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Static and dynamic non-reciprocity in bi-linear structures

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Non-reciprocal acoustic and elastic wave propagation has been shown to enable a plethora of effects analogous to phenomena seen in quantum physics and electromagnetics, such as immunity from back-scattering, unidirectional band gaps, and topologically protected states. These phenomena are of interest because they could lead to the design of direction-dependent acoustic devices that could be used to augment acoustic sensing and transmitting capabilities and may provide insight in the design of materials and structures for vibration and impact isolation. In the present work, we show that static and dynamic non-reciprocity can be achieved in structures composed of bi-linear springs, which have different, amplitude-independent moduli in tension and compression, by investigating the non-reciprocal response of a simple bi-linear structure composed of three springs and two masses in series. Non-reciprocity is demonstrated by calculating the response of each mass to forcing applied to the opposite mass, i.e. by applying Betti's reciprocity theorem to analytical force-displacement relationships in the static sense, and by numerically calculating the response to harmonic forcing in the dynamic sense. Non-reciprocity can be identified via examination of the frequency response curves, making this system a promising candidate for experiments and the design of non-reciprocal devices.

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1. INTRODUCTION

Bi-linear constitutive relations are characterized by a stiffness that transitions between two values at some critical load and have been used to model contact forces,[1–3] nonlinear beams,[4–7] nonlinear elastic solids containing cracks,[8] and many other systems incorporating abrupt, deformation-dependent changes in stiffness. The non-smoothness of these piecewise constitutive relations gives rise to complex, nonlinear dynamics, including strong harmonic generation and chaotic behavior.[1,4,9] Past works have developed theories for continuous bi-linear structures[10,11] and infinite bi-linear spring-mass chains.[12]

One application of mechanical nonlinearity, which has recently received significant attention, is the design of non-reciprocal materials and devices. Reciprocity is a fundamental physical principle that requires that a received signal is unchanged when the positions of a source and receiver are interchanged, and is generally obeyed except for certain specific scenarios (see Ref.,[13] Ref.,[14] and Refs.[15,16] for reciprocity theorems in the contexts of elasticity, elastodynamics, and acoustics, respectively). In the acoustical domain, breaking reciprocity allows tailored wave propagation with a directional dependence, including possibilities for one-way sound propagation,[16–18] and could lead to the design of acoustic devices aiding in numerous applications, such as vibration isolation, signal processing, acoustic communications, and energy harvesting. In past works, nonlinearity has been used to break reciprocity via harmonic generation.[19–22] In another recent study, static non-reciprocity was broken in a geometrically-nonlinear structure.[23]

In this work, we leverage the stiffness transition of the bi-linear force-displacement relationship to break reciprocity in mechanical structures. Specifically, we present a spring-mass model with two degrees of freedom, where some or all of the springs are bi-linear. The stiffness transition of the bi-linear springs occurs at equilibrium, making the nonlinearity amplitude-independent. In the static limit, non-reciprocity is shown using Betti's reciprocity theorem, a well-known theorem of elasticity.[13] In the dynamic regime, non-reciprocity is evident in the response of the structure to harmonic forcing, which could be accessible in future experimental realizations. This work aids in the design of mechanically non-reciprocal materials and structures by providing a very simple non-reciprocal model (e.g. in comparison to Ref.[23]), and could be used as a building block for more complex systems.

2. THEORETICAL MODEL

The discrete model system under consideration is composed of two masses m_1 and m_2 connected to each other and to ground by nonlinear springs, as shown in Fig. 1(a). The springs obey the bi-linear force-displacement relation

$$F(x) = kx + \alpha \operatorname{sgn}(x)x = k(x + \alpha|x|), \tag{1}$$

as shown in Fig. 1(b), such that the stiffnesses in tension and compression are $k(1+\alpha)$ and $k(1-\alpha)$, respectively. Thus, k can be interpreted as a mean stiffness and α determines the strength of the bi-linearity. The displacements of the masses are denoted x_1 and x_2, dissipation is introduced via the linear dashpots c_1 and c_2, and the two masses are subjected to external forces $f_1(t)$ and $f_2(t)$.

