LIMITS OF QUANTUM GRAPH OPERATORS WITH SHRINKING EDGES

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Abstract. We address the question of convergence of Schrödinger operators on metric graphs with general self-adjoint vertex conditions as lengths of some of graph's edges shrink to zero. We determine the limiting operator and study convergence in a suitable norm resolvent sense. It is noteworthy that, as edge lengths tend to zero, standard Sobolev-type estimates break down, making convergence fail for some graphs. We use a combination of functional-analytic bounds on the edges of the graph and Lagrangian geometry considerations for the vertex conditions to establish a sufficient condition for convergence. This condition encodes an intricate balance between the topology of the graph and its vertex data. In particular, it does not depend on the potential, on the differences in the rates of convergence of the shrinking edges, or on the lengths of the unaffected edges.

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1. Introduction

Continuous dependence of eigenvalues on edge lengths is a fundamental issue in the spectral theory of quantum graphs [BK, M14]. In particular, it is vital to spectral shape optimization problems which have received much attention recently (see for example [F05, EJ, KKM, BRV, KKMM, DR, BL, BKKD, R17, Ar] and references therein). In such optimization problems achieving extremum often requires redistribution of volume (edge length) from one edge to another. It is thus important to determine the limit of a quantum graph operator as one or more of the graph's edges shrink to zero.

We answer this question in a very general setting: Schrödinger operators on graphs with general self-adjoint vertex conditions. The question naturally breaks into three parts. First, one has to determine the domain of the putative limiting operator; this is simple to do on an intuitive level. We recall that any set of self-adjoint vertex conditions is determined by a system of linear relations

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between the values of the function f and its derivative f' at the vertices. Heuristically, the values of the function (and its derivative) at the end points of an infinitesimally short edge should match. Hence, it is natural to conjecture that the vertex conditions for the limiting operator stem from the augmented linear system

$$f \text{ satisfies the original vertex conditions and} \tag{1.1}$$

$$f_e(0) = f_e(\ell_e),\ f_e'(0) = f_e'(\ell_e), \quad \text{for every edge } e \text{ of length } \ell_e \to 0. \tag{1.2}$$

Eliminating from this system the variables corresponding to the edges of vanishing length, one obtains the new set of the limiting vertex conditions on the reduced graph.

The second step is to determine if the vertex conditions obtained through the above procedure *always define a self-adjoint operator* on the new graph. We answer this question in the positive by reformulating it in terms of Lagrangian geometry. It is well known that self-adjoint extensions of a symmetric operator with equal deficiency indices are in one-to-one correspondence with the Lagrangian planes in some symplectic Hilbert space [AS80, BF, KS99, Ha00, LS, LSS, McS]. The question of restricting self-adjoint vertex conditions from the original graph to its reduced version — with some edges shrunk to zero — is reframed in terms of the so-called linear symplectic reduction (see, for example, [McS]) allowing us to show that (1.1)-(1.2) indeed define a valid self-adjoint limiting operator.

We now give two simple but illuminating examples of the limiting vertex conditions. Consider the graph displayed in the left part of Figure 1. We impose $\delta-$type boundary conditions (cf. (2.18)) with coupling constants α_- and α_+ at the end points of the vanishing (middle) edge. Then the limiting vertex condition, in the right part of Figure 1, is also of $\delta-$type but with the coupling constant $\alpha_- + \alpha_+$. An interesting dichotomy arises when we contract a loop with $\theta-$periodic conditions (cf. (2.23)) as shown in Figure 2. If $\theta \neq 0 \pmod{2\pi}$, shrinking results in two separated vertices with the Dirichlet conditions (cf. (2.22)), whereas contracting a periodic loop (i.e. $\theta = 0$) preserves the conditions at the connecting vertex. More examples are considered in Section 3.

The final third step is to investigate convergence of approximating operators to the limiting operator. This turns out to be the most difficult part since the *convergence does not always hold*. In Section 3 we construct several examples of increasing sophistication that illustrate the problem. Perhaps the most striking example is that of a sequence of graphs, each with -1 as an eigenvalue, whose supposed limit is a positive operator, see Example 3.13. It turns out to be a delicate job to craft a condition which excludes all counter-examples and yet includes all known cases when the convergence does occur. This is achieved in Condition 3.2 ("Non-resonance Condition") which, informally, does not allow eigenfunctions of the approximating operators to be supported exclusively on the vanishing edges. We also show that in some settings which often arise in applications, this sufficient condition also turns out to be necessary. Condition 3.2 is formulated entirely in terms of the easily accessible information: the vertex conditions $\mathcal{L}$ on one hand and the topological connectivity information from equation (1.2) on the other. A weaker but more technical sufficient condition (which follows from Condition 3.2.) is that the norms of resolvents on the approximating graphs, considered as operators from L^2 to L^∞, remain uniformly bounded as lengths of some edges shrink to zero. We point out that such a boundedness does not hold in general since the standard Sobolev estimates break down as edge lengths go to zero.

It is important to elaborate on the notion of convergence appropriate for the operators we consider. The approximating and the limiting operators are defined on significantly different spaces making direct comparison impossible. Instead we use the notion of *generalized norm resolvent convergence*, formulated by O. Post [P06, P11, P12] and P. Exner [EP] to study the convergence of differential operators on thin structures to differential operators on graphs. The core of the method is to intertwine the spaces of functions supported on the thick and thin structures by means of

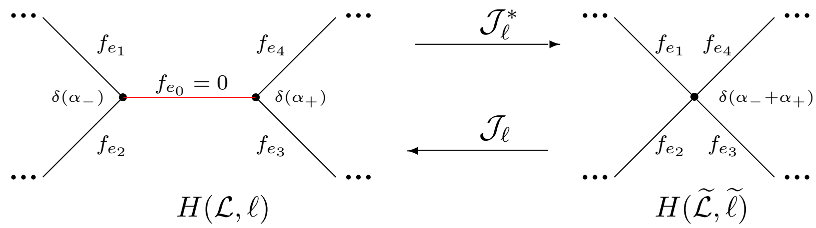

FIGURE 1. A vanishing edge (horizontal) e_0 connecting two vertices equipped with the δ-type boundary conditions. Quasi-unitary operators map the spaces of functions supported on respective graphs and "almost" intertwines the operators on the corresponding graphs.

quasi-unitary operators. For illuminating discussion of this subject we refer to [P12, Chapter 4]. In our model, the quasi-unitary operators $\mathcal{J}_\ell$, formally defined in (3.5), simply extend by zero the functions defined on the reduced graph. This action of the operators $\mathcal{J}_\ell$ and their quasi-inverses $\mathcal{J}_\ell^*$ is schematically illustrated in Figure 1.

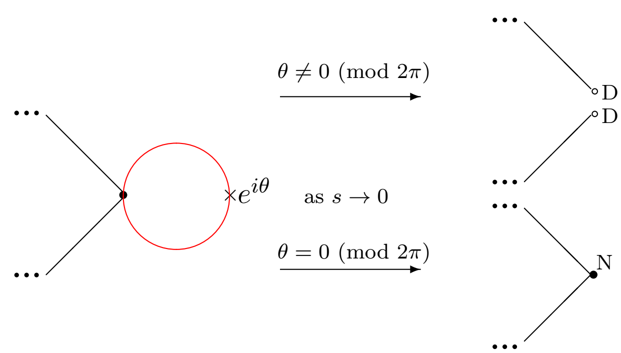

FIGURE 2. Loop of length s.

We now summarize previous related work. In [BK12] it was shown that the eigenvalues of the Schrödinger operator with arbitrary vertex conditions depend analytically on the edge lengths, as long as they remain strictly positive. On the opposite side of the spectrum are the results of [HS], where the behavior of the eigenvalues of the Schrödinger operators with matrix valued potential on $[0, s]$ was studied as $s \to 0$. The case of diagonal potential here is equivalent to a bipartite graph with all edges of the same length. Band and Levy [BL, App. A] gave an informal argument for eigenvalue convergence for the case of shrinking to zero edges that link vertices with Neumann–Kirchhoff, i.e. δ-type with zero coupling constant, conditions. They approached the problem via a secular determinant which is only viable for scale invariant boundary conditions and zero potential. Finally, perhaps the most directly related reference is Cheon, Exner and Turek [CET], which resolves a longstanding open problem about approximating a vertex with arbitrary conditions by a graph with internal structure but only δ-type conditions. As the approximating graph is shrunk to a point, the authors allow δ-couplings to vary, calculate Green's function explicitly and thus establish convergence. We note that the case of a graph with fixed δ-type conditions is covered by our results via Lemma 3.4. Finally we remark that our methods are not incremental extensions of the above mentioned works but a new combination of functional-analytic estimates and Lagrangian geometry considerations.

This paper is organized as follows. In Section 2 we discuss a one-to-one correspondence between Lagrangian planes and self-adjoint boundary conditions on metric graphs. Section 3 summarizes main results of this paper illustrated by numerous examples. Section 4 reviews relevant definitions and results from linear symplectic geometry, proves self-adjointness of the limiting operator, and explores the geometrical meaning of Condition 3.2. Functional-analytic estimates producing the main result are presented in Section 5.

Notation. We denote by I_n the $n\times n$ identity matrix. For an $n\times m$ matrix $A=(a_{ij})_{i=1,j=1}^{n,m}$ and a $k\times\ell$ matrix $B=(b_{ij})_{i=1,j=1}^{k,\ell}$, we denote by $A\otimes B$ the Kronecker product, that is, the $nk\times m\ell$ matrix composed of $k\times\ell$ blocks $a_{ij}B$, $i=1,\dots n$, $j=1,\dots m$. We let $\langle\cdot\,,\cdot\rangle_{\mathbb{C}^n}$ denote the complex scalar product in the space $\mathbb{C}^n$ of $n\times 1$ vectors. We denote by $\mathcal{B}(\mathcal{X})$ the set of linear bounded operators and by $\operatorname{Spec}(T)$ the spectrum of an operator T on a Hilbert space $\mathcal{X}$. Given a subspace $S\subset\mathcal{X}$ we denote ${}^dS:=S\oplus S$. Given an operator T acting in $\mathcal{X}$ we denote ${}^dT:=T\oplus T$, then ${}^dT$ acts in ${}^d\mathcal{X}$. Given two subspace $U,V\subset\mathcal{X}$, we write $\mathcal{X}:=U\dot{+}V$ if $U\cap V=\{0\}$ and $U+V=\mathcal{X}$.

We denote $\mathbb{R}^d_{>0}:=(0,\infty)^d$, $\mathbb{R}^d_{\geq 0}:=[0,\infty)^d$, $d\in\mathbb{N}$. Given two positive quantities x,y we write $x\lesssim_\alpha y$ if there exits a positive constant $c=c(\alpha)>0$ depending only on α such that $x\leq c(\alpha)y$, likewise $x\lesssim y$ if and only if $x\leq Cy$ for some absolute constant $C>0$. Given an edge e incident to a vertex v we write $e\sim v$.

