]]n_i\} dS$$

(35)

The integrals in (35) can be easily shown to be path-independent of the shape of the surface surrounding the boundary S^d as it shrinks onto it, and, it may be noted, that the above expression of J_k^{dyn} is valid both in linear and nonlinear anisotropic elasticity.

Due to equation (33), eqtn (35) (with total loading) yields

$$\delta \dot{E}^{tot} = \int_{S^d} \{[[(W+T)]]\upsilon_n + n_j[[\sigma_{ij}\dot{u}_i]]\} dS = 0 \tag{36}$$

where υ_n is the normal boundary velocity on a contour surrounding the defect (Freund, 1972) and moving with it, same as $\dot{l}$ in the notation in Appendix A where the jump of the displacement gradient across the boundary of the expanding ellipsoid of phase change under pre-stress is obtained along the lines of Markenscoff (2015).

The interpretation of equation (36), ***and the criterion for growth is that the self-similarly expanding boundary will assume a shape such that the total energy flux through the moving boundary (with all loading included, external plus due to self-stresses) is zero***. Equivalently, we can state ***that the moving boundary will assume a shape for which, under total loading, it will not become a sink or source of energy.*** The jump quantities in (36) for self-similar expansion of an ellipsoidal inclusion are independent of time and depend only on the direction of the normal to the boundary. The expression for the jump in the displacement gradient across the expanding ellipsoid of phase change under pre-stress is obtained in Appendix A by generalizing to dynamics the Hill jump conditions as in Markenscoff (2015). Consequently, the integral in equation (36) with the self-stresses only scales in time as t^2. The scaling of the integral is also valid in the limiting "flat" cases of the ellipsoid as discussed further below.

With the Hadamard jump conditions (11) equation (36) under total loading yields

$$\delta \dot{E}^{tot} = \lim_{S^d \to 0} \int_{S^d} \dot{l}([[W]] - \langle \sigma_{ij} \rangle [[u_{i,j}]]) dS = 0 \tag{37}$$

There are two possibilities that the integrand in (37) is zero,

$$\text{either } \dot{l} = 0 \text{ , or } [[W]] - \langle \sigma_{ij} \rangle [[u_{i,j}]] = 0 \tag{38}$$

corresponding to two different "modes" of expansion governed by the boundary velocity $\dot{l}$.

The first possibility of the normal boundary velocity $\dot{l} = 0$ is satisfied on the upper and lower surfaces of the ellipsoid if they become flattened ($1/s_3 << 1/s_1, 1/s_2$) and $1/s_3 \to 0$ (circular penny-shape, or elliptical penny-shape, Figure 2B), and in the two-dimensional limit of the ellipsoid becoming an elliptic cylinder, which, in turn, becomes flattened, $1/s_2 \to \infty, s_1/s_3 \to 0$, as a propagating band (Figure 2C). The band may contain any type of eigenstrain, longitudinal or shear. Thus no energy is expended to move the upper and lower surfaces, and this is the minimum energy lost mode.

This **flattened mode** $\dot{l}=0$ is one of a "dynamic collapse/cavitation instability". It is the *instability* that **breaks the spherical symmetry of the input**, and corresponds to the minimum energy to move the boundary. We may note that in the static counterpart of (37) in Eshelby 1970, a possibility for the integrand to vanish is that $\delta\xi_k=\mathbf{0}$, as in very thin bands in martensitic transformations, which, though, should carry finite energy within the vanishingly small volume, as in dynamics. The second possibility is the second term in (38) to be zero on the whole boundary, which for volumetric symmetry of the loading would require energy to be spent to move the whole spherically expanding boundary, which it may not have, as discussed in the sequel for model B of Knopoff and Randall (1970) of drop of bulk modulus under pressure. For ch

Considering eqtn (37) **among all the possible orientations in the three dimensional space the ellipsoid will choose to flatten on that plane where the second factor becomes zero, i.e., where on the line part of the boundary the energy release rate becomes zero, so that** $\delta\dot{E}^{tot}=\lim_{S^d\to 0}\int_{S^d}\dot{l}([[W]]-<\sigma_{ij}>[[u_{i,j}]])dS=0$ **is satisfied over the whole boundary, and the energy rate to move the boundary will be the minimum among all other orientations within the solid angle**. If the integral in (37) along the line boundary is negative, the boundary will not move. **It will expand on the plane on which equation (38b) first becomes zero, and this will be the one of maximum Peach-Koehler type of configurational force applied on the moving boundary as to overcome the self-stress to move it**. **This will determine both the magnitude and directions of the axes speeds.**

