Group trisections and smooth 4–manifolds

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A trisection of a smooth, closed, oriented 4–manifold is a decomposition into three 4–dimensional 1–handlebodies meeting pairwise in 3–dimensional 1–handlebodies, with triple intersection a closed surface. The fundamental groups of the surface, the 3–dimensional handlebodies, the 4–dimensional handlebodies, and the closed 4–manifold, with homomorphisms between them induced by inclusion, form a commutative diagram of epimorphisms, which we call a trisection of the 4–manifold group. A trisected 4–manifold thus gives a trisected group; here we show that every trisected group uniquely determines a trisected 4–manifold. Together with Gay and Kirby's existence and uniqueness theorem for 4–manifold trisections, this gives a bijection from group trisections modulo isomorphism and a certain stabilization operation to smooth, closed, connected, oriented 4–manifolds modulo diffeomorphism. As a consequence, smooth 4–manifold topology is, in principle, entirely group theoretic. For example, the smooth 4–dimensional Poincaré conjecture can be reformulated as a purely group theoretic statement. *

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Let g and k be integers with $g \geq k \geq 0$. We fix the following groups, described explicitly by presentations:

- $S_0 = \{1\}$ and, for $g > 0$, $S_g = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \ldots [a_g, b_g] \rangle$, i.e. the standard genus g surface group with standard labelled generators. We identify this in the obvious way with $\pi_1(\#^g S^1 \times S^1, *)$.
- $H_0 = \{1\}$ and, for $g > 0$, $H_g = \langle x_1, \ldots, x_g \rangle$, i.e. a free group of rank g with g labelled generators. We identify this in the obvious way with $\pi_1(\natural^g S^1 \times B^2, *)$. Note that, if $g < g'$, then $H_g \subset H_{g'}$.
- $Z_0 = \{1\}$ and, for $k > 0$, $Z_k = \langle z_1, \ldots, z_k \rangle$, i.e. a free group of rank k with k labelled generators. We identify this in the obvious way with $\pi_1(\natural^k S^1 \times B^3, *)$. Again, if $k < k'$ then $Z_k \subset Z_{k'}$.

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Let V denote the set of vertices of a cube, and let E denote the set of edges.

Definition 1 A (g, k)–trisection of a group G is a commutative cube of groups as shown below, such that each homomorphism is surjective and each face is a pushout.

$$\begin{tikzcd}
 & H_g \arrow[r] \arrow[dr] & Z_k \arrow[dr] & \\
S_g \arrow[ur] \arrow[r] \arrow[dr] & H_g \arrow[ur] \arrow[dr] & Z_k \arrow[r] & G \\
 & H_g \arrow[r] \arrow[ur] & Z_k \arrow[ur] &
\end{tikzcd}$$

We label the groups $\{G_v \mid v \in V\}$ and the maps $\{f_e \mid e \in E\}$, so that a trisection of G is the pair $(\{G_v\}, \{f_e\})$. A trisected isomorphism from a trisection $(\{G_v\}, \{f_e\})$ of G to a trisection $(\{G'_v\}, \{f'_e\})$ of G' is a collection of isomorphisms $h_v : G_v \to G'_v$, for all $v \in V$, commuting with the f_e's and f'_e's. A trisected isomorphism is orientation preserving if the isomorphism $h : S_g \to S_g$ induces an isomorphism on the abelianizations $h_* : \mathbb{Z}^{2g} \to \mathbb{Z}^{2g}$ which has determinant $+1$.

Because all maps after the initial three f_e's are pushout maps, a trisection of the group G is determined by these $f_e : S_g \to H_g$. More generally, given any triple of group homomorphisms $\alpha_i : A \to B_i, i = 1, 2, 3$, epimorphisms or not, one can define C_{ij} as the pushout of the maps α_i and α_j and D_i as the pushout of the maps $B_i \to C_{ij}$ and $B_i \to C_{ik}$. It becomes apparent that the $D_i, i = 1, 2, 3$ are canonically isomorphic when one writes down presentations for A and the B_i and then sees what happens. Thus any triple of maps $f_e : S_g \to H_g$ with rank k free pushouts uniquely determines a group trisection. (Even more generally, Peter Teichner has pointed out that in any category with colimits, a triple of morphisms $A \to B_i, i = 1, 2, 3$ determines a cube of pushout maps whose far corner is the colimit of the triple of morphisms.)