2. Preliminaries and Notation

2.1. Schrödinger Operators on Graphs With Fixed Edge Lengths.

We begin by discussing differential operators on metric graphs. To set the stage, let us fix a discrete graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$ where $\mathcal{V}$ and $\mathcal{E}$ denote the set of vertices and edges correspondingly. We assume that $\mathcal{G}$ consists of finite number $|\mathcal{V}|$ of vertices and finite number $|\mathcal{E}|$ of edges. Each edge $e\in\mathcal{E}$ is assigned positive length $\ell_e\in(0,\infty)$ and some direction. The corresponding metric graph is denoted by Γ. The boundary $\partial\Gamma$ of the metric graph is defined as follows,

$$\partial\Gamma:=\cup_{e\in\mathcal{E}}\{a_e,b_e\},\tag{2.1}$$

where a_e,b_e denote the end points of edge e. Then, one has

$$L^2(\partial\Gamma)\cong\mathbb{C}^{2|\mathcal{E}|},\tag{2.2}$$

where the space $L^2(\partial\Gamma)=\bigoplus_{e\in\mathcal{E}}\left(L^2(\{a_e\})\oplus L^2(\{b_e\})\right)$ corresponds to the discrete Dirac measure with support $\cup_{e\in\mathcal{E}}\{a_e,b_e\}$. Let us introduce the following spaces of functions

$$L^2(\Gamma):=\bigoplus_{e\in\mathcal{E}}L^2(e),\ \ \widehat{H}^k(\Gamma):=\bigoplus_{e\in\mathcal{E}}H^k(e),\ \ k\in\mathbb{N},$$

where $H^k(e)$ is the standard L^2 based Sobolev space of order $k\in\mathbb{N}$. The Dirichlet and Neumann trace operators are defined by the formulas

$$\gamma_D:\widehat{H}^2(\Gamma)\to L^2(\partial\Gamma),\ \ \gamma_D f:=f|_{\partial\Gamma}, f\in\widehat{H}^2(\Gamma),\tag{2.3}$$

$$\gamma_N:\widehat{H}^2(\Gamma)\to L^2(\partial\Gamma),\ \ \gamma_N f:=\partial_\nu f|_{\partial\Gamma}, f\in\widehat{H}^2(\Gamma),\tag{2.4}$$

where $\partial_\nu f$ denotes the inward derivative of f. The trace operator is a bounded, linear operator given by

$$\operatorname{tr}:=\begin{bmatrix}\gamma_D\\ \gamma_N\end{bmatrix},\ \operatorname{tr}:\widehat{H}^2(\Gamma)\to L^2(\partial\Gamma)\oplus L^2(\partial\Gamma)\cong\mathbb{C}^{4|\mathcal{E}|}.\tag{2.5}$$

This notation gives rise to the following form of the second Green's identity,

$$\int_\Gamma\overline{f''}g-\overline{f}g''=-\int_{\partial\Gamma}\overline{\partial_\nu f}g-\overline{f}\partial_\nu g=\langle\operatorname{tr}f,[J\otimes I_{2|\mathcal{E}|}]\operatorname{tr}g\rangle_{\mathbb{C}^{4|\mathcal{E}|}}.\tag{2.6}$$

Finally, the Sobolev space of functions vanishing on the boundary $\partial\Gamma$ together with their derivatives is denoted by

$$\widehat{H}^2_0(\Gamma):=\left\{f\in\widehat{H}^2(\Gamma):\operatorname{tr}f=0\right\}.\tag{2.7}$$

Next, we introduce the minimal Schrödinger operator H_{min} and its adjoint H_{max}. To this end, let us fix a bounded real-valued potential $q \in L^\infty(\Gamma;\mathbb{R})$. Then the linear operator

$$H_{min} := -\frac{d^2}{dx^2} + q, \quad \operatorname{dom}(H_{min}) = \widehat{H}^2_0(\Gamma), \tag{2.8}$$

is symmetric in $L^2(\Gamma)$. Its adjoint $H_{max} := H^*_{min}$ is given by the formulas

$$H_{max} := -\frac{d^2}{dx^2} + q, \quad \operatorname{dom}(H_{max}) = \widehat{H}^2(\Gamma). \tag{2.9}$$

Moreover, the deficiency indices of H_{min} are finite and equal, that is,

$$0 < \dim\ker(H_{max} - \mathbf{i}) = \dim\ker(H_{max} + \mathbf{i}) < \infty. \tag{2.10}$$

By the standard von-Neumann theory, the self-adjoint extensions of H_{min} exist and every self-adjoint extension H satisfies $H_{min} \subset H = H^* \subset H_{max}$. There are various possible parameterizations of all self-adjoint extensions of the minimal operator. In this paper we utilize the one stemming from symplectic geometry. Namely, we use the fact that the self-adjoint extensions of the minimal operator are in one-to-one correspondence with the Lagrangian planes in some symplectic Hilbert space [AS80, McS, Pa]. This relation was noted by many authors in different forms, cf., e.g, [BF, Ha00, KS99, LS]. For the sake of completeness we provide its proof in Section 4 after recalling the definition of Lagrangian subspaces of a symplectic space.

Proposition 2.1 (cf. [Ha00, KS99, KS06])**.** *Assume that $q \in L^\infty(\Gamma;\mathbb{R})$. Then the self-adjoint extensions of H_{min} (cf. (2.8)) are in one-to-one correspondence with the Lagrangian planes in ${}^dL^2(\partial\Gamma)$ equipped with the symplectic form ω given by*

$$\omega: \ {}^dL^2(\partial\Gamma) \times {}^dL^2(\partial\Gamma) \to \mathbb{C}, \tag{2.11}$$

$$\omega((\phi_1,\phi_2),(\psi_1,\psi_2)) := \int_{\partial\Gamma} \overline{\phi_2}\psi_1 - \overline{\phi_1}\psi_2, \tag{2.12}$$

$$(\phi_1,\phi_2),(\psi_1,\psi_2) \in {}^dL^2(\partial\Gamma). \tag{2.13}$$

Namely, the following two assertions hold.

1) If H is a self-adjoint extension of H_{min} then

$$\mathcal{L}(H) := \operatorname{tr}\big(\operatorname{dom}(H)\big) \text{ is a Lagrangian plane in } {}^dL^2(\partial\Gamma).$$

Moreover, the mapping $H \mapsto \mathcal{L}(H)$ is injective.

2) Conversely, if $\mathcal{L} \subset {}^dL^2(\partial\Gamma)$ is a Lagrangian plane then the operator

$$H(\mathcal{L}) := -\frac{d^2}{dx^2} + q(x),\ \operatorname{dom}\big(H(\mathcal{L})\big) = \{f \in \widehat{H}^2(\Gamma) : \operatorname{tr} f \in \mathcal{L}\}, \tag{2.14}$$

is a self-adjoint extension of H_{min}.

We recall a related description of the domain of $H(\mathcal{L})$: There exist three orthogonal projections P_D, P_N, P_R acting in $L^2(\partial\Gamma)$, referred to as the Dirichlet, Neumann, and Robin projections respectively, such that

$$L^2(\partial\Gamma) = \operatorname{ran}(P_D) \oplus \operatorname{ran}(P_N) \oplus \operatorname{ran}(P_R), \tag{2.15}$$

and an invertible, self-adjoint matrix Q such that

$$\operatorname{dom}(H(\mathcal{L})) = \left\{ f \in \widehat{H}^2(\Gamma) \middle| \begin{matrix} P_D\gamma_D f = 0,\ P_N\gamma_N f = 0, \\ P_R\gamma_N f = QP_R\gamma_D f \end{matrix} \right\}, \tag{2.16}$$

cf., e.g., [BK, Theorem 1.1.4]. In this notation for arbitrary $f \in \operatorname{dom}(H(\mathcal{L},\ell))$ one has

$$\langle f, H(\mathcal{L},\ell)f\rangle_{L^2(\Gamma(\ell))} = \|f'\|^2_{L^2(\Gamma(\ell))} + \langle f, q^\ell f\rangle_{L^2(\Gamma(\ell))} + \langle P_R\gamma^\ell_D f, QP_R\gamma^\ell_D f\rangle_{L^2(\partial\Gamma)}. \tag{2.17}$$

The vertex conditions are called *scale invariant* if $P_R = 0$, cf. [BK, Section 1.4.2]. Conditions are scale invariant if and only if the corresponding Lagrangian plane $\mathcal{L} \subset L^2(\partial\Gamma) \oplus L^2(\partial\Gamma)$ decomposes as $\mathcal{L} = \mathcal{L}_D \oplus \mathcal{L}_N$, see Proposition 4.5 in Section 4.

Next, we list some standard conditions at a vertex v (here $\partial_\nu f$ denotes the inward derivative of f):

- δ-type condition with coupling constant $\alpha \in \mathbb{R}$:

$$\begin{cases} f \text{ is continuous at } v, \\ \sum_{v\sim e} \partial_\nu f(v) = \alpha f(v), \end{cases} \tag{2.18}$$

- Neumann–Kirchhoff condition is given by (2.18) with $\alpha = 0$,

$$\begin{cases} f \text{ is continuous at } v, \\ \sum_{v\sim e} \partial_\nu f(v) = 0, \end{cases} \tag{2.19}$$

- δ'-type condition with coupling constant $\alpha \in \mathbb{R}$:

$$\begin{cases} \partial_\nu f \text{ is continuous at } v, \\ \sum_{v\sim e} f(v) = \alpha \partial_\nu f(v), \end{cases} \tag{2.20}$$

- anti-Kirchhoff condition is given by (2.20) with $\alpha = 0$,

$$\begin{cases} \partial_\nu f \text{ is continuous at } v, \\ \sum_{v\sim e} f(v) = 0, \end{cases} \tag{2.21}$$

- Dirichlet conditions

$$f_e(v) = 0, \qquad \text{for all } e \sim v, \tag{2.22}$$

- θ-periodic (magnetic) condition at a vertex of degree 2 with incident edges e_1 and e_2 is given by

$$\begin{cases} f_{e_1}(v) = e^{i\theta} f_{e_2}(v), \qquad \theta \in \mathbb{R}, \\ \partial_\nu f_{e_1}(v) = -e^{i\theta} \partial_\nu f_{e_2}(v). \end{cases} \tag{2.23}$$

2.2. **Schrödinger Operators on Graphs With Vanishing Edges.** The main purpose of this paper is to investigate convergence of the spectral projections of the Schrödinger operators on $\Gamma(\ell)$, where $\ell = (\ell_e)_{e\in\mathcal{E}}$ denotes the vector of edge lengths, as

$$\ell \to \widetilde{\ell} \text{ in } \mathbb{R}^{|\mathcal{E}|}, \qquad \text{where} \quad \ell \in \mathbb{R}^{|\mathcal{E}|}_{>0} \quad \text{and} \quad \widetilde{\ell} = (\widetilde{\ell}_e)_{e\in\mathcal{E}} \in \mathbb{R}^{|\mathcal{E}|}_{\geq 0} \setminus \{0\}. \tag{2.24}$$

Note that the components of ℓ are all positive, whereas some, but not all, components of $\widetilde{\ell}$ are equal to zero. The "limiting" metric graph $\Gamma(\widetilde{\ell})$ is based on the discrete graph $\widetilde{\mathcal{G}}$ obtained from $\mathcal{G}$ by contracting the edges with $\widetilde{\ell}_e = 0$.

We emphasize that the main difficulty is in dealing with the edges whose lengths tend to zero. For notational convenience we label edges of the graph $\mathcal{G}$ so that the first m ones are rescaled but not completely shrunk to zero, and the remaining $|\mathcal{E}| - m$ edges are being shrunk to zero as $\ell \to \widetilde{\ell}$, that is, we write

$$\widetilde{\ell} = (\widetilde{\ell}_{e_1}, ..., \widetilde{\ell}_{e_m}, 0, ...0)^\top, \tag{2.25}$$

where the first $m \geq 1$ components of $\widetilde{\ell}$ are positive. To simplify notation we denote the set of the *non-vanishing* edges of $\Gamma(\ell)$ by

$$\mathcal{E}_+ := \{e_1, ..., e_m\}, \tag{2.26}$$

and the *vanishing* ones by

$$\mathcal{E}_0 := \{e_{m+1}, ..., e_{|\mathcal{E}|}\}. \tag{2.27}$$

Let $\Gamma_+(\ell)$ be the subgraph of $\Gamma(\ell)$ with the set of edges $\mathcal{E}_+$, and let $\Gamma_0(\ell)$ be the subgraph of $\Gamma(\ell)$ with the set of edges $\mathcal{E}_0$. In particular, one has

$$\Gamma(\ell) = \Gamma_+(\ell) \cup \Gamma_0(\ell). \tag{2.28}$$

Let ℓ_+ denote the vector of edge lengths of graph $\Gamma_+(\ell)$, and ℓ_0 denote the vector of edge lengths of $\Gamma_0(\ell)$, that is, $\ell = (\ell_+, \ell_0)$. Next, since all components of ℓ are positive, the spaces $\partial\Gamma(\ell)$, $\partial\Gamma_+(\ell)$, and $\partial\Gamma_0(\ell)$ do not depend on ℓ. We therefore drop ℓ and write $\partial\Gamma$, $\partial\Gamma_+$, and $\partial\Gamma_0$ respectively. Then, in particular, $\partial\Gamma = \partial\Gamma_+ \cup \partial\Gamma_0$ and

$$L^2(\partial\Gamma) = L^2(\partial\Gamma_+) \oplus L^2(\partial\Gamma_0), \tag{2.29}$$

where L^2 spaces correspond to the discrete Dirac measure with support $\cup_{e\in\mathcal{E}}\{a_e, b_e\}$. We notice that all spaces in (2.29) are finite-dimensional since Γ is a compact graph. Let P_+ be the orthogonal projection acting in $L^2(\partial\Gamma)$ with $\operatorname{ran}(P_+) = L^2(\partial\Gamma_+) \oplus \{0\}$, and let $P_0 := I_{L^2(\partial\Gamma)} - P_+$. We recall the notation ${}^dL^2(\partial\Gamma) := L^2(\partial\Gamma) \oplus L^2(\partial\Gamma)$ and we write ${}^dP$ for the operator $P \oplus P$ acting in ${}^dL^2(\partial\Gamma)$. In particular, for the symplectic form (2.11), one has

$$\omega(u, v) = \omega({}^dP_+u, {}^dP_+v) + \omega({}^dP_0u, {}^dP_0v), \tag{2.30}$$

for all $u, v \in {}^dL^2(\partial\Gamma)$.