IV. Model of DFEs: Nucleation and growth

Nucleation

Based on the self-similarly expanding ellipsoidal inclusion we present a model for deep-focus earthquakes (DFEs). Nucleation is assumed to occur as the material undergoes phase change (a drop in the moduli and volume collapse) under high pressure .We will assume that an instability of phase change starts instantaneously from zero dimension and that the initiation of phase transformation may induce the instability to continue by the same amount per unit volume and time. Moreover, we assume that the expanding region of "phase change" is ellipsoidal in shape, expanding with constant axes speeds (to be determined) for some time t, until self-similarity is broken (due to interactions).

Let $\Delta W(C_{ijkl},C^{*}_{ijkl},\varepsilon^{*vc}_{kk},\varepsilon^{0}_{ij},1/s_i)$ be the strain energy density drop inside a self-similarly (subsonically) expanding ellipsoidal area with volume collapse equivalent eigenstrain ε^{*vc}_{ij} and moduli change from C_{ijkl} to C^{*}_{ijkl} under uniform applied strain ε^{0}_{ij}. The strain energy density drop is constant due to the Eshelby property and computable based on the interior strains $\varepsilon_{ij}=S^{dyn}_{ijkl}\varepsilon^{**}_{kl}$ with $\varepsilon^{**}_{ij}=\varepsilon^{*}_{ij}+\varepsilon^{*vc}_{ij}$, with ε^{*}_{ij} due to the change in moduli under prestress and ε^{*vc}_{ij} due to volume collapse, according to (22), so that

$$\Delta W = 1/2C_{ijkl}\varepsilon^0_{ij}\varepsilon^0_{kl} - 1/2C^*_{ijkl}(\varepsilon^0_{kl} + S^{dyn}_{klmn}\varepsilon^{**}_{mn} - \varepsilon^{*vc}_{kl})(\varepsilon^0_{ij} + S^{dyn}_{ijmn}\varepsilon^{**}_{mn} - \varepsilon^{*vc}_{ij}) \quad (39)$$

where the second term is the stress times the linear part of the strain in the expanding inhomogeneous inclusion. We can then write the total *energy rate input to the phenomenon* of the DFE which is the rate of the loss of strain energy inside the volume of the expanding ellipsoid, i.e.,

$$\Delta\dot{W}^{tot} = d/dt\{\Delta W(4\pi/3)t^3/s_1s_2s_3\} = \Delta W\, 4\pi t^2/(s_1s_2s_3) \quad (40)$$

where ΔW is the constant strain energy density drop given by (39). We may note that the energy rate input in (40) grows in time as t^2 (while self-similarity holds).

A particular case will arise if the drop in strain energy density ΔW is very large (tends to infinity) so that in equation (40) we will have

$$\Delta W/(s_1s_2s_3) = \lim_{1/s_3\to 0}(\Delta W/s_3)/s_1s_2 = const \quad (41)$$

which will allow finite energy (and not infinitesimal) to propagate in a flat ellipsoid ($1/s_3 \to 0$) or in a band (flattened elliptical cylinder, $1/s_2 \to \infty, s_1/s_3 \to 0$). It may be noted here that for the possibility that it becomes a needle (a question raised by R. Jeanloz) we would need to have in equation (40)

$$\Delta W/(s_1s_2s_3) = \lim_{s_2\to 0}\lim_{1/s_3\to 0}(\Delta W/s_3s_2)s_1 = const \quad (42)$$

which means that the strain energy density ΔW would need to tend to infinity faster than in (41).

Criterion of growth for an ellipsoidal region of phase change and determination of the axes speeds as a minimization problem:

At a given time *t*, given an input rate of strain energy drop in the volume of the self-similarly expanding ellipsoid (with unknown axes speeds), such that $\Delta W(1/s_i)/(s_1s_2s_3) = K$ (a constant), the expanding boundary will assume a shape (which determines the expansion speeds) such that it *minimizes* the total energy rate expended to move the inhomogeneity position, and, for this shape, the boundary will be neither a sink nor a source of energy as it expands, with the rate to move the boundary $\dot{E}^{bndry}$ quantified by the dynamic *J* integral (due to self-stresses), and satisfying $\dot{E}^{bndry}/K < 1$ (smaller than the available energy rate). In the presence of dissipation an additional experimentally determined term should be added to the rate of energy needed to move the boundary (as in equation (46) below).