Applying the balance of linear momentum to each mass, the equations of motion are found as

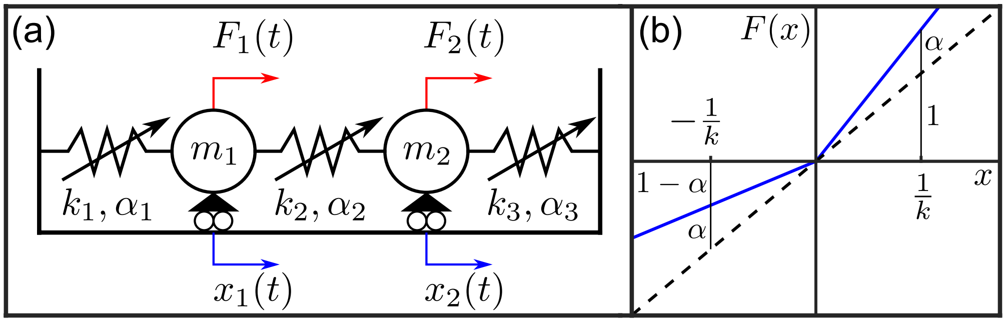

Figure 1: (a) Schematic of the two-mass, bi-linear oscillator model. (b) Force-displacement relationship of the bi-linear springs.

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + (k_1+k_2)x_1 - k_2x_2 + k_1\alpha_1|x_1| - k_2\alpha_2|x_2-x_1| \;=\; f_1(t) \tag{2}$$
$$m_2\ddot{x}_2 + c_2\dot{x}_2 + (k_2+k_3)x_2 - k_2x_1 - k_3\alpha_3|x_2| + k_2\alpha_2|x_2-x_1| \;=\; f_2(t), \tag{3}$$

where the over-dots denote differentiation with respect to time t. To increase the applicability of our results to real systems in the future, we non-dimensionalize this model by introducing the re-scaled variables $u_{1,2} = x_{1,2}/x_0$ and $\tau = \sqrt{k_2/m_2}t$, where x_0 is an arbitrary characteristic amplitude, and substituting them into Eqs. (2) and (3). In matrix form, the dimensionless equations of motion are

$$\mathbf{M}\ddot{\vec{u}} + \mathbf{C}\dot{\vec{u}} + \mathbf{K}(\vec{u})\vec{u} = \begin{bmatrix} F_1(\tau) \\ F_2(\tau) \end{bmatrix}, \tag{4}$$

with mass, damping, and stiffness matrices given by

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix}, \tag{5}$$
$$\mathbf{C} = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}, \tag{6}$$
$$\mathbf{K}(\vec{u}) = \begin{bmatrix} 1+\kappa_1(1+\alpha_1\,\mathrm{sgn}(u_1)) + \alpha_2\,\mathrm{sgn}(u_2-u_1) & -(1+\alpha_2\,\mathrm{sgn}(u_2-u_1)) \\ -(1+\alpha_2\,\mathrm{sgn}(u_2-u_1)) & 1+\kappa_3(1+\alpha_3\,\mathrm{sgn}(u_2)) + \alpha_2\,\mathrm{sgn}(u_2-u_1) \end{bmatrix}, \tag{7}$$

respectively, where $\mu = m_2/m_1$, $\gamma_{1,2} = c_{1,2}/\sqrt{k_2m_1}$, $\kappa_{1,3} = k_{1,3}/k_2$, $F_{1,2} = f_{1,2}/(k_2x_0)$, and $\Omega = \omega\sqrt{m_1/k_2}$ are dimensionless system parameters.