To complete the setting, let us define the Schrödinger operators corresponding to each lengths vector $\ell \in \mathbb{R}^{|\mathcal{E}|}_{>0}$. To this end, let us fix a family of potentials $q^\ell \in L^\infty(\Gamma(\ell); \mathbb{R})$ corresponding to the graphs with positive edge lengths, and the limiting potential $q^{\widetilde{\ell}} \in L^\infty(\Gamma(\widetilde{\ell}); \mathbb{R})$ satisfying

$$\begin{gathered}\|q^\ell\|_{L^\infty(\Gamma(\ell);\mathbb{R})} = \mathcal{O}(1) \text{ as } \ell \to \widetilde{\ell},\\ \sup_{y\in[0,1]} |q^\ell_e(\ell_e y) - q^{\widetilde{\ell}}_e(\widetilde{\ell}_e y)| = o(1) \text{ as } \ell_e \to \widetilde{\ell}_e, \text{ for all } e \in \mathcal{E}_+.\end{gathered} \tag{2.31}$$

These conditions hold, for example, if the family q^ℓ is obtained by rescaling a fixed potential. Next, we fix a Lagrangian plane

$$\mathcal{L} \subset {}^dL^2(\partial\Gamma). \tag{2.32}$$

For $\ell \in \mathbb{R}^{|\mathcal{E}|}_{>0}$ let $H(\mathcal{L}, \ell)$ denote the self-adjoint Schrödinger operator acting in $L^2(\Gamma(\ell))$ and given by the formulas

$$\begin{gathered}H(\mathcal{L}, \ell) := -\frac{d^2}{dx^2} + q^\ell(x),\\ \operatorname{dom}\big(H(\mathcal{L}, \ell)\big) = \{f \in \widehat{H}^2(\Gamma(\ell)) : \operatorname{tr}^\ell f \in \mathcal{L}\},\end{gathered} \tag{2.33}$$

where the trace operator $\operatorname{tr}^\ell = (\gamma_D^\ell, \gamma_N^\ell)^\top$ acts from $\widehat{H}^2(\Gamma(\ell))$ to $L^2(\partial\Gamma) \oplus L^2(\partial\Gamma)$ as indicated in (2.5). In particular, the norm of tr^ℓ depends on ℓ. The resolvent of $H(\mathcal{L}, \ell)$ is denoted by

$$R(\mathcal{L}, \ell, z) := (H(\mathcal{L}, \ell) - z)^{-1},\ z \in \mathbb{C} \setminus \operatorname{Spec}(H(\mathcal{L}, \ell)). \tag{2.34}$$

3. Main Results

In this section we collect the statements of our main results together with examples that illustrate their application. The proofs will be provided in subsequent sections.

First, we define an operator $H(\widetilde{\mathcal{L}}, \widetilde{\ell})$ on the graph $\Gamma(\widetilde{\ell})$, which will serve as the limiting operator for $H(\mathcal{L}, \ell)$ as $\ell \to \widetilde{\ell}$. The definition is motivated by the heuristic observation, made in the Introduction, that the limiting boundary conditions should be of the form (1.2).

Theorem 3.1. *Assume that* $\mathcal{L} \subset {}^dL^2(\partial\Gamma)$ *is a Lagrangian plane with respect to symplectic form* ω, *cf.* (2.11)-(2.13). *Let*

$$\widetilde{\mathcal{L}} := \{(\phi_1|_{\partial\Gamma+}, \phi_2|_{\partial\Gamma+}) : (\phi_1, \phi_2) \in \mathcal{L} \cap (D_0 \oplus N_0)\}, \tag{3.1}$$

where

$$D_0 = \{\phi_1 \in L^2(\partial\Gamma) : \phi_1(a_e) = \phi_1(b_e),\ e \in \mathcal{E}_0\}, \tag{3.2}$$

$$N_0 = \{\phi_2 \in L^2(\partial\Gamma) : \phi_2(a_e) = -\phi_2(b_e),\ e \in \mathcal{E}_0\}. \tag{3.3}$$

Then $\widetilde{\mathcal{L}}$ *is a Lagrangian plane in* ${}^dL^2(\partial\Gamma_+)$ *with respect to the symplectic form* ω_{Γ_+} *obtained by restricting* ω *to* ${}^dL^2(\partial\Gamma_+)$. *Therefore, the operator* $H(\widetilde{\mathcal{L}}, \widetilde{\ell})$ *acting in* $L^2(\Gamma(\widetilde{\ell}))$ *and given by*

$$\begin{aligned} &H(\widetilde{\mathcal{L}}, \widetilde{\ell}) := -\frac{d^2}{dx^2} + q^{\widetilde{\ell}}, \\ &\operatorname{dom}\big(H(\widetilde{\mathcal{L}}, \widetilde{\ell})\big) = \{f \in \widehat{H}^2(\Gamma(\widetilde{\ell})) : \operatorname{tr}^{\widetilde{\ell}}(f) \in \widetilde{\mathcal{L}}\}, \end{aligned} \tag{3.4}$$

is self-adjoint.

Proof. The proof, based on linear symplectic reduction, is provided on page 16. □

The main result of this paper is the convergence of the spectral projections of the self-adjoint Schrödinger operators $H(\mathcal{L}, \ell)$ to those of $H(\widetilde{\mathcal{L}}, \widetilde{\ell})$. It will be established under the following condition.

Condition 3.2 (Non-resonance Condition)**.** *Suppose that for all* $(\phi_1, \phi_2) \in \mathcal{L} \cap (D_0 \oplus N_0)$ *such that* $\phi_1|_{\partial\Gamma+} = \phi_2|_{\partial\Gamma+} = 0$ *one has* $\phi_1 = 0$.

Informally, this condition says that if a function from the domain of H is small on non-vanishing edges, then its value (but not, necessarily, its derivative) should also be small on the vanishing edges. Let us emphasize the striking similarity between Condition 3.2 and the definition of $\widetilde{\mathcal{L}}$ in equation (3.1). As explained in Remark 4.4, this condition is generic among all self-adjoint Schrödinger operators on Γ (as parameterized by the Lagrangian planes $\mathcal{L}$).

Condition 3.2 is also easy to check on important classes of graphs. The first class consists of the graphs with scale invariant conditions, in which case Condition 3.2 is also necessary.

Lemma 3.3. *Suppose that the Robin part of* $H(\mathcal{L}, \ell)$ *is absent, that is,* $P_R = 0$ *in* (2.16). *Then Condition* 3.2 *holds if and only if the zero function is the only function satisfying the boundary conditions* $\operatorname{tr}(f) \in \mathcal{L}$, *that is constant on each edge of* Γ_0 *and vanishes on* Γ_+.

Proof. On page 17. □

The second class includes connected graphs with a continuity condition imposed at every vertex.

Lemma 3.4. *Suppose that every vanishing edge* $e \in \mathcal{E}_0$ *belongs to a path* $\mathcal{P}_e$ *that contains at least one non-vanishing edge, and along which the function* $|f|$ *is continuous for every* $f \in \operatorname{dom}\big(H(\mathcal{L}, \ell)\big)$. *Then Condition* 3.2 *holds.*

Proof. On page 17. □

In order to formulate our results on spectral convergence, let us introduce quasi-unitary operators $\mathcal{J}_\ell$ which lift the functions defined on the limiting graph $\Gamma(\widetilde{\ell})$ to the approximating graph $\Gamma(\ell)$.

This is achieved by linear scaling on the edges of Γ_+ and by extending functions by zero on the edges of Γ_0, that is by defining $\mathcal{J}_\ell \in \mathcal{B}\left(L^2(\Gamma(\widetilde{\ell})), L^2(\Gamma(\ell))\right)$ as follows:

$$(\mathcal{J}_\ell f)(x) = \sum_{e\in\mathcal{E}_+} \chi_e(x)\sqrt{\frac{\widetilde{\ell}_e}{\ell_e}}\, f\left(\frac{x\widetilde{\ell}_e}{\ell_e}\right), \quad x \in \Gamma(\ell), \tag{3.5}$$

where $\chi_e(\cdot)$ is the characteristic function of $e \subset \Gamma(\ell)$. We remark that $\mathcal{J}_\ell^*\mathcal{J}_\ell$, where $\mathcal{J}_\ell^*$ denotes the adjoint operator, is identity on $L^2(\Gamma(\widetilde{\ell}))$, see Theorem 5.4 for details.

Theorem 3.5 (Convergence of resolvents)**.** *Assume Condition* 3.2*. Then, as* $\ell \to \widetilde{\ell}$*,*

$$\begin{aligned} &\left\|\mathcal{J}_\ell R(\widetilde{\mathcal{L}}, \widetilde{\ell}, z) - R(\mathcal{L}, \ell, z)\mathcal{J}_\ell\right\|_{\mathcal{B}\left(L^2(\Gamma(\widetilde{\ell})), L^2(\Gamma(\ell))\right)} \to 0, \\ &\left\|(I_{L^2(\Gamma(\mathcal{G};\ell))} - \mathcal{J}_\ell\mathcal{J}_\ell^*)R(\mathcal{L}, \ell, z)\right\|_{\mathcal{B}(L^2(\Gamma(\mathcal{G};\ell)))} \to 0, \end{aligned} \tag{3.6}$$

where $R(\mathcal{L}, \ell, z)$ *and* $R(\widetilde{\mathcal{L}}, \widetilde{\ell}, z)$ *denote the resolvents of the respective operators.*

Proof. On page 22 as a combination of Theorems 5.4 and 5.5. □

An immediate corollary of the convergence of resolvents is convergence of spectra.

Theorem 3.6 (Spectral convergence)**.** *Assume Condition* 3.2 *holds. Then*

$$\mathrm{Spec}(H(\mathcal{L}, \ell)) \to \mathrm{Spec}(H(\widetilde{\mathcal{L}}, \widetilde{\ell})) \ \textit{as}\ \ell \to \widetilde{\ell}, \tag{3.7}$$

in the Hausdorff sense for multisets. Namely, if λ_0 *has multiplicity* $m \in \{0, 1, 2, \ldots\}$ *in the multiset* $\mathrm{Spec}(H(\widetilde{\mathcal{L}}, \widetilde{\ell}))$ *then for all sufficiently small* $\varepsilon > 0$ *there exists* $\delta = \delta(\varepsilon, \lambda_0) > 0$ *such that*

$$\mathrm{card}\left(\mathrm{Spec}(H(\mathcal{L}, \ell)) \cap B(\lambda_0, \varepsilon)\right) = m \ \textit{whenever}\ |\ell - \widetilde{\ell}| < \delta. \tag{3.8}$$

Furthermore, eigenspaces converge in the following sense,

$$\begin{aligned} &\left\|\mathcal{J}_\ell\chi(H(\widetilde{\mathcal{L}}, \widetilde{\ell})) - \chi(H(\mathcal{L}, \ell))\mathcal{J}_\ell\right\|_{\mathcal{B}\left(L^2(\Gamma(\widetilde{\ell})), L^2(\Gamma(\ell))\right)} \to 0, \\ &\left\|\left(I_{L^2(\Gamma(\mathcal{G};\ell))} - \mathcal{J}_\ell\mathcal{J}_\ell^*\right)\chi(H(\mathcal{L}, \ell))\right\|_{\mathcal{B}(L^2(\Gamma(\mathcal{G};\ell)))} \to 0, \end{aligned} \tag{3.9}$$

where $\chi(H(\mathcal{L}, \ell))$ *and* $\chi(H(\widetilde{\mathcal{L}}, \widetilde{\ell}))$ *denote the spectral projections of the respective operators onto an interval* (a, b) *with* $a, b \in \mathbb{R} \setminus \mathrm{Spec}(H(\widetilde{\mathcal{L}}, \widetilde{\ell}))$*.*

Proof. On page 25 □

In the case of the Laplace operator with scale invariant vertex conditions we show that Condition 3.2 is not only sufficient but also necessary for the spectral convergence to hold.

Theorem 3.7. *Assume that the Robin part of* $H(\mathcal{L}, \ell)$ *is absent , that is,* $P_R = 0$ *in* (2.16) *and that* $q^\ell \equiv 0$ *. Then* (3.7) *holds* if and only if *Condition* 3.2 *is fulfilled.*

Proof. On page 26. □

While Condition 3.2 is convenient to use (see numerous examples below), it will not be used directly in the proofs. Instead we will need a more technical result: a uniform bound on the resolvent of $H(\mathcal{L}, \ell)$ as an operator from $L^2(\Gamma(\ell))$ to $L^\infty(\Gamma(\ell))$ which follows from Condition 3.2. In fact, it is this bound that implies the conclusion of Theorem 3.5. We explore this bound in the following two theorems.