The above variational formulation provides the equations for the determination of the axes speeds, which will determine $\dot{E}^{bndry}$ and simultaneously the outside radiated energy for a given energy rate input *K*. We note that all self-energy rates scale with t^2 in self-similar motion. The flattened ellipsoid will have an elliptical line edge L, of unknown length depending on the axes speeds, that need to be determined, and are free variables. In the sharp limit at the edge, in the limit from the three-dimensional solution (eqtns (31)-(32)) the fields become square root singular, i.e., as $(t/x)^{1/2}$ and the energy release in the integrand varies per unit length as the square of it, i.e., as t/x. The total contribution of the integral around the ellipse of length L (growing as *t*) to the self-energy rate to move the elliptical boundary will grow as t^2, same as the growth of the input strain energy in the volume. For general eigenstrain eqtn (37) cannot be satisfied for an ellipsoidal shape. According to the criterion postulated above, the total variation of the energy rate to move the boundary will be minimized by varying the axes speeds. This is analogous to Mura (1982) eqtns (27.25), obtaining two equations for the axes speeds by setting the two derivatives (with respect to the axes speeds $1/s_1$ and $1/s_2$) of the total J^{dyn} integral around the ellipse equal to zero, while keeping constant the strain energy in the limit (41)

$$\Delta W(1/s_1, 1/s_2)4\pi/(s_1 s_2) = K \tag{43}$$

accounted for with a Lagrange multiplier and also with one for the constraint $\dot{E}^{bndry}/K < 1$.

In the presence of dissipation per unit length of the moving boundary, equation (37) will be modified by an additional term in the line integral, a function of the normal boundary velocity $\dot{l}$. A criterion for growth can be written as

$$\delta(\dot{E}^{tot} + \int_L D(\dot{l})\,dL) = 0 \tag{44}$$

Equation (44) constitutes the counterpart of the Griffith criterion of fracture (e.g., Mura (1982) eqtn (34.5) but does not involve a characteristic length, and so it preserves the self-similarity. The specific expression of $\delta\dot{E}^{tot}$ in eqtn (44) has to be obtained by evaluating the energy-release rates for all the contributing components of the eigenstrain in the expression (37), and this will involve a new quantity in the literature for an expanding thin disk with in-plane eigenstrains with stress intensity factors analogous to (29) and (31) for a crack.

We may note that, if the integrand is axisymmetric, i.e., only with normal eigenstrain in the vertical direction, then the variational problem above reduces to the one of finding the shape that among all shapes with constant surface area (here given by (43)) minimizes the perimeter (the energy release rate on the boundary, which, in this case, would not have angular dependence due to axisymmetry), and it is the circle. This means that under normal loading the elliptical crack will expand circularly (Craggs, 1966) and travel without loss of energy at the Rayleigh wave-speed. For non-axisymmetric loading (with shear) in the elliptical self-similarly expanding crack of Burridge and Willis (1969) the axes speeds will be determined analogously by minimizing the total energy release rate around the elliptical crack with respect to the two axes speeds.

Evolution equation for an expanding "band"(flattened elliptical cylinder) of phase change under pre-stress.

We consider now the case of bands (flattened elliptical cylinder (Figure 1(c)) where the condition that no sinks/sources of energy are produced is satisfied **point-wise**, while in the flattened ellipsoid case of Figure 2(b) it is only satisfied in the integral sense along the whole boundary as discussed above.

The expanding motion of an inclusion with phase change will start on the plane of the flattened elliptical cylinder when, according to (37), the "driving force" (defined as below) satisfies

$$[[W]] - < \sigma_{ij} > [[u_{i,j}]] = 0 \tag{45}$$

The quantities in this expression include the externally applied ones. In the presence of dissipation the right-hand-side of the equation of (45) will be not be equal to zero but to a dissipative term, so that point-wise

$$[[W]] - < \sigma_{ij} > [[u_{i,j}]] = f(\dot{l}) \tag{46}$$

which is the equation that drives the expansion (equation of motion), relating the pre-stress and the phase change to the axes expansion speeds.

