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Information and trading targets in a dynamic market equilibrium☆

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ABSTRACT

This paper describes equilibrium interactions between dynamic portfolio rebalancing given a private end-of-day trading target and dynamic trading on long-lived private information. Order-splitting for portfolio rebalancing injects multifaceted dynamics in the market. These include autocorrelated order flow, sunshine trading, endogenous learning, and short-term speculation. The model has testable implications for intraday patterns in volume, liquidity, price volatility, order-flow autocorrelation, differences between informed-investor and rebalancer trading strategies, and for how these patterns comove with trading-target volatility and other market conditions.

1. Introduction

Trading via dynamic order-splitting algorithms is a pervasive fact in today's financial markets.[1] Informed investors use dynamic order-splitting to increase trading profits by slowing the public revelation of their private information. Order-splitting is not, however, limited to informed investors. Less informed investors — index mutual funds and comparatively more passive pensions and insurance companies — rely on order-splitting to minimize trading costs for hedging and portfolio rebalancing. As described in O'Hara (2015), portfolio managers transmit *parent orders* — specifying the total amount of a security to be bought

[1] Pension & Investments (2007) reported that in a survey of leading institutional investors, 72% said they used order-execution algorithms. Anecdotal evidence suggests that the use of order-execution algorithms has grown further in subsequent years. Order-execution algorithms are different from computer-based market making, latency arbitrage, and other high-frequency trading strategies.

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or sold over a fixed trading horizon — to brokers who use computer algorithms to break parent orders into sequences of smaller *child orders*.[2] While dynamic informed trading has been studied extensively (see, e.g., Kyle, 1985), order-splitting for portfolio rebalancing is less understood.

Our paper is the first to model a market equilibrium with dynamic trading given both long-lived private information and portfolio rebalancing. We consider a multi-period (Kyle, 1985) market in which there are two strategic investors with different trading motives who each follow optimal dynamic trading strategies. One investor is a standard Kyle strategic informed investor with long-lived private information. The other investor is a strategic portfolio rebalancer who trades over multiple rounds to minimize the cost of hitting a private parent terminal trading target.

Our model lets us investigate the economic motivations for dynamic order-splitting for portfolio rebalancing and its equilibrium effects. Our analysis leads to three main insights:

- Dynamic rebalancing and dynamic informed trading are structurally different from each other. Child orders for dynamic rebalancing, like informed trading, are timed to reduce the price-impact cost of trading, but rebalancing orders also have components driven by sunshine trading, endogenous learning, and constrained short-term speculative trading.
- Dynamic rebalancing affects the mix of information and trading noise in the arriving order flow and, thereby, affects equilibrium price discovery and liquidity provision.[3] There are direct effects given the mixture of noise and information in the rebalancer's parent trading target and also because the rebalancer learns endogenously through the trading process itself. In particular, the rebalancer can filter the aggregate order flow better than market makers by incorporating his knowledge about his own past order submissions. In addition, there are indirect effects due to the equilibrium response of the informed investor to the rebalancer's trading, i.e., how aggressively she trades on her private information given informational competition with the rebalancer and how she exploits additional noise in prices due to price pressure from the rebalancer's orders.
- Trading constraints induce autocorrelation in the aggregate order flow. In particular, dynamic rebalancing based on a parent target leads to autocorrelated child order flow that is different from unpredictable informed-investor orders and serially independent noise-trader orders. Autocorrelated rebalancing orders lead to a type of sunshine trading with market makers trying to forecast the remaining future latent trading demand of the rebalancer since predictable orders have no price impact.

In addition, an extensive battery of numerical experiments identifies a number of testable implications of dynamic rebalancing:

- Dynamic rebalancing induces *U*-shaped intraday patterns in expected trading volume, price volatility, and order-flow autocorrelation and twists the price impact of order flow over the day, where the magnitude of these intraday patterns is increasing in the volatility of the rebalancing target. Thus, daily time-variation in the volatility of rebalancing targets should induce comovement in a cross-section of multiple intraday price and volume patterns.
- Rebalancer and informed-investor orders tend to become negatively correlated over time as the informed investor trades against price pressure from past rebalancer orders.

Our analysis integrates two literatures on pricing and trading. The first literature is about price discovery. Kyle (1985) describes equilibrium pricing and dynamic trading in a market with a single investor with long-lived private information. Subsequent work by Holden and Subrahmanyam (1992); Foster and Viswanathan (1994, 1996); and Back et al. (2000) allows for multiple informed investors with long-lived information. Our model extends Foster and Viswanathan, (1996) — who were the first to model a dynamic equilibrium with multiple investors with heterogeneous information and to solve the "forecasting the forecasts of others" problem — to allow for trading-target constraints. Given our interest in information aggregation and intraday order-flow dynamics, the Kyle set-up lets us abstract from the arms race for speed (Hoffmann, 2014; Biais et al., 2015), intermediation chains linking multiple market makers (Weller, 2013), limit order cancelation and flickering quotes (Hasbrouck and Saar, 2009; Baruch and Glosten, 2013), market fragmentation and latency (Kumar and Seppi, 1994; Menkveld et al., 2017), and other millisecond-level high-frequency trading (HFT) phenomena.

A second literature studies optimal dynamic order execution for uninformed investors with trading targets. This includes (Bertsimas and Lo, 1998; Almgren and Chriss, 1999; 2000; Gatheral and Schied, 2011; Engle et al., 2012; Predoiu et al., 2011; Boulatov et al. 2016) as well as models of predatory trading in Brunnermeier and Pedersen (2005) and Carlin et al. (2007). This research takes the price impact function for orders as an exogenous model input. In contrast, we model optimal order execution in an equilibrium setting that endogenizes the effect of dynamic rebalancing on pricing.[4] A partial equilibrium analysis misses these equilibrium effects. In addition, unlike in

[2] Keim and Madhavan (1995) is the first empirical study of dynamic order-splitting by institutional investors. Recently, van Kervel and Menkveld (2018) estimate an average of 156 child trades per parent order for four large institutions trading on Nasdaq OMX. Korajczyk and Murphy (2018) estimate an average of between 327 and 604 child orders per large parent order depending on whether the parent order is nonstressful (lower three quartiles of large trades) or stressful (top quartile) for Canadian equities. See (Johnson, 2010) for more on specific dynamic trading algorithms. The (Securities and Exchange Commission, 2010) report also discusses the role of trading algorithms in the current market landscape.

[3] Uninformed trading noise plays a critical role in markets with adverse selection. See Akerlof, (1970); Grossman and Stiglitz, (1980); Kyle, (1985); and Glosten and Milgrom (1985).

[4] In our model, order flow has a price impact due to adverse selection. Alternatively, price impacts can be due to inventory costs and imperfect competition in liquidity provision (see Choi et al., 2018).

the predatory trading models, our rebalancer's trading target is not publicly known ex ante, but is random and private information. This is arguably the usual situation on normal trading days, as opposed to special days (e.g., futures rolls and index reconstitutions) on which the direction of rebalancing is predictable.

Models combining both informed trading and optimized uninformed rebalancing have largely been restricted to static settings or to multi-period settings with short-lived information and/or exogenous restrictions on rebalancer trading. Admati and Pfleiderer (1988) study a series of repeated one-period trading rounds with short-lived information and uninformed discretionary traders who only trade once but who decide when to time their trading. An exception is Seppi (1990) who models an informed investor and a strategic uninformed investor with a trading target who both can trade dynamically. He solves for separating and partial pooling equilibria with upstairs block trading for a restricted set of model parameterizations.

Our paper is related to Degryse et al. (2014). Both papers model dynamic order-splitting by an uninformed rebalancer. Consequently, both models have autocorrelated order flows. Order-flow autocorrelation is empirically significant but absent in previous Kyle models.[5] However, there are two differences between our model and Degryse et al. (2014). First, informed investors in Degryse et al. (2014) have short-lived private information; i.e., they only have one chance to trade on intraday signals before they become public. In contrast, our informed investor trades on long-lived information over multiple intraday time periods. Consequently, it is harder to distinguish cumulative order imbalances due to rebalancing from imbalances due to information trading. This reduces the value of sunshine trading. Second, our rebalancer orders depend adaptively on the realized path of aggregate order flow over the day in addition to the trading target. Adaptive trading is absent in Degryse et al. (2014) where the rebalancer trades deterministically over time to reach his target. In particular, our rebalancer learns endogenously about the informed investor's information, because he can filter the aggregate order flow better than the market makers. Our analysis is possible because we adapt the approach of Foster and Viswanathan (1996) to circumvent the large state-space problem mentioned in Degryse et al. (2014).

Our analysis includes three types of sunshine trading. The first is the previously discussed zero-price impact of predictable orders. In our model and in Degryse et al. (2014), predictable orders have no incremental information content and, thus, absent frictions in the supply of liquidity, no price impact.[6] A second type of sunshine trading exploits predictable market dynamics as liquidity is temporarily depleted and then replenished over time (see, e.g., Predoiu et al., 2011). In our model, the informed-investor trading corrects price pressure from past rebalancer orders, which lowers the rebalancer's subsequent trading costs. The third type of sunshine trading exploits predictable intraday variation in liquidity.

2. Model

We model a multi-period discrete-time market for a risky stock. A trading day is normalized to the interval [0,1] during which there are $N \in \mathbb{N}$ time points at which trading occurs where $\Delta := \frac{1}{N} > 0$ is the time step. As in Kyle (1985), the stock's terminal value $\tilde{v}$ becomes publicly known at time $N+1$ after the market closes at the end of the day. The value $\tilde{v}$ is normally distributed with mean zero and volatility $\sigma_{\tilde{v}} > 0$. Additionally, there is a money market account that pays a zero interest rate.

Four types of investors trade in the model:

- An informed investor (who we call a *hedge fund portfolio manager*) knows the terminal stock value $\tilde{v}$ at the beginning of trading and has zero initial positions in the stock and the money market account. The hedge fund manager is risk-neutral and maximizes the expected value of her fund's final wealth. The hedge fund's order for the stock at time n, $n = 1, \dots, N$, is denoted by $\Delta\theta_n^I$ where θ_n^I is its accumulated total stock position at time n with $\theta_0^I := 0$ initially.
- A constrained investor (who we call the *rebalancer*) needs to rebalance his portfolio by buying or selling stock to reach a parent terminal trading-target constraint $\tilde{a}$ on his final stock position θ_N^R by the end of the trading day. For example, he might be a portfolio manager for a large index fund or a passive pension plan or an insurance company, who needs to rebalance his portfolio or to respond to fund inflows/outflows. The parent target $\tilde{a}$ is private knowledge of the rebalancer. In practice, such investors trade dynamically using optimal order-execution algorithms to minimize their trading costs. He starts the day with zero initial positions in the stock ($\theta_0^R := 0$) and his money market account.[7] The target $\tilde{a}$ is jointly normally distributed with the terminal stock value $\tilde{v}$ and has a mean of zero, a volatility $\sigma_{\tilde{a}} > 0$, and a correlation $\rho \in [0, 1]$ with $\tilde{v}$. When ρ is 0, the rebalancer is initially uninformed. If $\rho > 0$, we think of the rebalancer as being initially informed about $\tilde{v}$ but subject to random binding non-public risk limits.[8] Importantly, our rebalancer rationally understands the extent to which he is uninformed.[9] The rebalancer is risk-neutral and maximizes the expected value of his final wealth subject to the parent-target constraint. The rebalancer's child order for the stock at time n,

[5] For early empirical evidence on order-flow autocorrelation in equity markets, see (Hasbrouck, 1991a; 1991b). More recently, Brogaard et al. (2016) find autocorrelation in orders from non-HFT investors (which is our focus) as well as in HFT orders.

[6] Predictable sunshine trading is statistically inferred in our model rather than publicly preannounced as in Admati and Pfleiderer (1991).

[7] This normalization simplifies the notation for their objective functions but is without loss of generality. Both the hedge fund and the rebalancer finance their stock trading by borrowing/lending.

[8] The fact that the terminal value $\tilde{v}$ is measured in dollars while the trading target $\tilde{a}$ is measured in shares is not problematic for $\tilde{v}$ and $\tilde{a}$ being correlated random variables.

[9] Alternatively, if some investors trade under the mistaken belief that they are informed, but the signals they condition on are in fact just noise, then their orders should have the same functional form as actual informed-investor orders (see Kyle and Obizhaeva, 2016). In our model, informed investors and rebalancers trade differently because their trading motives are different.

$n = 1, \ldots, N$, is denoted by $\Delta\theta_n^R$, and the terminal constraint requires $\Delta\theta_N^R = \tilde{a} - \theta_{N-1}^R$ at time N.

- Noise traders (who we think of as small non-strategic retail investors) submit net stock orders at times n, $n = 1, \ldots, N$, that are exogenous Brownian motion increments Δw_n. These increments are normally distributed with zero means and variances $\sigma_w^2\Delta$ for a constant $\sigma_w > 0$ and are independent of $\tilde{v}$ and $\tilde{a}$.
- Competitive risk-neutral market makers observe the aggregate net order flow y_n at times n, $n = 1, \ldots, N$, where

$$y_n := \Delta\theta_n^I + \Delta\theta_n^R + \Delta w_n, \quad y_0 := 0. \tag{1}$$

Given competition and risk-neutrality, market makers clear the market (i.e., trade $-y_n$) at a stock price

$$p_n = \mathbb{E}[\tilde{v} \,|\, y_1, \ldots, y_n], \quad n = 1, 2, \ldots, N, \quad p_0 := 0. \tag{2}$$

In the past, market makers were dealers on the floor of an exchange. Today, market making is performed by high-frequency trading firms running algorithms on servers colocated near an exchange's market-crossing engine. These market-making algos process order-flow information in real-time when setting prices.

The presence of the rebalancer with a parent trading constraint is the main difference between our model and Kyle (1985) and the multi-agent settings in Holden and Subrahmanyam (1992) and Foster and Viswanathan (1994, 1996). In particular, at each time n, the rebalancer has a latent demand to trade the remaining $\tilde{a} - \theta_{n-1}^R$ shares over the rest of the day. Previous microstructure theory says very little about markets with daily latent trading demand. As we shall see, this latent trading demand produces new stylized market features such as autocorrelated order flow.

The hedge fund trades strategically to maximize its expected terminal wealth

$$\mathbb{E}\left[\theta_N^I(\tilde{v} - p_N) + \theta_{N-1}^I \Delta p_N + \ldots + \theta_1^I \Delta p_2 \,\middle|\, \tilde{v}\right]$$
$$= \sum_{n=1}^{N} \mathbb{E}\left[(\tilde{v} - p_n)\Delta\theta_n^I \,\middle|\, \tilde{v}\right], \tag{3}$$

where $\Delta p_n := p_n - p_{n-1}$. Although at time $n = 1$, the hedge fund only knows $\tilde{v}$ in (3), it knows that its orders at later times $n \in \{2, \ldots, N\}$ will also be able to incorporate information about the then-past aggregate orders $y_1, \ldots, y_{n-1}$. Thus, the hedge fund maximizes (3) over measurable functions $\Delta\theta_1^I$ in the sigma algebra $\sigma(\tilde{v})$ induced by $\tilde{v}$ at time $n = 1$ and measurable functions $\Delta\theta_n^I$ in the sigma algebras $\sigma(\tilde{v}, y_1, \ldots, y_{n-1})$ at times $n \in \{2, \ldots, N\}$ where, as in Kyle (1985), the contemporaneous aggregate order flow y_n is not publicly known at time n but is publicly known starting at time $n + 1$.[10]

The rebalancer also trades strategically to maximize his expected terminal wealth

$$\mathbb{E}\left[\tilde{a}(\tilde{v} - p_N) + \theta_{N-1}^R \Delta p_N + \ldots + \theta_1^R \Delta p_2 \,\middle|\, \tilde{a}\right]$$
$$= \frac{\rho\sigma_{\tilde{v}}}{\sigma_{\tilde{a}}}\tilde{a}^2 - \sum_{n=1}^{N} \mathbb{E}\left[(\tilde{a} - \theta_{n-1}^R)\Delta p_n \,\middle|\, \tilde{a}\right], \tag{4}$$

but with the difference that now there is the terminal rebalancing constraint $\theta_N^R = \tilde{a}$ relative to his initial position $\theta_0^R = 0$. The equality in (4) follows from $p_N = \sum_{n=1}^{N} \Delta p_n$, $p_0 = 0$, and $\mathbb{E}[\tilde{v} \,|\, \tilde{a}] = \frac{\rho\sigma_{\tilde{v}}}{\sigma_{\tilde{a}}}\tilde{a}$. The rebalancer's problem in (4) is conditioned on the rebalancer's initial private information (here, the target $\tilde{a}$), but the rebalancer also understands that his later orders can be conditioned on future aggregate order flows. Thus, (4) is maximized over measurable functions $\Delta\theta_1^R$ in the sigma algebra $\sigma(\tilde{a})$ at time 1 and $\Delta\theta_n^R$ in $\sigma(\tilde{a}, y_1, \ldots, y_{n-1})$ at times $n \in \{2, \ldots, N\}$.

There are two points to note here: First, the information sets of the hedge fund $\sigma(\tilde{v}, y_1, \ldots, y_{n-1})$, the rebalancer $\sigma(\tilde{a}, y_1, \ldots, y_{n-1})$, and market makers $\sigma(y_1, \ldots, y_{n-1}, y_n)$ at time $n \in \{1, 2, \ldots, N\}$ do not nest. Second, Appendix A shows that in equilibrium the hedge fund's problem (3) and the rebalancer's problem (4) are both quadratic in the investors' respective orders.

Definition 1. A *Bayesian Nash* equilibrium is a collection of functions $\{\theta_n^I, \theta_n^R, p_n\}_{n=1}^N$ such that:

(i) Given $\{\theta_n^R, p_n\}_{n=1}^N$, the strategy $\{\theta_n^I\}_{n=1}^N$ maximizes the hedge fund's objective (3).
(ii) Given $\{\theta_n^I, p_n\}_{n=1}^N$, the strategy $\{\theta_n^R\}_{n=1}^N$ maximizes the rebalancer's objective (4).
(iii) Given $\{\theta_n^I, \theta_n^R\}_{n=1}^N$, the pricing rule $\{p_n\}_{n=1}^N$ satisfies (2).[11]

We construct a Bayesian Nash equilibrium with the following linear structure: First, the rebalancer's and hedge fund's optimal trading strategies are[12]

$$\Delta\theta_n^R = \beta_n^R(\tilde{a} - \theta_{n-1}^R) + \alpha_n^R q_{n-1}, \quad \theta_0^R := 0, \tag{5}$$

$$\Delta\theta_n^I = \beta_n^I(\tilde{v} - p_{n-1}), \quad \theta_0^I := 0, \tag{6}$$

where $\{\beta_n^I, \beta_n^R, \alpha_n^R\}_{n=1}^N$ are constants with $\beta_N^R = 1$ and $\alpha_N^R = 0$ and the process q_n is the market makers' expectation $q_n = \mathbb{E}[\tilde{a} - \theta_n^R \,|\, y_1, \ldots, y_n]$ of the rebalancer's latent trading demand $\tilde{a} - \theta_n^R$ for the rest of the day conditional on the history of aggregate order flows up through time n. The rebalancer and hedge fund are not restricted to use linear strategies, but they optimally choose linear strategies in the equilibrium we construct.

[10] Alternatively, we can require $\Delta\theta_n^I$ to be in the sigma algebra $\sigma(\tilde{v}, p_1, \ldots, p_{n-1})$ and then use the one-to-one mapping between prices p_n and aggregate order flows y_n in Definition 1 below to infer the aggregate order flows. This approach is taken in, e.g., Back (1992). Since in equilibrium the orders $y_1, \ldots, y_{n-1}$ can be inferred from the prices $p_1, \ldots, p_{n-1}$ provided that $\lambda_1, \ldots, \lambda_{n-1}$ are non-zero and vice versa, the sigma algebras $\sigma(\tilde{v}, y_1, \ldots, y_{n-1})$ and $\sigma(\tilde{a}, y_1, \ldots, y_{n-1})$ are equivalent to $\sigma(\tilde{v}, p_1, \ldots, p_{n-1})$ and $\sigma(\tilde{a}, p_1, \ldots, p_{n-1})$. However, our model simply assumes that aggregate order flows are directly observable to non-market-makers via high-speed market data-feeds with a one-period lag.

[11] The Doob-Dynkin lemma clarifies Definition 1: For any random variable B and any $\sigma(B)$-measurable random variable A, there is a deterministic function f such that $A = f(B)$. Therefore, we can write $\theta_n^R = f_n^R(\tilde{a}, y_1, \ldots, y_{n-1})$, $\theta_n^I = f_n^I(\tilde{v}, y_1, \ldots, y_{n-1})$, and $p_n = f_n^p(y_1, \ldots, y_n)$ for three deterministic functions f_n^R, f_n^I, and f_n^p. The functions f_n^R, f_n^I, and f_n^p are fixed whereas the realization of the aggregate order-flow variables $y_1, \ldots, y_n$ vary with the controls θ^I and θ^R.

[12] If an additional $\alpha_n^I q_{n-1}$ term is included in the hedge fund's strategy in (6), then $\alpha_n^I = 0$ in equilibrium. Contact the authors for a proof of this result.

Second, the q_n process in (5) is a structural consequence of the rebalancing constraint in equilibrium. Much like p_n gives the market-maker beliefs about the stock valuation, q_n gives the market-maker beliefs at time n about how much the rebalancer still needs to trade to reach his parent target. The presence of q_n in (5) means that the rebalancer's orders are not limited to be deterministic functions of his target $\tilde{a}$. Rather, they can depend adaptively on the prior order-flow history, which is in contrast to the deterministic rebalancer orders in Degryse et al. (2014). It follows from (5) that the market makers' expectation at time $n-1$ of the rebalancer's next order at time n is

$$\mathbb{E}[\Delta\theta_n^R \,|\, y_1, \dots, y_{n-1}] = (\alpha_n^R + \beta_n^R) q_{n-1}. \tag{7}$$

Consequently, the aggregate order flow is autocorrelated in this market:[13]

$$\begin{aligned}\mathbb{E}[y_n \,|\, y_1, \dots, y_{n-1}] &= \mathbb{E}[\Delta\theta_n^I + \Delta\theta_n^R + \Delta w_n \,|\, y_1, \dots, y_{n-1}] \\ &= (\alpha_n^R + \beta_n^R)\, q_{n-1}.\end{aligned} \tag{8}$$

The dynamics of q_n are[14]

$$\begin{aligned}\Delta q_n :&= \mathbb{E}[\tilde{a} - \theta_n^R \,|\, y_1, \dots, y_n] - q_{n-1} \\ &= \mathbb{E}[\tilde{a} - \theta_n^R \,|\, y_1, \dots, y_{n-1}] \\ &\quad + r_n(y_n - \mathbb{E}[y_n \,|\, y_1, \dots, y_{n-1}]) - q_{n-1} \\ &= \mathbb{E}[-\Delta\theta_n^R \,|\, y_1, \dots, y_{n-1}] \\ &\quad + r_n(y_n - \mathbb{E}[y_n \,|\, y_1, \dots, y_{n-1}]) \\ &= r_n y_n - (1 + r_n)(\alpha_n^R + \beta_n^R) q_{n-1},\end{aligned} \tag{9}$$

for $q_0 := 0$ and constants $\{r_n\}_{n=1}^N$.

Third, the pricing rule in our linear equilibrium has dynamics

$$\begin{aligned}\Delta p_n &= \lambda_n\big(y_n - \mathbb{E}[y_n \,|\, y_1, \dots, y_{n-1}]\big) \\ &= \lambda_n\big(y_n - (\alpha_n^R + \beta_n^R) q_{n-1}\big),\end{aligned} \tag{10}$$

for $n = 1, \dots, N$ where $\{\lambda_n\}_{n=1}^N$ are constants.[15] The price at time n is not affected by the part of the order flow at time n that is predictable given past orders. Thus, the $(\alpha_n^R + \beta_n^R) q_{n-1}$ term in (10) represents a type of predictable sunshine trading.

Optimal trading for portfolio rebalancing reflects a number of considerations: First, the rebalancer needs to reach his parent trading target $\tilde{a}$ at time N. Second, he wants to reach this target at the lowest cost possible. Cost minimization occurs through several channels:

- The rebalancer splits up his child orders to spread their price impact over time taking into account intraday patterns of the price-impact coefficients λ_n.
- The rebalancer takes advantage of sunshine trading. Early orders signal predictable future orders at later dates, which, from (10), have no price impact.
- The rebalancer trades strategically on information about the stock value $\tilde{v}$ to reduce his costs and even, sometimes, to earn a trading profit. If $\rho > 0$, the rebalancer starts out with private stock-valuation information. However, even if the rebalancer is initially uninformed about $\tilde{v}$ (i.e., $\rho = 0$), he still learns information endogenously over time via the trading process (see (12) below).
- The rebalancer reduces his trading costs using the fact that, on average, the hedge fund trades against price pressure induced by the rebalancer's past orders. If, for example, early uninformed rebalancer buy orders raise prices, then, in expectation, the hedge fund should buy less/sell more in the future, thereby putting downward pressure on later prices which, in turn, reduces the expected cost of subsequent rebalancer buying.