In view of this, one could define an *abstract (g, k)–group trisection* as a triple of epimorphisms $f_i : S_g \to H_g$ $(i = 1, 2, 3)$, whose pairwise pushouts are rank k free groups. By taking the colimit, an abstract group trisection then uniquely determines a group trisection of a particular group. This parallels the distinction between an abstract group presentation, which is a list of generators and relators but which doesn't include the group itself in the notation, and a presentation of a particular group G, in which G is identified with the abstract group being presented. In any case, in this paper we work with (g, k)–trisections of a group G.

There is a unique $(0, 0)$–trisection of the trivial group. Figure 1 illustrates a $(3, 1)$–trisection of the trivial group, which we will call "the standard trivial $(3, 1)$–trisection."

Figure 2 illustrates the same diagram more topologically. (For trisections with $g = 1$ and $g = 2$, see the basic 4–manifold trisection examples in [1]; in fact, the 4–dimensional uniqueness results in [6], together with Theorem 5 below, give uniqueness statements for group trisections with $g \leq 2$.)

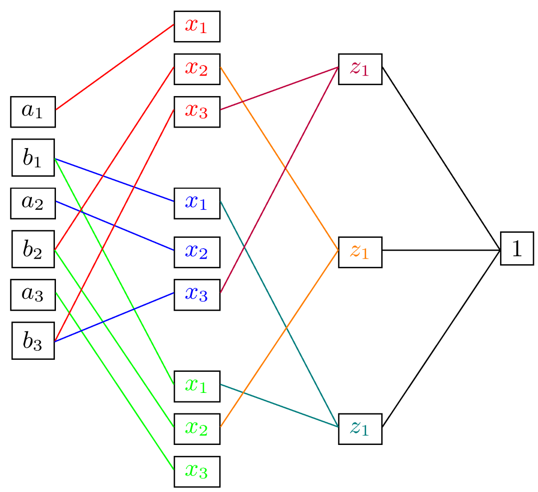

Figure 1: A $(3,1)$–trisection of the trivial group. All maps send generators to generators or to 1; the diagram shows where each map sends each generator, with the understanding that generators not shown to be mapped anywhere are mapped to 1.

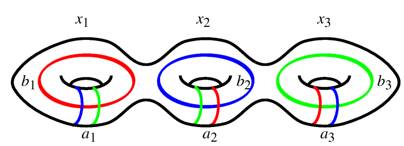

Figure 2: The trivial $(3,1)$–trisection illustrated topologically; each color describes a handlebody filling of the genus 3 surface, so that the curves specify the kernels of the homomorphisms.

Definition 2 Given a (g,k)–trisection $(\{G_v\},\{f_e\})$ of G and a (g',k')–trisection $(\{G'_v\},\{f'_e\})$ of G', there is a natural "connected sum" $(g'' = g + g', k'' = k + k')$–trisection $(\{G''_v\},\{f''_e\})$ of $G'' = G * G'$ defined by first shifting all the indices of the generators for the G'_v's by either g (when $G'_v = S_{g'}$ or $G'_v = H_{g'}$) or k (when $G'_v = Z_{k'}$) and then, for each generator y of G''_v, declaring $f''_e(y)$ to be either $f_e(y)$ or $f'_e(y)$ according to whether y is in G_v or G'_v.

Definition 3 The stabilization of a group trisection is the connected sum of the given trisection with the standard trivial $(3,1)$–trisection. Thus the stabilization of a (g,k)–trisection of G is a $(g+3,k+1)$–trisection of the same group $G = G * \{1\}$.