For an expanding homogeneous inclusion with eigenstrain in a homogeneous elastic material we have obtained for the jumps in Markenscoff and Ni, 2010, 2016), Markenscoff (2010), and in statics Eshelby (1978)

$$[[W]] - < \sigma_{ij} > [[u_{i,j}]] = - < \sigma_{ij} > [[\varepsilon^*_{ij}]] = < \sigma_{ij} > \varepsilon^*_{ij} \tag{47}$$

so that (38)b is written for the self-stress we have (analogous to Truskinovsky, 1987)

$$< \sigma_{ij} > [[\varepsilon^*_{ij}]] = 0 \tag{48}$$

where the stress is the self-stress σ^{dyn}_{ij} due to dynamically moving boundary of phase discontinuity. Due to linearity, the presence of pre-stress σ^0_{ij} can be added to the self-stress so that σ_{ij} in (48) is the total stress , $\sigma_{ij} = \sigma^0_{ij} + \sigma^{dyn}_{ij}$, and (48) yields

$$\sigma^0_{ij}\varepsilon^*_{ij} + < \sigma^{dyn}_{ij} > \varepsilon^*_{ij} = 0 \tag{48b}$$

Equation (48b) expresses the fact that **a Peach-Koehler type of configurational force applied on the phase boundary by the pre-stress is balancing the self-force needed to move the boundary and this determines the boundary velocity** involved in the quantities in (48b) which are calculated from the dynamic Eshelby solution presented above.

In the case that the pre-stress is not hydrostatic pressure, but, due to some tension added in some direction the principal applied stresses become unequal, there will be planes of maximum shear at 45 degrees to the principal directions so that the flattened ellipsoid will orient itself on them

(Figure 2). Equation (48b) will express that the maximum in- plane shear acting on the flattened band/ellipsoid will produce the maximum Peach-Koehler force acting on a dislocation (line or loop), as **in the very thin ellipsoid the eigenstrain will give in the limit with the thickness a displacement discontinuity (Burgers vector). Thus, suddenly emitted gliding dislocations will be produced to radiate out with minimum dissipative loss** with the strain energy produced by the phase change under pre-stress condensed into the core of the Burgers vector.

The components of the eigenstrain ε^*_{ij} are six, the three of which will produce a dislocation loop (with the edge components giving a "double couple"), while the longitudinal components will produce screw-pairs (according to Weertman in Knopoff and Randall(1970)). These eigenstrain components manifest themselves in the radiation patterns given by equations (18), and this topic is being further investigated as it relates to models in seismology.

We may note that for general time dependent eigenstrains Willis (1965) found that inclusions do not yield a dislocation as Eshelby showed in statics, but it has to be investigated whether this is the case for self-similar motion. It is conjectured that the limit gives a dislocation in self-similarly expanding motion based on the fact that the static inclusion is a limit of the self-similar (Ni and Markenscoff, 2016c). The motion of dislocations jumping to constant velocity (but starting from rest) were analyzed in Markenscoff (1980) and Markenscoff and Clifton (1981), and the self-force, or "drag-force", or energy-release rate, needed to be provided for a dislocation to jump from rest to a constant velocity was obtained in Clifton and Markenscoff (1981), with the difference here being that they are generated instantaneously.

V. The dynamic Eshelby Tensor for the penny-shape ellipsoid

Both cases of penny-shape ellipse and flattened elliptical cylinder (band) can be computed by means of the Dynamic Eshelby Tensor (DET) for the particular limiting planar cases of the ellipsoid, which are of the same form as the static ones (see Mura, 1982), but with the ratio of axes speeds in place of the ratio of the axes lengths, which has been also confirmed numerically from the full three dimensional expression of the DET in Ni and Markenscoff, 2016 c). This is because the static Eshelby Tensor is obtained from the dynamic one only from the contributions of the *M* waves. For example, Figure 4 plots the component S^{dyn}_{1313} for the ellipsoid given in [Ni and Markenscoff, 2016c, eqtn 3.9] as an integral evaluated numerically, and compares it with the analytical (asymptotic) penny-shape value for it from the expression in Mura , 1982, eqtn, (11.23) for the penny-shape inclusion, but with ratio of the axes speeds in the place of the lengths, i.e.,

$$S^{dyn(penny-shape)}_{1313} = (1/2)\{1+(\nu-2)/(1-\nu)(\pi/4)s_1/s_3\} \tag{49}$$