Despite the complexity of the multiple drivers of rebalancing trading, the rebalancer's equilibrium orders take the simple linear form in (5). To gain intuition, we rearrange the rebalancer's order at time n from (5) as follows:

$$\Delta\theta_n^R = (\alpha_n^R + \beta_n^R)\, q_{n-1} + \beta_n^R\, (\tilde{a} - \theta_{n-1}^R - q_{n-1}). \tag{11}$$

The first component, $(\alpha_n^R + \beta_n^R) q_{n-1}$, as noted in (7), is the market makers' expectation of the rebalancer's order at time n. From the sunshine-trading property in (10), this amount is traded with no price impact at time n. The second component, $\beta_n^R(\tilde{a} - \theta_{n-1}^R - q_{n-1})$, in (11) is due to two effects: First, $\tilde{a} - \theta_{n-1}^R - q_{n-1}$ is mechanically the amount the rebalancer still needs to trade beyond the market makers' expectation of his remaining latent trading demand in order to reach his parent target $\tilde{a}$. Second, $\tilde{a} - \theta_{n-1}^R - q_{n-1}$ summarizes the private information of the rebalancer provided that the lagged rebalancer strategy coefficients $\beta_1^R, \dots, \beta_{n-1}^R$ are all different from 1. The proviso about the rebalancer coefficients is a knife-edge technical condition that ensures information about $\tilde{a}$ is not lost when θ_{n-1}^R is subtracted from $\tilde{a}$.[16] Given this proviso, $\tilde{a} - \theta_{n-1}^R - q_{n-1}$ is informative about two factors that allow the rebalancer to speculate on future price changes. The first is current stock-price misvaluation after trading at time $n-1$ in the

[13] The second equality in (8) follows from i) the independence of $\tilde{v} - p_{n-1}$, and, thus, $\Delta\theta_n^I$ from (6), and the past aggregate order flows, ii) the assumption that the noise-trader orders are zero–mean, independent, and identically distributed over time, and iii) the expression for expected rebalancer orders in (7).

[14] The second equality in (9) follows from the definition of q_n and the projection theorem where r_n is a projection coefficient. The third equality follows from $\mathbb{E}[\tilde{a} - \theta_n^R | y_1, \dots, y_{n-1}] = \mathbb{E}[\tilde{a} - \theta_{n-1}^R - \Delta\theta_n^R | y_1, \dots, y_{n-1}] = q_{n-1} - \mathbb{E}[\Delta\theta_n^R | y_1, \dots, y_{n-1}]$. The fourth equality follows from (7) and (8).

[15] The first equality in (10) follows because conditional expectations are linear projections given the jointly Gaussian structure of the linear equilibrium. In particular, the projection theorem is used to update price p_n relative to price p_{n-1} given the innovation in the aggregate order flow y_n relative to its expectation given past orders. The second equality follows from (8).

[16] We require that the equilibrium paths produce $\sigma(\tilde{a}, y_1, \dots, y_{n-1}) = \sigma(\tilde{a} - \theta_{n-1}^R - q_{n-1}, y_1, \dots, y_{n-1})$. For $n = 2$, we have $\tilde{a} - \theta_1^R = \tilde{a} - \beta_1 \tilde{a}$ and the desired property holds when $\beta_1 \neq 1$. For $n = 3$, we have $\tilde{a} - \theta_2^R = \tilde{a} - \tilde{a}(\beta_2(1 - \beta_1) + \beta_1) - \alpha_2 q_1$ and the desired property holds when $\beta_1 \neq 1$ and $\beta_2 \neq 1$. The general case for n arbitrary is similar.

market:[17]

$$\begin{aligned}&\mathbb{E}[\tilde{v}-p_{n-1}\,|\,\tilde{a},y_1,\dots,y_{n-1}]\\&=\mathbb{E}[\tilde{v}-p_{n-1}\,|\,\tilde{a}-\theta^R_{n-1}-q_{n-1},y_1,\dots,y_{n-1}]\\&=\mathbb{E}[\tilde{v}-p_{n-1}\,|\,\tilde{a}-\theta^R_{n-1}-q_{n-1}].\end{aligned}\tag{12}$$

In general, $\tilde{a}-\theta^R_{n-1}-q_{n-1}$ is informative about $\tilde{v}-p_{n-1}$ at times $n-1\geq 2$, even if $\rho=0$ (i.e., $\tilde{a}$ and $\tilde{v}$ are ex ante independent), because knowledge about his own past orders lets the rebalancer filter the prior order-flow history to learn about $\tilde{v}$ better than the market makers. This dynamic learning is absent from deterministic rebalancing as in Degryse et al. (2014). The second speculative factor is that $\tilde{a}-\theta^R_{n-1}-q_{n-1}$ is also informative about forecast errors in market-maker sunshine-trading expectations $(\alpha^R_k+\beta^R_k)q_{k-1}$ for dates $k\geq n$ given that

$$\begin{aligned}&\mathbb{E}[q_k-\mathbb{E}[q_k|y_1,\dots,y_{n-1}]\,|\,\tilde{a},y_1,\ \dots y_{n-1}]\\&=\mathbb{E}[q_k-\mathbb{E}[q_k\,|\,y_1,\dots,y_{n-1}]\,|\,\tilde{a}-\theta^R_{n-1}-q_{n-1}]\end{aligned}\tag{13}$$

which, via (10), lets the rebalancer forecast the next price p_n at time n and also subsequent prices p_k at $k>n$. The predictability of the order-flow impacts on these prices is important — in addition to the predictability of $\tilde{v}$ — because the rebalancer cannot hold stock positions to time $N+1$ and liquidate them at $\tilde{v}$. Rather, his speculative positions must be liquidated at endogenous future market prices at time N or earlier to satisfy the parent target $\tilde{a}$ at time N.

Turning to the informed investor, the term $\tilde{v}-p_{n-1}$ in (6) plays two roles in the hedge fund's strategy: It is private information about both the stock value and also, in equilibrium, about the rebalancer's remaining latent trading demand $\tilde{a}-\theta^R_{n-1}$:[18]

$$\begin{aligned}&\mathbb{E}[\tilde{a}-\theta^R_{n-1}\,|\,\tilde{v},y_1,\dots,y_{n-1}]\\&=q_{n-1}+\mathbb{E}[\tilde{a}-\theta^R_{n-1}-q_{n-1}\,|\,\tilde{v}-p_{n-1},y_1,\dots,y_{n-1}]\\&=q_{n-1}+\mathbb{E}[\tilde{a}-\theta^R_{n-1}-q_{n-1}\,|\,\tilde{v}-p_{n-1}].\end{aligned}\tag{14}$$

Summary: There are three qualitative ways in which the equilibrium structure of a market changes with order-splitting from dynamic portfolio rebalancing. First, rebalancer child orders are structurally different from informed-investor orders. For example, the rebalancer orders in (5) have a two-factor structure depending on q_{n-1} and $\tilde{a}-\theta^R_{n-1}$ whereas the informed-investor orders in (6) have a one-factor structure depending on $\tilde{v}-p_{n-1}$. Second, aggregate order flow becomes autocorrelated. Third, the aggregate order flow now has two components, one predictable and one a random innovation. Only the latter has a price impact.

3. Equilibrium

In this section we give sufficient conditions for a linear Bayesian Nash equilibrium as in (5) through (10). Our analysis extends the logic of Foster and Viswanathan (1996) to allow for a trading constraint. Their approach solves the "forecasting the forecasts of others" problem when showing deviations from equilibrium strategies are suboptimal. Appendix A presents the analysis in greater detail.

To begin, consider a set of possible candidate values for the equilibrium constants

$$\lambda_n,r_n,\beta^R_n,\alpha^R_n,\beta^I_n,\quad n=1,\dots,N,\tag{15}$$

with

$$\beta^R_1\neq 1,\dots,\beta^R_{N-1}\neq 1,\tag{16}$$

$$\beta^R_N=1,\quad \alpha^R_N=0.\tag{17}$$

The restrictions in (16) for times $1,\dots,N-1$ are the technical proviso discussed in regards to the representation of the rebalancer's information in (12), and the restrictions in (17) at time N follow because the rebalancer must reach his target $\tilde{a}$ after his last round of trade. Given a set of candidate constants (15)–(17), we define a system of "hat" price and order-flow processes

$$\Delta\hat{\theta}^I_n:=\beta^I_n(\tilde{v}-\hat{p}_{n-1})\quad \hat{\theta}^I_0:=0,\tag{18}$$

$$\Delta\hat{\theta}^R_n:=\beta^R_n(\tilde{a}-\hat{\theta}^R_{n-1})+\alpha^R_n\hat{q}_{n-1},\quad \hat{\theta}^R_0:=0,\tag{19}$$

$$\hat{y}_n:=\Delta\hat{\theta}^I_n+\Delta\hat{\theta}^R_n+\Delta w_n,\quad \hat{y}_0:=0,\tag{20}$$

$$\Delta\hat{p}_n:=\lambda_n(\hat{y}_n-(\alpha^R_n+\beta^R_n)\hat{q}_{n-1}),\quad \hat{p}_0:=0,\tag{21}$$

$$\Delta\hat{q}_n:=r_n\hat{y}_n-(1+r_n)(\alpha^R_n+\beta^R_n)\hat{q}_{n-1},\quad \hat{q}_0:=0,\tag{22}$$

which denote the processes that agents conjecture that other agents follow. In equilibrium, conjectured beliefs must be correct in that $p_n=\hat{p}_n$ (the price process is the conjectured price process), $\theta^R_n=\hat{\theta}^R_n$ (the rebalancer orders follow the conjectured strategy), etc. The conjectured processes (18)–(22) make problems (3) and (4) analytically tractable in that the hedge-fund and rebalancer problems can both be described with low-dimensional state variable processes (see (35) and (42) below).

The conjectured system $\{\Delta\hat{\theta}^I_n,\Delta\hat{\theta}^R_n,\hat{y}_n,\Delta\hat{p}_n,\Delta\hat{q}_n\}$ is fully specified (autonomous) by the coefficients (15). Given the zero-mean and joint normality of $\tilde{v}$, $\tilde{a}$, and w, the conjectured system (18)–(22) is zero-mean and jointly normal. The variances and covariance for the conjectured dynamics over time are denoted[19]

$$\Sigma^{(1)}_n:=\mathbb{V}\big[\tilde{a}-\hat{\theta}^R_n-\hat{q}_n\big],\tag{23}$$

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[17] The first equality in (12) follows from $q_{n-1},\theta^R_{n-1}\in\sigma(\tilde{a},y_1,\dots,y_{n-1})$, which produces $\sigma(\tilde{a},y_1,\dots,y_{n-1})=\sigma(\tilde{a}-\theta^R_{n-1}-q_{n-1},y_1,\dots,y_{n-1})$ given the proviso about the rebalancer strategy coefficients and using (5) to compute θ^R_{n-1}. The independence, given multivariate normality, between $\tilde{v}-p_{n-1}$ and $(y_1,\dots,y_{n-1})$ and between $\tilde{a}-\theta^R_{n-1}-q_{n-1}$ and $(y_1,\dots,y_{n-1})$ allow us to discard $(y_1,\dots,y_{n-1})$ when computing $\mathbb{E}[\tilde{v}-p_{n-1}\,|\,\tilde{a}-\theta^R_{n-1}-q_{n-1},y_1,\dots,y_{n-1}]$ in the second equality in (12). As a practical matter, the strategy coefficient proviso was never relevant in the numerical analysis presented in Section 4.

[18] The logic for (14) is similar to the logic for (12) in footnote 17. The only difference is that no proviso is needed on the informed-investor strategy coefficients since here there is no analogue in (14) to the subtraction of θ^R_{n-1} in (12).

[19] The variance $\Sigma^{(2)}_n$ of $\tilde{v}$ and the conditional variance of $\tilde{a}$ by itself are, by definition, non-increasing over time. However, the variance $\Sigma^{(1)}_n$ of the latent trading demand $\tilde{a}-\theta^R_n$ might not be monotonely decreasing. The stock positions θ^R_n in $\tilde{a}-\theta^R_n$ are random variables that change stochastically over different times n rather than a fixed random variable. In particular, the possibility of speculative trading means that θ^R_n can, at some dates, move randomly away from $\tilde{a}$ before eventually moving towards $\tilde{a}$ later in the day and thereby driving $\tilde{a}-\theta^R_n$ to zero.

$$\Sigma_n^{(2)} := \mathbb{V}[\tilde{v} - \hat{p}_n], \tag{24}$$

$$\Sigma_n^{(3)} := \mathbb{E}\Big[\big(\tilde{a} - \hat{\theta}_n^R - \hat{q}_n\big)(\tilde{v} - \hat{p}_n)\Big]. \tag{25}$$

These moments are "post-trade" at time n in that they reflect trading up-through and including the time-n order flow y_n. In other words, they are inputs for trading decisions and pricing in round $n+1$. The initial variances and covariance at $n = 0$ are exogenously given by

$$\Sigma_0^{(1)} = \sigma_{\tilde{a}}^2, \quad \Sigma_0^{(2)} = \sigma_{\tilde{v}}^2, \quad \Sigma_0^{(3)} = \rho\, \sigma_{\tilde{a}}\sigma_{\tilde{v}}. \tag{26}$$

In equilibrium, the constants (15) must satisfy consistency restrictions, which we explain in two steps:

Step 1: The first set of restrictions on the pricing coefficients $\{\lambda_n, r_n\}_{n=1}^N$ is that in equilibrium $\hat{p}_n$ and $\hat{q}_n$ must be consistent with Bayesian updating. For the conjectured prices $\hat{p}_n$ to be conditional expectations $\mathbb{E}[\tilde{v}\,|\,\hat{y}_1, \ldots, \hat{y}_n]$ for the conjectured system, the same logic as for the equilibrium prices p_n in (10), implies

$$\begin{aligned}\Delta \hat{p}_n &= \lambda_n\big(\hat{y}_n - \mathbb{E}[\hat{y}_n\,|\,\hat{y}_1, \ldots, \hat{y}_{n-1}]\big)\\ &= \lambda_n\big(\hat{y}_n - (\alpha_n^R + \beta_n^R)\hat{q}_{n-1}\big),\end{aligned} \tag{27}$$

for $n = 1, \ldots, N$ where λ_n equals the projection coefficient

$$\frac{\mathrm{Cov}(\tilde{v} - \hat{p}_{n-1}, \hat{y}_n - \mathbb{E}[\hat{y}_n\,|\,\hat{y}_1, \ldots, \hat{y}_{n-1}])}{\mathbb{V}(\hat{y}_n - \mathbb{E}[\hat{y}_n\,|\,\hat{y}_1, \ldots, \hat{y}_{n-1}])}. \tag{28}$$

This is a restriction on the price-process coefficients in terms of the hedge-fund and rebalancer strategy coefficients. A related logic gives restrictions on r_n for $\hat{q}_n$ to be the conditional expectation $\mathbb{E}[\tilde{a} - \hat{\theta}_n^R\,|\,\hat{y}_1, \ldots, \hat{y}_n]$ over time. The resulting two restrictions on the equilibrium constants for $n = 1, \ldots, N$ (see Lemma 1 in A.1) are

$$\lambda_n = \frac{\beta_n^I\Sigma_{n-1}^{(2)} + \beta_n^R\Sigma_{n-1}^{(3)}}{(\beta_n^I)^2\Sigma_{n-1}^{(2)} + (\beta_n^R)^2\Sigma_{n-1}^{(1)} + 2\beta_n^I\beta_n^R\Sigma_{n-1}^{(3)} + \sigma_w^2\Delta}, \tag{29}$$

$$r_n = \frac{(1-\beta_n^R)\big(\beta_n^I\Sigma_{n-1}^{(3)} + \beta_n^R\Sigma_{n-1}^{(1)}\big)}{(\beta_n^I)^2\Sigma_{n-1}^{(2)} + (\beta_n^R)^2\Sigma_{n-1}^{(1)} + 2\beta_n^I\beta_n^R\Sigma_{n-1}^{(3)} + \sigma_w^2\Delta}. \tag{30}$$

The conditional variances and covariance in (23)–(25) are computed recursively as

$$\Sigma_n^{(1)} = (1-\beta_n^R)\big((1-\beta_n^R - r_n\beta_n^R)\Sigma_{n-1}^{(1)} - r_n\beta_n^I\Sigma_{n-1}^{(3)}\big), \tag{31}$$

$$\Sigma_n^{(2)} = (1-\lambda_n\beta_n^I)\Sigma_{n-1}^{(2)} - \lambda_n\beta_n^R\Sigma_{n-1}^{(3)}, \tag{32}$$

$$\Sigma_n^{(3)} = (1-\beta_n^R)\big((1-\lambda_n\beta_n^I)\Sigma_{n-1}^{(3)} - \lambda_n\beta_n^R\Sigma_{n-1}^{(1)}\big). \tag{33}$$

Note the "block" structure here: The updating coefficients λ_n and r_n just depend on the strategy coefficients β_n^R and β_n^I and the prior variances and covariance from time $n-1$ (along with the exogenous noise-trading variance σ_w^2). The post-trade variances and covariance $\Sigma_n^{(1)}$, $\Sigma_n^{(2)}$, and $\Sigma_n^{(3)}$ just depend on the updating coefficients λ_n and r_n, the strategy coefficients β_n^R and β_n^I, and the prior variances and covariance from time $n-1$.

Step 2: The second set of restrictions is that the coefficients $\{\beta_n^I, \beta_n^R, \alpha_n^R\}_{n=1}^N$ give optimal trading strategies for the hedge fund and the rebalancer.

Consider first the hedge fund at a generic time n. For a conjectured strategy $\hat{\theta}^I$ to be the hedge fund's equilibrium strategy, deviations from $\hat{\theta}^I$ cannot be profitable. Proving this requires modeling the effects of possible past suboptimal play. The hedge fund knows not only the terminal stock value $\tilde{v}$, but also, as in Foster and Viswanathan (1996), the extent to which the actual prices, quantity expectations, and rebalancer positions (i.e., p_n, q_n, and θ_n^R in (10), (9), and (5) given its actual orders $\Delta\theta_1^I, \ldots, \Delta\theta_n^I$) deviate from their conjectured values (i.e., $\hat{p}_n$, $\hat{q}_n$, and $\hat{\theta}_n^R$ from (21), (22), and (19) given the conjectured orders $\Delta\hat{\theta}_1^I, \ldots, \Delta\hat{\theta}_n^I$ in (18)). In particular, the actual "un-hatted" processes depend on actual past orders whereas the conjectured "hat" processes depend on conjectured past orders. Although the rebalancer's strategy is fixed by the sequences of coefficients $\beta_1^R, \ldots, \beta_N^R$ and $\alpha_1^R, \ldots, \alpha_N^R$ in (5), its actual holdings θ_n^R are subject to the hedge fund's choice of θ^I because the aggregate order flows affect the rebalancer's orders. Similar statements apply to the prices p_n and latent trading-demand expectations q_n.

A natural set of state variables to consider for the hedge fund's problem in (3) is

$$\begin{aligned}&\tilde{v} - \hat{p}_n, \quad \hat{q}_n, \quad \hat{\theta}_n^I - \theta_n^I,\\ &\hat{\theta}_n^R - \theta_n^R, \quad \hat{q}_n - q_n, \quad \hat{p}_n - p_n.\end{aligned} \tag{34}$$

The first two quantities in (34) describe market pricing errors (given the hedge fund's private valuation information) and the predicted future latent rebalancer trading demand (given market information) in the conjectured equilibrium. The next four quantities describe the hedge fund's private information about its actual holdings and about deviations its actual past orders have induced in the rebalancer's holdings, market expectations about the future rebalancer latent trading demand, and market prices relative to the conjectured processes in (18)–(22). However, the state space for the hedge fund can be simplified, because in equilibrium some of these state variables only matter in combination for the hedge fund's optimization problem. Appendix A shows that two composite state variables are sufficient for the hedge fund's value function:

$$\begin{aligned}X_n^{(1)} &:= \tilde{v} - p_n, \quad X_n^{(2)} := (\hat{\theta}_n^R - \theta_n^R) + (\hat{q}_n - q_n)\\ &\quad + \tfrac{\Sigma_n^{(3)}}{\Sigma_n^{(2)}}\big(\tilde{v} - \hat{p}_n\big), \quad n = 0, \ldots, N.\end{aligned} \tag{35}$$

From a technical point of view, this is a substantial reduction from the six state variables in (34). Two seems likely to be the minimum number of state variables necessary for the hedge fund's problem. Lemma 2 in Appendix A ensures that the $X_n^{(1)}$ and $X_n^{(2)}$ processes are observable for the hedge fund. In equilibrium, with $\theta_n^I = \hat{\theta}_n^I$ and, thus, $p_n = \hat{p}_n$, $q_n = \hat{q}_n$, and $\theta_n^R = \hat{\theta}_n^R$, it follows from (35) that

$$X_n^{(2)} = \tfrac{\Sigma_n^{(3)}}{\Sigma_n^{(2)}}X_n^{(1)}, \quad n = 0, 1 \ldots, N. \tag{36}$$

Thus, on the equilibrium path, the hedge fund's state space reduces to just $\tilde{v} - p_n$, which is consistent with the form of its equilibrium orders in (6).

Lemma 2 in Appendix A shows that the hedge fund's value function at each time $n = 1, \ldots, N$ has the quadratic form

$$\max_{\substack{\Delta\theta_k^I\in\sigma(\tilde v,y_1,\dots,y_{k-1})\\ n+1\le k\le N}} \mathbb{E}\left[\sum_{k=n+1}^{N}(\tilde v-p_k)\Delta\theta_k^I\,\middle|\,\tilde v,y_1,\dots,y_n\right]$$
$$=I_n^{(0)}+I_n^{(1,1)}(X_n^{(1)})^2+I_n^{(1,2)}X_n^{(1)}X_n^{(2)}+I_n^{(2,2)}(X_n^{(2)})^2, \quad (37)$$

where $I_n^{(0)}, I_n^{(1,1)}, I_n^{(1,2)}$, and $I_n^{(2,2)}$ are constants. Lemma 2 also shows that the hedge fund's problem (37) is quadratic in its orders $\Delta\theta_n^I$. The first-order condition for (37) gives the hedge fund's optimal orders

$$\Delta\theta_n^I=\gamma_n^{(1)}X_{n-1}^{(1)}+\gamma_n^{(2)}X_{n-1}^{(2)},\quad n=1,\dots,N, \quad (38)$$

where the coefficients $\gamma_n^{(1)}$ and $\gamma_n^{(2)}$ depend on the hedge-fund value-function coefficients and on the parameters of the conjectured price, latent trading demand, and rebalancer strategy processes given in (A.28) and (A.29) in Appendix A. The second-order condition for the strategy in (38) to be optimal for the hedge fund is

$$I_n^{(2,2)}r_n^2+I_n^{(1,2)}r_n\lambda_n+I_n^{(1,1)}\lambda_n^2<\lambda_n,\quad n=1\dots,N. \quad (39)$$

By inserting the hedge fund's candidate strategy (38) and (A.28)–(A.29) into the expectation in (37), we can determine the hedge fund's value-function coefficients recursively as in Eqs. (A.42)–(A.44) in Section A.5.

Equating the coefficients in (38) with (6) and using the equilibrium condition (36) gives the following restriction on the hedge fund's strategy coefficients:

$$\beta_n^I=\gamma_n^{(1)}+\gamma_n^{(2)}\frac{\Sigma_{n-1}^{(3)}}{\Sigma_{n-1}^{(2)}},\quad n=1\dots,N. \quad (40)$$

For fixed moments $\Sigma_n^{(1)}, \Sigma_n^{(2)}$, and $\Sigma_n^{(3)}$, we can use the linear Eqs. (31)–(33) to express $\Sigma_{n-1}^{(1)}, \Sigma_{n-1}^{(2)}$, and $\Sigma_{n-1}^{(3)}$ in terms of $r_n, \lambda_n, \beta_n^I, \beta_n^R$. Eqs. (A.28)–(A.29) and (29)–(30) can then be used to see that (40) is a fifth–degree polynomial in $\{\beta_n^R,\beta_n^I\}$ whenever $\Sigma_n^{(i)}$, $i=1,2,3$, and $I_n^{(i,j)}$, $i=1,2$ and $i\le j\le 2$, are fixed.