3. STATIC ANALYSIS

In the static limit ($\ddot{\vec{u}} = \dot{\vec{u}} = \Omega = 0$ and F_1 and F_2 constant in time), the dimensionless equations of motion reduce to

$$\mathbf{K}(\vec{u})\vec{u} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}. \tag{8}$$

Let $u_{b,a}$ be the displacement of mass b due to a force F_a applied to mass a. According to Betti's reciprocity theorem, for a reciprocal structure,

$$F_b \cdot u_{b,a} = F_a \cdot u_{a,b}. \tag{9}$$

Thus, we must compare $u_{2,1}$ and $u_{1,2}$, which requires that we solve Eq. (8) for cases where $F_2 = 0$ or $F_1 = 0$. For the sake of illustration, we assume in this example that F_1 and F_2 are both oriented in the positive direction, and that only the middle spring is bi-linear (i.e. $\alpha_1 = \alpha_3 = 0$).

For the case $F_2 = 0$, we observe that the left spring is in tension, while the middle and right springs are in compression; this implies $\mathrm{sgn}(u_1) = \mathrm{sgn}(u_2) = 1$ and $\mathrm{sgn}(u_2 - u_1) = -1$, which eliminates the $\vec{u}$-dependence of $\mathbf{K}$. Equation (8) can then be easily inverted, giving the displacement of interest,

$$u_{2,1} = \frac{(1-\alpha_2)F_1}{(1-\alpha_2)(\kappa_1+\kappa_3)+\kappa_1\kappa_3}. \tag{10}$$

Following a similar process for the case $F_1 = 0$, for which the left and middle springs are in tension and the right spring is in compression (implying $\mathrm{sgn}(u_1) = \mathrm{sgn}(u_2) = \mathrm{sgn}(u_2 - u_1) = 1$), we find

$$u_{1,2} = \frac{(1+\alpha_2)F_2}{(1+\alpha_2)(\kappa_1+\kappa_3)+\kappa_1\kappa_3}. \tag{11}$$

Comparing Eq. (10) and Eq. (11) for the case $F_1 = F_2$, we find that $u_{2,1} \neq u_{1,2}$ unless $\alpha_2 = 0$. Thus, Betti's reciprocity theorem is broken when the middle spring is bi-linear. Finally, we remark that this analysis can be carried out in the same manner for more complicated scenarios, e.g. with F_1 and F_2 oriented in the negative direction and/or with more than one bi-linear spring.

4. DYNAMIC ANALYSIS

In this section, we extend the static analysis into the dynamic regime and observe non-reciprocity for cases of harmonic forcing. In all the following numerical results, we use the parameters $\mu = 0.5$, $\kappa_1 = 0.5$, $\kappa_3 = 0.75$, $\alpha_1 = 0$, $\alpha_2 = -(1/2)$, and $\alpha_3 = -(1/3)$. These parameters were chosen so that all three springs having the same stiffness in tension and different stiffnesses in compression.

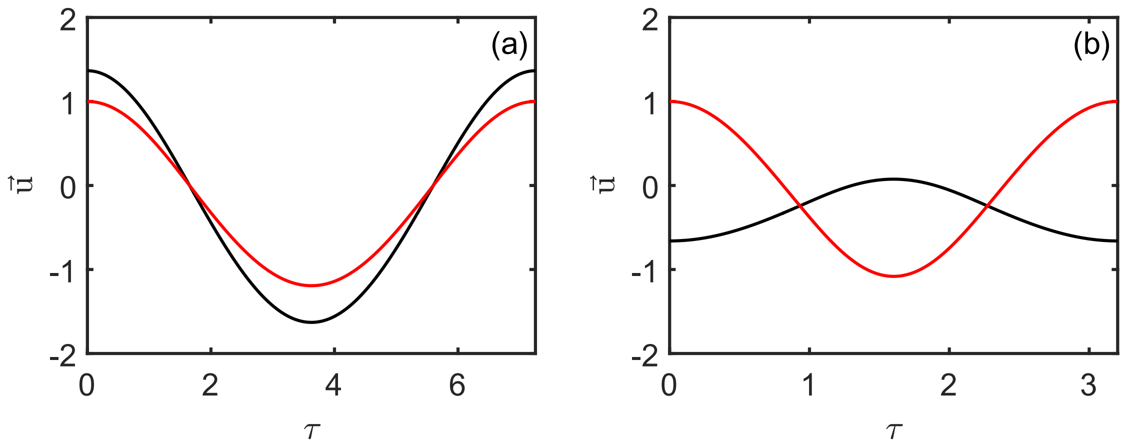

Figure 2: Time histories of the displacements for (a) in-phase nonlinear normal mode with frequency $\Omega_1 = 0.87$ and (b) out-of-phase nonlinear normal mode with frequency $\Omega_2 = 1.96$. Black and red curves correspond to $u_1(\tau)$ and $u_2(\tau)$, respectively.