Definition 4 [1] A (g,k)–trisection of a smooth, closed, oriented, connected 4–manifold X is a decomposition $X = X_1 \cup X_2 \cup X_3$ such that:

- Each X_i is diffeomorphic to $\natural^k S^1 \times B^3$.
- Each $X_i \cap X_j$, with $i \neq j$, is diffeomorphic to $\natural^g S^1 \times B^2$.
- $X_1 \cap X_2 \cap X_3$ is diffeomorphic to $\#^g S^1 \times S^1 = \Sigma_g$.

If X is equipped with a base point p, a based trisection of (X,p) is a trisection with $p \in X_1 \cap X_2 \cap X_3$. A parametrized based trisection of (X,p) is a based trisection equipped with fixed diffeomorphisms (the "parametrizations") from the (X_i,p)'s to $(\natural^k S^1 \times B^3, *)$, from the $(X_i \cap X_j, p)$'s to $(\natural^g S^1 \times B^2, *)$ and from $X_1 \cap X_2 \cap X_3$ to $(\#^g S^1 \times S^1 = \Sigma_g, *)$, where $*$ in each case indicates a standard fixed base point, respected by the standard inclusions $(\#^g S^1 \times S^1 = \Sigma_g, *) \hookrightarrow (\natural^g S^1 \times B^2, *) \hookrightarrow (\natural^k S^1 \times B^3, *)$. A trisected diffeomorphism between trisected 4-manifolds is simply a diffeomorphism that respects the decomposition, and a trisected diffeomorphism is orientation preserving if it preserves orientations on each piece.

Henceforth, all manifolds are smooth, oriented and connected, and all diffeomorphisms preserve orientation. Until further notice, trisected 4–manifolds are closed.

There is an obvious map from the set of parametrized based trisected 4–manifolds to the set of trisected groups, which we will call $\mathcal{G}$; the groups are the fundamental groups of the X_i's and their intersections, after identification with standard models via the parametrizations, and the maps are those induced by inclusions composed with parametrizations. Changing the parametrizations (but respecting orientations) and base point will change the group trisection by an orientation preserving isomorphism of trisected groups, and thus we will also see $\mathcal{G}$ as a map from trisected 4–manifolds to trisected groups up to orientation preserving isomorphism.

The main result of this paper is that $\mathcal{G}$ induces a bijection between trisected 4–manifolds up to orientation preserving trisected diffeomorphism and trisected groups up to orientation preserving trisected isomorphism, and that this bijection respects stabilizations in both categories.

Theorem 5 *There exists a map $\mathcal{M}$ from the set of trisected groups to the set of trisected 4–manifolds such that $\mathcal{M} \circ \mathcal{G}$ is the identity up to orientation preserving trisected diffeomorphism and $\mathcal{G} \circ \mathcal{M}$ is the identity up to orientation preserving trisected isomorphism. The unique $(0,0)$–trisection of $\{1\}$ maps to the unique $(0,0)$–trisection of S^4, the standard $(3,1)$–trisection of $\{1\}$ maps to the standard $(3,1)$–trisection of S^4, and connected sums of group trisections map to connected sums of 4–manifold* *trisections. Thus $\mathcal{M}$ induces a bijection between the set of trisected groups modulo orientation preserving trisected isomorphism and stabilization and the set of smooth, closed, connected, oriented* 4*–manifolds modulo orientation preserving diffeomorphism.*

Though it might not be obvious from a purely group-theoretic point of view, it follows from [1] that every finitely presented group admits a trisection, because every finitely presented group is the fundamental group of a closed, orientable 4–manifold. Even more striking, perhaps, is that by Theorem 5 the collection of trisections of any particular group contains all the complexity of smooth 4–manifolds with the given fundamental group, including not just their homotopy types but also their diffeomorphism types. In particular there is a subset of the trisections of the trivial group corresponding to the countably many exotic smooth structures on a given simply connected topological 4–manifold, e.g. the K3 surface. (To get the full countable collection, it seems likely that g must be unbounded.) An interesting problem is to understand the equivalence relation on group trisections that corresponds to *homeomorphisms* between 4–manifolds.