Similarly, six natural state variables for the rebalancer's problem in (4) are

$$\tilde a-\hat\theta_n^R,\quad \hat q_n,\quad \hat\theta_n^R-\theta_n^R,\quad \hat\theta_n^I-\theta_n^I,\quad \hat q_n-q_n,\quad \hat p_n-p_n. \quad (41)$$

The first two quantities in (41) describe the rebalancer's latent trading demand (given his private information about his target and past orders) and the market-maker prediction of his future latent trading demand (given the public order-flow history) in a conjectured equilibrium. The next four quantities describe the rebalancer's private information about its own past orders and how they caused the hedge fund's holdings, the market's latent trading demand predication, and prices to deviate from the conjectured equilibrium. However, the rebalancer's state space can also be simplified. Just three composite state variables are sufficient for the rebalancer's value function:

$$Y_n^{(1)}:=\tilde a-\theta_n^R,\quad Y_n^{(2)}:=(\hat p_n-p_n)+\frac{\Sigma_n^{(3)}}{\Sigma_n^{(1)}}(\tilde a-\hat\theta_n^R-\hat q_n),$$
$$Y_n^{(3)}:=q_n,\quad n=0,1,\dots,N. \quad (42)$$

Lemma 3 in Appendix A ensures these processes are observable for the rebalancer. In equilibrium, with $p_n=\hat p_n$, $q_n=\hat q_n$, and $\theta_n^I=\hat\theta_n^I$, it follows from (42) that

$$Y_n^{(2)}=\frac{\Sigma_n^{(3)}}{\Sigma_n^{(1)}}(Y_n^{(1)}-Y_n^{(3)}),\quad n=1,\dots,N. \quad (43)$$

Thus, on the equilibrium path, the state space for the rebalancer at time n reduces to just two state variables, $\tilde a-\theta_n^R$ and q_n, which is consistent with (5). When the hedge fund's strategy is fixed as in (6), Lemma 3 in Appendix A shows that the rebalancer's value function is quadratic in the rebalancer state variables

$$\max_{\substack{\Delta\theta_k^R\in\sigma(\tilde a,y_1,\dots,y_{k-1})\\ n+1\le k\le N-1}} -\mathbb{E}\left[\sum_{k=n+1}^{N}(\tilde a-\theta_{k-1}^R)\Delta p_k\,\middle|\,\tilde a,y_1,\dots,y_n\right]$$
$$=L_n^{(0)}+\sum_{1\le i\le j\le 3}L_n^{(i,j)}Y_n^{(i)}Y_n^{(j)}, \quad (44)$$

where $L_n^{(0)},\dots,L_n^{(3,3)}$ are constants. Lemma 3 also ensures that the rebalancer's problem (44) is quadratic in his orders $\Delta\theta_n^R$. The corresponding first-order condition gives the rebalancer's optimal orders

$$\Delta\theta_n^R=\delta_n^{(1)}Y_{n-1}^{(1)}+\delta_n^{(2)}Y_{n-1}^{(2)}+\delta_n^{(3)}Y_{n-1}^{(3)},\quad n=1,\dots,N, \quad (45)$$

where the coefficients $\delta_n^{(1)}$, $\delta_n^{(2)}$, and $\delta_n^{(3)}$ depend on the rebalancer's value-function coefficients, and the parameters of the conjectured price, latent trading demand, and hedge fund's strategy processes given in (A.38)–(A.40) in Appendix A. The associated second-order condition for the rebalancer's optimal strategy is

$$L_n^{(1,1)}+L_n^{(3,3)}r_n^2+L_n^{(1,2)}\lambda_n+L_n^{(2,2)}\lambda_n^2<L_n^{(1,3)}r_n+L_n^{(2,3)}r_n\lambda_n,$$
$$n=1,\dots,N. \quad (46)$$

Similar to the hedge fund's problem, by inserting the rebalancer's candidate strategy (45) and (A.38)–(A.40) into the expectation in (44), we can find the rebalancer's value-function coefficients recursively as in Eqs. (A.45)–(A.50) in Section A.5.

By equating the coefficients in (45) with (5) and using the equilibrium condition (43), we get two restrictions:

$$\beta_n^R=\delta_n^{(1)}+\delta_n^{(2)}\frac{\Sigma_{n-1}^{(3)}}{\Sigma_{n-1}^{(1)}},\quad \alpha_n^R=\delta_n^{(3)}-\delta_n^{(2)}\frac{\Sigma_{n-1}^{(3)}}{\Sigma_{n-1}^{(1)}},\quad n=1,\dots,N. \quad (47)$$

Similarly to (40), the first equation in (47) is a fifth–degree polynomial in $\{\beta_n^R,\beta_n^I\}$ whenever $\Sigma_n^{(i)}$, $i=1,2,3$, and $L_n^{(i,j)}$, $i=1,2,3$ and $i\le j\le 3$, are fixed. The second equation in (47) is a linear equation in α_n^R once all of the other parameters are determined.

Our main theoretical result is the following:

Theorem 1. Constants $\{\lambda_n,r_n,\beta_n^R,\alpha_n^R,\beta_n^I\}_{n=1}^N$ satisfying restrictions (16)–(17) describe a linear Bayesian Nash equilibrium of the form in (5), (6), (9), and (10) if, for all times $n=1,\dots,N$, the following restrictions hold:

i) *The pricing and latent-trading prediction restrictions in (29)–(30) hold where the moments $\Sigma_n^{(1)}, \Sigma_n^{(2)}$, and $\Sigma_n^{(3)}$ are given in (26) and (31)–(33).*
ii) *The equilibrium strategy conditions (40) and (47) are satisfied with the second-order conditions (39) and (46) holding where the value-function coefficients $\{I_n^{(i,j)}\}_{1\le i\le j\le 2}$ and $\{L_n^{(i,j)}\}_{1\le i\le j\le 3}$ for $n=1,\dots,N-1$*

are computed via the recursions (A.42)–(A.44) and (A.45)–(A.50) in Section A.5.

Theorem 1 is a verification result for a set of model parameters to constitute a linear equilibrium. It extends Proposition 1 in Foster and Viswanathan (1996) to allow for an investor with a trading constraint. As with most discrete-time Kyle models, including (Foster and Viswanathan, 1996), we do not have analytic expressions for the equilibrium. Equilibria must be computed numerically. Section A.6 describes our numerical algorithm. However, there is an existence and comparative-static result for the asset-value variance $\sigma^2_{\tilde{v}}$.

Proposition 1. Assume the linear equilibrium of Theorem 1 exists with coefficients

$$\lambda_n, r_n, \beta^I_n, \beta^R_n, \alpha^R_n, \Sigma^{(1)}_{n-1}, \Sigma^{(2)}_{n-1}, \Sigma^{(3)}_{n-1}, \quad n = 1, \ldots, N, \quad (48)$$

for a given parameterization

$$\Sigma^{(1)}_0 = \sigma^2_{\tilde{a}}, \quad \Sigma^{(2)}_0 = \sigma^2_{\tilde{v}}, \quad \Sigma^{(3)}_0 = \rho\, \sigma_{\tilde{a}}\, \sigma_{\tilde{v}} \quad (49)$$

where $\sigma_{\tilde{a}} > 0, \sigma_{\tilde{v}} > 0, \rho \geq 0$. Then for any parameterization

$$\Sigma^{(1)}_0 = \sigma^2_{\tilde{a}}, \quad \Sigma^{(2)}_0 = h^2\sigma^2_{\tilde{v}}, \quad \Sigma^{(3)}_0 = \rho\, \sigma_{\tilde{a}}\, h\, \sigma_{\tilde{v}} \quad (50)$$

for any constant $h > 0$, an equilibrium exists with

$$h\,\lambda_n, r_n, \frac{\beta^I_n}{h}, \beta^R_n, \alpha^R_n, \Sigma^{(1)}_{n-1}, h^2\,\Sigma^{(2)}_{n-1}, h\,\Sigma^{(3)}_{n-1}, \quad n = 1, \ldots, N. \quad (51)$$

This result follows immediately from verifying that, if the set of equations and inequalities for the equilibrium conditions hold for (48), then they also hold for (51).[20] As is expected, greater asset-value volatility makes prices more sensitive to order flow (λ_n is increasing in $\sigma^2_{\tilde{v}}$), and this reduction in the absolute level of liquidity causes informed investors to trade less aggressively (β^I_n is decreasing in $\sigma^2_{\tilde{v}}$). Perhaps more surprisingly, the rebalancer's strategy coefficients are unaffected by $\sigma^2_{\tilde{v}}$. One piece of intuition is the following: The rebalancer has to reach his target $\tilde{a}$ at time N and relative trade-offs between liquidity at different dates ($\lambda_n/\lambda_{n'}$) are unaffected by $\sigma^2_{\tilde{v}}$.

4. Numerical results

Our analysis in this section investigates two quantitative questions: What do dynamic rebalancing strategies look like in our market? And what are the equilibrium effects of the rebalancing constraint on price discovery, liquidity, and order flow? To answer these questions, we conduct an extensive battery of numerical experiments over the model parameter space.

Our numerical specification has $N = 10$ rounds of trading and the total variance of the Brownian motion noise-trading order flow over the day (N periods) is normalized at $\sigma^2_w = 1$. The variance of the terminal stock value $\tilde{v}$ is set to $\sigma^2_{\tilde{v}} = 1$. We do not numerically vary $\sigma^2_{\tilde{v}}$, because Proposition 1 gives analytic comparative statics. In particular, intraday patterns in the strategy and price coefficients are either invariant to $\sigma^2_{\tilde{v}}$ or scale proportionally with $\sigma_{\tilde{v}}$, $\sigma^2_{\tilde{v}}$, or $1/\sigma_{\tilde{v}}$. The target variance $\sigma^2_{\tilde{a}}$ and target informativeness ρ are varied over a 2×2 grid with $\sigma^2_{\tilde{a}}$ taking values $0.2, 0.4, \ldots, 2$ (i.e., from one-fifth up to twice the daily noise-trading variance) and with ρ taking values $0, 0.05, 0.10, \ldots, 0.45$. Over this range of $\sigma^2_{\tilde{a}}$ and ρ parameters, our results are numerically well-behaved. However, when ρ is greater than 0.45 and the target variance $\sigma^2_{\tilde{a}}$ is small (e.g., typically 0.2 or 0.4), our numerical results are sometimes less well-behaved.[21] Our discussion focuses on results in the numerically well-behaved region. While our numerical findings are not necessarily global properties, they hold for a large portion of the parameter space. Moreover, given the prevalence of order-splitting in real-world markets by passive and less informed institutions, a low ρ and a high $\sigma^2_{\tilde{a}}$ are, arguably, the empirically relevant cases.

Most of our analysis is presented visually in figures showing intraday patterns. In our standard template, Figure "A" is for the case of uninformative targets ($\rho = 0$) with target variances $\sigma^2_{\tilde{a}}$ equal to 0.2, 1, and 2. Figure "B" is for the case of informative targets ($\rho = 0.45$) with the same three target variances. The various intraday patterns are qualitatively similar for other parameterizations in between those shown here, and the patterns change relatively smoothly in the target variance $\sigma^2_{\tilde{a}}$ and correlation ρ. Thus, one can interpolate between the cases in the figures to infer the patterns for other variances $\sigma^2_{\tilde{a}}$ and correlations ρ. These patterns are also qualitatively similar outside of our parameter grid for correlations $\rho > 0.45$ so long as $\sigma^2_{\tilde{a}}$ is not too small.

We assess the impact of portfolio rebalancing by comparing our model with two alternative models. For $\rho = 0$, we compare our equilibrium with Kyle (1985). For $\rho > 0$, we compare our model with a variant of the Foster and Viswanathan (1994) model, which we call the *modified FV model*. In the modified FV model, one investor has superior information in that she knows the terminal stock value $\tilde{v}$, while a less-informed investor receives a noisy signal $\tilde{a}$ with a correlation $\rho > 0$ with $\tilde{v}$.[22] The signal $\tilde{a}$ in the modified FV model has the same distribution as the target $\tilde{a}$ in our rebalancing model. However, in the modified FV model there is no trading constraint. The one difference between our modified FV model and the original Foster and Viswanathan (1994) model is that our better-informed investor does not know the less-informed investor's information whereas in Foster and Viswanathan (1994) the better-informed investor knows both $\tilde{v}$ and $\tilde{a}$. Hence, our dynamic rebalancing model and the modified FV model have identical information structures. Comparing equilibria in the two models shows the effect of the parent-target constraint when $\rho > 0$. The modified FV model is described in more detail in Appendix B and in the Internet Appendix. One feature of the modified FV model to note is that the

[20] The value function coefficients change from $I^{(1,1)}_n$, $I^{(1,2)}_n$, $I^{(2,2)}_n$, $L^{(1,1)}_n$, $L^{(1,2)}_n$, $L^{(1,3)}_n$, $L^{(2,2)}_n$, $L^{(2,3)}_n$, and $L^{(3,3)}_n$ to $\frac{I^{(1,1)}_n}{h}$, $I^{(1,2)}_n$, $h\,I^{(2,2)}_n$, $h\,L^{(1,1)}_n$, $L^{(1,2)}_n$, $h\,L^{(1,3)}_n$, $\frac{L^{(2,2)}_n}{h}$, $L^{(2,3)}_n$, and $h\,L^{(3,3)}_n$ for $n = 1, \ldots, N$.

[21] Some variables occasionally take large values quite different from their values at adjacent times n and $n+1$.

[22] The modified FV model reduces to the Kyle (1985) model when $\rho = 0$ since then the less-informed investor has no private information and, thus, in equilibrium does not trade.

signal variance $\sigma_{\tilde{a}}^2$ does not affect the informativeness of the less-informed investor's signal. Thus, many properties of the modified FV model are unaffected by changes in $\sigma_{\tilde{a}}^2$. In contrast, changing $\sigma_{\tilde{a}}^2$ has an effect in our rebalancing model because $\sigma_{\tilde{a}}^2$ is an ex ante measure of the size of the parent constraint on the rebalancer's trading.

4.1. Overview of numerical results

Our numerical analysis produces a variety of empirical predictions. One set of results describes quantitative properties of equilibrium rebalancing orders and the importance of various economic considerations in the rebalancer trading strategy.

- The mean and standard deviation of rebalancer orders have intraday patterns that are *U*-shaped for parameterizations within our numerically well-behaved set. In addition, the magnitude of the *U*-shape increases when the target variance $\sigma_{\tilde{a}}^2$ increases and becomes skewed when the correlation ρ increases. Thus, the model not only predicts the existence of *U*-shaped intraday rebalancer order-flow patterns, but also predicts how these intraday patterns vary with time-variation in the volatility of rebalancing targets.
- The realized parent target $\tilde{a}$ has a large effect on the rebalancer child orders relative to adaptive trading. In addition, predictable interactions with informed-investor orders have an important impact on the *U*-shaped timing of optimal rebalancing orders.

A second set of findings describes the equilibrium effects of dynamic rebalancing on the price and order-flow processes.

- Trading volume, price volatility, and order-flow autocorrelation have *U*-shaped intraday patterns that are increasing in target variance $\sigma_{\tilde{a}}^2$.[23] This prediction is testable by looking at whether these intraday patterns increase for stocks on days for which rebalancing-target uncertainty is greater (e.g., days with highly volatile mutual fund inflows/outflows).
- Daily order-flow autocorrelation (estimated using intraday data) can be used — given its low sensitivity to changes in ρ and insensitivity to $\sigma_{\tilde{v}}^2$ — as an empirical proxy to track time-variation in rebalancing volatility. Thus, time-variation in the size of the various intraday patterns and in the aggregate order-flow autocorrelation level should be positively correlated.
- Autocorrelation of the aggregate order flow is linked to autocorrelation in the orders of individual investors who are rebalancing. This is in contrast to order-flow autocorrelation due to cross-autocorrelation across different investors due to front-running and back-running (see Yang and Zhu, 2015).

We have a few more observations about testability. First, the aggregate order-flow and pricing predictions are testable using standard intraday price and order-flow data. On the other hand, predictions about rebalancing strategies and structural differences between rebalancers and informed investors require investor-level order-flow data (e.g., from the Investment Industry Regulatory Organization of Canada). Second, since rebalancer order flows are autocorrelated, while informed-investor orders are not, this difference can be used to identify individual institutions in an investor-level order database as being *likely rebalancers* (if their orders have above-average autocorrelation) or *likely informed investors* (if their orders are less autocorrelated). Third, we can use a direct (or inferred as above) classification of individual investors to test whether the orders of likely rebalancers become more negatively correlated with orders from likely informed investors over the day. Fourth, our comparative static predictions are not just about individual patterns, but rather about the co-movement of a cross-section of multiple intraday patterns. Fifth, our predications about time-variation in the volatility (i.e., second moment) of non-public portfolio-rebalancing trading demand are different from predictions about changing means (i.e., first moments) of publicly predictable trading demand investigated in Bessembinder et al. (2016)

4.2. Dynamic rebalancing

The rebalancer's orders are described by the strategy coefficients β_n^R and α_n^R. Figs. 1A and 1B show intraday patterns for these coefficients. The fact that β_n^R is positive means, from (11), that the rebalancer trades in the direction of his private information $\tilde{a}-\theta_{n-1}^R-q_{n-1}$. Intuitively, the larger $\tilde{a}-\theta_{n-1}^R$ is relative to q_{n-1}, the more the rebalancer must trade mechanically to achieve his target. It is also intuitive that the β_n^R coefficient increases as the end of the day (and the binding rebalancing deadline) approaches. Fig. 1B shows that there is an interaction between the target variance $\sigma_{\tilde{a}}^2$ and the informativeness ρ of the target. When $\sigma_{\tilde{a}}^2$ is small (e.g., 0.2), the information content of a given magnitude of target realization $\tilde{a}$ is large, and, thus, the rebalancer in Fig. 1B scales his trades aggressively early in the day to exploit the information in $\tilde{a}$. Consequently, the β_n^R coefficients are larger in Fig. 1B (with $\rho=0.45$) than in Fig. 1A. In contrast, when $\sigma_{\tilde{a}}^2$ is large (e.g., 1.0 or 2.0), the magnitudes of the rebalancer's trades are already large given the β_n^R coefficients when $\rho=0$ (as in Fig 1A), and so the impact of informativeness $\rho=0.45$ on the order size in Fig. 1B is negligible.

Next, consider the sunshine-trading component $(\alpha_n^R+\beta_n^R)q_{n-1}$ from (11). The fact that $\alpha_n^R+\beta_n^R$ is positive (if the two coefficients in Fig. 1 are added together) means that, on average, the rebalancer buys more when market makers believe he has a large latent buying demand. Again, this is intuitive. The sum $\alpha_n^R+\beta_n^R$ is small for most of the day, but increases towards the end of the day, when the rebalancer engages in more sunshine trading to close the predictable part q_{n-1} (as well as the unpredictable part $\tilde{a}-\theta_{n-1}^R-q_{n-1}$) of his remaining gap $\tilde{a}-\theta_n^R$.

[23] Order-splitting is certainly not the only cause of *U*-shaped intraday patterns, since many of the empirically documented intraday patterns predate the widespread use of order-splitting algorithms. However, the magnitude of these *U*-shaped patterns should co-vary with rebalancing volatility.

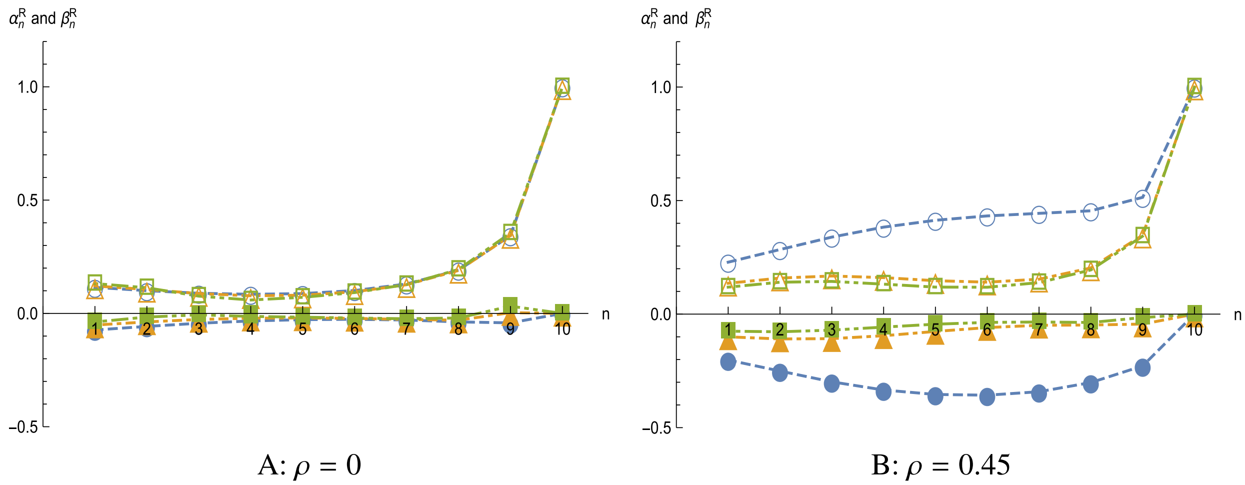

Fig. 1. Intraday patterns for the rebalancer strategy coefficients α_n^R (lines with •, ▲, ■) and β_n^R (lines with ○, △, □) for times $n = 1, 2, \ldots, 10$. The parameters are $N = 10$, $\sigma_{\tilde{v}}^2 = 1$, $\sigma_w^2 = 1$, and $\sigma_{\tilde{a}}^2 = 0.2$ (– – with ○ or •), 1.0 (– · –with △ or ▲), 2.0 (– · · · – with □ or ■).

The coefficient α_n^R captures the incremental impact of trading motives that are present when trading on the private information $\tilde{a} - \theta_{n-1}^R - q_{n-1}$ but absent when trading on q_{n-1}. In particular, when trading on $\tilde{a} - \theta_{n-1}^R - q_{n-1}$, the rebalancer is motivated in part by opportunities for speculation and the fact that non-sunshine trading has price impacts in addition to the mechanical effects of trading towards his target. A negative value of α_n^R means that non-mechanical motives increase the rebalancer's trading on his information $\tilde{a} - \theta_{n-1}^R - q_{n-1}$ relative to his trading on q_{n-1}. Intuitively, the larger the rebalancer's actual future latent trading demand $\tilde{a} - \theta_{n-1}^R$ is relative to the market-maker forecast, q_{n-1}, the more the market underestimates future aggregate buying relative to the rebalancer's private information. This predictability causes the rebalancer to anticipate rising future prices and, thereby, leads him to buy more/sell less at time n. Since sunshine-trading forecast-error predictability causes the rebalancer to trade more in the direction of his information $\tilde{a} - \theta_n^R - q_{n-1}$, it makes α_n^R smaller or negative. In contrast, the intuition for predictability about current mispricing $\tilde{v} - p_{n-1}$ is more complicated.[24] However, it can be shown that the impact of current mispricing predictability potentially can have the opposite sign of the impact of sunshine-trading forecast-error predictability. As a result, the net impact measured by α_n^R cannot be signed unambiguously a priori. In our numerical analysis, however, α_n^R is consistently negative, even when $\rho > 0$. This suggests that the sunshine-trading forecast-error motive is dominant here.

The rebalancer's strategy coefficients α_n^R and β_n^R reflect the combined effects of the economic considerations described in Section 2. We disentangle these various economic considerations and assess their quantitative importance using two different decompositions. The first decomposes the rebalancer orders into their dependence on the underlying variables. The second, considered in the Internet Appendix, is based on a set of ad hoc strategies that include and omit various economic considerations.

Decomposition into underlying variables: The latent trading-demand expectation q_n and cumulative holdings θ_{n-1}^R in (11) are endogenous processes, so we further decompose the rebalancer's orders into linear functions of the underlying exogenous random variables — the rebalancing target $\tilde{a}$, the terminal value $\tilde{v}$, and noise-trader orders Δw_j — in the market:

$$\Delta\theta_n^R = A_n^R\,\tilde{a} + B_n^R\,\tilde{v} + \sum_{j=1,\ldots n-1} c_{j,n}^R\,\Delta w_j. \tag{52}$$

This decomposition follows from the joint linearity of prices, orders, and the q_n process. The dependence on $\tilde{v}$ and the noise-trader orders Δw_j comes through the q_n process and its dependence on lagged aggregate orders. The dependence on the target $\tilde{a}$ is both direct and also indirect through the lagged θ_{n-1}^R and q_{n-1} terms in (11). This decomposition is then used to relate statistical properties of the rebalancer child orders to the statistical properties of $\tilde{a}$, $\tilde{v}$, and the noise-trader orders.

Fig. 2 shows the linear decomposition coefficients from (52) for the rebalancer orders over time for our six reference parameterizations. Similar patterns hold for other parameterizations in our parameter-space analysis. One fact affecting these intertemporal patterns is the terminal parent constraint ($\theta_N^R = \tilde{a}$), which, by construction, requires $\sum_{n=1,\ldots N} A_n^R = 1$, $\sum_{n=1,\ldots N} B_n^R = 0$, and $\sum_{n=j+1,\ldots N} c_{j,n}^R = 0$ for $j = 1, \ldots, N-1$. Thus, the rebalancer trades on price effects from $\tilde{v}$ and noise-trader orders but then must eventually unwind these positions. Note that the coefficients $c_{j,n}^R$ on noise-trader orders Δw_j in the lower two plots in Fig. 2 do not start until one period after time j when an order Δw_j arrives and is cleared in the market.