A. NONLINEAR NORMAL MODES

Before considering the system with harmonic driving, it is useful to consider the case of free, undamped response. Specifically, we calculate the nonlinear normal modes (NNMs) of the system, which are periodic (but not necessarily synchronous) motions that generally have frequency-energy dependence.[24] While the NNMs do not generally form a basis for all configurations of the structure (and thus cannot be used for modal analysis in the classical sense), they can still provide a lot of insight about the nonlinear dynamic behavior of a system under consideration. In particular, when subjected to harmonic forcing, the resonances of the system typically occur near the NNMs.[24,25]

To compute the NNMs of our system, we use a shooting method,[26] which finds roots of the cost function $\vec{u}(T, \vec{u}_0) - \vec{u}_0$. Here, $\vec{u}_0 = [u_1(\tau = 0), u_2(\tau = 0)]$ is a vector of initial conditions, T is the period of the periodic solution, and $\vec{u}(T, \vec{u}_0)$ is the state of the system at time T (i.e. after time evolution through Eq. (4)). In other words, the shooting algorithm finds a set of initial conditions $\vec{u}_0$ and period T that result in a periodic solution of the equations of motion. The cost function is evaluated using direct numerical integration of Eq. (4), using a standard fourth-order Runge-Kutta method, and convergence is achieved using Newton-Raphson iterations.

Time histories of computed NNMs of our system are shown in Fig. 2. Since the transitions between the two stiffnesses of our bi-linear springs occur at their respective equilibrium lengths (and thus there are no stiffening or softening effects with amplitude), the nonlinearity is amplitude-independent, and the NNMs do not have frequency-energy dependence. Thus, the continuations of the two NNMs shown into higher amplitudes have the same frequencies and mode shapes.

B. FORCED RESPONSE

Here we consider harmonic forcing, such that the right-hand side of Eq. (4) is

$$\begin{bmatrix} \tilde{F}_1 \\ \tilde{F}_2 \end{bmatrix} \cos(\Omega\tau), \tag{12}$$