Theorem 3.8. *Recall* (2.32)–(2.34)*. Then the following statements are equivalent:*

(i) There exists a constant $c > 0$, independent of ℓ, such that

$$\|R(\mathcal{L}, \ell, \mathbf{i})\|_{\mathcal{B}(L^2(\Gamma(\ell)), L^\infty(\Gamma(\ell)))} < c, \tag{3.10}$$

for all ℓ sufficiently close $\widetilde{\ell}$.

(ii) There exists a constant $c > 0$, independent of ℓ, such that

$$\left\|\chi_e R(\mathcal{L}, \ell, \mathbf{i})\right\|_{\mathcal{B}(L^2(\Gamma(\ell)))} < c\sqrt{\ell_e} \quad \text{for each } e \in \mathcal{E}_0,$$

for all ℓ sufficiently close $\widetilde{\ell}$.

(iii) There exists a constant $c > 0$, independent of ℓ and f, such that

$$\|f\|^2_{L^\infty(\Gamma(\ell))} \leq c \left(\|f\|^2_{L^2(\Gamma(\ell))} + \|f''\|^2_{L^2(\Gamma(\ell))} \right), \ f \in \operatorname{dom}(H(\mathcal{L}, \ell)), \tag{3.11}$$

for all ℓ sufficiently close $\widetilde{\ell}$

Moreover, if one of the above statements holds then for some constant $c > 0$, independent of ℓ and f, one has

$$\|f'\|^2_{L^2(\Gamma(\ell))} \leq c \left(\|f\|^2_{L^2(\Gamma(\ell))} + \|f''\|^2_{L^2(\Gamma(\ell))} \right), \ f \in \operatorname{dom}(H(\mathcal{L}, \ell)), \tag{3.12}$$

and

$$\|R(\mathcal{L}, \ell, \mathbf{i})\|_{\mathcal{B}\left(L^2(\Gamma(\ell)), \widehat{H}^2(\Gamma(\ell))\right)} < c, \tag{3.13}$$

for all ℓ sufficiently close $\widetilde{\ell}$.

Proof. On page 18. □

Theorem 3.9. *Condition 3.2 implies statements (i)-(iii) of Theorem 3.8. Furthermore, if the Robin part of $H(\mathcal{L}, \ell)$ is absent, that is, $P_R = 0$ in (2.16), then (i)-(iii) of Theorem 3.8 are equivalent to Condition 3.2*

Proof. On page 19. □

To illustrate our results we will now discuss several examples of graphs with shrinking edges. We start with the most basic example where there is no spectral convergence.

Example 3.10 (Shrinking Neumann interval)**.** In this example we consider a disconnected two edge graph $\Gamma = \{e_N, e_D\}$ and the Laplace operator subject to Neumann boundary conditions on e_N and to Dirichlet boundary conditions on e_D. The spectrum of such quantum graph is given by

$$\{0\} \cup \left\{ \left(\frac{\pi k_1}{\ell_D}\right)^2, \left(\frac{\pi k_2}{\ell_N}\right)^2 : k_1 \in \mathbb{N}, k_2 \in \mathbb{N} \right\}. \tag{3.14}$$

Now let $\ell_N \to 0$ while $\ell_D = 1$. Condition 3.2 (in the form of Lemma 3.3) fails: the function equal to 1 on e_N and 0 on e_D satisfies the boundary conditions for all ℓ_N. This function gives rise to eigenvalue 0 which is *not* present in the spectrum of $H(\widetilde{\mathcal{L}}, \widetilde{\ell})$ defined according to (3.1). The latter operator is simply the Dirichlet Laplacian on the interval e_D whose spectrum is

$$\left\{ \left(\frac{\pi k_1}{\ell_D}\right)^2 : k_1 \in \mathbb{N} \right\}. \tag{3.15}$$

A slight variation of this example is the same graph with $\ell_D \to 0$, $\ell_N = 1$, which does satisfy Condition 3.2. The limiting operator is the Neumann Laplacian on the interval e_N whose spectrum is the limit of the set in (3.14) as $\ell_D \to 0$.

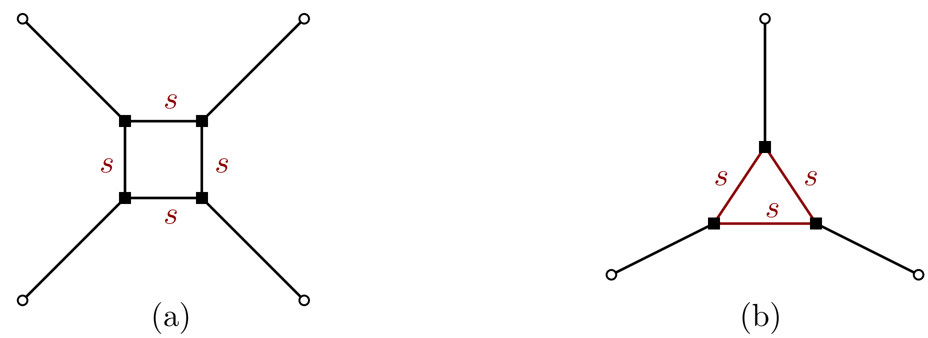

FIGURE 3. Two similar graphs with different convergence outcomes as $s \to 0$: the spectrum of (a) does not converge while that of (b) does. Empty circles denote Dirichlet conditions and full squares denote anti-Kirchhoff conditions, cf. (2.21).

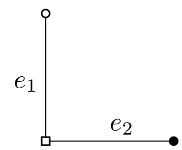

FIGURE 4. $\circ$ denotes Dirichlet vertex conditions, $\bullet$ denotes Neumann–Kirchhoff conditions, cf. (2.19), $\square$ denotes vertex conditions given by (3.16)

Despite its simplicity, the above example illustrates the common mechanism of convergence failure: presence of an eigenfunction whose support is shrinking to zero. The following example shows that there are connected graphs with similar features.

Example 3.11. Consider the graph in Fig. 3(a), equipped with anti-Kirchhoff conditions, cf. (2.21). Condition 3.2 fails by Lemma 3.3 since there is a function equal to $+1$ on vertical vanishing edges, -1 on horizontal vanishing edges and zero on all non-vanishing edges. This is an eigenfunction with eigenvalue zero whose support is the vanishing part.

We point out that the spectral convergence (or lack thereof) depends not only on the boundary conditions but also on the topology of the graph. It is easy to see that the graph in Fig. 3(b), despite having the same vertex conditions as Fig. 3(a), satisfies the conditions of Lemma 3.3: the only function, constant on each edge and equal to zero on the non-vanishing edges must be zero on the whole graph.

Are the eigenfunctions of eigenvalue 0 the only ones to cause such problems? In general, the answer is no. Let us start with a related question: suppose the whole graph is scaled by s as $s \to 0$. Weyl's law dictates that the bulk of the eigenvalues grow at the rate $1/s^2$. If the vertex conditions are scale invariant, all eigenvalues of the Laplacian get multiplied by $1/s^2$ and grow (except the eigenvalue 0). But is the same true in general?

Example 3.12. Consider the graph consisting of two edges of length $\ell_1 = \ell_2 = s$ connected at one endpoint, see Figure 4. Impose Dirichlet and Neumann conditions at endpoints of degree one of edges e_1 and e_2, correspondingly. At the vertex of degree 2 impose the conditions that we will call *hyperbolic*,

$$\begin{cases} \partial_\nu f_{e_1}(v) = -f_{e_2}(v), \\ \partial_\nu f_{e_2}(v) = -f_{e_1}(v). \end{cases} \tag{3.16}$$

This graph has vanishing volume but -1 remains an eigenvalue independently of s. The corresponding eigenfunction is

$$f_{e_1}(x) = \sinh(x), \qquad f_{e_2}(x) = \cosh(x), \tag{3.17}$$

where on both edges the point $x = 0$ is at the vertex of degree one.

We now turn this into an example of a *connected* graph with some non-vanishing edges.

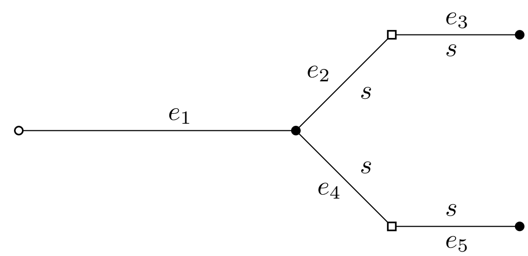

FIGURE 5. e_1 is of length 1, e_k is of length s for $k \in \{2, 3, 4, 5\}$, $\circ$ denotes Dirichlet conditions, $\bullet$ denotes Neumann–Kirchhoff conditions, $\square$ denotes vertex given by (3.16).

Example 3.13. Consider the graph shown in Fig. 5. The lengths of the edges are $\ell_1 = 1$ and $\ell_2 = \ell_3 = \ell_4 = \ell_5 = s \to 0$. There is an eigenfunction with eigenvalue -1 for every $s > 0$:

$$f_{e_1}(x) \equiv 0,\ f_{e_2}(x) = \sinh(x),\ f_{e_3}(x) = \cosh(x),\ f_{e_4}(x) = -\sinh(x),\ f_{e_5}(x) = -\cosh(x). \tag{3.18}$$

Using (1.2), it is easy to see that $H(\widetilde{\mathcal{L}}, \widetilde{\ell})$ is the edge e_1 with Dirichlet conditions. Thus every approximating graph has an eigenvalue -1 while its would-be limit is a strictly positive operator.

Let us now explore some examples where the spectral convergence holds.

Example 3.14 (Tadpole graph with a vanishing loop)**.** Consider the graph consisting of an edge and a loop attached to one of its endpoints, see Figure 6. We impose Neumann–Kirchhoff conditions at the attachment point and the Dirichlet condition at the other endpoint. We assume the magnetic flux α is threading the loop. The magnetic field is realized as the condition

$$f(c+) = e^{i\alpha} f(c-) \qquad \partial_\nu f(c+) = -e^{i\alpha} \partial_\nu f(c-) \tag{3.19}$$

at an arbitrary point c on the loop. The derivative is taken in the direction away from c according to our convention; this leads to the minus sign in (3.19). Let $\ell_1 = 1$ be the length of the edge and $\ell_2 = s$ be the length of the loop. The spectral convergence, as $s \to 0$, holds by Lemma 3.4 and Theorem 3.6. However, the limiting operator depends on whether $\alpha = 0$ or not.

It is interesting to explore this difference from the point of view of the secular manifold. Following a well-known procedure [B17], the eigenvalues $\lambda = k^2 > 0$ of this graph can be found as the solutions of the secular equation $F(kl_1, kl_2; \alpha) = 0$, where, in this case, the secular function F is given by

$$F(x_1, x_2; \alpha) = -2 \sin x_1 \left(\cos x_2 - \cos(\alpha)\right) - \cos x_1 \sin x_2. \tag{3.20}$$

To understand the behavior of eigenvalues, we follow Barra–Gaspard, cf. [BG], and visualize them as the intersections of the straight line $[kl_1, kl_2]$, $k \in (0, \infty)$ with the analytic variety

$$\Sigma_\alpha = \{(x_1, x_2) \in \mathbb{R}^2 : F(x_1, x_2; \alpha) = 0\}, \tag{3.21}$$

usually referred to as *secular manifold*. This convenient characterization is available only for graphs with scale invariant vertex conditions and zero potential. Both the line and the secular manifold Σ_α for two values of α (zero and non-zero) are illustrated in Figure 6. Since we are setting $l_1 = 1$, the values of k can be read as the x-coordinate of the intersection points.