Quantitatively, the target $\tilde{a}$ is a major driver of the rebalancer's orders. In addition, the trajectory of the A_n^R decomposition coefficients on $\tilde{a}$ have a *U*-shaped intraday

[24] For example, at time 1, the direction of the mispricing predictability is determined by $\text{cov}[\tilde{v} - p_1, \tilde{a} - \theta_1^R - q_1] = -(1 - \beta_1^R)\lambda_1\beta_1^R\sigma_{\tilde{a}}^2 + (1 - \beta_1^R)(1 - \lambda_1\beta_1^I)\sigma_{\tilde{a}}\sigma_{\tilde{v}}\rho$. If $\rho = 0$ and given $0 < \beta_1^R < 1$ and $\lambda_1 > 0$, then $\text{cov}[\tilde{v} - p_1, \tilde{a} - \theta_1^R - q_1]$ is negative.

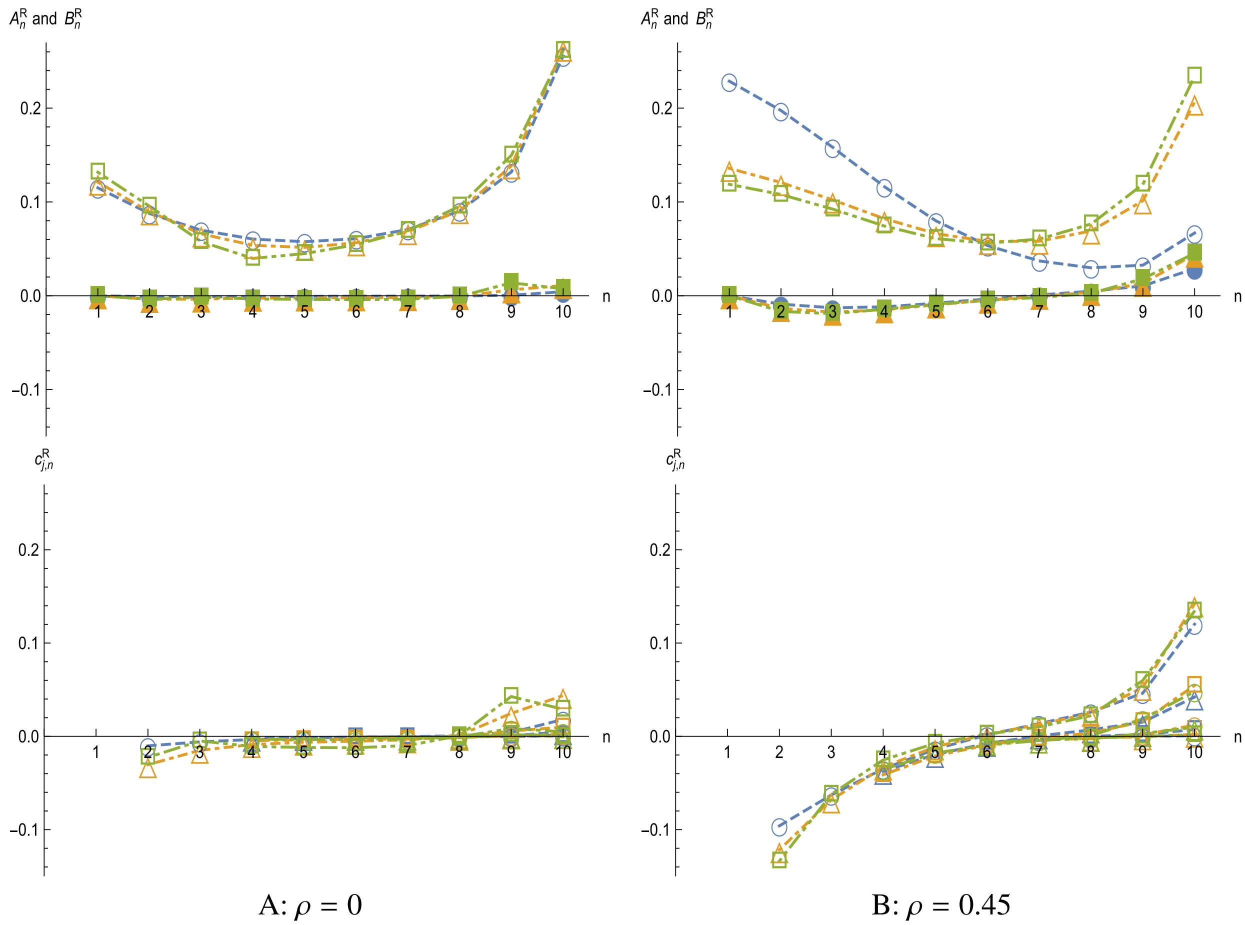

Fig. 2. Intraday patterns for the linear-decomposition coefficients for the rebalancer orders for times $n = 1, 2, \ldots, 10$. The top figures show the coefficients A_n^R on the target $\tilde{a}$ (lines with ○, △, □) and B_n^R on the asset value $\tilde{v}$ (lines with •, ▲, ■), and the lower figures show the coefficients $c_{j,n}^R$ on the noise-trader orders Δw_j with arrival times $j = 1, 3, 5,$ and 7. The parameters are $N = 10$, $\sigma_{\tilde{v}}^2 = 1$, $\sigma_w^2 = 1$, and $\sigma_{\tilde{a}}^2 = 0.2$ (– – with ○ or •), 1.0 (– · –with △ or ▲), 2.0 (– · · · – with □ or ■).

pattern. These *U*-shaped coefficients for rebalancer orders mean that the trading target induces *U*-shapes in both the mean volume and volatility of rebalancer trading over the day. Perhaps surprisingly, the decomposition coefficient on $\tilde{v}$ is initially negative at time 2. A partial intuition follows from the rebalancer order in (5). At time $n = 2$, the loadings on $\tilde{v}$ and Δw_1 come from the dependence of $\Delta\theta_2^R$ on q_1. Given a positive informed-investor strategy coefficient β_1^I, the sign of the rebalancer loadings on $\tilde{v}$ and Δw_1 are, by construction, the same and are controlled by the coefficient α_2^R in (5). Since α_2^R at time 2 is consistently negative in all of our numerical examples, the rebalancer trades against price pressure from the noise traders rather than with the informed investor and $\tilde{v}$, and so the decomposition coefficients on $\tilde{v}$ and Δw_1 are both negative. Later in the day, the coefficient on $\tilde{v}$ switches sign when the rebalancer unwinds his speculative positions given his trading-target constraint.

Deterministic and adaptive components: The decomposition in (52) lets us break the rebalancer's orders into a deterministic component,[25]

$$\mathbb{E}[\Delta\theta_n^R \mid \tilde{a}] = A_n^R \tilde{a} + B_n^R \mathbb{E}[\tilde{v} \mid \tilde{a}] = \left(A_n^R + B_n^R \rho \frac{\sigma_{\tilde{v}}}{\sigma_{\tilde{a}}}\right) \tilde{a} \tag{53}$$

that depends on the target $\tilde{a}$ and a separate random adaptive component

$$\Delta\theta_n^R - \mathbb{E}[\Delta\theta_n^R \mid \tilde{a}] = B_n^R (\tilde{v} - \mathbb{E}[\tilde{v} \mid \tilde{a}]) + \sum_{j=1,\ldots,n-1} c_{j,n}^R \Delta w_j \tag{54}$$

that depends on the portion of $\tilde{v}$ that the rebalancer cannot predict given $\tilde{a}$ and on the noise orders $\{\Delta w_1, \ldots, \Delta w_{n-1}\}$. The deterministic component is due to price-impact smoothing, predictable sunshine trading, and predictable interactions with orders from the informed investor who, on average, trades to reverse price pressure caused by the rebalancer orders. The adaptive component

[25] The expectation $\mathbb{E}[\Delta\theta_n^R \mid \tilde{a}]$ in (53) is taken over the stock value $\tilde{v}$ (which is mean-zero but can be correlated with $\tilde{a}$) and the noise-trader orders Δw_j (which are mean-zero and uncorrelated with $\tilde{a}$). The second equality follows from the joint normality of $\tilde{a}$ and $\tilde{v}$.

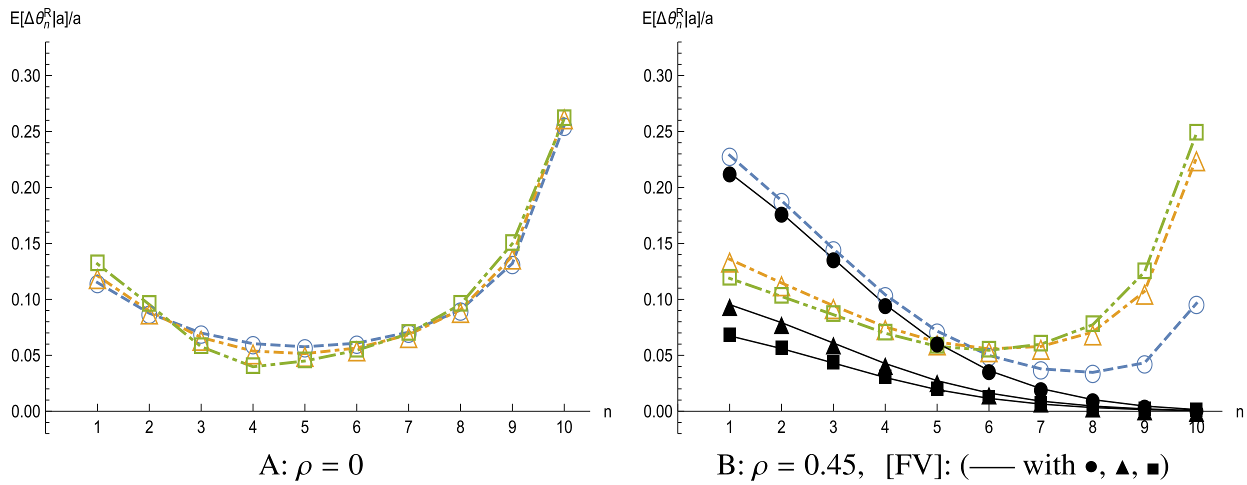

Fig. 3. Intraday patterns for the ratio $\mathbb{E}[\Delta\theta_n^R\,|\,\tilde{a}]/\tilde{a}$ of the conditional expected rebalancer order relative to the target $\tilde{a}\neq 0$ at times $n=1,\dots,10$. The parameters are $N=10$, $\sigma_{\tilde{v}}^2=1$, $\sigma_w^2=1$, and $\sigma_{\tilde{a}}^2=0.2$ (– – with ○), 1.0 (– · –with △), 2.0 (– · · – with □).

comes from the q_n term in (11) after controlling for the target $\tilde{a}$. This component reflects real-time sunshine trading (i.e., reactions to fluctuations in q_n induced by the arriving aggregate order flow over the day) and speculation on information learned through the trading process. Here, we use the decomposition in (53) and (54) to identify direct effects of rebalancing trading on market volume. Later, in Section 4.3, it is used to understand the equilibrium effects of rebalancing on pricing and on the informed-investor orders.

The separation here is not just algebraic; rather it has meaning in terms of separability of the rebalancer's optimization problem in (4). The deterministic expected orders in (53) give the optimal strategy for a rebalancer who precommits at time 0 to using deterministic child orders given by functions $\{x_n(\tilde{a})\}_{n=1}^N$. The proof of the next result is in Appendix C.

Proposition 2. Assume the linear equilibrium of Theorem 1 exists. Then the conditional expected equilibrium orders $\mathbb{E}[\Delta\theta_n^R|\,\tilde{a}]$ are the optimal orders x_n^* for a rebalancer who is constrained to trade deterministically over time.

The adaptive order component in (54) shows how our rebalancer, who is not constrained to trade deterministically, optimally deviates over time from the optimal deterministic strategy x_n^* to respond to changes in market beliefs due to the arriving aggregate order flow.

Fig. 3 shows the expected rebalancer orders over the day conditional on the target $\tilde{a}$ scaled as a ratio $\mathbb{E}[\Delta\theta_n^R\,|\,\tilde{a}]/\tilde{a}$ relative to the target $\tilde{a}\neq 0$. From the linearity in (53), the ratio does not depend on the realized target $\tilde{a}$. If $\rho=0$, then the ratio $\mathbb{E}[\Delta\theta_n^R\,|\,\tilde{a}]/\tilde{a}$ has the identical intraday pattern as the decomposition coefficients A_n^R on $\tilde{a}$ (e.g., compare the *U*-shaped pattern for the ratio in Fig. 3A with the A_n^R coefficients in Fig. 2A).[26] If $\rho>0$, then, the ratios are shifted by the $B_n^R\rho\frac{\sigma_{\tilde{v}}}{\sigma_{\tilde{a}}}$ term in (53). This *U*-shape volume pattern is common to all of the parameterizations we consider. For example, Fig. 3B shows the *U*-shaped pattern for the $\rho=0.45$ case. These intraday patterns for rebalancer orders are conceptually different from those for the less-informed investor's orders in the modified FV model.[27] Because of the rebalancing constraint and the dynamics of sunshine trading, the rebalancer trading has an upturn in expected volume at the end of the day. In the Internet Appendix, the rebalancer orders are decomposed further to identify the specific portion due to predictable sunshine trading.

The second component of the rebalancer orders is the adaptive component in (54) that responds to fluctuations in the aggregate order flow over the trading day. The randomness is due to speculative trading by the rebalancer (given his endogenous learning through trading over time) and real-time sunshine trading (given fluctuations in the market-maker expectations q_n). The size of adaptive trading is measured using the standard deviation $\mathrm{SD}[\Delta\theta_n^R|\,\tilde{a}]$ given the target $\tilde{a}$. Figs. 4A-B show that the standard deviation is initially zero at time $n=1$ (when the rebalancer only knows $\tilde{a}$ and has not yet observed any lagged aggregate order flows) but then is roughly *U*-shaped over the rest of the trading day (i.e., higher at times 2 and 10). The *U*-shape becomes more pronounced when the correlation ρ is large. In contrast, the standard deviation is hump-shaped in the modified FV model. Our parameter-space analysis finds that the *U*-shape increases as $\sigma_{\tilde{a}}^2$ increases. This is consistent with increased endogenous learning (since the rebalancer's orders are a larger part of the noise in the aggregated order flow and since the

[26] Degryse et al. (2014) obtain a similar *U*-shaped pattern but with both short-lived information and deterministic rebalancing. Optimal order execution models can also have *U*-shaped optimal strategies (see, e.g., Predoiu et al., 2011), but our model endogenizes the liquidity resilience and replenishment dynamics that drive this result.

[27] Unlike other plots in which the modified FV model is insensitive to $\sigma_{\tilde{a}}^2$, the ratio here is decreasing in $\sigma_{\tilde{a}}^2$ because the order size in the numerator of this ratio is invariant to how the information in the signal $\tilde{a}$ is scaled, but the denominator in this ratio is the scaled signal $\tilde{a}$.

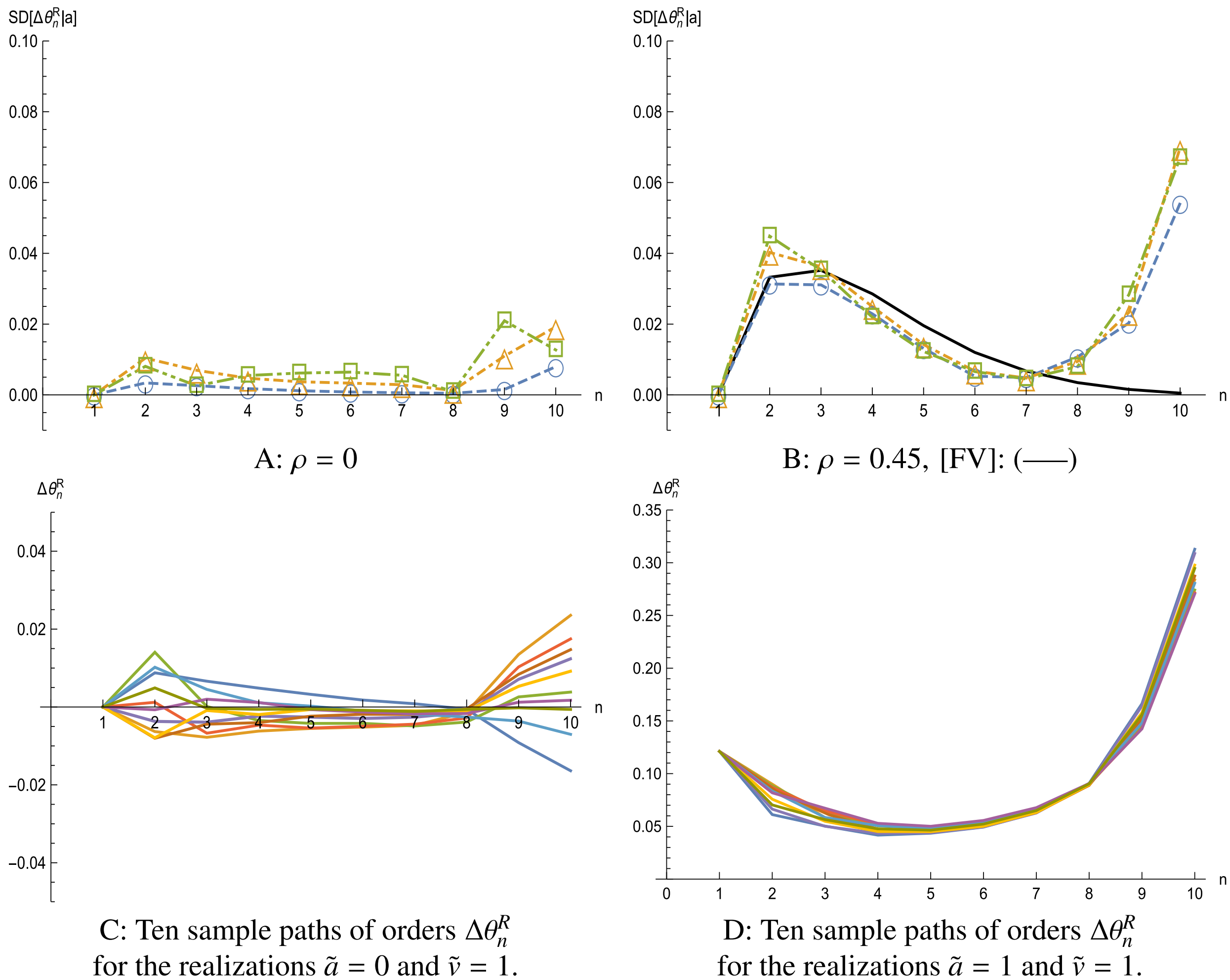

Fig. 4. Plots A and B show intraday patterns for the conditional standard deviation $\mathrm{SD}[\Delta\theta_n^R|\tilde{a}]$ of the rebalancer's orders at times $n = 1, \ldots, 10$. Plots C and D show examples of ten sample paths of rebalancer orders $\Delta\theta_n^R$ for two particular target values. The parameters are $N = 10$, $\sigma_{\tilde{v}}^2 = 1$, $\sigma_w^2 = 1$, and $\sigma_{\tilde{a}}^2 = 0.2$ (– – with ○), 1.0 (– · –with △), 2.0 (– · · · – with □) (A and B only). For C and D, $\sigma_{\tilde{a}}^2 = 1$ and $\rho = 0$.

rebalancer can filter his larger orders better than the market makers) and a larger real-time sunshine component (as market-maker expectations about the rebalancer's latent trading demand become more sensitive to aggregate order-flows).[28]

Fig. 4C shows an example of ten simulated paths of the rebalancer's order flows over time in the case of $\sigma_{\tilde{a}}^2 = 1$ and $\rho = 0$. The realized stock value $\tilde{v}$ here is one, and the realized trading target $\tilde{a}$ is zero, but the noise-trader order paths are random. Along these paths, the rebalancer buys/sells more than his terminal parent target $\tilde{a} = 0$ at early times (e.g., $n = 2$) and then unwinds his position later to achieve his trading target. This is not manipulation. Rather, it is constrained short-term speculation due to the combination of endogenous learning about $\tilde{v}$ and the trading constraint $\tilde{a}$. The rebalancer does not trade at time $n = 1$ because, given $\tilde{a} = 0$, he does not need to rebalance, and because, initially, he does not have any valuation information given $\rho = 0$. However, at time $n = 2$ the rebalancer trades based on whether — given the stock-value information gleaned from filtering the order flow y_1 better than the market makers — the stock appears over- or under-valued. Later, however, he unwinds these positions to achieve his parent target $\theta_N^R = \tilde{a} = 0$ at the end of the day. The dispersion across the paths is consistent with the intraday pattern of the rebalancer order-flow standard deviation. Paths for non-zero targets $\tilde{a}$ involve shifting the means of these paths from zero to the appropriate deterministic path given $\tilde{a}$.[29] This is illustrated in Fig. 4D for a target $\tilde{a} = 1$.

[28] The rebalancer and the informed trader acquire information at different times than each other (as in Foucault et al., 2016), and the rebalancer endogenously engages in short-term speculation (as in Froot et al., 1992), since he must unwind his speculative positions before the definitive public value revelation of $\tilde{v}$ at time $N + 1$.

[29] When the realized target $\tilde{a}$ is large, the rebalancer's orders tend to be in the same direction over time (e.g., a large positive target $\tilde{a}$ is associated with a series of buy orders). Randomness in his orders due to the q_n process (connected with sunshine trading and endogenous learning) usually just causes the rebalancer to speed up or slow down his trading relative to his expected orders given his target.

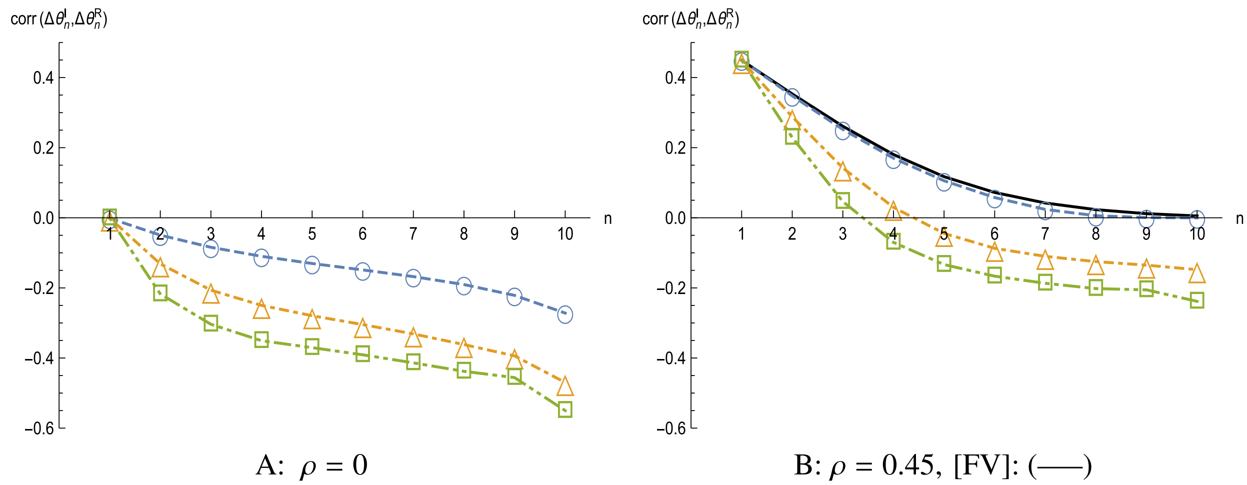

Fig. 5. Intraday patterns for the unconditional $\mathrm{corr}(\Delta\theta^I_n, \Delta\theta^R_n)$ for times $n = 1, \ldots, 10$. The parameters are $N = 10$, $\sigma^2_{\tilde{v}} = 1$, $\sigma^2_w = 1$, and $\sigma^2_{\tilde{a}} = 0.2$ (– – with ○), 1.0 (– · –with △), 2.0 (– · · – with □).

Interactions with informed-investor orders: Another factor that reduces rebalancing costs is that the rebalancer's orders tend to become negatively correlated with the hedge fund's orders over time. Fig. 5A shows that, if $\rho = 0$, then the correlation between the hedge fund's orders and the rebalancer's orders is negative at times $n > 1$. This negative correlation is mutually beneficial for the rebalancer and the hedge fund. By trading in opposite directions (in expectation), they symbiotically provide liquidity to each other (i.e., their orders partially offset each other). If $\rho > 0$, then, as illustrated in Fig. 5B, the order correlation starts out positive, but later turns negative. In contrast, orders for better-informed and less-informed investors in the modified FV model are always positively correlated.[30]

Additional analysis in the Internet Appendix shows that the predictable interaction with the informed-investor orders has a significant impact on the rebalancer's trading. First, a large part of the negative correlation between informed-investor and rebalancer orders is due to the informed investor trading against price pressure due to the rebalancer's orders. As the rebalancer trades towards his (uninformative or imperfectly informative) target $\tilde{a}$, the hedge fund trades opposite the noise that rebalancing induces in prices.[31] Second, the predictable interactions with the informed-investor orders are a quantitatively important driver of the *U*-shape in the deterministic component of the rebalancer's orders. The intuition is that the rebalancer trades less during the middle of the day to give the informed investor time to offset price pressure from the rebalancer's orders early in the day before the rebalancer then trades again later in the day.

