Cracks as limits of Eshelby inclusions

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As limiting behaviors of Eshelby ellipsoidal inclusions with transformation strain, crack solutions can be obtained both in statics and dynamics (for self-similarly expanding ones). Here is presented the detailed analysis of the static tension and shear cracks, as distributions of vertical centers of eigenstrains and centers antisymmetric shear, respectively, inside the ellipse being flattened to a crack, so that the singular external field is obtained by the analysis, while the interior is zero. It is shown that a distribution of eigenstrains that produces a symmetric center of shear cannot produce a crack. A possible model for the Barenblatt crack is proposed by the superposition of two elliptical inclusions by adjusting their small axis and strengths of eigenstrains so that the singularity cancels at the tip.

Keywords: Eshelby inclusions, transformation strain, cracks, Barenblatt crack, elasticity

1. Introduction

Mura [1] has presented a method for solving crack problems as Eshleby [2] ellipsoidal inclusion problems. Based on the constant stress Eshelby property Mura cancels the applied tractions on the faces of an ellipsoidal inclusion as the "vertical" axis of the ellipsoid tends to zero, so that the inclusion is flattened to a crack. The limit of the product of the eigenstrain times the vertical axis length, as the eigenstrain tends to infinity and the axes length tends to zero, is a finite quantity, the crack opening displacement.

Here is provided a complete analysis for the tension and shear Griffith cracks based on distributing centers of eigenstrain. In addition to the eigenstrains considered by Mura, are included eigenstrains so that all stresses vanish in the interior of the crack. The external field is obtained analytically, while Mura [1] only obtains the singularity based on the energy-release rate known independently. We show that a crack cannot be produced as a symmetric center of shear with eigenstrains $\varepsilon^*_{xx} = -\varepsilon^*_{yy}$, because the internal stresses cannot be cancelled by an applied stress field in this case.

Analogously to the static, Markenscoff [3] has shown that the self-similarly expanding elliptical crack Burridge and Willis [4] can be obtained from the limit of the self-similarly expanding ellipsoidal inclusion with transformation strain [5–7], since the constant stress Eshelby property is valid also in self-similarly expanding Eshelby inclusions. It may be noted that the dynamic Eshelby fields in respective limits give both the static Eshelby inclusion (and hence the static cracks) and also Rayleigh waves as the axis expansion speed of the elliptically expanding crack tends to the Rayleigh wave speed.

2. The "flattened" elliptical cylinder with transformation strain as a crack

We consider that the crack will be a distribution of eigenstrains inside a flattened elliptical cylinder. The interior stresses of a flattened elliptical cylinder $(a_3 \to \infty,\ a_2/a_1 = \eta)$ with transformation strains ε^*_{ij} are given by (also, [1])

$$\begin{aligned}
\sigma^{\mathrm{I}}_{xx} &= \frac{\mu}{1-\nu}\left[-\frac{2+\eta}{(1+\eta)^2}\varepsilon^*_{xx} - \frac{\eta}{(1+\eta)^2}\varepsilon^*_{yy} - 2\nu\frac{1}{1+\eta}\varepsilon^*_{zz}\right],\\
\sigma^{\mathrm{I}}_{yy} &= \frac{\mu}{1-\nu}\left[-\frac{\eta}{(1+\eta)^2}\varepsilon^*_{xx} - \frac{\eta(1+2\eta)}{(1+\eta)^2}\varepsilon^*_{yy} - 2\nu\frac{\eta}{1+\eta}\varepsilon^*_{zz}\right],\\
\sigma_{zz} &= \frac{\mu}{1-\nu}\left[-2\nu\frac{1}{1+\eta}\varepsilon^*_{xx} - 2\nu\frac{\eta}{1+\eta}\varepsilon^*_{yy} - 2\varepsilon^*_{zz}\right].
\end{aligned} \tag{1}$$

To obtain the crack $|x| < a$ under tension $\sigma_{yy} = T,$ the interior stresses in the inclusion should be $\sigma_{xx} = \sigma_{xy} = 0,\ \sigma_{yy} = -T.$

From Eqs. (1) the leading terms in the interior stresses as $\eta \to 0$ are

$$\begin{aligned}
\frac{1-\nu}{\mu}\sigma^{\mathrm{I}}_{xx} &= -2\varepsilon^*_{xx} - \eta\varepsilon^*_{yy},\\
\frac{1-\nu}{\mu}\sigma^{\mathrm{I}}_{yy} &= -\eta\varepsilon^*_{xx} - \eta\varepsilon^*_{yy} + \mathrm{O}(\eta^2).
\end{aligned} \tag{2}$$