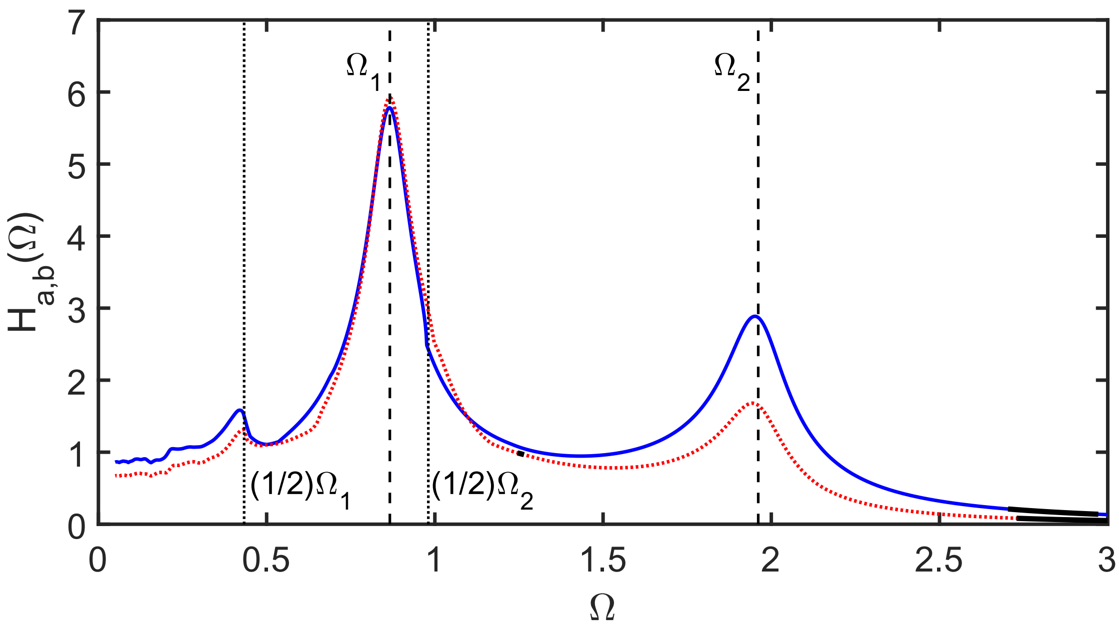

Figure 3: Frequency response curves. Blue solid and red dotted curves correspond to $H_{1,2}(\Omega)$ and $H_{2,1}(\Omega)$, respectively. Black dashed vertical lines indicate the frequencies Ω_1 and Ω_2 of the nonlinear normal modes shown in Fig. 2(a) and Fig. 2(b), respectively. Black dotted vertical lines indicate the frequencies whose second harmonics are Ω_1 and Ω_2. Thick black curve segments indicate unstable solutions.

where $\tilde{F}_1$ and $\tilde{F}_2$ are constant driving amplitudes and Ω is a dimensionless frequency. Our goal is to find the response of each mass to harmonic forcing applied to the opposite mass, as functions of frequency. We define the frequency response functions

$$H_{a,b}(\Omega) = \max_{0 \leq \tau < \frac{2\pi}{\Omega}} \left(|u_{a,b}(\Omega, \tau)|\right) / \tilde{F}_b, \tag{13}$$

or "the maximum magnitude of the response u_a to harmonic forcing $\tilde{F}_b \cos(\Omega\tau)$ applied to m_b, over one period." In contrast to the frequency response functions of linear systems, which are typically defined as ratios of amplitudes of harmonic signals, our definition in terms of absolute value is needed because nonlinearity gives rise to multiple harmonic components in the response. Weak viscous damping $\gamma_1 = \gamma_2 = 0.1$ is included to keep the response bounded at resonance.

To compute these frequency response curves, we use the same shooting algorithm as was used to compute the NNMs, except that the Jacobian of the cost function must include additional terms to account for the non-autonomous forcing.[9] The frequency dependence is found using a pseudo-arclength numerical continuation technique.[26]

Computed frequency response curves are shown in Fig. 3. It is clear that the two frequency response curves $H_{1,2}(\Omega)$ and $H_{2,1}(\Omega)$ are not equal, which indicates non-reciprocity (in a reciprocal system, these two curves would be identical by definition). While these curves show strong nonlinearity, as evidenced by resonances at the sub-harmonics of the two main resonance peaks and unstable branches, the response is still amplitude independent.

5. CONCLUSION

In this work, we have demonstrated mechanical non-reciprocity in a model for a bi-linear spring-mass structure with two degrees of freedom. In the static limit, non-reciprocity is shown in terms of Betti's reciprocity theorem and arises due to different tensile and compressive stiffnesses. When subjected to harmonic driving forces, non-reciprocity can be observed in the frequency response curves. While the bi-linear constitutive relationship includes strong nonlinearity, the dynamics herein are amplitude-independent because the stiffness transitions occur when the springs are un-deformed.

Experimental realization is well within reach for this model, as bi-linear structures with desired stiffness ratios can be fabricated by incorporating contacting parts in additively-manufactured lattice structures,[27] and frequency response curves can be obtained using standard vibration testing techniques.

Opportunities for future work include more in-depth analysis of the static non-reciprocal response, particularly strategies for maximizing non-reciprocity when the forces act in multiple directions; finding methods to predict the strength of dynamic non-reciprocity *a priori*, without complicated numerical methods; design of non-reciprocal structures with more degrees of freedom; and extension to wave propagation in infinite chains of the structures considered here.

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