Summary: The rebalancer orders have a large deterministic component — that depends on the parent target $\tilde{a}$ — that reflects price-impact smoothing, predictable sunshine trading, and anticipated reactions from the informed investor's trading. There is also an adaptive component due to learning and real-time sunshine trading. The adaptive component is relatively small except when the target variance and informativeness are high. These observations follow from the large rebalancer decomposition loadings on $\tilde{a}$ (in Fig. 2A) and the rebalancer order standard deviations $\mathrm{SD}[\Delta\theta^R_n | \tilde{a}]$ (in Fig. 4). They are confirmed further in the Internet Appendix based on a second decomposition using ad hoc strategies. Thus, equilibrium rebalancing strategies are more complicated than simple TWAP (time-weighted average price) strategies.[32] These features of rebalancing orders also drive the equilibrium impact of dynamic rebalancing on prices, liquidity, informed-investor trading, and the aggregate order flow discussed next in Section 4.3. In particular, the *U*-shaped patterns in the deterministic and adaptive components of the rebalancer orders are connected with *U*-shaped patterns in prices and market volume.

4.3. Equilibrium effects

Stock markets have a variety of significant empirical intraday patterns in prices and order flows.[33] We now consider how dynamic rebalancing affects the equilibrium properties of pricing and the trading behavior of other investors and, thus, the resulting intraday patterns in prices, liquidity, and order flows in our model.

30 In the modified FV model, iterated expectations gives

$$\mathrm{cov}(\Delta\hat{\theta}^I_n, \Delta\hat{\theta}^R_n) = \beta^I_n \beta^R_n \mathbb{E}[(\tilde{v} - \hat{p}_{n-1})(\hat{s}_{n-1} - \hat{p}_{n-1})] = \beta^I_n \beta^R_n \mathbb{V}[\hat{s}_{n-1} - \hat{p}_{n-1}],$$

which is positive given $\beta^I_n > 0$ and $\beta^R_n > 0$, where $\hat{s}_{n-1} = \mathbb{E}[\tilde{v}|\tilde{a}, y_1, \ldots, y_{n-1}]$.

31 Foster and Viswanathan (1996) also has negative cross-investor order correlation when both investors have symmetric noisy signals. However, our price-pressure correction mechanism is different from their Bayesian learning mechanism.

32 The ability to reduce trading costs on parent orders benchmarked to TWAP and VWAP (value-weighted average price) is part of the business model for agency order execution.

33 Intraday patterns are robust properties of volume and price volatility in equity markets that were first documented in Wood et al. (1985) and Jain and Joh (1988).

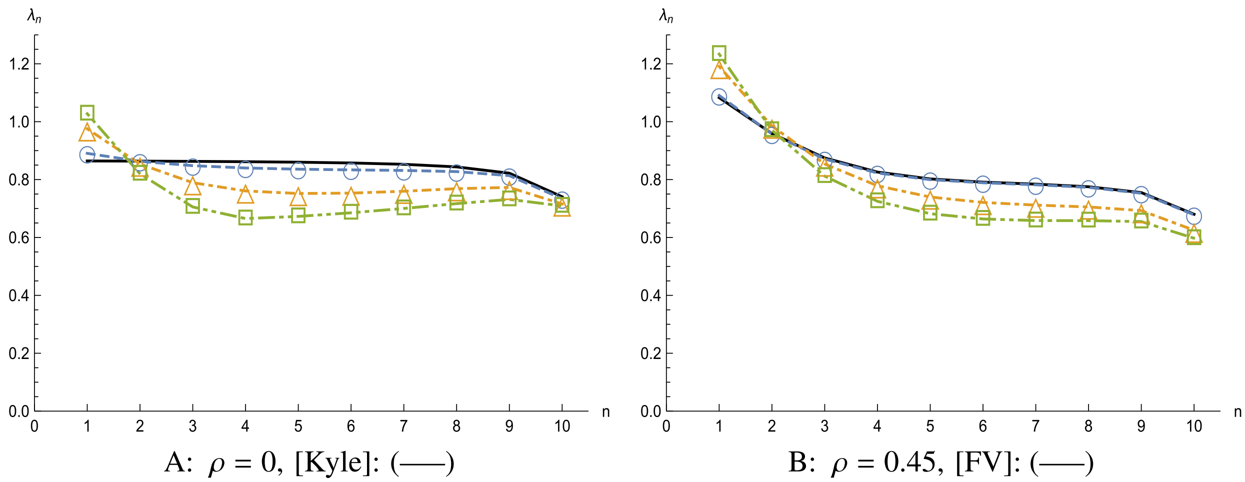

Fig. 6. Intraday patterns for price impacts λ_n for times $n = 1, \dots, 10$. The parameters are $N = 10$, $\sigma_{\tilde{v}}^2 = 1$, $\sigma_w^2 = 1$, and $\sigma_{\tilde{a}}^2 = 0.2$ (– – with ○), 1.0 (– · –with △), 2.0 (– · · · – with □).

The economics underlying these equilibrium effects follows from how dynamic rebalancing affects the mix of information and noise in the aggregate order flow. There are two direct channels for this effect: First, the trading target $\tilde{a}$ can be written as a combination of noise and valuation information

$$\tilde{a} = \sigma_{\tilde{a}} \left[\frac{\rho}{\sigma_{\tilde{v}}} \tilde{v} + \sqrt{1 - \rho^2}\, \tilde{\epsilon} \right] \tag{55}$$

where (given the joint multivariate normality) $\tilde{\epsilon}$ is a standard Normal random variable that is independent of $\tilde{v}$, and where ρ controls the information content in $\tilde{a}$, and $\sigma_{\tilde{a}}$ scales the volatility of $\tilde{a}$ (but not its informativeness) and, thus, scales the magnitude of the constraint on the rebalancer's trading. The second direct channel is that the rebalancer speculates on private information about $\tilde{v}$ learned endogenously over time by filtering the aggregate order flow better than market makers. There are also indirect channels due to equilibrium effects of information-competition and rebalancing noise on how the hedge fund trades on its private information about $\tilde{v}$.

Price dynamics: Fig. 6 shows how dynamic rebalancing affects the price impact of order-flow parameter λ_n over the trading day. This relation is complicated because it is the net effect of all of the direct and indirect channels through which rebalancing affects the order-flow mix of information and noise. It is further complicated because the relation between λ_n and the aggressiveness β_n^I of informed trading in (29) is non-monotone. However, individual channels can be isolated in a few special cases. Consider the $\rho = 0$ case in Fig. 6A. At time $n = 1$, there has been no endogenous learning by the rebalancer, and, given $\rho = 0$, the target $\tilde{a}$ is uninformative noise. From (29), the direct effect of the rebalancing noise at $n = 1$ is, therefore, to lower the price impact λ_1. Hence, the fact that the equilibrium λ_1 with rebalancing (non-black dashed lines) actually increases relative to Kyle (solid black line) is entirely due to the indirect effect of rebalancing on the informed-investor trading at time $n = 1$. At later times $n > 2$, the price impacts in Fig. 6A are lower than in Kyle. The result is a twist in the slope of λ_n over time. Fig. 6B shows similar twists relative to the modified FV model (same black line given the independence from $\sigma_{\tilde{a}}^2$ in the modified FV model) when $\rho > 0$. The twist in λ_n consistently increases when there is more trading-target volatility $\sigma_{\tilde{a}}^2$, as shown in both Figs. 6A and 6B. The price-impact twist in our model differs from Degryse et al. (2014) in which intraday price impacts have an inverted *U*-shape (see their Fig. 1). This difference is due to the direct and indirect effects of endogenous learning given long-lived information and, when $\rho > 0$, of the fact that rebalancing targets in our model are then also informative.

Fig. 7 shows the variance $\Sigma_n^{(2)}$ of the market pricing errors $\tilde{v} - p_n$ over time, which measures the quality of price discovery. When $\rho = 0$, more information is revealed at early times compared to the Kyle model (due to more aggressive informed trading by the hedge fund, see below), but pricing accuracy is reduced later in the day. When $\rho > 0$ (so that $\tilde{a}$ is ex ante informative), the trading target constrains the aggressiveness of the rebalancer's orders relative to the unconstrained purely informational orders of the less-informed investor in the modified FV model. This constraint, depending on the parameterization, can cause the rebalancer's orders to be larger or smaller than in the modified FV model. For example, holding fixed the informativeness of the target at $\rho = 0.45$, a larger target variance $\sigma_{\tilde{a}}^2$ increases the size of the rebalancer's orders induced by a target realization with a given amount of asset-value information, which leads to faster information aggregation in Fig. 7. This is due to both the direct effect of larger information-based rebalancer trades and also an indirect information-competition race-to-trade effect that increases the aggressiveness of the informed hedge funds' orders (see Fig. 9 below). The Internet Appendix shows further that these price-discovery dynamics lead to *U*-shaped intraday patterns in price volatility that are increasing in the rebalancing target variance $\sigma_{\tilde{a}}^2$.

A novel feature of dynamic rebalancing is sunshine trading, since predictable orders do not have price impacts (see (10)). The key variable here is the market-maker

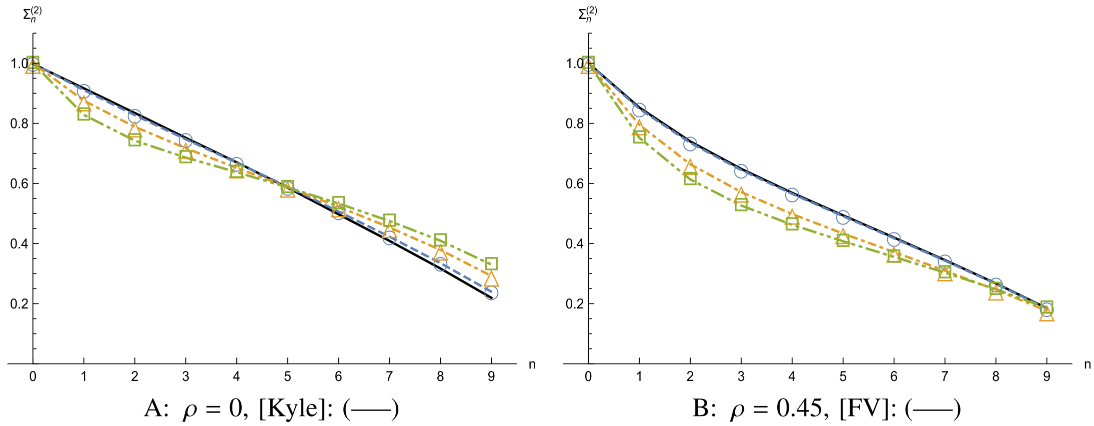

Fig. 7. Intraday patterns for the variance $\Sigma_n^{(2)}$ of the pricing error $\tilde{v} - p_n$ for times $n = 0, 1, \ldots, 9$. The parameters are $N = 10$, $\sigma_{\tilde{v}}^2 = 1$, $\sigma_w^2 = 1$, and $\sigma_{\tilde{a}}^2 = 0.2$ (– – with ○), 1.0 (– · –with △), 2.0 (– · · – with □).

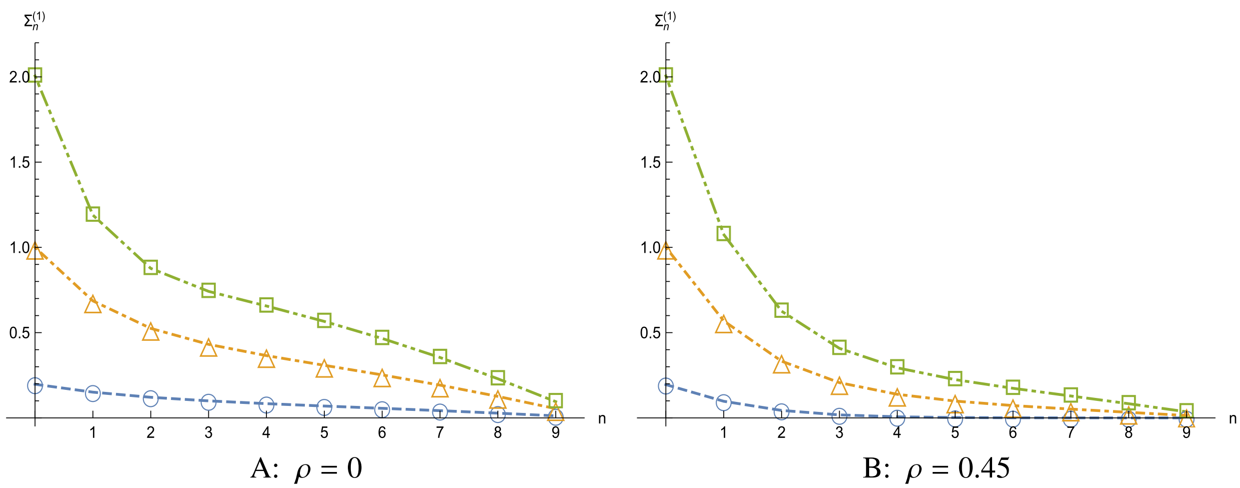

Fig. 8. Intraday patterns for the variance $\Sigma_n^{(1)}$ of uncertainty about remaining latent trading-demand $\tilde{a} - \theta_n^R - q_n$ for times $n = 0, 1, \ldots, 9$. The parameters are $N = 10$, $\sigma_{\tilde{v}}^2 = 1$, $\sigma_w^2 = 1$, and $\sigma_{\tilde{a}}^2 = 0.2$ (– – with ○), 1.0 (– · –with △), 2.0 (– · · – with □).

expectation q_n of the rebalancer's remaining latent trading demand $\tilde{a} - \theta_{n-1}^R$. Fig. 8 shows the market makers' uncertainty $\Sigma_n^{(1)} = \mathbb{V}[\tilde{a} - \theta_n^R - q_n]$ about the rebalancer's remaining latent trading demand. Although a priori $\Sigma_n^{(1)}$ need not be monotone over time (see footnote 19), Fig. 8 shows that uncertainty about the remaining latent trading demand is monotonely decreasing for a wide range of values of $\sigma_{\tilde{a}}^2$ and ρ.

Informed investor: Fig. 9 shows the hedge fund's strategy coefficients β_n^I, which determine how aggressively the hedge-fund manager trades on her private information $\tilde{v} - p_{n-1}$ over time. As in Kyle (1985), the intensity of informed trading in our model increases as time approaches the terminal time N. This is consistent with the fact that the incentive to delay trading on information becomes weaker later in the day as the remaining time left for trading becomes shorter. We also see that the effect of increased rebalancing target variance $\sigma_{\tilde{a}}^2$ on informed trading is *U*-shaped. Increasing $\sigma_{\tilde{a}}^2$ increases β_n^I (i.e., causes the informed investor to trade more aggressively) earlier and later in the day but leaves β_n^I relatively unchanged in the middle of the day. In addition, if $\rho > 0$, hedge-fund trading aggressiveness increases somewhat due to the information-competition effect. The apparent size of the changes in β_1^I — which are on the order of 10% — is visually understated in Fig. 9 because of the vertical scaling (due to the size of β_{10}^I).

A linear decomposition for the informed hedge fund's orders,

$$\Delta\theta_n^I = A_n^I \tilde{a} + B_n^I \tilde{v} + \sum_{j=1,\ldots n-1} c_{j,n}^I \Delta w_j, \tag{56}$$

lets us break the hedge fund's orders into a deterministic component given the signal $\tilde{v}$ and an adaptive component that depends on the rebalancer target $\tilde{a}$ and the noise-trader orders. This decomposition is considered further in the Internet Appendix.

Aggregate order-flow and volume: Autocorrelation in the aggregate order flow is one of the novel effects of dynamic

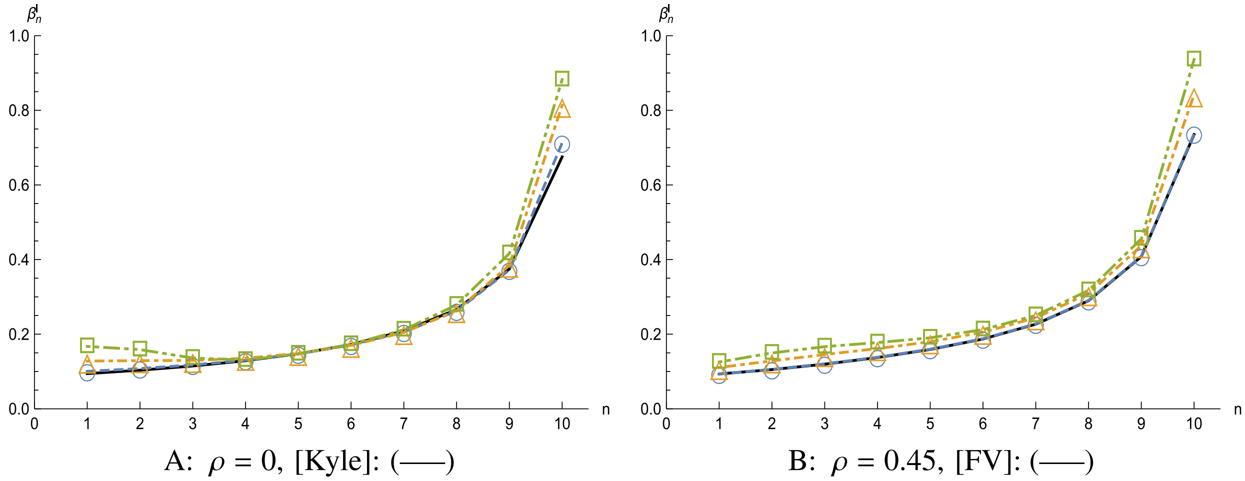

Fig. 9. Intraday patterns for the hedge-fund strategy coefficient β_n^I at times $n = 1, \ldots, 10$. The parameters are $N = 10$, $\sigma_{\tilde{v}}^2 = 1$, $\sigma_w^2 = 1$, and $\sigma_{\tilde{a}}^2 = 0.2$ (– – with ○), 1.0 (– · –with △), 2.0 (– · · – with □).

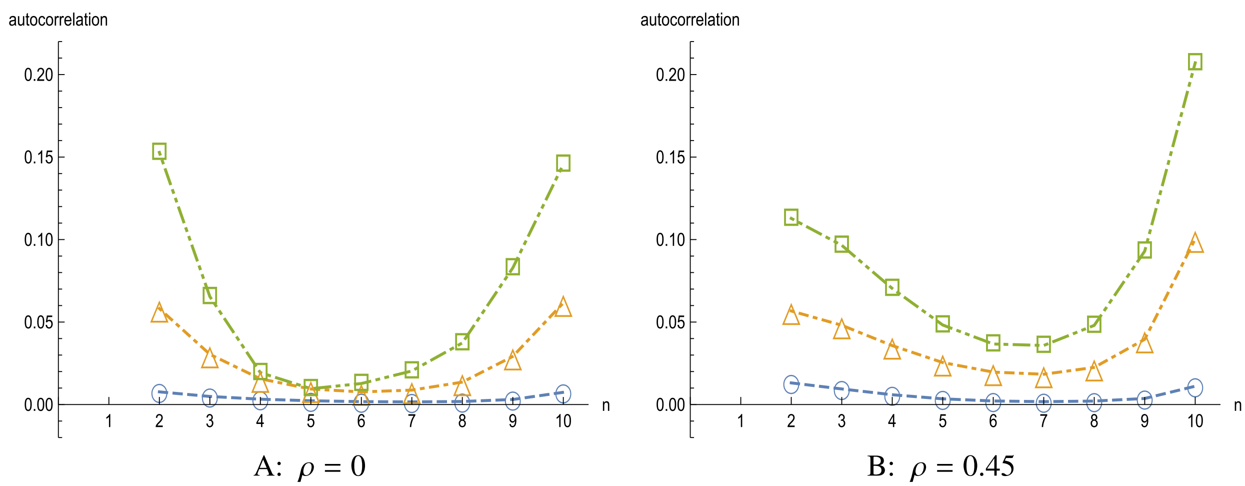

Fig. 10. Intraday patterns for the aggregate order-flow autocorrelation $\frac{\mathbb{E}[y_{n-1}y_n]}{\sqrt{\mathbb{E}[y_{n-1}^2]\mathbb{E}[y_n^2]}}$ for times $n = 2, 3, \ldots, 10$. The parameters are $N = 10$, $\sigma_{\tilde{v}}^2 = 1$, $\sigma_w^2 = 1$, and $\sigma_{\tilde{a}}^2 = 0.2$ (– – with ○), 1.0 (– · –with △), 2.0 (– · · – with □).

rebalancing. Fig. 10 shows the unconditional autocorrelation of the (signed) aggregate order flow over the trading day. Although the absolute level of autocorrelation is not high, there is a clear *U*-shaped pattern of higher order-flow autocorrelation at the beginning and end of the day (when, from Fig. 3, the rebalancer trades more) and lower autocorrelation during the middle of the day (when the rebalancer trades less). Our parameter-space analysis shows that the order-flow autocorrelation level and the magnitude of the *U*-shape are both increasing in the target variance $\sigma_{\tilde{a}}^2$.

Market trading volume over the day is also affected by dynamic rebalancing. Our proxy for trading volume is

$$\max(0, \Delta\theta_n^R) + \max(0, \Delta\theta_n^I) + \max(0, \Delta w_n) + \max(0, -y_n), \quad (57)$$

which is buy-side volume except that it ignores crosses among the noise traders. Fig. 11 confirms that the *U*-shaped intraday patterns of rebalancer volume carry over and induce *U*-shaped patterns in the intraday means and standard deviations of market volume in the rebalancing model and also relative to Kyle (1985) and the modified FV model. As can be seen, these *U*-shaped volume patterns are increasing in the parent-target variance $\sigma_{\tilde{a}}^2$.

4.4. Asset-value variance and intraday patterns

A variety of intraday patterns in pricing and order flows is documented in the previous sections. Proposition 1 can be extended to show that these intraday patterns are either insensitive to asset-value volatility or scale simply relative to $\sigma_{\tilde{v}}$.

Proposition 3. Given a market parameterization $\{\sigma_{\tilde{a}}^2, \sigma_{\tilde{v}}^2, \rho\,\sigma_{\tilde{a}}\,\sigma_{\tilde{v}}\}$ *as in (49) with an equilibrium, if the parameterization changes to* $\{\sigma_{\tilde{a}}^2, h^2\sigma_{\tilde{v}}^2, \rho\,\sigma_{\tilde{a}}\,h\,\sigma_{\tilde{v}}\}$, *then in the new equilibrium the intraday patterns in market characteristics change as follows:*

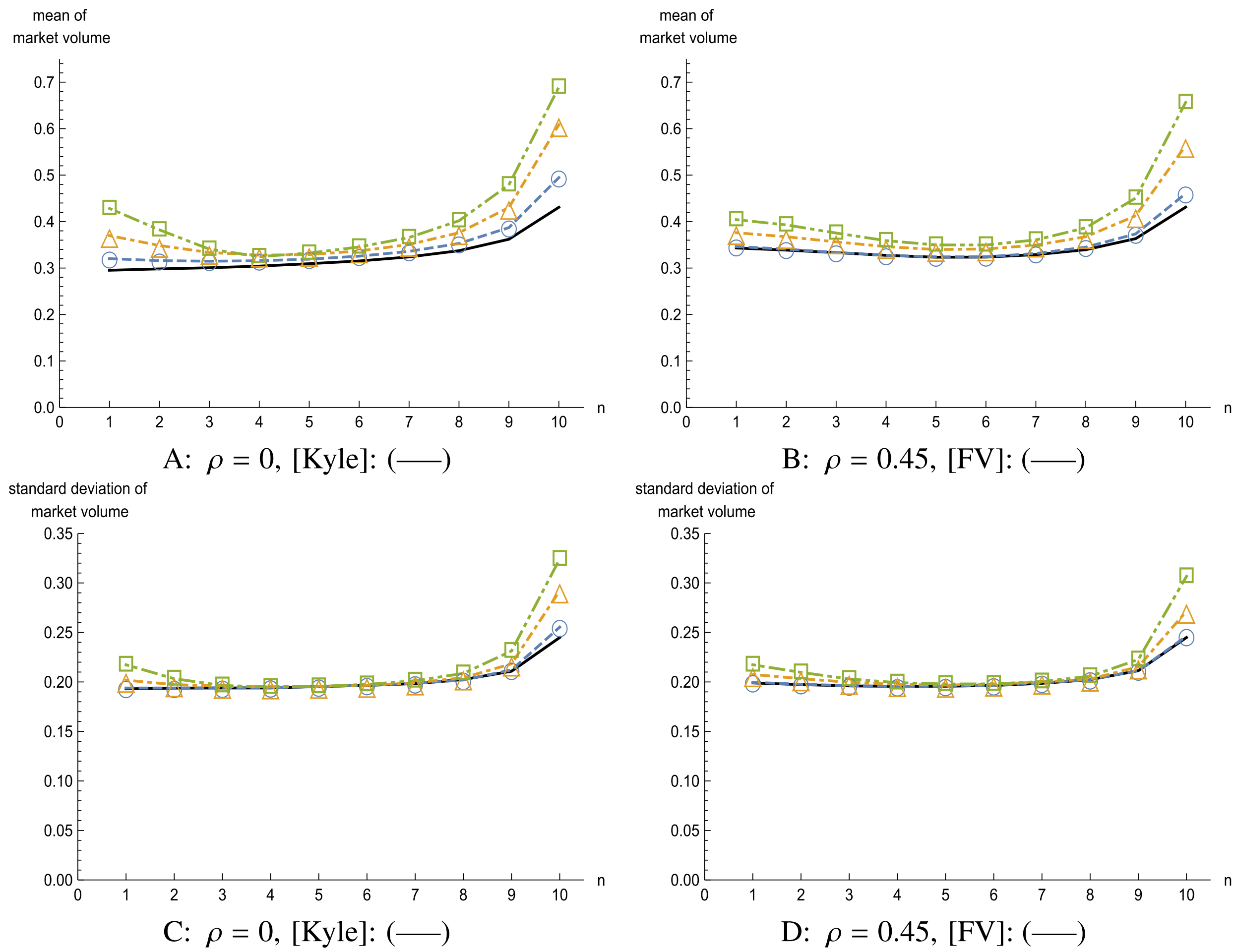

Fig. 11. Plots A and B show intraday patterns in the unconditional means of market volume, and plots C and D show the unconditional standard deviations of market volume at times $n = 1, \ldots, 10$. The parameters are $N = 10$, $\sigma_{\tilde{v}}^2 = 1$, $\sigma_w^2 = 1$, and $\sigma_{\tilde{a}}^2 = 0.2$ (– – with ○), 1.0 (– · –with △), 2.0 (– · · – with □).