In order for $\sigma_{yy} = -T,$ we set $\varepsilon^*_{yy} = e \to \infty,\ \ \eta \to 0,$ and $\eta e = T(1-\nu)/\mu.$ Moreover, for $\sigma_{xx} = 0,$ from Eq. (2) there should exist eigenstrain $\varepsilon^*_{xx} = -\eta\varepsilon^*_{yy}/2.$ Such eigenstrains were not considered by Mura, but they need to be considered to obtain all stress and displacement fields correctly. We give below the fields for a vertical center of eigenstrain $\varepsilon^*_{yy} = e,$ $\varepsilon^*_{xx} = \varepsilon^*_{xy} = \varepsilon^*_{zz} = 0,$ and $\lim\limits_{a\to 0}(\varepsilon^*_{yy}\pi a^2) = q$ obtained by means of the Airy stress function

$$U(r,\theta) = -\mu q/(\pi(\kappa+1))(2\log r + \cos(2\theta)),$$

from which we obtain

$$
\begin{aligned}
&u_x(x,y)=\frac{q}{2\pi(\kappa+1)}\left[(5-\kappa)\frac{x}{r^2}-4\frac{x^3}{r^4}\right],\\
&u_y(x,y)=(\kappa-1)\frac{y}{r^2}+4\frac{y^3}{r^4},\\
&u_x(x,0)=-\frac{q(\kappa-1)}{2\pi(\kappa+1)}\frac{1}{x},\\
&u_y(x,0)=\frac{q}{2}\delta(x)\operatorname{sgn} y,\\
&\sigma_{xx}(x,y)=\frac{2\mu q}{\pi(\kappa+1)}\left[\frac{1}{r^2}-8\frac{x^2}{r^4}+8\frac{x^4}{r^6}\right],\\
&\sigma_{yy}(x,y)=\frac{2\mu q}{\pi(\kappa+1)}\left[-3\frac{1}{r^2}+12\frac{x^2}{r^4}-8\frac{x^4}{r^6}\right],\\
&\sigma_{xy}(x,y)=\frac{4\mu q}{\pi(\kappa+1)}\left[\frac{xy}{r^4}-4\frac{xy^3}{r^6}\right],\\
&\sigma_{xx}(x,0)=\sigma_{yy}(x,0)=\frac{2\mu q}{\pi(\kappa+1)}\frac{1}{x^2},\ \sigma_{xy}(x,0)=0.
\end{aligned}
\tag{3}
$$

3. The stress field of a crack in tension as a distribution of vertical centers of eigenstrain

The crack stress field is obtained by distributing in the interior of the ellipse (crack) vertical centers of eigenstrain $\varepsilon^*_{yy}=e$ with density proportional to the thickness x_2 of the ellipse, so that $q(x)=q_0\, a_2/a_1\sqrt{a_1^2-x^2}$ so that from Eqs. (3) we have

$$
\begin{aligned}
&\sigma_{xx}(x,0)=\sigma_{yy}(x,0)=\\
&=\frac{2\mu}{\pi(\kappa+1)}\frac{a_2}{a_1}q_0\int_{-a_1}^{a_1}\frac{\sqrt{a_1^2-\xi^2}}{(\xi^2-x^2)^2}\mathrm{d}\xi=\\
&=-\frac{2\mu}{\kappa+1}\frac{a_2}{a_1}q_0\left[1-\frac{|x|}{\sqrt{x^2-a_1^2}}H(|x|-a_1)\right],
\end{aligned}
\tag{4}
$$

where κ denotes the Kolosov constant and the integral in the sense of principal value was evaluated according to Kaya and Erdogan [8]. Requiring that on $|x|<a_1$, $\sigma_{yy}(x,0)=-T$, in the limit as $a_2\to 0$ and $e\to\infty$, we set $2\mu a_2 q_0/(a_1(\kappa+1))=T$, superpose $\sigma_{yy}=T$ and obtain the total stresses

$$
\begin{aligned}
&\sigma_{xx}(x,0+)=T\left[-1+\frac{|x|}{\sqrt{x^2-a_1^2}}H(|x|-a_1)\right],\\
&\sigma_{yy}(x,0)=T\frac{|x|}{\sqrt{x^2-a_1^2}}H(|x|-a_1),\\
&\sigma_{xy}(x,0)=0.
\end{aligned}
\tag{5}
$$

4. The stress field of a crack in shear as a distribution of antisymmetric centers of eigenstrain

For the shear crack centers of eigenstrain $\varepsilon^*_{xy}=e, \varepsilon^*_{xx}=\varepsilon^*_{yy}=0$ (anti-symmetric centers of shear) are distributed similarly with the stress fields derived from the stress function $U(r,\theta)=2\mu q\sin(2\theta)/(\pi(\kappa+1))$ we have the fields

$$
\begin{aligned}
&u_x(x,y)=\frac{q}{\pi(\kappa+1)}\left[(\kappa+3)\frac{y}{r^2}-4\frac{y^3}{r^4}\right],\\
&u_y(x,y)=\frac{q}{\pi(\kappa+1)}\left[(\kappa+3)\frac{x}{r^2}-4\frac{x^3}{r^4}\right],\\
&u_x(x,0)=q\delta(x)\operatorname{sgn} y,\ u_y(x,0)=\frac{q(\kappa-1)}{\pi(\kappa+1)}\frac{1}{x},
\end{aligned}
\tag{6}
$$

$$\sigma_{xx}(x,y)=\frac{8\mu q}{\pi(\kappa+1)}\left[-3\frac{xy}{r^4}+4\frac{xy^3}{r^6}\right],$$

$$\sigma_{yy}(x,y)=\frac{8\mu q}{\pi(\kappa+1)}\left[\frac{xy}{r^4}-4\frac{xy^3}{r^6}\right],$$

$$\sigma_{xy}(x,y)=\frac{4\mu q}{\pi(\kappa+1)}\left[\frac{1}{r^2}-8\frac{x^2}{r^4}+8\frac{x^4}{r^6}\right],$$

$$\sigma_{xy}(x,0)=\frac{4\mu q}{\pi(\kappa+1)}\frac{1}{x^2},\ \ \sigma_{yy}(x,0)=0.$$