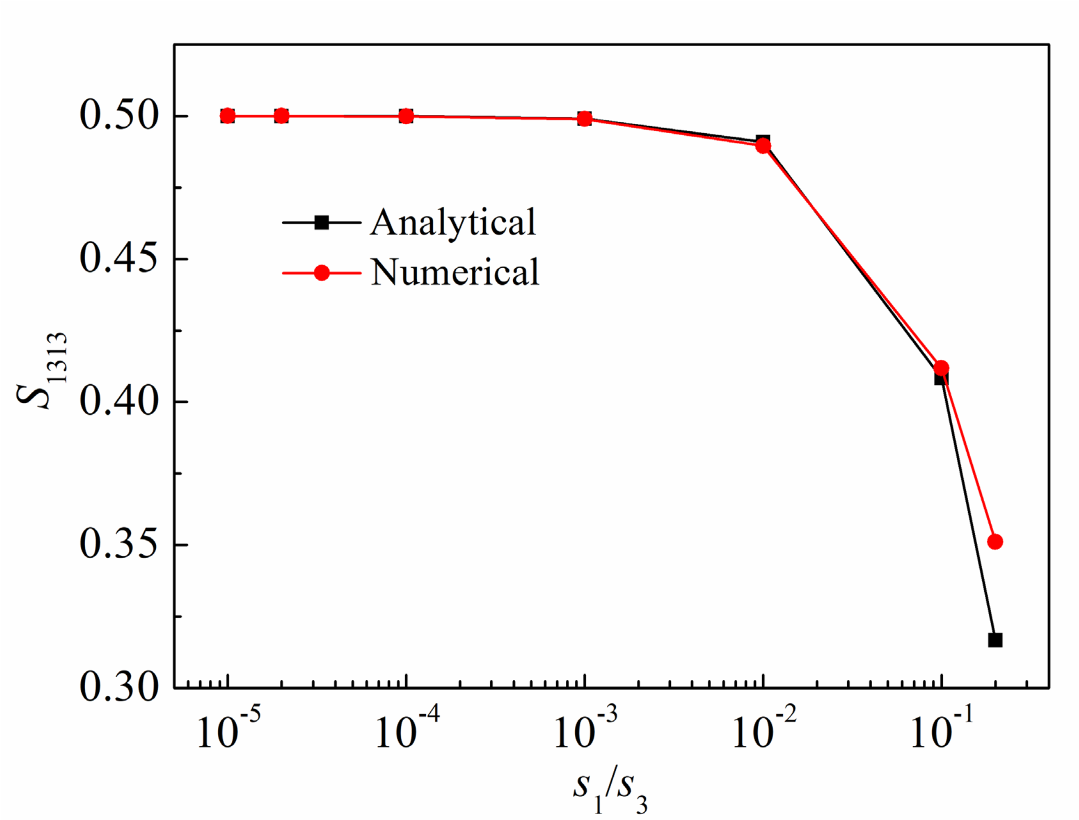

Figure 3. Dynamic Eshelby Tensor component for penny-shape ellipsoid ($\nu = 0.25$)

The agreement is very good for the expansion speed of the third axis (flattened ellipsoid) up to 10^{-2} of the in-plane speed. All computations of the expanding flattened ellipsoids, as in Figures 1b, 1c, can be computed with the expressions for the Dynamic Eshelby Tensor obtained from the corresponding static penny-shape (e.g., Mura, 1982)

VI. The energetics for models proposed in the literature of deep earthquakes

The energetics to move the boundary of phase discontinuity as dictated by Noether's theorem have not been considered in the seismology literature before, and, in view of that, we are reassessing whether some models proposed in the literature are possible. Equation (48b) allows to determine whether the inclusion boundary has enough energy to propagate in self-similar expansion, and at what speed, given the input quantities (applied pressure ε_{ij}^{0}, moduli drop), so that the deep focus earthquake becomes possible in this mode.