- *The order-decomposition coefficients A_n^R and $c_{j,n}^R$ for the rebalancer and A_n^I and $c_{j,n}^I$ for the informed investor on $\tilde{a}$ and Δw_j are unaffected by h.*
- *The order-decomposition coefficients B_n^R and B_n^I on $\tilde{v}$ become B_n^R/h and B_n^I/h.*
- *The expectation $\mathbb{E}[\Delta\theta_n^R|\tilde{a}]$, sunshine-trading ratio, $SD[\Delta\theta_n^R|\tilde{a}]$, order correlation $corr[\Delta\theta_n^I, \Delta\theta_n^R]$, and aggregate order-flow autocorrelation are all unaffected by h.*
- *The informed-investor expected volume $\mathbb{E}[|\Delta\theta_n^I||\tilde{v}]$ becomes $\frac{1}{h}\,\mathbb{E}[|\Delta\theta_n^I||\tilde{v}]$.*
- *The price-change volatility $SD[\Delta p_n]$ becomes $h\ SD[\Delta p_n]$.*

The proposition follows from algebraic substitution of the scaling factor h in the expressions for the various quantities of interest. As in Proposition 1, the rebalancer's trading strategy is relatively unaffected by the stock-value variance. One exception is the coefficient B_n^R, but this is just a pass-through from the informed-investor orders in the aggregate order flow.

5. Robustness

The qualitative properties of our model are likely to be robust to relaxing our modeling assumptions. First, our model assumes a hard rebalancing constraint. Alternatively, the rebalancing constraint could be soft with a quadratic penalty for deviations from the target, or investors could have a random private value for the asset that is decreasing in their terminal holdings. In either case, the rebalancer should still engage in order-splitting to reduce their trading costs. These alternative rebalancing motives should result in some amount of price elasticity in the total amount traded by rebalancers. This should increase the importance of the adaptive part of rebalancer orders that responds to the prior order-flow history.

Second, informed investors and rebalancers only use market orders in our model. In practice, however, order-splitting algorithms also use limit orders (see O'Hara, 2015). While the mathematics of the dynamic programming problems and the rational-expectations fixed point would be complicated, we still expect rebalancing to result in order-flow autocorrelation and for predictable components of market and limit order flows to have no persistent price impacts. Empirically, limit order flows are also autocorrelated (see Biais et al., 1995).

Third, our market makers are competitive, risk-neutral, and have no order processing costs. As a result, prices are martingales in our model. We do not expect market-making frictions and transitory price effects to eliminate the informational aspects of rebalancing. It would be interesting to investigate empirically how transitory market frictions and persistent informational aspects of order-splitting interact.

6. Conclusion

This paper has explored dynamic order-splitting for portfolio rebalancing and its equilibrium interactions with price discovery, order-flow dynamics, and market liquidity. Our paper is the first to investigate these issues with both long-lived information and dynamic rebalancing given a terminal parent trading target. Dynamic rebalancing does not just inject additional trading noise in the market; rather it affects the structure of the market equilibrium. Order flow becomes autocorrelated and liquidity and price-discovery dynamics change because of sunshine trading. In addition, dynamic rebalancing affects equilibrium prices and also the process for arriving orders from the informed investor. Our model has a variety of empirically testable implications for intraday market patterns and their co-movement with rebalancing target volatility.

Our model has many interesting possible extensions for future theory. One possible extension is to model dynamic rebalancing in continuous-time. Another extension is to relax the assumption that all investors are risk-neutral. For example, exponential utility is a natural specification to consider. Finally, our model could be extended to include multiple informed investors and rebalancers.

Appendix A. Proofs and algorithm

A1. Kalman filtering

Lemma 1. Consider the conjectured system (18)–(22) corresponding to arbitrary coefficients $\{\beta_n^I, \beta_n^R, \alpha_n^R\}_{n=1}^N$. *Whenever (29)–(30) hold, we have*

$$\hat{p}_n = \mathbb{E}[\tilde{v}\,|\,\hat{y}_1,\dots,\hat{y}_n], \tag{A.1}$$

$$\hat{q}_n = \mathbb{E}[\tilde{a}-\hat{\theta}_n^R\,|\,\hat{y}_1,\dots,\hat{y}_n], \tag{A.2}$$

where $\hat{p}$ *is defined by (21) and* $\hat{q}$ *is defined by (22). Furthermore, the recursions for the variances and covariance (31)–(33) hold.*

Proof. For $n=1,\dots,N$, we have the moment definitions in (23)–(25) where the starting values are given in (26). We then define the process $\hat{z}_n^M$ as

$$\begin{aligned}\hat{z}_n^M &:= \hat{y}_n - (\alpha_n^R+\beta_n^R)\hat{q}_{n-1}\\ &= \beta_n^I(\tilde{v}-\hat{p}_{n-1}) + \beta_n^R(\tilde{a}-\hat{\theta}_{n-1}^R-\hat{q}_{n-1}) + \Delta w_n.\end{aligned} \tag{A.3}$$

These Gaussian variables $\hat{z}_1^M, \hat{z}_2^M, \dots, \hat{z}_N^M$ are mutually independent and satisfy $\sigma(\hat{z}_1^M,\dots,\hat{z}_n^M) = \sigma(\hat{y}_1,\dots\hat{y}_n)$. The projection theorem for Gaussian random variables gives

$$\begin{aligned}\Delta\hat{p}_n &= \mathbb{E}[\tilde{v}\,|\,\hat{z}_1^M,\dots,\hat{z}_n^M] - \mathbb{E}[\tilde{v}\,|\,\hat{z}_1^M,\dots,\hat{z}_{n-1}^M]\\ &= \frac{\mathbb{E}[\tilde{v}\,\hat{z}_n^M]}{\mathbb{V}[\hat{z}_n^M]}\hat{z}_n^M,\end{aligned} \tag{A.4}$$

$$\begin{aligned}\Delta\hat{q}_n &= \mathbb{E}[\tilde{a}-\hat{\theta}_n^R\,|\,\hat{z}_1^M,\dots\hat{z}_n^M] - \mathbb{E}[\tilde{a}-\hat{\theta}_{n-1}^R\,|\,\hat{z}_1^M,\dots,\hat{z}_{n-1}^M]\\ &= \mathbb{E}[\tilde{a}-\hat{\theta}_{n-1}^R\,|\,\hat{z}_1^M,\dots\hat{z}_n^M] - \mathbb{E}[\tilde{a}-\hat{\theta}_{n-1}^R\,|\,\hat{z}_1^M,\dots,\hat{z}_{n-1}^M]\\ &\quad - \mathbb{E}[\Delta\hat{\theta}_n^R\,|\,\hat{z}_1^M,\dots,\hat{z}_n^M]\\ &= \frac{\mathbb{E}[(\tilde{a}-\hat{\theta}_{n-1}^R)\hat{z}_n^M]}{\mathbb{V}[\hat{z}_n^M]}\hat{z}_n^M\\ &\quad - \mathbb{E}\Big[\beta_n^R(\tilde{a}-\hat{\theta}_{n-1}^R-\hat{q}_{n-1})\\ &\quad + (\alpha_n^R+\beta_n^R)\hat{q}_{n-1}\,\Big|\,\hat{z}_1^M,\dots,\hat{z}_n^M\Big]\\ &= \frac{\mathbb{E}[(\tilde{a}-\hat{\theta}_{n-1}^R-\hat{q}_{n-1})\hat{z}_n^M]}{\mathbb{V}[\hat{z}_n^M]}\hat{z}_n^M\\ &\quad - \beta_n^R\mathbb{E}[\tilde{a}-\hat{\theta}_{n-1}^R-\hat{q}_{n-1}\,|\,\hat{z}_n^M] - (\alpha_n^R+\beta_n^R)\hat{q}_{n-1}\\ &= (1-\beta_n^R)\frac{\mathbb{E}[(\tilde{a}-\hat{\theta}_{n-1}^R-\hat{q}_{n-1})\hat{z}_n^M]}{\mathbb{V}[\hat{z}_n^M]}\hat{z}_n^M\\ &\quad - (\alpha_n^R+\beta_n^R)\hat{q}_{n-1}.\end{aligned} \tag{A.5}$$

To proceed, we first compute

$$\begin{aligned}\mathbb{V}[\hat{z}_n^M] &= \mathbb{E}\Big[\Big(\beta_n^I(\tilde{v}-\hat{p}_{n-1})\\ &\quad + \beta_n^R(\tilde{a}-\hat{\theta}_{n-1}^R-\hat{q}_{n-1}) + \Delta w_n\Big)^2\Big]\\ &= (\beta_n^I)^2\Sigma_{n-1}^{(2)} + (\beta_n^R)^2\Sigma_{n-1}^{(1)} + 2\beta_n^I\beta_n^R\Sigma_{n-1}^{(3)} + \sigma_w^2\Delta,\end{aligned} \tag{A.6}$$

$$\begin{aligned}\mathbb{E}[\tilde{v}\hat{z}_n^M] &= \mathbb{E}[(\tilde{v}-\hat{p}_{n-1})\hat{z}_n^M]\\ &= \mathbb{E}\Big[(\tilde{v}-\hat{p}_{n-1})\Big(\beta_n^I(\tilde{v}-\hat{p}_{n-1})\\ &\quad + \beta_n^R(\tilde{a}-\hat{\theta}_{n-1}^R-\hat{q}_{n-1}) + \Delta w_n\Big)\Big]\\ &= \beta_n^I\Sigma_{n-1}^{(2)} + \beta_n^R\Sigma_{n-1}^{(3)},\end{aligned} \tag{A.7}$$

$$\begin{aligned}&\mathbb{E}[(\tilde{a}-\hat{\theta}_{n-1}^R-\hat{q}_{n-1})\hat{z}_n^M]\\ &= \mathbb{E}\Big[(\tilde{a}-\hat{\theta}_{n-1}^R-\hat{q}_{n-1})\Big(\beta_n^I(\tilde{v}-\hat{p}_{n-1})\\ &\quad + \beta_n^R(\tilde{a}-\hat{\theta}_{n-1}^R-\hat{q}_{n-1}) + \Delta w_n\Big)\Big]\\ &= \beta_n^I\Sigma_{n-1}^{(3)} + \beta_n^R\Sigma_{n-1}^{(1)}.\end{aligned} \tag{A.8}$$

Combining these expressions and by matching coefficients with (21) and (22), we find the lemma's statement equivalent to the restrictions (29)–(30). Based on these expressions, the recursion for $\Sigma_n^{(1)}$, $n=1,\dots,N$, in (31) is

$$\begin{aligned}\Sigma_n^{(1)} &:= \mathbb{V}[\tilde{a}-\hat{\theta}_n^R-\hat{q}_n]\\ &= \mathbb{V}[\tilde{a}-\hat{\theta}_{n-1}^R-\hat{q}_{n-1}-\Delta\hat{\theta}_n^R-\Delta\hat{q}_n]\\ &= \mathbb{V}[\tilde{a}-\hat{\theta}_{n-1}^R-\hat{q}_{n-1}-\Delta\hat{\theta}_n^R-r_n\hat{y}_n\\ &\quad + (1+r_n)(\alpha_n^R+\beta_n^R)\hat{q}_{n-1}]\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{V}\Big[\tilde a - \hat\theta^R_{n-1} - (1-(1+r_n)(\alpha^R_n+\beta^R_n))\hat q_{n-1} \\
&\quad - (1+r_n)(\beta^R_n(\tilde a - \hat\theta^R_{n-1}) + \alpha^R_n \hat q_{n-1}) \\
&\quad - r_n\Big(\beta^I_n(\tilde v - \hat p_{n-1})\Big) - r_n\Delta w_n\Big] \\
&= \mathbb{V}\Big[\Big(1-(1+r_n)\beta^R_n\Big)(\tilde a - \hat\theta^R_{n-1}) \\
&\quad - \Big(1-(1+r_n)\beta^R_n\Big)\hat q_{n-1} \\
&\quad - r_n\beta^I_n(\tilde v - \hat p_{n-1}) - r_n\Delta w_n\Big] \\
&= \mathbb{V}\Big[\Big(1-(1+r_n)\beta^R_n\Big)(\tilde a - \hat\theta^R_{n-1} - \hat q_{n-1}) \\
&\quad - r_n\beta^I_n(\tilde v - \hat p_{n-1}) - r_n\Delta w_n\Big] \\
&= \Big(1-(1+r_n)\beta^R_n\Big)^2\Sigma^{(1)}_{n-1} + (r_n\beta^I_n)^2\Sigma^{(2)}_{n-1} \\
&\quad + r_n^2\sigma_w^2\Delta - 2\Big(1-(1+r_n)\beta^R_n\Big)r_n\beta^I_n\Sigma^{(3)}_{n-1} \\
&= (1-\beta^R_n)\Big((1-\beta^R_n - r_n\beta^R_n)\Sigma^{(1)}_{n-1} - r_n\beta^I_n\Sigma^{(3)}_{n-1}\Big),
\end{aligned} \tag{A.9}
$$

where the last equality uses (30). The recursions for $\Sigma^{(2)}_n$ and $\Sigma^{(3)}_n$, $n=1,\dots,N$, in (32) and (33) are found similarly. □

A2. Informed investor's optimization problem

We start with the following lemma which contains most of the calculations we will need later. Recall the hedge fund's state processes $\{X^{(1)}_n, X^{(2)}_n\}$ are defined by (35).

Lemma 2. Fix the constants (15) subject to the pricing-coefficient restrictions (29)–(30) holding and use them to define $\Delta\theta^R_n$ by (5) and to define the moments (31)–(33) with initial values (26). Let $\Delta\theta^I_n \in \sigma(\tilde v, y_1,\dots,y_{n-1})$, $n=1,\dots,N$, be arbitrary for the hedge fund. We can then define the Gaussian random variables

$$
\hat z^I_n := \hat y_n - \Delta\hat\theta^I_n - (\alpha^R_n+\beta^R_n)\hat q_{n-1} - \beta^R_n\tfrac{\Sigma^{(3)}_{n-1}}{\Sigma^{(2)}_{n-1}}(\tilde v - \hat p_{n-1}), \quad n=1,\dots,N \tag{A.10}
$$

where the conjectured "hat" processes are defined in (18)–(22). The variable $\hat z^I_k$ is independent of $\{\tilde v, \hat y_1, \dots, \hat y_{k-1}\}$ for $k \le N$, and the following measurability properties are satisfied:

$$
\hat\theta^R_n - \theta^R_n \in \sigma(\tilde v, y_1,\dots,y_n) = \sigma(\tilde v, \hat y_1,\dots,\hat y_n) = \sigma(\tilde v, \hat z_1, \dots \hat z_n), \quad n \in 1,\dots,N. \tag{A.11}
$$

Furthermore, the state variables $X^{(1)}_n$ and $X^{(2)}_n$ defined in (35) for $n=1,\dots,N$ have Markovian dynamics

$$
\Delta X^{(1)}_n = -\lambda_n\Big(\Delta\theta^I_n + \beta^R_n X^{(2)}_{n-1}\Big) - \lambda_n \hat z^I_n, \quad X^{(1)}_0 = \tilde v, \tag{A.12}
$$

$$
\Delta X^{(2)}_n = -r_n\Delta\theta^I_n - (1+r_n)\beta^R_n X^{(2)}_{n-1} - \frac{\Sigma^{(3)}_n}{\Sigma^{(2)}_n}\lambda_n \hat z^I_n, \quad X^{(2)}_0 = \frac{\rho\sigma_{\tilde a}}{\sigma_{\tilde v}}\tilde v. \tag{A.13}
$$

Finally, for any constants $I^{(1,1)}_n, I^{(1,2)}_n$, and $I^{(2,2)}_n$, we have the conditional expectation

$$
\begin{aligned}
&\mathbb{E}\Big[(\tilde v - p_n)\Delta\theta^I_n + I^{(1,1)}_n\Big(X^{(1)}_n\Big)^2 + I^{(1,2)}_n X^{(1)}_n X^{(2)}_n \\
&\quad + I^{(2,2)}_n\Big(X^{(2)}_n\Big)^2 \,\Big|\, \tilde v, y_1,\dots,y_{n-1}\Big] \\
&= X^{(1)}_{n-1}\Delta\theta^I_n - (\Delta\theta^I_n)^2\lambda_n - \Delta\theta^I_n\lambda_n\beta^R_n X^{(2)}_{n-1} \\
&\quad + I^{(1,1)}_n\Big(\Big(X^{(1)}_{n-1}\Big)^2 - 2\lambda_n X^{(1)}_{n-1}\Big(\Delta\theta^I_n + \beta^R_n X^{(2)}_{n-1}\Big) \\
&\quad + \lambda_n^2\Big(\Delta\theta^I_n + \beta^R_n X^{(2)}_{n-1}\Big)^2 + \lambda_n^2\mathbb{V}[\hat z^I_n]\Big) \\
&\quad + I^{(1,2)}_n\Big(X^{(1)}_{n-1}X^{(2)}_{n-1} - X^{(1)}_{n-1}\Big(r_n\Delta\theta^I_n + (1+r_n)\beta^R_n X^{(2)}_{n-1}\Big) \\
&\quad - X^{(2)}_{n-1}\lambda_n\Big(\Delta\theta^I_n + \beta^R_n X^{(2)}_{n-1}\Big) \\
&\quad + \lambda_n\Big(\Delta\theta^I_n + \beta^R_n X^{(2)}_{n-1}\Big)\Big(r_n\Delta\theta^I_n + (1+r_n)\beta^R_n X^{(2)}_{n-1}\Big) \\
&\quad + \lambda_n^2\tfrac{\Sigma^{(3)}_n}{\Sigma^{(2)}_n}\mathbb{V}[\hat z^I_n]\Big) \\
&\quad + I^{(2,2)}_n\Big(\Big(X^{(2)}_{n-1}\Big)^2 - 2X^{(2)}_{n-1}\Big(r_n\Delta\theta^I_n + (1+r_n)\beta^R_n X^{(2)}_{n-1}\Big) \\
&\quad + \Big(r_n\Delta\theta^I_n + (1+r_n)\beta^R_n X^{(2)}_{n-1}\Big)^2 + \lambda_n^2\Big(\tfrac{\Sigma^{(3)}_n}{\Sigma^{(2)}_n}\Big)^2\mathbb{V}[\hat z^I_n]\Big),
\end{aligned} \tag{A.14}
$$

which is quadratic in $\Delta\theta^I_n$, and where the variance $\mathbb{V}[\hat z^I_n]$ can be computed to be

$$
\mathbb{V}[\hat z^I_n] = (\beta^R_n)^2\left(\Sigma^{(1)}_{n-1} - \frac{\Big(\Sigma^{(3)}_{n-1}\Big)^2}{\Sigma^{(2)}_{n-1}}\right) + \sigma_w^2\Delta. \tag{A.15}
$$

Proof. The joint normality claim follows by an induction argument. To see the independence claim for $\hat z^I_n$ in (A.10), notice that $\hat z^I_n$ is the order-flow innovation process for the informed investor

$$
\begin{aligned}
&\hat y_n - \mathbb{E}[\hat y_n | \tilde v, \hat y_1, \dots, \hat y_{n-1}] \\
&= \beta^R_n\Big(\tilde a - \hat\theta^R_{n-1} - \hat q_{n-1} \\
&\quad - \mathbb{E}\Big[\tilde a - \hat\theta^R_{n-1} - \hat q_{n-1} \,|\, \tilde v, \hat y_1,\dots,\hat y_{n-1}\Big]\Big) + \Delta w_n \\
&= \beta^R_n\Big(\tilde a - \hat\theta^R_{n-1} - \hat q_{n-1} - \tfrac{\Sigma^{(3)}_{n-1}}{\Sigma^{(3)}_{n-1}}(\tilde v - \hat p_{n-1})\Big) \\
&\quad + \alpha^R_n\hat q_{n-1} - \alpha^R_n\hat q_{n-1} + \Delta w_n \\
&= \hat y_n - \Delta\hat\theta^I_n - (\alpha^R_n+\beta^R_n)\hat q_{n-1} - \beta^R_n\tfrac{\Sigma^{(3)}_{n-1}}{\Sigma^{(3)}_{n-1}}\Big(\tilde v - \hat p_{n-1}\Big) \\
&= \hat z^I_n.
\end{aligned} \tag{A.16}
$$

Let $k \le n-1$ be arbitrary, and then iterated expectations produce the zero-correlation property:

$$
\begin{aligned}
\mathbb{E}[\hat y_k \hat z^I_n] &= \mathbb{E}[\mathbb{E}[\hat y_k \hat z^I_n \,|\, \tilde v, \hat y_1,\dots,\hat y_k]] \\
&= \mathbb{E}[\hat y_k \mathbb{E}[\hat z^I_n \,|\, \tilde v, \hat y_1,\dots,\hat y_k]] = 0.
\end{aligned} \tag{A.17}
$$

Independence follows then from the joint normality.

Next, we observe that the last equality in (A.11) follows directly from (A.10). We proceed by induction and observe

$$
\sigma(\tilde v, y_1) = \sigma(\tilde v, \beta^R_1\tilde a + \Delta w_1) = \sigma(\tilde v, \hat y_1), \tag{A.18}
$$

$$\hat{\theta}_1^R - \theta_1^R = 0, \tag{A.19}$$

which follows from $\hat{\theta}_1^I, \theta_1^I \in \sigma(\tilde{v})$. Suppose that (A.11) holds for n. Then,

$$\begin{aligned}\hat{\theta}_{n+1}^R - \theta_{n+1}^R &= (1-\beta_{n+1}^R)(\hat{\theta}_n^R - \theta_n^R)\\ &\quad + \alpha_{n+1}^R(\hat{q}_n - q_n) \in \sigma(\tilde{v}, y_1, \dots, y_n),\\ \sigma(\tilde{v}, \hat{y}_1, \dots, \hat{y}_{n+1}) &= \sigma(\tilde{v}, y_1, \dots, y_n, \hat{y}_{n+1})\\ &= \sigma(\tilde{v}, y_1, \dots, y_n, y_{n+1} + \Delta\hat{\theta}_{n+1}^I - \Delta\theta_{n+1}^I\\ &\quad + \Delta\hat{\theta}_{n+1}^R - \Delta\theta_{n+1}^R)\\ &= \sigma(\tilde{v}, y_1, \dots, y_{n+1}),\end{aligned} \tag{A.20}$$

which proves (A.11). The dynamics (A.12) can be seen as follows

$$\begin{aligned}\Delta X_n^{(1)} &= -\Delta p_n\\ &= -\lambda_n\Big(\Delta\theta_n^I + \beta_n^R(\tilde{a} - \theta_{n-1}^R) + \alpha_n^R q_{n-1} + \Delta w_n\Big)\\ &\quad + \lambda_n(\alpha_n^R + \beta_n^R)q_{n-1}\\ &= -\lambda_n\Big(\Delta\theta_n^I + \beta_n^R(\tilde{a} - \theta_{n-1}^R) + \alpha_n^R q_{n-1} + \hat{y}_n\\ &\quad - \Delta\hat{\theta}_n^I - \Delta\hat{\theta}_n^R\Big) + \lambda_n(\alpha_n^R + \beta_n^R)q_{n-1}\\ &= -\lambda_n\Big(\Delta\theta_n^I + \beta_n^R(\hat{\theta}_{n-1}^R - \theta_{n-1}^R)\\ &\quad + \hat{z}_n^I + \beta_n^R(\hat{q}_{n-1} - q_{n-1}) + \beta_n^R \tfrac{\Sigma_{n-1}^{(3)}}{\Sigma_{n-1}^{(2)}}\big(\tilde{v} - \hat{p}_{n-1}\big)\Big)\\ &= -\lambda_n\Big(\Delta\theta_n^I + \beta_n^R X_{n-1}^{(2)} + \hat{z}_n^I\Big).\end{aligned} \tag{A.21}$$

The dynamics (A.13) follow similarly using expressions (29)–(30) and (32)–(33).