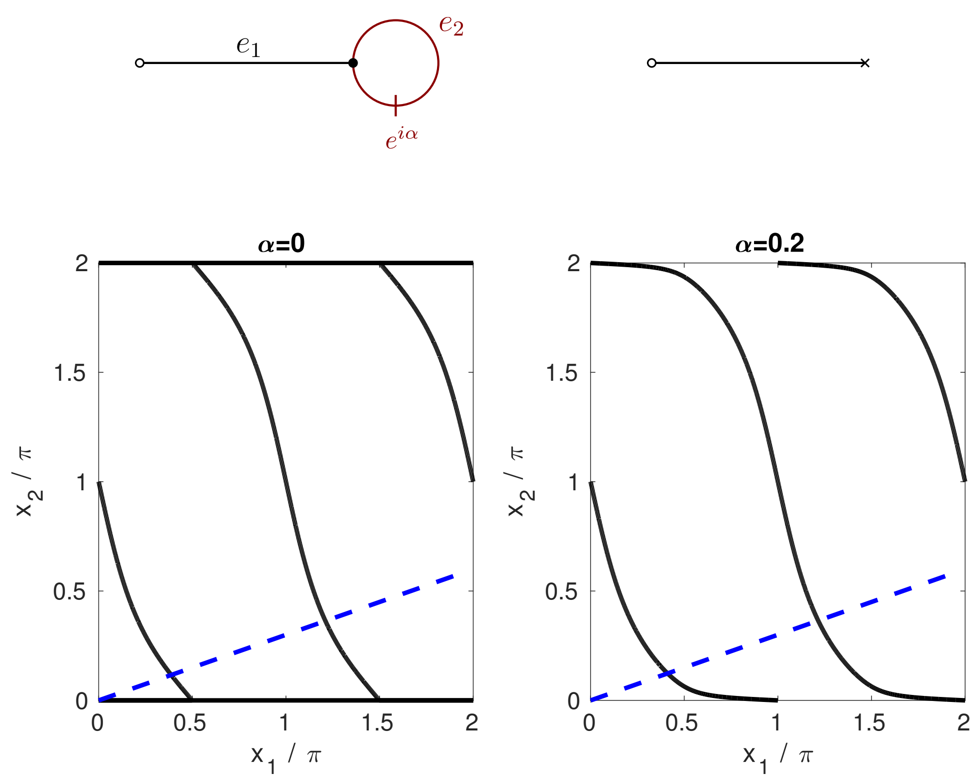

FIGURE 6. Top panel: $\circ$ denotes Dirichlet vertex conditions, $\bullet$ denotes Kirchhoff conditions , $\times$ denotes Dirichlet conditions if $\alpha \neq 0 \bmod (2\pi)$ and Kirchhoff conditions if $\alpha = 0 \bmod (2\pi)$.
Bottom panel: The secular manifold and the Barra–Gaspard flow for two values of α.

The structure of secular manifold undergoes a significant change from $\alpha = 0$ to $\alpha \neq 0$. When $\alpha = 0$, the secular manifold on the torus is a union of a smooth curve and the line $x_2 = 0$. When $\alpha \neq 0$, there are two smooth curves (which related by a shift of π in x_1 direction).

Suppose that the slope of the dashed lines in Figure 6 is equal to s. Then as $s \to 0$, the first intersection point converges to $(\pi/2, 0)$ when $\alpha = 0 \bmod (2\pi)$ and to $(\pi, 0)$ otherwise. That is, the first intersection point tends to the first eigenvalue of the Neumann–Dirichlet interval if $\alpha = 0 \bmod (2\pi)$ and to the first eigenvalue of Dirichlet–Dirichlet interval otherwise.

If, instead of contracting the loop, we contract the edge, our results dictate that the loop will get the Dirichlet conditions at the (former) attachment point. This disconnects the loop into an interval of length l_2 with Dirichlet endpoints and the spectrum $k_n = \pi n/l_2$. The result is independent of α (the magnetic field on an interval can be removed by a gauge transformation) and can be seen both from Figure 6 (the dashed line is getting close to vertical) or from setting $x_1 = 0$ in the secular function, equation (3.20).

Finally, we remark that simply setting the relevant $x = 0$ does not always produce the correct secular function for the limiting problem: as observed in [ABB], we get identically zero if we set $x_2 = 0$ for the loop with no magnetic field ($\alpha = 0$).

Example 3.15 (A vanishing cycle in a graph with Neumann–Kirchhoff conditions)**.** Consider the tetrahedron graph (complete graph on 4 vertices, K_4) with one vertex turned into a triangle. We will be contracting the triangle into a single vertex, see Figure 7, scaling it by $s \to 0$. We notice that the assumption of Lemma 3.4 is satisfied, hence, the spectral convergence holds.

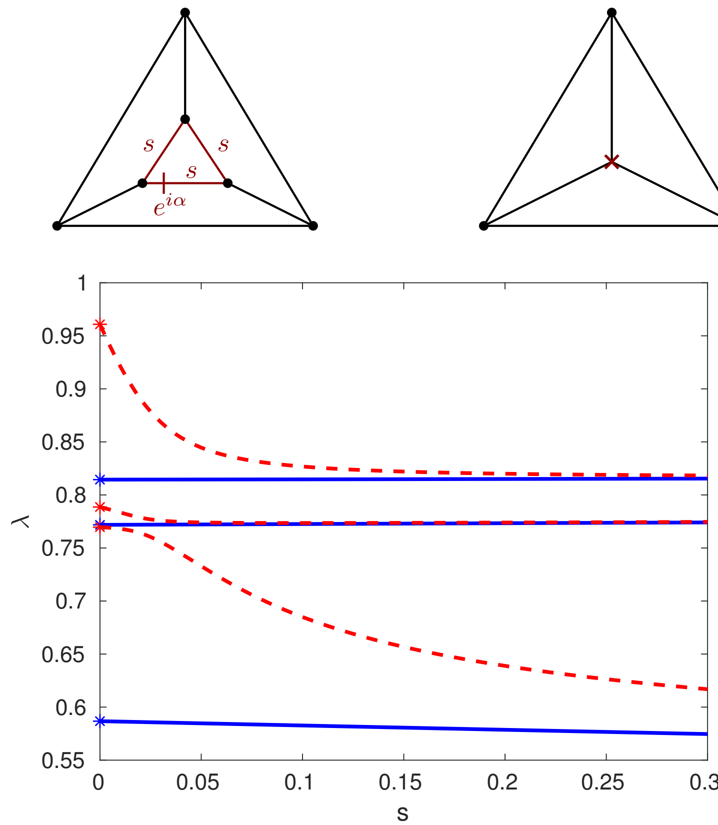

FIGURE 7. Bottom panel: Numerical calculation of the spectrum of a graph with a cycle of length 3 contracting into a single vertex. Blue curves correspond to no magnetic field, red lines correspond to a small flux threading the cycle. The limiting eigenvalue displayed as stars at $s = 0$ were calculated from the limiting vertex being supplied with Neumann–Kirchhoff (solid blue line) and Dirichlet (dashed red line) conditions.

We will thread magnetic flux α through the small triangle, realized as imposing conditions (3.19) on one of its edges. The limit predicted by our results depends on the value of the flux. For zero flux we simply recover Neumann–Kirchhoff conditions at the limiting vertex. When flux is non-zero (modulo 2π), the limiting conditions are Dirichlet which effectively disconnects the three edges at the central vertex. These results are confirmed by the agreement between the results for small s and the limiting graph computations, shown in Figure 7.

Example 3.16. In a slight modification of Example 3.12, we consider the graph displayed in Figure 4, but with edge e_2 now having constant length 1 while e_1 is shrinking. In this setting Condition 3.2 is satisfied and the spectral convergence holds.

4. Lagrangian and Symplectic Subspaces

The purpose of this section is to provide proof of the results that make heavy use of symplectic geometry, namely Proposition 2.1, Theorem 3.1 and Lemmas 3.3 and 3.4. We start by collecting the basic facts and definitions (see, for example, [McS] for further information).

Definition 4.1. Let $n \in \mathbb{N}$ and S be a complex linear space of dimension $2n$. A form $\omega : S \times S \to \mathbb{C}$ is called symplectic if the following holds:

(i) ω is *sesquilinear*, that is, $\omega(\alpha x+\beta y,z)=\overline{\alpha}\omega(x,z)+\overline{\beta}\omega(y,z)$ and $\omega(z,\alpha x+\beta y)=\alpha\omega(z,x)+\beta\omega(z,y)$, for all $x,y,z\in S$ and $\alpha,\beta\in\mathbb{C}$

(ii) ω is *skew-Hermitian*, that is, $\omega(x,y)=-\overline{\omega(y,x)}$, for all $x,y\in S$,

(iii) ω is *nondegenerate*, that is, if $\omega(x,y)=0$ for all $y\in S$, then $x=0$.

The pair (S,ω) is called a *symplectic space*.

Let ω be a symplectic form on $\mathbb{C}^{2n}$ and $V\subset\mathbb{C}^{2n}$ be a linear subspace. The *annihilator* of V is denoted by V° and defined by the formula

$$V^\circ:=\{x\in\mathbb{C}^{2n}:\omega(x,y)=0\text{ for all }y\in V\}.\tag{4.1}$$

Since the form ω is nondegenerate one has [McS, Lemma 2.2]

$$\dim(V)+\dim(V^\circ)=2n,\qquad (V^\circ)^\circ=V.\tag{4.2}$$

Definition 4.2. Let ω be a symplectic form on $\mathbb{C}^{2n}\times\mathbb{C}^{2n}$ and let $S,V,W,\mathcal{L}\subset\mathbb{C}^{2n}$ be linear subspaces. Then S is called *symplectic* if $S^\circ\cap S=\{0\}$, V is called *isotropic* if $V\subset V^\circ$, W is called *co-isotropic* if $W^\circ\subset W$, and $\mathcal{L}$ is called *Lagrangian* if $\mathcal{L}^\circ=\mathcal{L}$.

A subspace S of a symplectic space $(\mathbb{C}^{2n},\omega)$ is symplectic if and only if the restriction $\omega|_S$ of ω on S is a symplectic form on the linear space S (in other words, S is a symplectic subspace if and only if $(S,\omega|_S)$ is a symplectic space).

The main use of the Lagrangian theory in this paper is to characterize self-adjoint vertex conditions on a graph.

Proof of Proposition 2.1. The second Green's identity yields

$$\langle H_{max}f,g\rangle_{L^2(\Gamma)}-\langle f,H_{max}g\rangle_{L^2(\Gamma)}=\omega(\operatorname{tr}f,\operatorname{tr}g).\tag{4.3}$$

where $f,g\in\widehat{H}^2(\Gamma)$.

Let us assume that H is a self-adjoint extension of H_{min}. The subspace

$$\operatorname{tr}\big(\operatorname{dom}(H)\big)\subset L^2(\partial\Gamma)\oplus L^2(\partial\Gamma),$$

is isotropic since $\omega(\operatorname{tr}f,\operatorname{tr}g)=0$ whenever $f,g\in\operatorname{dom}(H)$. In order to show that it is maximal, we recall that $\operatorname{ran}(\operatorname{tr})=L^2(\partial\Gamma)\oplus L^2(\partial\Gamma)$. Assume that $w\in\big(\operatorname{tr}(\operatorname{dom}(H))\big)^\circ$, then there exists $f\in\widehat{H}^2(\Gamma)$ such that $w=\operatorname{tr}f$. Then for any $g\in\operatorname{dom}(H)$, one has

$$\langle Hf,g\rangle_{L^2(\Gamma)}-\langle f,Hg\rangle_{L^2(\Gamma)}=\omega(\operatorname{tr}f,\operatorname{tr}g)=0,$$

hence, $f\in\operatorname{dom}(H^*)=\operatorname{dom}(H)$. Therefore, the subspace $\operatorname{tr}\big(\operatorname{dom}(H)\big)$ is Lagrangian. We complete the proof of injectivity in the first part of the statement of the proposition by noticing that if $H_k=H_k^*, k=1,2$ are two self-adjoint extensions of H_{min} satisfying

$$\operatorname{tr}\big(\operatorname{dom}(H_1)\big)=\operatorname{tr}\big(\operatorname{dom}(H_2)\big),\tag{4.4}$$

then

$$H_k\subset H^*|_{\operatorname{dom}(H_1)+\operatorname{dom}(H_2)}=(H^*|_{\operatorname{dom}(H_1)+\operatorname{dom}(H_2)})^*, k=1,2.\tag{4.5}$$

Since $H_k, k=1,2$ are self-adjoint operators, (4.5) yields

$$\operatorname{dom}(H_1)=\operatorname{dom}(H_2),\text{ hence, }H_1=H_2.\tag{4.6}$$

To prove the second assertion in the proposition, let us fix a Lagrangian plane $\mathcal{L}\subset L^2(\partial\Gamma)\oplus L^2(\partial\Gamma)$. Clearly, the operator given by (2.14) is symmetric. Furthermore, for arbitrary $h\in\operatorname{dom}(H(\mathcal{L})^*)$ and $g\in\operatorname{dom}(H(\mathcal{L}))$ one has

$$0=\langle H(\mathcal{L})^*h,g\rangle_{L^2(\Gamma)}-\langle h,H(\mathcal{L})g\rangle_{L^2(\Gamma)}=\omega(\operatorname{tr}h,\operatorname{tr}g).$$

Therefore, $\operatorname{tr}h\in\mathcal{L}^\circ=\mathcal{L}$ and $h\in\operatorname{dom}(H(\mathcal{L}))$. Hence, $H(\mathcal{L})$ is a self-adjoint operator. □

To establish Theorem 3.1 we use a technique sometimes called *linear symplectic reduction.*

Proposition 4.3 (see, for example, [McS, Lemma I.2.7] or [LM, Proposition I.8.4])**.** *Let W be a co-isotropic subspace of the symplectic space $(\mathbb{C}^{2n},\omega)$. The* reduced symplectic space *associated with W is the space*

$$\dot{W} = W/W^{\circ} \tag{4.7}$$

with the symplectic form naturally induced by ω.