In the case of drop in the bulk modulus under pressure we have checked the assumption that the growth would be spherical keeping the symmetry as in model B of Knopoff and Randall (1970). A calculation yielded the result that the energy to move the boundary would exceed the available, and, thus, the spherical dynamic growth is shown not to be possible in model B: We solve (22) for the equivalent eigenstrain with $S^{dyn}_{kkll} = (\frac{1+\nu}{1-\nu})\frac{(1+2V/a)}{(1+V/a)^2}$ (on the basis on Ni and Markenscoff, 2016b), and equation (48b) yields

$$\sigma^0_{ij}\varepsilon^*_{ij} - 2\mu(3\lambda+2\mu)/(\lambda+2\mu)\varepsilon^{*2}$$
$$-2\mu(3\lambda+2\mu)/(\lambda+2\mu)(V/a)^2(3-V/a)/[(1+V/a)(1-(V/a)^2]\varepsilon^{*2} = 0 \quad (50)$$

where the first term is of the Peach-Kohler configurational type, and the second term is the static term (same as in Eshelby, 1978) that would need to be overcome before the motion starts. The third term is obtained in Markenscoff and Ni (2016) for an inclusion with eigenstrain in a homogeneous material, which we are applying for the eigenstrain being the equivalent eigenstrain for an expanding inhomogeneity. However, this may be approximate since it may differ from (48b) for inhomogeneity. Considering that $\sigma^0_{ij}\varepsilon^*_{ij} = 3K\varepsilon^0_{kk}\varepsilon^*$, and in view of (22) relating the eigenstrain to the applied strain ε^0_{kk} , the term ε^{*2} will factor out in (50) , and numerical calculations have shown that the energy to move the boundary would substantially exceed the input energy.

Essentially, in the Eshelby approach the equivalent eigenstrain due to drop in moduli under pressure is proportional to the pressure (eqtn (22). If you increase the pressure, the eigenstrain increases, so the pressure increase does not give you enough driving force to move the boundary because the eigenstrain induced by pressure also has increased proportionally. However, if you have change in density, this is an equivalent eigenstrain which in magnitude is independent of the pressure. So more pressure will provide more energy to move the phase boundary that has eigenstrain independent of the pressure. (This does not include the deformation due to the pressure itself). So, volume collapse under high enough pressure has enough energy to expand spherically and also simultaneously with bulk modulus drop (it would provide the difference). However, Noether dictates that in self-similar expansion the inclusion will take the shape that extremizes the energy to move the boundary, and for stability we may assume minimization so that planarity will be favored, and the model of Randall (1964) with spherical expansion for change in density may not happen corresponding to maximum of the self-energy. The volume collapse propagating planarly is treated through the DET for the expanding penny-shape as an equivalent eigenstrain in a very thin expanding elliptical disk region possessing in-plane eigenstrains. This is a “new” dynamic defect that is different than a crack and may manifest itself in this phenomenon.

VII. Conclusions

In conclusion, we have treated the problem of a self-similarly expanding region of phase change (change in density, i.e., volume collapse, and change in moduli) under pre-stress, with the phase change being an equivalent eigenstrain (obtained through the Dynamic Eshelby Tensor) in the dynamic generalization of the Eshelby ellipsoidal inclusion problem. By dimensional analysis alone for self-similar expansion of the ellipsoid, the Cauchy-Kowalewskaya theorem dictates zero particle velocity in the interior domain. As a result no kinetic energy is radiated to the interior and dissipated focusing at the origin. Moreover, for self-similar expansion, Noether’s theorem dictates that the axes expansion speeds will take the values that, given the input energy rate due to phase change, *minimize the energy-rate spent to move the boundary* of phase discontinuity (dynamic *J* integral), so that it does not become a sink or source of energy. This minimization problem determines fully the axes speeds (in magnitude and direction) in terms of the input parameters of phase change under pre-stress. A particular possibility given by the expression derived from Noether's theorem is that the ellipsoid is "flattened" making the expanding region of phase change *planar*, which constitutes *breaking of the symmetry* of the input and a newly discovered mode of *dynamic cavitational instability*. The equivalent to the phase change eigenstrain in the flattened very thin ellipsoid will give in the limit dislocations that will glide out at maximum speed and minimum loss on the orientation of the flattened ellipsoid on planes where the Peach-Koehler type configurational force is maximum (planes of maximum shear stress). The analysis shows how these two fundamental theorems of mathematical physics taken together maximize the energy available to radiate outside, and may have immediate application to the problem of Deep Focus Earthquakes considered to be due to phase transformation under pre-stress. Noether's theorem (total dyn *J* integral) dictates the kinetics (providing a criterion for propagation) and they will occur when the pre-stress is big enough so that the configuration force on them (Peach-Koehler type) can overcome the self-stress due to inertia (plus a dissipation) to move the defect of the boundary of phase discontinuity.