By distributing the centers of shear eigenstrain inside the ellipse and integrating as before we obtain the stresses

$$\begin{aligned}
&\sigma_{xy}(x,0)=-\frac{4\mu q_0}{\kappa+1}\frac{a_2}{a_1}\begin{cases}1, & |x|<a_1,\\ 1-\dfrac{|x|}{\sqrt{x^2-a_1^2}}, & a_1<|x|<\infty,\end{cases}\\
&\sigma_{xx}(x,0)=-\frac{8\mu q_0}{\kappa+1}\frac{a_2}{a_1}\frac{x}{\sqrt{a_1^2-x^2}}\operatorname{sgn} y,\ \ |x|<a_1,\\
&\sigma_{yy}(x,0)=0
\end{aligned}\tag{7}$$

and the displacements

$$\begin{aligned}
&u_x(x,0^+)-u_x(x,0^-)=\frac{2q_0a_2}{a_1}\sqrt{a_1^2-x^2},\ \ |x|<a_1,\\
&u_y(x,0)=\frac{q_0a_2}{a_1}\frac{\kappa-1}{\kappa+1}\begin{cases}x, & |x|<a_1,\\ x-\sqrt{x^2-a_1^2}\operatorname{sgn} x, & a_1<x<\infty,\end{cases}
\end{aligned}\tag{8}$$

where we set $4\mu/(\kappa+1)(a_2/a_1)q_0=S$, and we superpose $\sigma_{xy}=S$ to obtain the total stresses for the Griffith shear crack. We may note that $u_y(x,0)=x$ for $|x|<a_1$ means that the shear crack rotates.

5. Symmetric centers of shear do not produce a crack

A symmetric center of shear consists of eigenstrains $\varepsilon^*_{xx}=-\varepsilon^*_{yy}=e,\ \varepsilon^*_{xy}=0$.

As Eqs. (1) indicate as $e\to\infty$ so that $e\eta\to$ const, $\sigma_{xx}\to\infty$ in the interior of the crack. It is not possible to cancel these stresses by remote tractions as to produce a cavity for the crack. However, for a self-similarly expanding inclusion with transformation strain containing symmetric centers of shear, dynamic fields would be emitted outside and the shear stress would be singular at the tips. The stress σ_{xx} would be symmetric in x, while σ_{xy} would be antisymmetric.

6. Concluding remarks for the Barenblatt crack

A detailed analysis shows how a crack can be obtained as the limit of an Eshelby ellipsoidal (in 2D elliptical) inclusion as the eigenstrain tends to infinitity, the small axis length tends to zero, while their product to a length that gives the crack opening displacement. This is due to the Eshelby constant stress property so that the constant applied traction on the crack faces can be cancelled. This property is true also in the dynamic generalization of the Eshelby inclusion problem where the Burridge and Willis [4] self-similarly expanding elliptical crack is obtained, and at the corresponding axis speed Rayleigh waves, as M waves emitted by the expanding ellipsoid.

A question may arise whether the Barenblatt crack can be produced by distribution of eigenstrains [9]. We may consider the superposed fields of two elliptical inclusions with axes lengths $(a_1,b_1),(a_2,b_2)$ as in the Fig. 1, so that all stresses in the interior of the inner ellipsoid vanish as to produce a crack. Only in the case $a_1=a_2$ we can adjust the eigenstrain densities q_1 and q_2 that the singularity cancels at that point. It would be interesting to obtain the fields and compare with the Barenblatt crack.

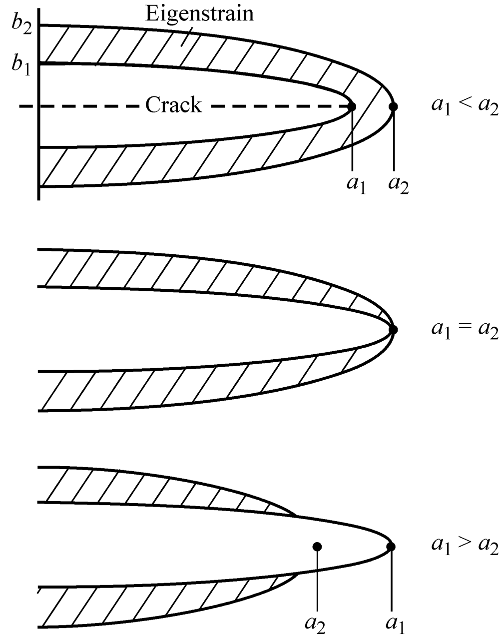

Fig. 1. Superposition of two elliptical inclusions to produce a crack with vanishing stress at the tip

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