If $\mathcal{L}$ is a Lagrangian subspace of $(\mathbb{C}^{2n},\omega)$, then the projection of $\mathcal{L}\cap W$ onto $\dot{W}$ is Lagrangian in $\dot{W}$.

Proof of Theorem 3.1. In order to prove that $\widetilde{\mathcal{L}}$ is a Lagrangian plane we let $W = D_0 \oplus N_0$ and investigate $\dot{W}$. We recall that

$$W = \{(\phi_1,\phi_2)\in {}^dL^2(\partial\Gamma) : \phi_1(a_e)=\phi_1(b_e),\ \phi_2(a_e)=-\phi_2(b_e),\ e\in\mathcal{E}_0\}. \tag{4.8}$$

Importantly, ϕ_1 and ϕ_2 take arbitrary values on the edges $e\in\mathcal{E}_+$. Explicit calculation shows that

$$\begin{aligned} W^{\circ} &= \{(\phi_1,\phi_2)\in {}^dL^2(\partial\Gamma) : \phi_1|_{\partial\Gamma+}=\phi_2|_{\partial\Gamma+}=0;\ \phi_1(a_e)=\phi_1(b_e),\ \phi_2(a_e)=-\phi_2(b_e),\ e\in\mathcal{E}_0\} \\ &= \{(\phi_1,\phi_2)\in {}^dL^2(\partial\Gamma) : \phi_1|_{\partial\Gamma+}=\phi_2|_{\partial\Gamma+}=0;\}\cap W. \end{aligned} \tag{4.9}$$

This shows that W is co-isotropic and W/W° is naturally identified with ${}^dL^2(\partial\Gamma_+)$. By the second part of Proposition 4.3, $\widetilde{\mathcal{L}}$ which is defined in (3.1) as the projection of $\mathcal{L}\cap W$ to ${}^dL^2(\partial\Gamma_+)$ is Lagrangian in ${}^dL^2(\partial\Gamma_+)$.

□

Remark 4.4. Using equation (4.9) we can succinctly write Condition 3.2 as

$$\mathcal{L}\cap(D_0\oplus N_0)^{\circ} \subset 0\oplus L^2(\partial\Gamma_0). \tag{4.10}$$

Note the similarity to the condition of transversality of $\mathcal{L}$ and $D_0\oplus N_0$, namely $\mathcal{L}\cap(D_0\oplus N_0)^{\circ}=0$ (we used that $\mathcal{L}=\mathcal{L}^{\circ}$). Transversality is generic in the Grassmanian of all Lagrangian planes $\mathcal{L}$. Therefore, our less restrictive Condition 3.2 is also generic.

4.1. **Geometry of Condition 3.2.** In this section we delve deeper into the meaning of Condition 3.2 and prove Lemmas 3.3 and 3.4. To approach Lemma 3.3 we characterize scale invariant conditions in terms of the Lagrangian plane $\mathcal{L}$.

Proposition 4.5. *The vertex conditions,* (2.16)*, for the operator $H(\mathcal{L})$ are scale invariant, that is, $P_R=0$, if and only if there exist subspaces $\mathcal{L}_D\subset L^2(\partial\Gamma)$ and $\mathcal{L}_N\subset L^2(\partial\Gamma)$ such that*

$$\mathcal{L}=\left\{(\phi_1,\phi_2)\in {}^dL^2(\partial\Gamma) : \phi_1\in\mathcal{L}_D, \phi_2\in\mathcal{L}_N\right\}. \tag{4.11}$$

Proof. If $P_R=0$ then (4.11) holds with $\mathcal{L}_D := \ker(P_D)$, $\mathcal{L}_N := \ker(P_N)$. Conversely, assuming (4.11) we will first establish that

$$\mathcal{L}_D=\mathcal{L}_N^{\perp}. \tag{4.12}$$

Let us pick arbitrary $f\in\mathcal{L}_N^{\perp}$ and notice that for all $\phi_1\in\mathcal{L}_D$, $\phi_2\in\mathcal{L}_N$ one has

$$\omega((f,0),(\phi_1,\phi_2)) = -\int_{\partial\Gamma}\overline{f}\phi_2 = 0. \tag{4.13}$$

Since $\mathcal{L}$ is Lagrangian, this yields $(f,0)\in\mathcal{L}$ and, in particular, $f\in\mathcal{L}_D$. Next, to prove $\mathcal{L}_D\subset\mathcal{L}_N^{\perp}$ we observe that $(\phi_1,0),(0,\phi_2)\in\mathcal{L}$ for all $\phi_1\in\mathcal{L}_D$, $\phi_2\in\mathcal{L}_N$, thus

$$0=\omega((\phi_1,0),(0,\phi_2)) = -\int_{\partial\Gamma}\overline{\phi_1}\phi_2. \tag{4.14}$$

Let P_D, P_N denote the orthogonal projections in $L^2(\partial\Gamma)$ with $\ker(P_D) = \mathcal{L}_D$ and $\ker(P_N) = \mathcal{L}_N$. Then

$$\operatorname{ran}(P_D) \oplus \operatorname{ran}(P_N) = \mathcal{L}_N \oplus \mathcal{L}_D = L^2(\partial\Gamma) \tag{4.15}$$

and by (2.14), (2.16) and (4.12)

$$\operatorname{dom}(H(\mathcal{L})) = \left\{ f \in \widehat{H}^2(\Gamma) \middle| P_D\gamma_D f = 0,\ P_N\gamma_N f = 0 \right\}. \tag{4.16}$$

Thus, $P_R = 0$. □

Proof of Lemma 3.3. Let us note that Condition 3.2 can be succinctly written as follows

$$(\phi_1, \phi_2) \in \mathcal{L} \cap (D_0 \oplus N_0) \cap \ker({}^dP_+) \quad \Rightarrow \quad \phi_1 = 0. \tag{4.17}$$

Suppose that the assumption of the Lemma holds yet Condition 3.2 is not satisfied. Then pick arbitrary $(\phi_1, \phi_2) \in \mathcal{L} \cap (D_0 \oplus N_0) \cap \ker({}^dP_+)$ with $\phi_1 \neq 0$ and define the following function

$$f := \sum_{e \in \mathcal{E}_0} \phi_1(a_e)\chi_e \not\equiv 0. \tag{4.18}$$

By construction, we have $\gamma_D f = \phi_1$. Also, since the function is constant on every edge, $\gamma_N f = 0$. Since our vertex conditions are scale invariant, by Proposition 4.5, $(\phi_1, \phi_2) \in \mathcal{L}$ implies $(\phi_1, 0) \in \mathcal{L}$ and therefore $f \in \operatorname{dom}(H(\mathcal{L}, \ell))$, in contradiction to the assumption.

Conversely, suppose that f is a nonzero function constant on each edge satisfying the boundary conditions and such that $\operatorname{supp}(f) \subset \Gamma_0$. Then $\operatorname{tr} f \in \mathcal{L} \cap (D_0 \oplus N_0) \cap \ker({}^dP_+)$ yet $\gamma_D f \neq 0$ and therefore the choice $(\phi_1, \phi_2) = \operatorname{tr} f$ falsifies Condition 3.2. □

Proof of Lemma 3.4. Due to the continuity assumption every function $f \in \operatorname{dom}(H(\mathcal{L}, \ell))$ satisfying

$$(\operatorname{tr} f) \restriction_{\partial\Gamma_+} = 0,$$

and

$$f(a_e) = f(b_e), \text{ for all } e \in \mathcal{E}_0$$

has zero Dirichlet trace: $\gamma_D f = 0$. □

5. Resolvent estimates and the spectral convergence

As mentioned in Section 3, in order to prove spectral convergence, Theorem 3.6, we will require some technical estimates listed in Theorem 3.8. Before we formally prove Theorems 3.8 and 3.9, we compare these estimates with standard functional-analytic results.

Part *(i)* of Theorem 3.8 gives a bound on the resolvent of a quantum graph operator. A well-known bound on the resolvent of a general self-adjoint operator H on a Hilbert space $\mathcal{H}$ gives

$$\|(H - zI)^{-1}\|_{\mathcal{B}(\mathcal{H})} \leq \frac{1}{|\operatorname{Im} z|}, \qquad \operatorname{Im} z \neq 0. \tag{5.1}$$

We immediately get, for any $\Gamma(\ell)$,

$$\|R(\mathcal{L}, \ell, \mathbf{i})\|_{\mathcal{B}(L^2(\Gamma(\ell)))} \leq 1. \tag{5.2}$$

We stress that this bound is weaker than part *(i)* of Theorem 3.8, which bounds $R(\mathcal{L}, \ell, \mathbf{i})$ as an operator from L^2 to L^∞.

On the other hand, part *(iii)* of Theorem 3.8 is reminiscent of the following standard Sobolev-type inequalities that hold for all edges e,

$$\|f_e\|_{L^\infty(e)} \leq \ell_e^{-1/2}\|f_e\|_{L^2(e)} + \ell_e^{1/2}\|f_e'\|_{L^2(e)}, \tag{5.3}$$

$$\|f_e'\|_{L^2(e)} \leq \ell_e^{-1}\|f_e\|_{L^2(e)} + \ell_e\|f_e''\|_{L^2(e)}, \tag{5.4}$$

$$\|f_e'\|_{L^\infty(e)} \leq \ell_e^{-1/2}\|f_e'\|_{L^2(e)} + \ell_e^{1/2}\|f_e''\|_{L^2(e)}, \tag{5.5}$$

(cf., e.g, [Bu, Theorem 4.2.4 and Corollary 4.2.7 part 1.]). However, in a situation when some edge lengths $\ell_e \to 0$, uniform bound (3.11) is a substantially stronger statement.

Proof of Theorem 3.8. By the resolvent identity it suffices to verify equivalency of the statements for the free resolvent, i.e. we may assume that $q^\ell \equiv 0$ for all ℓ. Indeed, denoting the free resolvent by $R_0(\mathcal{L}, \ell, \mathbf{i}) := (H_0(\mathcal{L}, \ell) - \mathbf{i})^{-1}$ one has

$$R(\mathcal{L}, \ell, \mathbf{i}) = R_0(\mathcal{L}, \ell, \mathbf{i}) - R_0(\mathcal{L}, \ell, \mathbf{i}) q^\ell R(\mathcal{L}, \ell, \mathbf{i}). \tag{5.6}$$

Next, we recall that by assumptions

$$\|q^\ell\|_{L^\infty(\Gamma(\ell);\mathbb{R})} \le c, \tag{5.7}$$

for some $c > 0$ and all ℓ sufficiently close to $\widetilde{\ell}$. Combining this bound with (5.2) and (5.6) one infers that parts (i) and (ii) hold if and only if they hold with $q^\ell \equiv 0$. In addition, since $\mathrm{dom}(H(\mathcal{L}, \ell)) = \mathrm{dom}(H_0(\mathcal{L}, \ell))$, part (iii) holds if and only it holds with $q^\ell \equiv 0$.

(i) $\Longrightarrow$ *(ii).* For arbitrary $e \in \mathcal{E}_0$ and $v \in L^2(\Gamma)$,

$$\|\chi_e R_0(\mathcal{L}, \ell, \mathbf{i}) v\|_{L^2(\Gamma(\ell))} \le \|\chi_e\|_{L^2(\Gamma(\ell))} \|R_0(\mathcal{L}, \ell, \mathbf{i}) v\|_{L^\infty(\Gamma(\ell))} \tag{5.8}$$

$$\le \ell_e^{1/2} \|R_0(\mathcal{L}, \ell, \mathbf{i})\|_{\mathcal{B}(L^2(\Gamma(\ell)), L^\infty(\Gamma(\ell)))} \|v\|_{L^2(\Gamma(\ell))}. \tag{5.9}$$

Combining this with *(i)* we infer *(ii)*.