Deep earthquakes are a long-standing open problem, with all evidence in nature and analog experiments that they propagate planarly (e.g., Wang *et al*, 2017). The model will provide a framework to relate seismic signals to the input parameters of phase change and pre-stress through the equivalent eigenstrains which also account for the geometry of the expanding region, very importantly how the planarity affects the emissions of the 3D phase change condensed into the propagating two-dimensional ones in the limit. The model opens the field for further investigation. Both theorems, the Cauchy-Kowalewskaya and Noether's on which the analysis is based are valid also for anisotropic elasticity, but for self-similarly expanding anisotropic inclusions we have not yet proven the Eshelby property for the interior domain and we do not have the Dynamic Eshelby Tensor, although an anisotropic interior in an isotropic matrix is obtainable by the isotropic Dynamic Eshelby Tensor as shown here. For full anisotropy in static inclusions the Eshelby property was proven by Willis (1970), and we conjecture that it is true also for self-similarly expanding anisotropic inclusions. The planarity (symmetry breaking) due to Noether's theorem (valid in nonlinear anisotropic elasticity) is independent of scale, valid from the nano to the very large (kilometers) one, and models analogous to the one for DFEs can be developed for other phenomena, such as the dynamic stress inducement of martensitic transformations (Escobar et al, 2000), dynamic shear banding, and amorphization (Zhao, et al, 2017).

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Appendix A

“Hill jump conditions” across the moving surface of the self-similarly expanding ellipsoid with phase change under pre-stress

We can generalize the Hill (1961) jump conditions to dynamics across the expanding ellipsoidal inclusion, as in Markenscoff (2015) (where a typo needs to be corrected with C^*_{ijkl} in eqtn (28)). We write as in Hill (1961)

$$[[u_{k,l}]] = \lambda_k n_l \tag{A1}$$

where the vector λ_k will be determined point-wise by satisfying the Hadamard jump conditions. Using the Hadamard equations (11)

$$[[\sigma^0_{ij} + \sigma_{ij}]]n_j = \rho \dot{l}[[\partial u_i / \partial t]] \tag{A2}$$

and with $\sigma^0_{ij} = C_{ijkl} u^0_{k,l}$ we obtain

$$\{C_{ijkl}[[u_{k,l}]] + \Delta C_{ijkl} u^0_{k,l} + \Delta C_{ijkl} S^{dyn}_{klmn} \varepsilon^{**}_{mn}\} n_j = \rho \dot{l}^2 [[\partial u_i / \partial x_j]] n_j \tag{A3}$$

(having used the other Hadamard condition in (11)), so that from (A3) we have an equation for λ_k

$$K_{ik}^{mb}\lambda_k = -\{\Delta C_{ijkl}\varepsilon_{kl}^{0} + \Delta C_{ijkl}S_{klmn}^{dyn}\varepsilon_{mn}^{**}\}n_j \qquad \text{(A4)}$$

with

$$K_{ik}^{mb} = \{C_{ijkl} - \rho \dot{l}^2 \delta_{ik}\delta_{jl}\}n_j n_l \quad .$$

We invert as in Mura, 1982,eqtn 6.8, and obtain

$$\lambda_i = -\{\Delta C_{jkmn}\varepsilon_{mn}^{0} + \Delta C_{jkmn}S_{mnrs}^{dyn}\varepsilon_{rs}^{**}\}n_k N_{ij}(\gamma,\vec{n}) / D\ (\gamma,\vec{n}) \qquad \text{(A5)}$$

with $\gamma^2 = \dot{l}^2 \xi_i \xi_i$

The exterior stress σ_{ij}^{+} is then computed and it depends on the applied strain ε_{ij}^{0}, the phase change ΔC_{ijkl}, the ellipsoid axes expansion speeds, the equivalent eigenstrain, the Dynamic Eshelby Tensor and the direction of the normal.