The expression for the variance (A.15) is found as follows:

$$\begin{aligned}\mathbb{V}[\hat{z}_n^I] &= \mathbb{V}\Big[\beta_n^R\Big(\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1}\\ &\quad - \mathbb{E}\Big[\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1} \,|\, \tilde{v}, \hat{y}_1, \dots, \hat{y}_{n-1}\Big]\Big) + \Delta w_n\Big]\\ &= \mathbb{V}\Big[\beta_n^R\Big(\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1} - \tfrac{\Sigma_{n-1}^{(3)}}{\Sigma_{n-1}^{(2)}}(\tilde{v} - \hat{p}_{n-1})\Big)\Big] + \sigma_w^2\Delta\\ &= (\beta_n^R)^2\Big(\Sigma_{n-1}^{(1)} - \frac{\big(\Sigma_{n-1}^{(3)}\big)^2}{\Sigma_{n-1}^{(2)}}\Big) + \sigma_w^2\Delta.\end{aligned} \tag{A.22}$$

To compute the conditional expectation (A.14), we compute the four individual terms. The first term in (A.14) equals

$$\begin{aligned}&\mathbb{E}[(\tilde{v} - p_n)\Delta\theta_n^I \,|\, \tilde{v}, y_1, \dots, y_{n-1}]\\ &= (\tilde{v} - p_{n-1})\Delta\theta_n^I - \Delta\theta_n^I\mathbb{E}[\Delta p_n \,|\, \tilde{v}, y_1, \dots, y_{n-1}]\\ &= X_{n-1}^{(1)}\Delta\theta_n^I\\ &\quad - \Delta\theta_n^I\lambda_n\mathbb{E}[\Delta\theta_n^I + \beta_n^R(\tilde{a} - \theta_{n-1}^R - q_{n-1}) \,|\, \tilde{v}, y_1, \dots, y_{n-1}]\\ &= X_{n-1}^{(1)}\Delta\theta_n^I - (\Delta\theta_n^I)^2\lambda_n\\ &\quad - \Delta\theta_n^I\lambda_n\beta_n^R\Big(\hat{\theta}_{n-1}^R - \theta_{n-1}^R + \hat{q}_{n-1} - q_{n-1}\\ &\quad + \mathbb{E}[\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1} \,|\, \tilde{v}, y_1, \dots, y_{n-1}]\Big)\\ &= X_{n-1}^{(1)}\Delta\theta_n^I - (\Delta\theta_n^I)^2\lambda_n - \Delta\theta_n^I\lambda_n\beta_n^R\Big(\hat{\theta}_{n-1}^R - \theta_{n-1}^R + \hat{q}_{n-1}\\ &\quad - q_{n-1} + \frac{\Sigma_{n-1}^{(3)}}{\Sigma_{n-1}^{(2)}}(\tilde{v} - \hat{p}_{n-1})\Big)\\ &= X_{n-1}^{(1)}\Delta\theta_n^I - (\Delta\theta_n^I)^2\lambda_n - \Delta\theta_n^I\lambda_n\beta_n^R X_{n-1}^{(2)}.\end{aligned} \tag{A.23}$$

The second term in (A.14) is

$$\begin{aligned}&\mathbb{E}\big[\big(X_n^{(1)}\big)^2 \,|\, \tilde{v}, y_1, \dots, y_{n-1}\big]\\ &= \big(X_{n-1}^{(1)}\big)^2 + 2X_{n-1}^{(1)}\mathbb{E}[\Delta X_n^{(1)} \,|\, \tilde{v}, y_1, \dots, y_{n-1}]\\ &\quad + \mathbb{E}\big[\big(\Delta X_n^{(1)}\big)^2 \,|\, \tilde{v}, y_1, \dots, y_{n-1}\big]\\ &= \big(X_{n-1}^{(1)}\big)^2 - 2\lambda_n X_{n-1}^{(1)}\Big(\Delta\theta_n^I + \beta_n^R X_{n-1}^{(2)}\Big)\\ &\quad + \lambda_n^2\Big(\Delta\theta_n^I + \beta_n^R X_{n-1}^{(2)}\Big)^2 + \lambda_n^2\mathbb{V}[\hat{z}_n^I].\end{aligned} \tag{A.24}$$

The third term in (A.14) is

$$\begin{aligned}&\mathbb{E}[X_n^{(1)}X_n^{(2)} \,|\, \tilde{v}, y_1, \dots, y_{n-1}]\\ &= X_{n-1}^{(1)}X_{n-1}^{(2)} + X_{n-1}^{(1)}\mathbb{E}[\Delta X_n^{(2)} \,|\, \tilde{v}, y_1, \dots, y_{n-1}]\\ &\quad + X_{n-1}^{(2)}\mathbb{E}[\Delta X_n^{(1)} \,|\, \tilde{v}, y_1, \dots, y_{n-1}]\\ &\quad + \mathbb{E}[\Delta X_n^{(1)}\Delta X_n^{(2)} \,|\, \tilde{v}, y_1, \dots, y_{n-1}]\\ &= X_{n-1}^{(1)}X_{n-1}^{(2)} - X_{n-1}^{(1)}\Big(r_n\Delta\theta_n^I + (1+r_n)\beta_n^R X_{n-1}^{(2)}\Big)\\ &\quad - X_{n-1}^{(2)}\lambda_n\Big(\Delta\theta_n^I + \beta_n^R X_{n-1}^{(2)}\Big)\\ &\quad + \lambda_n\Big(\Delta\theta_n^I + \beta_n^R X_{n-1}^{(2)}\Big)\Big(r_n\Delta\theta_n^I + (1+r_n)\beta_n^R X_{n-1}^{(2)}\Big)\\ &\quad + \lambda_n^2\tfrac{\Sigma_n^{(3)}}{\Sigma_n^{(2)}}\mathbb{V}[\hat{z}_n^I].\end{aligned} \tag{A.25}$$

Finally, the last term in (A.14) is

$$\begin{aligned}&\mathbb{E}\big[\big(X_n^{(2)}\big)^2 \,|\, \tilde{v}, y_1, \dots, y_{n-1}\big]\\ &= \big(X_{n-1}^{(2)}\big)^2 + 2X_{n-1}^{(2)}\mathbb{E}[\Delta X_n^{(2)} \,|\, \tilde{v}, y_1, \dots, y_{n-1}]\\ &\quad + \mathbb{E}\big[\big(\Delta X_n^{(2)}\big)^2 \,|\, \tilde{v}, y_1, \dots, y_{n-1}\big]\\ &= \big(X_{n-1}^{(2)}\big)^2 - 2X_{n-1}^{(2)}\Big(r_n\Delta\theta_n^I + (1+r_n)\beta_n^R X_{n-1}^{(2)}\Big)\\ &\quad + \Big(r_n\Delta\theta_n^I + (1+r_n)\beta_n^R X_{n-1}^{(2)}\Big)^2 + \lambda_n^2\Big(\tfrac{\Sigma_n^{(3)}}{\Sigma_n^{(2)}}\Big)^2\mathbb{V}[\hat{z}_n^I].\end{aligned} \tag{A.26}$$

□

Remark. The dynamics (A.12) and (A.13) show that the pair $(X^{(1)}, X^{(2)})$ form a Markov process. This implies that for any continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(X_n^{(1)}, X_n^{(2)})$ integrable, the conditional expectation

$$\mathbb{E}[f(X_n^{(1)}, X_n^{(2)}) | \tilde{v}, \hat{z}_1^I, \dots, \hat{z}_{n-1}^I] \tag{A.27}$$

is again a function g of $(X_{n-1}^{(1)}, X_{n-1}^{(2)})$. Furthermore, (A.14) shows i) if f is a second-degree polynomial, the resulting function g is also a second-degree polynomial, and ii) the conditional expectation of p_n is also a quadratic function of $(X_{n-1}^{(1)}, X_{n-1}^{(2)})$. In other words, the pair $(X^{(1)}, X^{(2)})$ is the state process for the informed investor's optimization problem.

Theorem 2. Fix the constants (15) subject to the pricing-coefficient restrictions (29)–(30) holding and use them to *define $\Delta\theta_n^R$ by (5), define the moments (31)–(33) with initial values (26), and compute the value-function coefficients $\{I_n^{(i,j)}\}_{1\le i\le j\le 2}$, $n=0,\dots,N$ using recursions (A.42)–(A.44) with $I_N^{(i,j)}=0$, subject to the second-order condition (39) holding. Then the hedge fund's value function has the quadratic form (37) where $X_n^{(1)}$ and $X_n^{(2)}$ are defined in (35) and Δp_n is defined by (10). Furthermore, the hedge fund's optimal trading strategy is given by (38) with coefficients*

$$\gamma_n^{(1)} := \frac{-1+I_n^{(1,2)}r_n+2I_n^{(1,1)}\lambda_n}{2(I_n^{(2,2)}r_n^2+\lambda_n(-1+I_n^{(1,2)}r_n+I_n^{(1,1)}\lambda_n))}, \tag{A.28}$$

$$\gamma_n^{(2)} := -\beta_n^R + \frac{-2I_n^{(2,2)}r_n(-1+\beta_n^R)+I_n^{(1,2)}\lambda_n-\beta_n^R\lambda_n(I_n^{(1,2)}+1)}{2(I_n^{(2,2)}r_n^2+\lambda_n(-1+I_n^{(1,2)}r_n+I_n^{(1,1)}\lambda_n))}. \tag{A.29}$$

Proof. We prove the theorem by backward induction. Suppose that (37) holds for time $n+1$. The hedge fund's value function in the n'th iteration then becomes

$$\max_{\substack{\Delta\theta_k^I\in\sigma(\tilde v,y_1,\dots,y_{k-1})\\ n\le k\le N}} \mathbb{E}\left[\sum_{k=n}^{N}(\tilde v-p_k)\Delta\theta_k^I\,\middle|\,\tilde v,y_1,\dots,y_{n-1}\right]$$
$$= \max_{\Delta\theta_n^I\in\sigma(\tilde v,y_1,\dots,y_{n-1})}\mathbb{E}\Bigg[(\tilde v-p_n)\Delta\theta_n^I+I_n^{(0)} + \sum_{1\le i\le j\le 2} I_n^{(i,j)}X_n^{(i)}X_n^{(j)}\,\Bigg|\,\tilde v,y_1,\dots,y_{n-1}\Bigg]. \tag{A.30}$$

Because (39) holds, Lemma 2 shows that the coefficient in front of $(\Delta\theta_n^I)^2$ appearing in (A.30) is strictly negative. Consequently, the first-order condition is sufficient for optimality and the maximizer is (38). By inserting the optimizer (38) into (A.30), we obtain the quadratic expression (37) for time n,

$$I_{n-1}^{(0)} + \sum_{1\le i\le j\le 2} I_{n-1}^{(i,j)}X_{n-1}^{(i)}X_{n-1}^{(j)}, \tag{A.31}$$

where the value-function coefficient recursions for $I_{n-1}^{(i,j)}$ are in (A.42)–(A.44). □

A3. Rebalancer's optimization problem

The following analogue of Lemma 2 uses the rebalancer's state variables $\{Y_n^{(1)},Y_n^{(2)},Y_n^{(3)}\}$ defined in (42).

Lemma 3. Fix constants (15) satisfying (16)–(17) and subject to the pricing-coefficient restrictions (29)–(30) holding and use them to define $\Delta\theta_n^I$ by (6) and define the moments (31)–(33) with initial values (26). Let $\Delta\theta_n^R\in\sigma(\tilde a,y_1,\dots,y_{n-1})$, $n=1,\dots,N$, be arbitrary for the rebalancer. We can then define the Gaussian random variables

$$\hat z_n^R := \hat y_n - \Delta\hat\theta_n^R - \beta_n^I\frac{\Sigma_{n-1}^{(3)}}{\Sigma_{n-1}^{(1)}}(\tilde a-\hat\theta_{n-1}^R-\hat q_{n-1}),\quad n=1,\dots,N \tag{A.32}$$

where the conjectured "hat" processes are defined in (18)–(22). The variable $\hat z_k^R$ is independent of $\{\tilde a,\hat y_1,\dots,\hat y_{k-1}\}$ for $k\le N$ and the following measurability properties are satisfied

$$\sigma(\tilde a,y_1,\dots,y_k)=\sigma(\tilde a,\hat y_1,\dots,\hat y_k)=\sigma(\tilde a,\hat z_1^R,\dots,\hat z_k^R). \tag{A.33}$$

Furthermore, the state variables $Y_n^{(1)}$, $Y_n^{(2)}$, and $Y_n^{(3)}$ defined in (42) $n=1,\dots,N$ have Markovian dynamics

$$\Delta Y_n^{(2)} = -\lambda_n\Big(\Delta\theta_n^R+\beta_n^IY_{n-1}^{(2)}-(\alpha_n^R+\beta_n^R)Y_{n-1}^{(3)}\Big)-r_n\frac{\Sigma_n^{(3)}}{\Sigma_n^{(1)}}\hat z_n^R,$$
$$Y_0^{(2)} = \frac{\sigma_{\tilde v}\rho}{\sigma_{\tilde a}}\tilde a, \tag{A.34}$$

$$\Delta Y_n^{(3)} = r_n\Big(\Delta\theta_n^R+\beta_n^IY_{n-1}^{(2)}\Big)-(1+r_n)(\alpha_n^R+\beta_n^R)Y_{n-1}^{(3)}+r_n\hat z_n^R,$$
$$Y_0^{(3)}=0. \tag{A.35}$$

For constants $L_n^{(1,1)},L_n^{(1,2)},L_n^{(1,3)},L_n^{(2,2)},L_n^{(2,3)}$, and $L_n^{(3,3)}$ we have the conditional expectation

$$\mathbb{E}[-(\tilde a-\theta_{n-1}^R)\Delta p_n+\sum_{1\le i\le j\le 3}L_n^{(i,j)}Y_n^{(i)}Y_n^{(j)}\mid\tilde a,y_1,\dots,y_{n-1}]$$
$$= -Y_{n-1}^{(1)}\Big(\lambda_n(\Delta\theta_n^R+\beta_n^IY_{n-1}^{(2)})-\lambda_n(\alpha_n^R+\beta_n^R)Y_{n-1}^{(3)}\Big)$$
$$+L_n^{(1,1)}\Big((Y_{n-1}^{(1)}-\Delta\theta_n^R)^2\Big)$$
$$+L_n^{(1,2)}(Y_{n-1}^{(1)}-\Delta\theta_n^R)\big(Y_{n-1}^{(2)}-\lambda_n\big(\Delta\theta_n^R+\beta_n^IY_{n-1}^{(2)}$$
$$-(\alpha_n^R+\beta_n^R)Y_{n-1}^{(3)}\big)\big)$$
$$+L_n^{(1,3)}(Y_{n-1}^{(1)}-\Delta\theta_n^R)\big(Y_{n-1}^{(3)}+r_n\big(\Delta\theta_n^R+\beta_n^IY_{n-1}^{(2)}\big)$$
$$-(1+r_n)(\alpha_n^R+\beta_n^R)Y_{n-1}^{(3)}\big)$$
$$+L_n^{(2,2)}\Big((Y_{n-1}^{(2)})^2-2Y_{n-1}^{(2)}\lambda_n\Big(\Delta\theta_n^R+\beta_n^IY_{n-1}^{(2)}$$
$$-(\alpha_n^R+\beta_n^R)Y_{n-1}^{(3)}\Big)$$
$$+\lambda_n^2\Big(\Delta\theta_n^R+\beta_n^IY_{n-1}^{(2)}-(\alpha_n^R+\beta_n^R)Y_{n-1}^{(3)}\Big)^2$$
$$+r_n^2\Big(\frac{\Sigma_n^{(3)}}{\Sigma_n^{(1)}}\Big)^2\mathbb{V}[\hat z_n^R]\Big)$$
$$+L_n^{(2,3)}\Big(Y_{n-1}^{(2)}Y_{n-1}^{(3)}+Y_{n-1}^{(2)}\Big(r_n\big(\Delta\theta_n^R+\beta_n^IY_{n-1}^{(2)}\big)$$
$$-(1+r_n)(\alpha_n^R+\beta_n^R)Y_{n-1}^{(3)}\Big)$$
$$-Y_{n-1}^{(3)}\lambda_n\Big(\Delta\theta_n^R+\beta_n^IY_{n-1}^{(2)}-(\alpha_n^R+\beta_n^R)Y_{n-1}^{(3)}\Big)$$
$$-r_n^2\frac{\Sigma_n^{(3)}}{\Sigma_n^{(1)}}\mathbb{V}[\hat z_n^R]-\lambda_n\Big(\Delta\theta_n^R+\beta_n^IY_{n-1}^{(2)}-(\alpha_n^R+\beta_n^R)Y_{n-1}^{(3)}\Big)$$
$$\times\Big(r_n\big(\Delta\theta_n^R+\beta_n^IY_{n-1}^{(2)}\big)-(1+r_n)(\alpha_n^R+\beta_n^R)Y_{n-1}^{(3)}\Big)\Big)$$
$$+L_n^{(3,3)}\Big((Y_{n-1}^{(3)})^2+2Y_{n-1}^{(3)}\Big(r_n\big(\Delta\theta_n^R+\beta_n^IY_{n-1}^{(2)}\big)$$
$$-(1+r_n)(\alpha_n^R+\beta_n^R)Y_{n-1}^{(3)}\Big)$$
$$+\Big(r_n\big(\Delta\theta_n^R+\beta_n^IY_{n-1}^{(2)}\big)-(1+r_n)(\alpha_n^R+\beta_n^R)Y_{n-1}^{(3)}\Big)^2$$
$$+r_n^2\mathbb{V}[\hat z_n^R]\Big), \tag{A.36}$$

which is quadratic in $\Delta\theta_n^R$, and where the variance $\mathbb{V}[\hat z_n^R]$ is given by

$$\mathbb{V}[\hat z_n^R]=(\beta_n^I)^2\Big(\Sigma_{n-1}^{(2)}-\frac{\big(\Sigma_{n-1}^{(3)}\big)^2}{\Sigma_{n-1}^{(1)}}\Big)+\sigma_w^2\Delta. \tag{A.37}$$

Proof. The proof is similar to the proof of Lemma 2 and is therefore omitted. The one difference is that restrictions (16)–(17) are used to change the sigma algebra $\sigma(\tilde{a}, y_1, \ldots, y_{n-1})$ into $\sigma(\tilde{a} - \hat{\theta}^R_{n-1} - \hat{q}_{n-1}, y_1, \ldots, y_{n-1})$ in the derivation of the expectation in (A.36). □

Theorem 3. Fix the constants (15) satisfying (16)–(17) and subject to the pricing-coefficient restrictions (29)–(30) holding and use them to define $\Delta\theta^I_n$ by (6), define the moments (31)–(33) with initial values (26), and compute the value-function coefficients $\{L^{(i,j)}_n\}_{1\le i\le j\le 3}$, $n = 0, \ldots, N$ using recursions (A.45)–(A.50) with $L^{(i,j)}_N = 0$ subject to the second-order-condition (46) holding. Then the rebalancer's value function has the quadratic form (44) where $\{Y^{(1)}_n, Y^{(2)}_n, Y^{(3)}_n\}$ are defined by (42) and Δp_n is defined by (10). Furthermore, the rebalancer's optimal trading strategy is given by (45) with coefficients

$$\delta^{(1)}_n := \frac{2L^{(1,1)}_n - L^{(1,3)}_n r_n + \lambda_n + L^{(1,2)}_n \lambda_n}{2\Big(L^{(1,1)}_n - L^{(1,3)}_n r_n + L^{(3,3)}_n r_n^2 + \lambda_n (L^{(1,2)}_n - L^{(2,3)}_n r_n + L^{(2,2)}_n \lambda_n)\Big)}, \tag{A.38}$$

$$\delta^{(2)}_n := -\beta^I_n + \frac{L^{(1,2)}_n - r_n(L^{(2,3)}_n + L^{(1,3)}_n \beta^I_n) + L^{(1,2)}_n \beta^I_n \lambda_n + 2(L^{(1,1)}_n \beta^I_n + L^{(2,2)}_n \lambda_n)}{2\Big(L^{(1,1)}_n - L^{(1,3)}_n r_n + L^{(3,3)}_n r_n^2 + \lambda_n (L^{(1,2)}_n - L^{(2,3)}_n r_n + L^{(2,2)}_n \lambda_n)\Big)}, \tag{A.39}$$

$$\delta^{(3)}_n := \frac{\begin{aligned}&\Big(- 2L^{(3,3)}_n r_n - L^{(1,3)}_n (-1 + \alpha^R_n + r_n \alpha^R_n + \beta^R_n + r_n \beta^R_n) + L^{(2,3)}_n \lambda_n \\ &+(\alpha^R_n + \beta^R_n)\big(2L^{(3,3)}_n r_n (1 + r_n) + \lambda_n (L^{(1,2)}_n - L^{(2,3)}_n - 2L^{(2,3)}_n r_n + 2L^{(2,2)}_n \lambda_n)\big)\Big)\end{aligned}}{2\Big(L^{(1,1)}_n - L^{(1,3)}_n r_n + L^{(3,3)}_n r_n^2 + \lambda_n (L^{(1,2)}_n - L^{(2,3)}_n r_n + L^{(2,2)}_n \lambda_n)\Big)}. \tag{A.40}$$

Proof. The proof is similar to the proof of Theorem 2 and is therefore omitted. □

A4. Remaining proof

Proof of Theorem 1. Part (iii) of Definition 1 holds from Lemma 1. Parts (i)–(ii) of Definition 1 hold from Theorem 2 and Theorem 3 as soon as we show that the optimizers (38) and (45) agree with (18) and (19). This, however, follows from the equilibrium conditions (40) and (47). □

A5. Value-function coefficients

Set the terminal coefficients

$$I^{(1,1)}_N := \ldots := I^{(2,2)}_N := L^{(1,1)}_N := \ldots := L^{(3,3)}_N := 0. \tag{A.41}$$

The recursion for the hedge fund's value-function coefficients is given by

$$I^{(1,1)}_{n-1} = \frac{-1 + r_n(2I^{(1,2)}_n - (I^{(1,2)}_n)^2 r_n + 4I^{(1,1)}_n I^{(2,2)}_n r_n)}{4(I^{(2,2)}_n r_n^2 + \lambda_n(-1 + I^{(1,2)}_n r_n + I^{(1,1)}_n \lambda_n))}, \tag{A.42}$$

$$I^{(1,2)}_{n-1} = -\frac{\begin{aligned}&\Big((-1 + I^{(1,2)}_n r_n)(I^{(1,2)}_n(-1 + \beta^R_n) + \beta^R_n)\lambda_n \\ &+2I^{(2,2)}_n r_n(-1 + \beta^R_n + r_n\beta^R_n - 2I^{(1,1)}_n(-1 + \beta^R_n)\lambda_n)\Big)\end{aligned}}{2(I^{(2,2)}_n r_n^2 + \lambda_n(-1 + I^{(1,2)}_n r_n + I^{(1,1)}_n \lambda_n))}, \tag{A.43}$$

$$I^{(2,2)}_{n-1} = \frac{\begin{aligned}&\lambda_n\Big(- (I^{(1,2)}_n(-1 + \beta^R_n) + \beta^R_n)^2 \lambda_n \\ &-4I^{(2,2)}_n(-1 + \beta^R_n)(-1 + I^{(1,1)}_n \lambda_n + \beta^R_n(1 + r_n - I^{(1,1)}_n \lambda_n))\Big)\end{aligned}}{4(I^{(2,2)}_n r_n^2 + \lambda_n(-1 + I^{(1,2)}_n r_n + I^{(1,1)}_n \lambda_n))}. \tag{A.44}$$