(ii) $\Longrightarrow$ *(iii).* Let $e \in \mathcal{E}_0$. For $f \in \mathrm{dom}(H_0(\mathcal{L}, \ell))$ put $-f'' - \mathbf{i} f = v$, then by *(ii)*,

$$\|f_e\|_{L^2(e)} \lesssim \sqrt{\ell_e} \|v\|_{L^2(\Gamma)} = \sqrt{\ell_e} \|f'' + \mathbf{i} f\|_{L^2(\Gamma(\ell))}. \tag{5.10}$$

Combining (5.10), (5.3), (5.4) we obtain that for every $e \in \mathcal{E}$,

$$\|f_e\|_{L^\infty(e)} \le 2\ell_e^{-1/2} \|f_e\|_{L^2(e)} + \ell_e^{3/2} \|f_e''\|_{L^2(e)} \tag{5.11}$$

$$\lesssim \|f'' + \mathbf{i} f\|_{L^2(\Gamma(\ell))} + \ell_e^{3/2} \|f_e''\|_{L^2(e)}, \tag{5.12}$$

$$\lesssim c(\ell) \left(\|f\|_{L^2(\Gamma(\ell))} + \|f''\|_{L^2(\Gamma(\ell))} \right), \tag{5.13}$$

where $c(\ell) = \mathcal{O}(1)$ as $\ell \to \widetilde{\ell}$. Note that we had to use (5.10) because $\ell_e \to 0$ for $e \in \mathcal{E}_0$.

(iii) $\Longrightarrow$ *(i)* Let $f \in \mathrm{dom}(H_0(\mathcal{L}, \ell))$ and let $-f'' - \mathbf{i} f = v$. Then by *(iii)*,

$$\|R_0(\mathcal{L}, \ell, \mathbf{i}) v\|^2_{L^\infty(\Gamma(\ell))} = \|f\|^2_{L^\infty(\Gamma(\ell))} \tag{5.14}$$

$$\lesssim c \left(\|f\|^2_{L^2(\Gamma(\ell))} + \|v + \mathbf{i} f\|^2_{L^2(\Gamma(\ell))} \right) \tag{5.15}$$

$$\lesssim c \left(\|R_0(\mathcal{L}, \ell, \mathbf{i}) v\|^2_{L^2(\Gamma(\ell))} + \|v\|^2_{L^2(\Gamma(\ell))} \right) \tag{5.16}$$

$$\lesssim c \|v\|^2_{L^2(\Gamma(\ell))}, \tag{5.17}$$

where in the last step we used (5.2). This proves *(i)*.

Next we prove (3.12) and (3.13). To this end, let $f \in \mathrm{dom}(H_0(\mathcal{L}, \ell))$ and let $-f'' - \mathbf{i} f = v$. Then using (2.17) and the Cauchy–Schwarz inequality, we obtain

$$\|f'\|^2_{L^2(\Gamma(\ell))} \le \left| \langle f, f'' \rangle_{L^2(\Gamma(\ell))} \right| + \left| \langle P_R \gamma_D^\ell f, Q P_R \gamma_D^\ell f \rangle_{L^2(\partial\Gamma)} \right| \tag{5.18}$$

$$\le \|f\|_{L^2(\Gamma(\ell))} \|f''\|_{L^2(\Gamma(\ell))} + \|Q\| \|\gamma_D^\ell f\|^2_{L^2(\partial\Gamma)} \tag{5.19}$$

$$\le \|f\|^2_{L^2(\Gamma(\ell))} + \|f''\|^2_{L^2(\Gamma(\ell))} + \|Q\| \|\gamma_D^\ell f\|^2_{L^2(\partial\Gamma)}. \tag{5.20}$$

Employing (3.11) we estimate the third term in (5.20) and infer (3.12).

Then, one has

$$\|R_0(\mathcal{L}, \ell, \mathbf{i}) v\|_{\widehat{H}^2(\Gamma(\ell))} = \|f\|^2_{\widehat{H}^2(\Gamma(\ell))} = \|f\|^2_{L^2(\Gamma(\ell))} + \|f'\|^2_{L^2(\Gamma(\ell))} + \|f''\|^2_{L^2(\Gamma(\ell))}$$

$$\lesssim c(\ell)\left(\|f\|^2_{L^2(\Gamma(\ell))}+\|f''\|^2_{L^2(\Gamma(\ell))}\right)\lesssim c(\ell)\|v\|^2_{L^2(\Gamma(\ell))},$$

where $c(\ell) \underset{\ell\to\widetilde{\ell}}{=} \mathcal{O}(1)$ and in the last step we proceeded as in (5.15)-(5.17). □

In the proof of Theorem 3.9 we will use the following geometric fact.

Proposition 5.1. *Suppose that A and B are closed linear subspaces of a Hilbert space X, and that at least one of them is finite dimensional. Let $\{b_n\}_{n=1}^\infty \subset B$ be such that* $\operatorname{dist}(b_n, A) \to 0$ *as $n\to\infty$. Then* $\operatorname{dist}(b_n, A\cap B)\to 0$ *as $n\to\infty$.*

As the following counterexample[1] shows, the proposition may not hold if both A and B are infinite dimensional. In the sequence space $X = \ell^2(\mathbb{N})$ we consider infinite dimensional subspaces

$$A = \{(x_1, x_2, x_2, x_3, x_3, \dots) \in \ell^2(\mathbb{N}) : x_k \in \mathbb{C}\} \tag{5.21}$$

$$B = \{(x_1, x_1, x_2, x_2, x_3, x_3, \dots) \in \ell^2(\mathbb{N}) : x_k \in \mathbb{C}\}, \tag{5.22}$$

and let

$$a_n = (1,\ 1-\tfrac{1}{n},\ 1-\tfrac{1}{n},\ 1-\tfrac{2}{n},\ 1-\tfrac{2}{n},\dots,\ \tfrac{1}{n},\ 0,\ 0,\dots) \in A, \tag{5.23}$$

$$b_n = (1,\ 1,\qquad 1-\tfrac{1}{n},\ 1-\tfrac{1}{n},\ 1-\tfrac{2}{n},\dots,\ \tfrac{1}{n},\ \tfrac{1}{n},\ 0,\dots) \in B, \tag{5.24}$$

for $n = 1, 2, \dots$. Then $\operatorname{dist}(b_n, A) \le \|b_n - a_n\| = n^{-1/2} \to 0$ while $A\cap B = \{0\}$ and $\operatorname{dist}(b_n, A\cap B) = \|b_n\| \to +\infty$ as $n\to\infty$.

Proof of Proposition 5.1. Let P denote the orthogonal projection onto $(A\cap B)^\perp$. We want to show that $\|Pb_n\| = \operatorname{dist}(b_n, A\cap B)\to 0$ as $n\to\infty$. Note that $Pb_n \in B$.

Consider the orthogonal decomposition

$$X = \Big((A\cap B)\oplus\big(A\cap(A\cap B)^\perp\big)\Big)\oplus A^\perp, \tag{5.25}$$

and split b_n accordingly, $b_n = x_n + y_n + z_n$. Applying P, we see that $Pb_n = y_n + Pz_n$. We know that $\|z_n\| = \operatorname{dist}(b_n, A) \to 0$, therefore $Pz_n \to 0$ and we conclude that either both sequences (Pb_n) and (y_n) converge to zero (and then the proof is finished), or else, may be by passing to a subsequence, they both are separated away from zero. Let us suppose that the latter holds. Then equality $\|y_n\|^{-1}Pb_n = \|y_n\|^{-1}y_n + \|y_n\|^{-1}Pz_n$ shows that the following two sequences,

$$\big(\|y_n\|^{-1}Pb_n\big)\subset B\cap(A\cap B)^\perp \quad\text{and}\quad \big(\|y_n\|^{-1}y_n\big)\subset A\cap(A\cap B)^\perp,$$

are bounded. Since at least one of the subspaces A or B is finite dimensional, passing to a subsequence, we may conclude that at least one of the two sequences converges. Then by $\|y_n\|^{-1}Pz_n\to 0$ both sequences must converge, and their common limit must be zero as it belongs to $A\cap B$ and $(A\cap B)^\perp$. Since $\|y_n\|^{-1}y_n$ is of unit length, the contradiction completes the proof. □

Proof of Theorem 3.9. Due to the resolvent identity, equation (5.6), it is enough to prove the statement for the free Laplacian. That is, we focus on the case of zero potential.

Seeking a contradiction we assume that condition(*iii*) from Theorem 3.8 does not hold and obtain sequences $\{\ell_n\}_{n=1}^\infty\subset\mathbb{R}^{|\mathcal{E}|}_{>0}$ and $\{\varphi_n\}_{n=1}^\infty\subset\operatorname{dom}(H(\mathcal{L},\ell_n))$ such that

$$\ell_n\to\widetilde{\ell}, \tag{5.26}$$

$$\|\varphi_n\|_{L^\infty(\Gamma(\ell_n))} = 1,\quad n\in\mathbb{N}, \tag{5.27}$$

$$\|\varphi_n\|_{L^2(\Gamma(\ell_n))}+\|\varphi_n''\|_{L^2(\Gamma(\ell_n))}\to 0,\quad n\to\infty \tag{5.28}$$

[1]Due to Th. Schlumprecht

From equation (2.17) one has

$$\|\varphi_n'\|^2_{L^2(\Gamma(\ell_n))} = \langle \varphi_n, \varphi_n'' \rangle_{L^2(\Gamma(\ell_n))} - \langle P_R\gamma_D\varphi_n, QP_R\gamma_D\varphi_n\rangle_{L^2(\partial\Gamma)}. \tag{5.29}$$

Thus, using (5.27), (5.28) we get

$$\|\varphi_n'\|_{L^2(\Gamma(\ell_n))} \underset{n\to\infty}{=} \mathcal{O}(1). \tag{5.30}$$

Using this, for each $e \in \mathcal{E}_0$ one obtains

$$|\varphi_n(a_e) - \varphi_n(b_e)| = \left|\int_e \partial_\nu \varphi_n\right| \lesssim \sqrt{\ell_{n,e}}\|\varphi_n'\|_{L^2(\Gamma(\ell_n))} \to 0, \quad n \to \infty. \tag{5.31}$$

Similarly, by (5.28), for each $e \in \mathcal{E}_0$ one has

$$|\varphi_n'(a_e) - \varphi_n'(b_e)| = \left|\int_e \varphi_n''\right| \lesssim \sqrt{\ell_{n,e}}\|\varphi_n''\|_{L^2(\Gamma(\ell_n))} \to 0, \quad n \to \infty. \tag{5.32}$$

That is,

$$\operatorname{dist}\big(\operatorname{tr}\varphi_n, D_0 \oplus N_0\big) \to 0,\ n \to \infty. \tag{5.33}$$

Next, using (5.28) and the standard Sobolev inequalities on Γ_+, cf. (5.3)–(5.5), we obtain

$$\|\varphi_n\|_{L^\infty(\Gamma_+(\ell_n))} \lesssim \|\varphi_n\|_{L^2(\Gamma_+(\ell_n))} + \|\varphi_n''\|_{L^2(\Gamma_+(\ell_n))} \underset{n\to\infty}{=} o(1), \tag{5.34}$$

$$\|\varphi_n'\|_{L^\infty(\Gamma_+(\ell_n))} \lesssim \|\varphi_n\|_{L^2(\Gamma_+(\ell_n))} + \|\varphi_n''\|_{L^2(\Gamma_+(\ell_n))} \underset{n\to\infty}{=} o(1). \tag{5.35}$$

In particular,

$$\lim_{n\to\infty}\|\varphi_n \restriction_{\partial\Gamma_+}\|_{L^\infty(\partial\Gamma_+)} = 0,\ \lim_{n\to\infty}\|\varphi_n' \restriction_{\partial\Gamma_+}\|_{L^\infty(\partial\Gamma_+)} = 0. \tag{5.36}$$

Moreover, one has

$$\liminf_{n\to\infty}\|\varphi_n \restriction_{\partial\Gamma_0}\|_{L^\infty(\partial\Gamma_0)} > 0. \tag{5.37}$$

Indeed, assuming the contrary and passing to a subsequence if necessary, one gets that for any $e \in \mathcal{E}_0$ and arbitrary $x \in e$

$$|\varphi_n(x)| \le |\varphi_n(a_e)| + \left|\int_e \partial_\nu\varphi_n\right| \le |\varphi_n(a_e)| + \sqrt{\ell_{n,e}}\|\varphi_n'\|_{L^2(\Gamma(\ell_n))} \to 0, \quad n \to \infty, \tag{5.38}$$

contradicting (5.27).

Next, using (5.33) and (5.36) we obtain

$$\operatorname{dist}\big(\operatorname{tr}\varphi_n, (D_0 \oplus N_0) \cap \ker({}^dP_+)\big) \to 0,\ n \to \infty. \tag{5.39}$$

Combining this with $\operatorname{tr}\varphi_n \in \mathcal{L}$ and Proposition 5.1, we obtain that

$$\operatorname{dist}\big((\phi_1^n, \phi_2^n), \mathcal{L} \cap (D_0 \oplus N_0) \cap \ker({}^dP_+)\big) \to 0,\ n \to \infty. \tag{5.40}$$

Interpreting Condition 3.2 as $\mathcal{L} \cap (D_0 \oplus N_0) \cap \ker({}^dP_+) \subset \{0\} \oplus L^2(\partial\Gamma)$, one has

$$\|\gamma_D\varphi_n\|_{L^2(\partial\Gamma)} = \operatorname{dist}\big(\operatorname{tr}\varphi_n, \{0\} \oplus L^2(\partial\Gamma)\big) \tag{5.41}$$

$$\le \operatorname{dist}\big(\operatorname{tr}\varphi_n, \mathcal{L} \cap (D_0 \oplus N_0) \cap \ker({}^dP_+)\big) \to 0,\ n \to \infty. \tag{5.42}$$

which contradicts (5.37).