Considering homotopy 4–spheres, we have

Corollary 6 *The smooth* 4*–dimensional Poincaré conjecture is equivalent to the following statement: "Every* $(3k, k)$*–trisection of the trivial group is stably equivalent to the trivial trisection of the trivial group."*

Proof A $(3k, k)$–trisection of the trivial group gives a $(3k, k)$–trisection of a simply connected 4–manifold. The Euler characteristic of a (g, k)–trisected 4–manifold is $2 - g + 3k$, so in this case we have an Euler characteristic 2 simply connected 4–manifold, i.e. a homotopy S^4. □

One approach to proving the Poincaré conjecture would be to prove first that there is a unique $(3, 1)$–trisection of $\{1\}$, or at least that every $(3, 1)$–trisection of $\{1\}$ gives a 4–manifold diffeomorphic to S^4, and then prove that, for any $(3k, k)$–trisection of $\{1\}$, there is a nontrivial group element in the intersection of the kernels of the three maps $S_g \to H_g$ which can be represented as an embedded curve in the corresponding surface Σ_g. This would give an inductive proof since such an embedded curve would give us a way to decompose the given trisection as a connected sum of lower genus trisections. In fact, this would prove more than the Poincaré conjecture; it would also prove a 4–dimensional analog of Waldhausen's theorem [**?**], to the effect that every trisection of S^4 is a stabilization of the trivial trisection and thus that any two trisections of S^4 of the same genus are isotopic. (This is not quite as strong as Conjecture 3.11 in [5] since [5] deals with *unbalanced* trisections and *unbalanced* stabilizations, in which each 4–dimensional piece X_i is diffeomorphic to some $\natural^{k_i} S^1 \times B^3$ but we do not assume that $k_1 = k_2 = k_3$. The theory of group trisections can naturally be extended to the unbalanced setting.) This strategy would be the exact 4–dimensional parallel to the strategy outlined in [9] for proving (or failing to prove) the 3–dimensional Poincaré conjecture.

Proof of Theorem 5 Given a (g,k)–trisection $(\{G_v\},\{f_e\})$ of G, we will construct $\mathcal{M}(\{G_v\},\{f_e\})$ beginning with $\Sigma_g = \#^g S^1 \times S^1$. For each of the three maps $f_e : S_g \to H_g$, because these are epimorphisms it is a standard fact that there is a diffeomorphism $\phi_e : \Sigma_g \to \partial(\natural^g S^1 \times B^2)$ such that $\imath \circ \phi_e : \Sigma_g \hookrightarrow \natural^g S^1 \times B^2$ induces f_e on π_1. See [4] for a proof; the sketch of the proof is as follows: Note that there is a map, well defined up to homotopy by f_e, from Σ_g to a wedge of g circles. Make this transverse to one point of each circle, not the base point. Then the inverse image of those points is a collection of embedded circles in Σ_g. Add a 2–handle to each circle, and then the new boundary is a collection of 2–spheres. Fill in each with 3–balls resulting in a handlebody.

Each ϕ_e is unique up to post-composing with a diffeomorphism of $\partial(\natural^g S^1 \times B^2)$ which extends over $\natural^g S^1 \times B^2$. To see this, suppose that we have two diffeomorphisms $\phi_e, \phi'_e : \Sigma_g \to \partial(\natural^g S^1 \times B^2)$ such that both $\imath \circ \phi_e$ and $\imath \circ \phi'_e$ induce f_e on π_1. Then in particular the kernels of $\imath \circ \phi_e$ and $\imath \circ \phi'_e$ coincide. So for any properly embedded disk D in $\natural^g S^1 \times B^2$, $\phi'(\phi_e^{-1}(\partial D))$ is a simple closed curve in $\partial(\natural^g S^1 \times B^2)$ which bounds a disk in $\natural^g S^1 \times B^2$ and thus, by Dehn's lemma [8], also bounds an embedded disk. Thus, thinking of ϕ_e and ϕ'_e as defining two handlebody fillings of Σ_g, we see that any simple closed curve that bounds an embedded disk in one handlebody bounds an embedded disk in the other handlebody, and thus the two fillings are diffeomorphic.