The recursion for the rebalancer's value-function coefficients is given by

$$L^{(1,1)}_{n-1} = -\frac{\begin{aligned}&\Big((L^{(1,3)}_n)^2 r_n^2 - 2(1 + L^{(1,2)}_n)L^{(1,3)}_n r_n \lambda_n + (1 + L^{(1,2)}_n)^2 \lambda_n^2 \\ &+4L^{(1,1)}_n(-L^{(3,3)}_n r_n^2 + \lambda_n + L^{(2,3)}_n r_n \lambda_n - L^{(2,2)}_n \lambda_n^2)\Big)\end{aligned}}{4\Big(L^{(1,1)}_n - L^{(1,3)}_n r_n + L^{(3,3)}_n r_n^2 + \lambda_n (L^{(1,2)}_n - L^{(2,3)}_n r_n + L^{(2,2)}_n \lambda_n)\Big)}, \tag{A.45}$$

$$
L_{n-1}^{(1,2)} = -\frac{\begin{array}{l}\Big((L_n^{(1,3)} r_n - \lambda_n)(L_n^{(2,3)} r_n + L_n^{(1,3)} r_n \beta_n^I - 2L_n^{(2,2)}\lambda_n) \\ +(L_n^{(1,2)})^2\lambda_n(-1+\beta_n^I\lambda_n) + L_n^{(1,2)}(r_n(L_n^{(1,3)} - 2L_n^{(3,3)} r_n) + \lambda_n \\ +r_n(L_n^{(2,3)} - 2L_n^{(1,3)}\beta_n^I)\lambda_n + \beta_n^I\lambda_n^2) + 2L_n^{(1,1)}(-r_n(L_n^{(2,3)} + 2L_n^{(3,3)} r_n\beta_n^I) \\ +(2L_n^{(2,2)} + \beta_n^I + 2L_n^{(2,3)} r_n\beta_n^I)\lambda_n - 2L_n^{(2,2)}\beta_n^I\lambda_n^2)\Big)\end{array}}{2\big(L_n^{(1,1)} - L_n^{(1,3)} r_n + L_n^{(3,3)} r_n^2 + \lambda_n(L_n^{(1,2)} - L_n^{(2,3)} r_n + L_n^{(2,2)}\lambda_n)\big)}, \tag{A.46}
$$

$$
L_{n-1}^{(1,3)} = \frac{\begin{array}{l}\Big[(L_n^{(1,3)})^2 r_n((1+r_n)(\alpha_n^R+\beta_n^R)-1) + (1+L_n^{(1,2)})\lambda_n\Big(2L_n^{(3,3)} r_n(1-\alpha_n^R-\beta_n^R) \\ -L_n^{(2,3)}\lambda_n + (L_n^{(1,2)} + L_n^{(2,3)})(\alpha_n^R+\beta_n^R)\lambda_n\Big) \\ +2L_n^{(1,1)}\big(2L_n^{(3,3)} r_n(1-(1+r_n)(\alpha_n^R+\beta_n^R)) \\ -L_n^{(2,3)}\lambda_n + (\alpha_n^R+\beta_n^R)\lambda_n(1+L_n^{(2,3)} + 2L_n^{(2,3)} r_n - 2L_n^{(2,2)}\lambda_n)\big) \\ +L_n^{(1,3)}\lambda_n\Big(\alpha_n^R - 1 + \beta_n^R + L_n^{(2,3)} r_n(\alpha_n^R+\beta_n^R-1) - (\alpha_n^R+\beta_n^R)(r_n + 2L_n^{(2,2)}\lambda_n) \\ -L_n^{(1,2)}(-1+\alpha_n^R + 2r_n\alpha_n^R + \beta_n^R + 2r_n\beta_n^R) + 2L_n^{(2,2)}\lambda_n\Big)\Big]\end{array}}{2(L_n^{(1,1)} - L_n^{(1,3)} r_n + L_n^{(3,3)} r_n^2 + \lambda_n(L_n^{(1,2)} - L_n^{(2,3)} r_n + L_n^{(2,2)}\lambda_n))}, \tag{A.47}
$$

$$
L_{n-1}^{(2,2)} = -\frac{\begin{array}{l}\Big[(L_n^{(1,2)})^2(-1+\beta_n^I\lambda_n)^2 - 2L_n^{(1,2)} r_n\big(L_n^{(2,3)} - L_n^{(2,3)}\beta_n^I\lambda_n \\ +\beta_n^I(-L_n^{(1,3)} + 2L_n^{(3,3)} r_n + L_n^{(1,3)}\beta_n^I\lambda_n)\big) + r_n\Big(\big((L_n^{(2,3)})^2 - 4L_n^{(2,2)}L_n^{(3,3)}\big) r_n \\ +(L_n^{(1,3)})^2 r_n(\beta_n^I)^2 + L_n^{(1,3)}(4L_n^{(2,2)} + 2L_n^{(2,3)} r_n\beta_n^I - 4L_n^{(2,2)}\beta_n^I\lambda_n)\Big) \\ -4L_n^{(1,1)}\big(L_n^{(2,2)}(-1+\beta_n^I\lambda_n)^2 + r_n\beta_n^I(L_n^{(2,3)} + L_n^{(3,3)} r_n\beta_n^I - L_n^{(2,3)}\beta_n^I\lambda_n)\big)\Big]\end{array}}{4(L_n^{(1,1)} - L_n^{(1,3)} r_n + L_n^{(3,3)} r_n^2 + \lambda_n(L_n^{(1,2)} - L_n^{(2,3)} r_n + L_n^{(2,2)}\lambda_n))}, \tag{A.48}
$$

$$
L_{n-1}^{(2,3)} = \frac{\begin{array}{l}\Big[\big(L_n^{(1,3)} r_n(L_n^{(2,3)} + L_n^{(1,3)}\beta_n^I) \\ -2L_n^{(1,1)}(L_n^{(2,3)} + 2L_n^{(3,3)} r_n\beta_n^I)\big)((1+r_n)(\alpha_n^R+\beta_n^R)-1) \\ +\big((L_n^{(2,3)})^2 r_n(\alpha_n^R+\beta_n^R-1) + 2L_n^{(1,1)}L_n^{(2,3)}\beta_n^I(\alpha_n^R + 2r_n\alpha_n^R + \beta_n^R + 2r_n\beta_n^R - 1) \\ +4L_n^{(2,2)}(-L_n^{(3,3)} r_n(-1+\alpha_n^R+\beta_n^R) + L_n^{(1,1)}(\alpha_n^R+\beta_n^R)) \\ +L_n^{(1,3)}(L_n^{(2,3)} r_n\beta_n^I(\alpha_n^R+\beta_n^R-1) - 2L_n^{(2,2)}(1-(1-r_n)(\alpha_n^R+\beta_n^R)))\big)\lambda_n \\ -2L_n^{(2,2)}\beta_n^I(L_n^{(1,3)}(-1+\alpha_n^R+\beta_n^R) + 2L_n^{(1,1)}(\alpha_n^R+\beta_n^R))\lambda_n^2 \\ +(L_n^{(1,2)})^2(\alpha_n^R+\beta_n^R)\lambda_n(-1+\beta_n^I\lambda_n) + L_n^{(1,2)}(L_n^{(2,3)}\lambda_n \\ -2L_n^{(3,3)} r_n(-1+\alpha_n^R + r_n\alpha_n^R + \beta_n^R + r_n\beta_n^R + \beta_n^I(-1+\alpha_n^R+\beta_n^R)\lambda_n) \\ +L_n^{(2,3)}\lambda_n((-1+r_n)(\alpha_n^R+\beta_n^R) + \beta_n^I(-1+\alpha_n^R+\beta_n^R)\lambda_n) \\ +L_n^{(1,3)}(-1+\alpha_n^R + r_n\alpha_n^R + \beta_n^R + r_n\beta_n^R + \beta_n^I\lambda_n - (1+2r_n)\beta_n^I(\alpha_n^R+\beta_n^R)\lambda_n))\Big]\end{array}}{2\big(L_n^{(1,1)} - L_n^{(1,3)} r_n + L_n^{(3,3)} r_n^2 + \lambda_n(L_n^{(1,2)} - L_n^{(2,3)} r_n + L_n^{(2,2)}\lambda_n)\big)}, \tag{A.49}
$$

$$
L_{n-1}^{(3,3)} = -\frac{\begin{array}{l}\Big[(L_n^{(1,3)})^2((1+r_n)(\alpha_n^R+\beta_n^R)-1)^2 + 2L_n^{(1,3)}\lambda_n\big(((1+r_n)(\alpha_n^R+\beta_n^R)-1)\times \\ \big(L_n^{(2,3)}(\alpha_n^R+\beta_n^R-1) - L_n^{(1,2)}(\alpha_n^R+\beta_n^R)\big) \\ -2L_n^{(2,2)}(\alpha_n^R+\beta_n^R-1)(\alpha_n^R+\beta_n^R)\lambda_n\big) \\ -4L_n^{(1,1)}\Big(L_n^{(3,3)}(-1+\alpha_n^R + r_n\alpha_n^R + \beta_n^R + r_n\beta_n^R)^2 + (\alpha_n^R+\beta_n^R)\lambda_n\times \\ \big(-L_n^{(2,3)}(-1+\alpha_n^R + r_n\alpha_n^R + \beta_n^R + r_n\beta_n^R) + L_n^{(2,2)}(\alpha_n^R+\beta_n^R)\lambda_n\big)\Big) \\ +\lambda_n\Big(\big((L_n^{(2,3)})^2 - 4L_n^{(2,2)}L_n^{(3,3)}\big)(-1+\alpha_n^R+\beta_n^R)^2\lambda_n + (L_n^{(1,2)})^2(\alpha_n^R+\beta_n^R)^2\lambda_n \\ -2L_n^{(1,2)}(\alpha_n^R+\beta_n^R-1)(2L_n^{(3,3)}((1+r_n)(\alpha_n^R+\beta_n^R)-1) \\ -L_n^{(2,3)}(\alpha_n^R+\beta_n^R)\lambda_n)\Big)\Big]\end{array}}{4\big(L_n^{(1,1)} - L_n^{(1,3)} r_n + L_n^{(3,3)} r_n^2 + \lambda_n(L_n^{(1,2)} - L_n^{(2,3)} r_n + L_n^{(2,2)}\lambda_n)\big)}. \tag{A.50}
$$

A6. Algorithm

This section describes an algorithm for searching numerically for a linear Bayesian Nash equilibrium. The algorithm is similar in logic to the algorithm in Section 5 in Foster and Viswanathan (1996), except that our algorithm requires three constants as inputs (due to the presence of two strategic agents) whereas (Foster and Viswanathan, 1996) only has one constant as an input.

The algorithm starts by taking as inputs three conjectured conditional moments for the final time N round of trading:[34]

$$\Sigma_{N-1}^{(1)} > 0, \quad \Sigma_{N-1}^{(2)} > 0, \quad \Sigma_{N-1}^{(3)} \in \mathbb{R} \quad \text{such that} \quad \left(\Sigma_{N-1}^{(3)}\right)^2 \le \Sigma_{N-1}^{(1)} \Sigma_{N-1}^{(2)}. \tag{A.51}$$

The algorithm then proceeds through backward induction.

Starting step for trading time N: We need $\{\lambda_N, \beta_N^I\}$ to satisfy (29) for $n = N$ where

$$\beta_N^I = \frac{1}{2\lambda_N} - \frac{\Sigma_{N-1}^{(3)}}{2\Sigma_{N-1}^{(2)}} \tag{A.52}$$

from the hedge fund's equilibrium strategy coefficient in (40) with $\lambda_N > 0$ in order to satisfy (39). Given those two constants $\{\lambda_N, \beta_N^I\}$, we set

$$\beta_N^R := 1, \quad \alpha_N^R := r_N := 0. \tag{A.53}$$

Because of the rebalancer's terminal constraint, his last round of trading (i.e., at time N) does not involve any optimization, and so we have

$$\mathbb{E}\left[-(\tilde{a} - \theta_{N-1}^R)\Delta p_N \,|\, \tilde{a}, y_1, \ldots, y_{N-1}\right] = -Y_{N-1}^{(1)}\left(\lambda_N (Y_{N-1}^{(1)} + \beta_N^I Y_{N-1}^{(2)}) - \lambda_N Y_{N-1}^{(3)}\right). \tag{A.54}$$

This relation implies the rebalancer's value-function coefficients for $n = N - 1$ are

$$L_{N-1}^{(1,1)} = -\lambda_N,\ L_{N-1}^{(1,2)} = -\lambda_N \beta_N^I,\ L_{N-1}^{(1,3)} = \lambda_N, \quad L_{N-1}^{(2,2)} = L_{N-1}^{(2,3)} = L_{N-1}^{(3,3)} = 0. \tag{A.55}$$

On the other hand, the hedge fund's problem in the last round of trading is similar to her problem in any other round of trading. By inserting the boundary conditions

$$I_N^{(1,1)} = I_N^{(1,2)} = I_N^{(2,2)} = 0 \tag{A.56}$$

into the recursions (A.42)–(A.44), we produce the value-function coefficients $I_{N-1}^{(i,j)}$.

Induction step: At each time n the algorithm takes the following terms as inputs:

$$\Sigma_n^{(1)}, \Sigma_n^{(2)}, \Sigma_n^{(3)}, \{I_n^{(i,j)}\}_{1\le i\le j\le 2}, \{L_n^{(i,j)}\}_{1\le i\le j\le 3}. \tag{A.57}$$

We first find the constants $\{\lambda_n, r_n, \Sigma_{n-1}^{(1)}, \Sigma_{n-1}^{(2)}, \Sigma_{n-1}^{(3)}, \beta_n^I, \beta_n^R\}$ by requiring that (29)–(30), (31)–(33) with $\Sigma_{n-1}^{(1)} > 0, \Sigma_{n-1}^{(2)} > 0$ and $\left(\Sigma_{n-1}^{(3)}\right)^2 \le \Sigma_{n-1}^{(1)}\Sigma_{n-1}^{(2)}$, monotonicity of $\Sigma_{n-1}^{(2)}$, (40), the first part of (47), as well as the second-order conditions (39)–(46) hold. These are seven polynomial equations in seven unknown constants. We can then subsequently define α_n^R by the second part of (47).

Next, the value-function coefficients $\{I_{n-1}^{(i,j)}\}_{1\le i\le j\le 2}$ and $\{L_{n-1}^{(i,j)}\}_{1\le i\le j\le 3}$ at time $n-1$ are found by the recursions (A.42)–(A.44) and (A.45)–(A.50).

Termination: The iteration above is continued back to time $n = 0$. If the resulting values at time $n = 0$ do not satisfy (26), then we adjust the conjectured starting input values in (A.51) and start the algorithm all over. If the resulting values at time $n = 0$ do satisfy (26), then the algorithm terminates. If the rebalancer coefficients satisfy (16), then the computed constants produce a linear Bayesian Nash equilibrium. Otherwise, no equilibrium was found.

[34] We do not take the post-trade time-N moments $(\Sigma_N^{(1)}, \Sigma_N^{(2)}, \Sigma_N^{(3)})$ as inputs because they are after the last round of trading. In addition, (31) and (33) together with the terminal condition $\beta_N^R = 1$ imply that $\Sigma_N^{(1)} = \Sigma_N^{(3)} = 0$.

Appendix B. Modified Foster and Viswanathan (1994)

Our modification of the Foster and Viswanathan (1994) model has N periods of trade after which the traded security pays off $\tilde{v} \sim N(0, \sigma_{\tilde{v}}^2)$ at time $N+1$. Four types of investors trade: First, a strategic risk-neutral investor who knows $\tilde{v}$ at time 0 and who trades dynamically over time using orders $\Delta\theta_n^I$. Second, a strategic risk-neutral less-informed investor who receives an initial signal $\tilde{a} \sim N(0, \sigma_{\tilde{a}}^2)$ with $\tilde{a}$ and $\tilde{v}$ being jointly normally distributed random variables with $\mathrm{corr}(\tilde{a}, \tilde{v}) = \rho \in (0, 1)$ and who trades dynamically using orders $\Delta\theta_n^L$. The "L" superscript here denotes that this second investor is "less" informed than the first (better-informed) investor with superscript "I". Third, noise traders submit random orders $\Delta w_n \sim N(0, \sigma_w^2 \Delta)$ which are independent of $(\tilde{v}, \tilde{a})$. Fourth, competitive risk-neutral market makers see the aggregate order flow at each time

$$y_n := \Delta\theta_n^I + \Delta\theta_n^L + \Delta w_n, \quad y_0 := 0, \tag{B.1}$$

and set prices p_n at which they then clear the market.

In our modified FV model, the better-informed investor does not know $\tilde{a}$, whereas in the original (Foster and Viswanathan, 1994) the better-informed investor knows both $\tilde{v}$ and $\tilde{a}$. Thus, except for the rebalancing constraint, the modified FV model has the identical information structure as in our model of strategic rebalancing.

A *Bayesian Nash equilibrium* for the modified FV model consists of: (i) Order strategies that, at each time n, maximize the expected profits of the better-informed and less-informed investors given their their respective information sets $\sigma(\tilde{v}, y_1, \ldots, y_{n-1})$ and $\sigma(\tilde{a}, y_1, \ldots, y_{n-1})$, and (ii) A pricing rule that sets prices to be conditional expectations

$$p_n = \mathbb{E}[\tilde{v} \,|\, y_1, \ldots, y_n], \quad n = 1, \ldots, N. \tag{B.2}$$

Our goal is to find a linear equilibrium in which the price dynamics are given by

$$\Delta p_n = \lambda_n y_n, \quad p_0 := 0. \tag{B.3}$$

The two informed investors' optimal orders take the form:

$$\Delta\theta_n^I = \beta_n^I(\tilde{v} - p_{n-1}), \quad \theta_0^I := 0, \tag{B.4}$$

$$\Delta\theta_n^L = \beta_n^L(s_{n-1} - p_{n-1}), \quad \theta_0^L := 0. \tag{B.5}$$

In (B.5) the process s_n denotes the less-informed investor's expectation of the stock payoff $\tilde{v}$ after trade at time n; that is,

$$s_n = \mathbb{E}[\tilde{v} \mid \tilde{a}, y_1, \ldots, y_n], \quad s_0 := \rho\frac{\sigma_{\tilde{v}}}{\sigma_{\tilde{a}}}\tilde{a}. \tag{B.6}$$

The dynamics of s_n are given by

$$\begin{aligned}\Delta s_n &= \phi_n\Big(y_n - \mathbb{E}[y_n \mid \tilde{a}, y_1, \ldots, y_{n-1}]\Big)\\ &= \phi_n\Big(y_n - (\beta_n^L + \beta_n^I)(s_{n-1} - p_{n-1})\Big)\\ &= \phi_n\Big(\Delta w_n + \beta_n^I(\tilde{v} - s_{n-1})\Big).\end{aligned} \tag{B.7}$$

In particular, the less-informed investor learns about $\tilde{v}$ by updating on the observed order flow. Because the better-informed investor knows $\tilde{v}$ initially, she does not update her expectations about $\tilde{v}$ over time. The Internet Appendix presents sufficient conditions for a linear Bayesian Nash equilibrium to exist in the modified FV model.

Finally, we remark that, unlike in our dynamic rebalancing model, there are no predictable components of the order-flow process (i.e., given the aggregate order-flow history) in the modified FV model. Consequently, no q_n process is present and the aggregate order-flow process becomes a martingale with respect to the flow of public information.

Appendix C. Expected rebalancer orders

Proof of Proposition 2. Let $\Delta\theta_n^I$ be defined by (6) throughout this proof, and let $\{\lambda_n, r_n\}$ be the linear equilibrium coefficients for $n = 1, \ldots, N$. The rebalancer's value function, when he is restricted to using only deterministic controls, is given by

$$V_m^{R,a} := \max_{\Delta\theta_n^R \in \sigma(\tilde{a}),\, m+1 \le n \le N-1} -\mathbb{E}\left[\sum_{n=m+1}^{N}(\tilde{a} - \theta_{n-1}^R)\Delta p_n \Big| \tilde{a}\right], \quad m = 0, \ldots, N. \tag{C.1}$$

This definition is the restriction of (44) to deterministic controls. It is straightforward to show that the value function in (C.1) is quadratic, and that the optimal deterministic control — denoted here as x_n^* — is linear in $\tilde{a}$ and is unique.

We define the sets of random variables $\mathcal{A}_n$ by

$$\mathcal{A}_n := \{Z \in \sigma(\tilde{a}, y_1, \ldots, y_{n-1}) : Z \text{ is independent of } \tilde{a}\}, \quad n = 1, 2, \ldots, N. \tag{C.2}$$

Given an arbitrary strategy $\Delta\theta_n^R = g_n\tilde{a} + Z_n$ with $g_n \in \mathbb{R}$ and $Z_n \in \mathcal{A}_n$, we define

$$p_n^a := \mathbb{E}[p_n|\tilde{a}], \quad p_n^Z := p_n - p_n^a, \quad \theta_n^{R,a} := \mathbb{E}[\theta_n^R|\tilde{a}], \quad \theta_n^{R,Z} := \theta_n^R - \theta_n^{R,a}. \tag{C.3}$$

We also define $q_n^a := \mathbb{E}[q_n|\tilde{a}]$ and $q_n^Z := q_n - q_n^a$. We then have the following recursive relations:

$$\Delta p_n^a = \lambda_n\Big(\beta_n^I\big(\tfrac{\rho\sigma_{\tilde{v}}}{\sigma_{\tilde{a}}}\tilde{a} - p_{n-1}^a\big) + g_n\tilde{a}\Big) - \lambda_n(\alpha_n^R + \beta_n^R)q_{n-1}^a, \tag{C.4}$$

$$\Delta p_n^Z = \lambda_n\Big(\beta_n^I\big(\tilde{v} - \tfrac{\rho\sigma_{\tilde{v}}}{\sigma_{\tilde{a}}}\tilde{a} - p_{n-1}^Z\big) + \Delta w_n + Z_n\Big) - \lambda_n(\alpha_n^R + \beta_n^R)q_{n-1}^Z, \tag{C.5}$$

$$\Delta q_n^a = r_n\Big(\beta_n^I\big(\tfrac{\rho\sigma_{\tilde{v}}}{\sigma_{\tilde{a}}}\tilde{a} - p_{n-1}^a\big) + g_n\tilde{a}\Big) - (1 + r_n)(\alpha_n^R + \beta_n^R)q_{n-1}^a, \tag{C.6}$$

$$\Delta q_n^Z = r_n\Big(\beta_n^I\big(\tilde{v} - \tfrac{\rho\sigma_{\tilde{v}}}{\sigma_{\tilde{a}}}\tilde{a} - p_{n-1}^Z\big) + \Delta w_n + Z_n\Big) - (1 + r_n)(\alpha_n^R + \beta_n^R)q_{n-1}^Z. \tag{C.7}$$

Expressions (C.4)–(C.7) imply that $p_n^a, q_n^a \in \sigma(\tilde{a})$, and that p_n^Z and q_n^Z are independent of $\tilde{a}$. This observation produces the following decomposition:

$$\max_{\substack{g_n\in\mathbb{R},\, Z_n\in\mathcal{A}_n,\\ \Delta\theta_n^R = g_n\tilde{a} + Z_n,\, 1\le n\le N-1}} -\mathbb{E}\left[\sum_{n=1}^{N}(\tilde{a} - \theta_{n-1}^R)\Delta p_n \Big| \tilde{a}\right] \tag{C.8}$$

$$= \left(\max_{g_n\in\mathbb{R},\, 1\le n\le N-1} -\sum_{n=1}^{N}(\tilde{a} - \theta_{n-1}^{R,a})\Delta p_n^a\right) + \left(\max_{Z_n\in\mathcal{A}_n,\, 1\le n\le N-1} \mathbb{E}\left[\sum_{n=1}^{N}\theta_{n-1}^{R,Z}\Delta p_n^Z\right]\right). \tag{C.8}$$

We know that the rebalancer's equilibrium optimal strategy is given by $\Delta\hat{\theta}_n^R$ in (19), which is linear in $\tilde{a}, y_1, \ldots, y_{n-1}$ and, therefore, can be written as $\Delta\hat{\theta}_n^R = \hat{g}_n\tilde{a} + \hat{Z}_n$ with $\hat{g}_n \in \mathbb{R}$ and $\hat{Z}_n \in \mathcal{A}_n$. Inserting the equilibrium optimal strategy into (C.8) we see that $(\hat{g}_n)_{n=1,\ldots,N}$ is the solution to

$$\max_{g_n\in\mathbb{R},\, 1\le n\le N-1} -\sum_{n=1}^{N}(\tilde{a} - \theta_{n-1}^{R,a})\Delta p_n^a. \tag{C.9}$$

Since (C.1) is equivalent to the optimization problem in (C.9), we conclude that $x_n^* = \hat{g}_n\tilde{a} = \mathbb{E}[\Delta\hat{\theta}_n^R|\tilde{a}]$, where, in equilibrium, $\Delta\hat{\theta}_n^R = \Delta\theta_n^R$. □

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