Use these three diffeomorphisms to attach three copies of $\natural^g S^1 \times B^2$, crossed with I, to $\partial\Sigma_g \times D^2$ in the standard way, giving a 4–manifold with three boundary components, each presented with a genus g Heegaard splitting. (Note that the cyclic ordering of the three handlebodies is essential to determine the orientation of the resulting 4–manifold, and that this is reflected in our definition of group trisection by the fact that the maps and groups are explicitly labelled by edges and vertices of a standard cube.)

Because each pushout from the initial three maps gives a free group of rank k, we know that the three boundary components mentioned above are closed 3–manifolds with rank k free fundamental groups. It is another well-known fact that each of these 3–manifolds is diffeomorphic to $\#^k S^1 \times S^2$. This follows from Kneser's conjecture (proved by Stallings [10]) that a free product decomposition of the fundamental group of a 3–manifold corresponds to a connected sum decomposition of the manifold, as well as Perelman's proof [7] of the 3–dimensional Poincaré conjecture that shows that no connected summand has trivial fundamental group. A prime connected summand (i.e., one that doesn't decompose further) therefore has fundamental group $\mathbb{Z}$, and a standard argument using the Loop and Sphere Theorems [8] and the Hurewicz theorem shows that an orientable prime 3-manifold with fundamental group $\mathbb{Z}$ must be $S^1 \times S^2$. See [2, Theorem 5.2] for further details.

Any two ways of filling in a connected sum of $S^1 \times S^2$'s with a 4–dimensional 1–handlebody differ by a diffeomorphism of the connected sum, and Laudenbach and Poenaru [3] proved that any such diffeomorphism extends to a diffeomorphism of the handlebody. Thus we can attach a copy $\natural^k S^1 \times B^3$ to each boundary component to produce a closed 4–manifold X which is uniquely determined up to diffeomorphism by this construction. As constructed, X comes with a trisection in which each X_i, $X_i \cap X_j$ and $X_1 \cap X_2 \cap X_3$ is by construction identified with the appropriate model manifold, with a standard base point in $X_1 \cap X_2 \cap X_3$. In other words, we have constructed a based, parametrized trisected 4–manifold uniquely determined up to trisected diffeomorphism by the given trisected group. This is the definition of $\mathcal{M}(\{G_v\}, \{f_e\})$. Note that the parametrizations are not uniquely determined, due to the indeterminacy associated to, first, filling in the 3–dimensional handlebodies associated with the surjections $\Sigma_g \to F_g$ and, second, attaching the 4–dimensional 1–handlebodies using the identification of each of the three closed 3–manifolds with $\#^k S^1 \times S^2$.

We have thus far proved that the map $\mathcal{M}$ is well defined. We now need to show that $\mathcal{M} \circ \mathcal{G}$ and $\mathcal{G} \circ \mathcal{M}$ are the identity maps on appropriate sets up to appropriate equivalences. The map $\mathcal{G}$ simply applies the π_1 functor to all pieces of a based, parameterized trisection of a 4–manifold, so clearly $\mathcal{G} \circ \mathcal{M}$ recovers the original trisected group up to isomorphism (one needs to choose parametrizations to apply $\mathcal{G}$, hence the isomorphism). Similarly, starting with a trisected 4–manifold and applying first $\mathcal{G}$ and then $\mathcal{M}$, the arguments above about the well definedness of $\mathcal{G}(\{G_v\}, \{f_e\})$ also show that the resulting trisected 4–manifold here is diffeomorphic to the initial one.

The main result of [1] is that every smooth, closed, connected, oriented 4–manifold has a trisection, and that any two trisections of the same 4–manifold become isotopic after performing some number of connected sums with the standard $(3, 1)$–trisection of S^4. The connected sum operation and the $(3, 1)$–trisection on the group side are constructed exactly to correspond to stabilization of manifolds via the map $\mathcal{M}$. This shows that $\mathcal{M}$ induces a bijection between trisected groups up to orientation preserving isomorphism and stabilization and oriented 4–manifolds up to orientation preserving diffeomorphism. □

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