To prove the last statement assume that $P_R = 0$. Then by Lemma 3.3 there exists a nonzero function f constant on each edge satisfying the boundary conditions and such that $\operatorname{supp}(f) \subset \Gamma_0$. Since $f'' = 0$ and $\|f\|^2_{L^2(\Gamma(\ell))} \to 0$ as $\ell \to \widetilde{\ell}$, the inequality (3.11) does not hold. □

As was pointed out in Introduction, our method of proving spectral convergence relies upon a technique developed by P. Exner and O. Post [EP, P06, P11, P12].

Definition 5.2. For each $t \in \mathbb{R}^n, n \in \mathbb{N}$, let H_t be a self-adjoint operator acting in the Hilbert space $\mathcal{H}_t$. Then H_t is said to converge in the *generalized norm resolvent sense* to $H_{\widetilde{t}}$, as $t \to \widetilde{t}$ if for each $t \in \mathbb{R}^n$ there exists a bounded linear operator $\mathcal{J}_t \in \mathcal{B}(\mathcal{H}_{\widetilde{t}}, \mathcal{H}_t)$ such that

$$\mathcal{J}_t^* \mathcal{J}_t = I_{\mathcal{H}_{\widetilde{t}}} \text{ for all } t \in \mathbb{R}^n, \tag{5.43}$$

$$\|(I_{\mathcal{H}_t} - \mathcal{J}_t\mathcal{J}_t^*)(H_t - zI_{\mathcal{H}_t})^{-1}\|_{\mathcal{B}(\mathcal{H}_t)} \underset{t\to\widetilde{t}}{=} o(1), \tag{5.44}$$

$$\|\mathcal{J}_t(H_{\widetilde{t}} - zI_{\mathcal{H}_{\widetilde{t}}})^{-1} - (H_t - zI_{\mathcal{H}_t})^{-1}\mathcal{J}_t\|_{\mathcal{B}(\mathcal{H}_{\widetilde{t}},\mathcal{H}_t)} \underset{t\to\widetilde{t}}{=} o(1), \tag{5.45}$$

for each $z \in \mathbb{C}$, $\operatorname{Im} z \neq 0$. In this case we write $H_t \xrightarrow{\text{gnr}} H_{\widetilde{t}}$, as $t \to \widetilde{t}$.

Assuming conditions *(i)-(iii)* of Theorem 3.8 we focus on showing that

$$H(\mathcal{L}, \ell) \xrightarrow{\text{gnr}} H(\widetilde{\mathcal{L}}, \widetilde{\ell}), \quad \ell \to \widetilde{\ell}. \tag{5.46}$$

As a first step, we show that, in the abstract setting, the generalized norm resolvent convergence is preserved under bounded perturbations.

Theorem 5.3. *Let H_t^0, $H_{\widetilde{t}}^0$ satisfy Definition* 5.2. *Let $A_t \in \mathcal{B}(\mathcal{H}_t)$ be a family of self-adjoint, bounded operators satisfying the relations*

$$\|\mathcal{J}_t A_{\widetilde{t}} - A_t \mathcal{J}_t\|_{\mathcal{B}(\mathcal{H}_{\widetilde{t}},\mathcal{H}_t)} \underset{t\to\widetilde{t}}{=} o(1) \quad \text{and} \quad \|A_t\|_{\mathcal{B}(\mathcal{H}_t)} \underset{t\to\widetilde{t}}{=} \mathcal{O}(1). \tag{5.47}$$

Then equations (5.44) *and* (5.45) *hold with $H_t = H_t^0 + A_t$.*

Proof. The proof relies on the resolvent identity

$$R(t) = R_0(t) - R(t)A_tR_0(t) \tag{5.48}$$

$$= R_0(t) + R_0(t)A_tR(t), \ t \in \mathbb{R}^n. \tag{5.49}$$

where

$$R(t) := (H_t^0 + A_t - zI_{\mathcal{H}_t})^{-1}, \qquad R_0(t) := (H_t^0 - zI_{\mathcal{H}_t})^{-1}, \quad \operatorname{Im} z \neq 0.$$

In order to verify (5.44) for $H_t = H_t^0 + A_t$, we combine (5.44) (for $R_0(t)$) and (5.49) and obtain

$$\begin{aligned}\|(I_{\mathcal{H}_t} - \mathcal{J}_t\mathcal{J}_t^*)R(t)\|_{\mathcal{B}(\mathcal{H}_t)} &\le \|(I_{\mathcal{H}_t} - \mathcal{J}_t\mathcal{J}_t^*)R_0(t)\|_{\mathcal{B}(\mathcal{H}_t)} + \|(I_{\mathcal{H}_t} - \mathcal{J}_t\mathcal{J}_t^*)R_0(t)A_tR(t)\|_{\mathcal{B}(\mathcal{H}_t)} \\ &\underset{t\to\widetilde{t}}{=} o(1)\left(1 + \|A_tR(t)\|_{\mathcal{B}(\mathcal{H}_t)}\right) \underset{t\to\widetilde{t}}{=} o(1),\end{aligned}$$

where we used the second equality in (5.47), and the general resolvent bound (5.1).

The identity

$$\begin{aligned}&\left(\mathcal{J}_tR(\widetilde{t}) - R(t)\mathcal{J}_t\right)\left(I_{\mathcal{H}_{\widetilde{t}}} + A_{\widetilde{t}}R_0(\widetilde{t})\right) \\ &\qquad = (I_{\mathcal{H}_t} - R(t)A_t)\left(\mathcal{J}_tR_0(\widetilde{t}) - R_0(t)\mathcal{J}_t\right) + R(t)\left(A_t\mathcal{J}_t - \mathcal{J}_tA_{\widetilde{t}}\right)R_0(\widetilde{t})\end{aligned}$$

may be verified by substituting (5.48) for $R(\widetilde{t})$ and $R(t)$ on the left-hand side and expanding. Using (5.45), (5.47) and (5.1), we arrive at

$$\left\|\left(\mathcal{J}_t\widetilde{R} - R(t)\mathcal{J}_t\right)\left(I_{\mathcal{H}_{\widetilde{t}}} + A_{\widetilde{t}}R_0(\widetilde{t})\right)\right\|_{\mathcal{B}(\mathcal{H}_{\widetilde{t}},\mathcal{H}_t)} \underset{t\to\widetilde{t}}{=} o(1). \tag{5.50}$$

Moreover, due to the identity

$$I_{\mathcal{H}_{\widetilde{t}}} + A_{\widetilde{t}}R_0(\widetilde{t}) = \left(H_{\widetilde{t}} - zI_{H_{\widetilde{t}}} + A_{\widetilde{t}}\right)R_0(\widetilde{t}),$$

the operator $I_{\mathcal{H}_{\widetilde{t}}} + A_{\widetilde{t}}\widetilde{R}_0$ is boundedly invertible on $\operatorname{Im} z \neq 0$. Thus (5.50) implies (5.45) for $H_t = H_t^0 + A_t$. ∎

In the following theorem we establish a version of (5.43) and (5.44) in the context of graphs with vanishing edges. Let us recall definition of $\mathcal{J}_\ell$ from (3.5).

Theorem 5.4. *Assume conditions (i)-(iii) of Theorem* 3.8 *hold. Then*

$$\mathcal{J}_\ell^*\mathcal{J}_\ell = I_{L^2(\Gamma(\widetilde{\ell}))},\ \ell\in\mathbb{R}^{|\mathcal{E}|}_{>0}, \tag{5.51}$$

and

$$\big\|(I_{L^2(\Gamma(\mathcal{G};\ell))}-\mathcal{J}_\ell\mathcal{J}_\ell^*)R(\mathcal{L},\ell,z)\big\|_{\mathcal{B}(L^2(\Gamma(\mathcal{G};\ell)))}\underset{\ell\to\widetilde{\ell}}{=}o(1), \tag{5.52}$$

for each $z\in\mathbb{C}$, $\operatorname{Im} z\neq 0$.

Proof. Using change of variables, one obtains

$$\mathcal{J}_\ell^*\in\mathcal{B}\big(L^2(\Gamma(\ell)),L^2(\Gamma(\widetilde{\ell}))\big), \tag{5.53}$$

$$(\mathcal{J}_\ell^* f)_e(x):=\sum_{e\in\mathcal{E}_+}\chi_e(x)\sqrt{\frac{\ell_j}{\tilde{\ell}_j}}f_e\left(\frac{x\ell_j}{\tilde{\ell}_j}\right),\quad f\in L^2(\Gamma(\ell)),\ x\in\Gamma(\widetilde{\ell}). \tag{5.54}$$

A direct computation shows that (3.5) and (5.54) yield (5.51). Moreover, one has

$$\mathcal{J}_\ell\mathcal{J}_\ell^*\in\mathcal{B}(L^2(\Gamma(\ell))),\quad \mathcal{J}_\ell\mathcal{J}_\ell^* f=\chi_{\Gamma_+(\ell)}f,\quad f\in L^2(\Gamma(\ell)), \tag{5.55}$$

where $\chi_{\Gamma_+(\ell)}$ denotes the characteristic function of $\Gamma_+(\ell)$.

By Theorem 3.8*(ii)* one has

$$\big\|(I_{L^2(\Gamma(\ell))}-\mathcal{J}_\ell\mathcal{J}_\ell^*)R(\mathcal{L},\ell,z)\big\|_{L^2(\Gamma(\ell))}=\big\|\sum_{e\in\mathcal{E}_0}\chi_e R(\mathcal{L},\ell,z)\big\|_{L^2(\Gamma(\ell))} \tag{5.56}$$

$$\leq\sum_{e\in\mathcal{E}_0}\|\chi_e R(\mathcal{L},\ell,z)\|_{L^2(\Gamma(\ell))}\underset{\ell\to\widetilde{\ell}}{=}\sum_{e\in\mathcal{E}_0}\mathcal{O}(\ell_e^{1/2})\underset{\ell\to\widetilde{\ell}}{=}o(1), \tag{5.57}$$

as asserted. ☐

In the following theorem we establish a version of (5.45) in the context of graphs with vanishing edges. Together with Theorem 5.4 this will conclude the proof of Theorem 3.5.

Theorem 5.5. *Assume conditions (i)-(iii) of Theorem* 3.8*, and recall the operator* $H(\widetilde{\mathcal{L}},\widetilde{\ell})$ *from Theorem* 3.1*. Then*

$$\Big\|\mathcal{J}_\ell R(\widetilde{\mathcal{L}},\widetilde{\ell},z)-R(\mathcal{L},\ell,z)\mathcal{J}_\ell\Big\|_{\mathcal{B}\left(L^2(\Gamma(\widetilde{\ell})),L^2(\Gamma(\ell))\right)}\underset{\ell\to\widetilde{\ell}}{=}o\,(1)\,, \tag{5.58}$$

for each $z\in\mathbb{C}$, $\operatorname{Im} z\neq 0$.

Proof. We split the proof into several natural steps. In the first step we prove (5.58) in the situation when the non-vanishing edges are being fixed while the vanishing edges tend to zero. This is the most challenging part of the proof. In the second step we deal with (5.58)when the vanishing edges are absent, while the non-vanishing edges rescale non-singularly. Finally, in the third step we put everything together, and obtain (5.58) as asserted.

Note that by Theorem 5.3 we may assume that $q^\ell\equiv 0$ for all ℓ.

Step 1. Let us denote $\ell=(\ell_+,\ell_0)$, $\widehat{\ell}:=(\ell_+,0)$. Then the scaling operator acting from $\Gamma(\widehat{\ell})$ to $\Gamma(\ell)$ is given by $\mathbb{J}_{\ell,\widehat{\ell}}\in\mathcal{B}\big(L^2(\Gamma(\widetilde{\ell})),L^2(\Gamma(\widehat{\ell}))\big)$,

$$(\mathbb{J}_{\ell,\widehat{\ell}}f)(x)=\sum_{e\in\mathcal{E}_+}\chi_e(x)f\,(x)\,,\quad x\in\Gamma(\ell), \tag{5.59}$$

The goal of this step is to prove a version of (5.58) with respect to this scaling operator. Namely, we will prove that

$$\left\|\mathbb{J}_{\ell,\widehat{\ell}}R(\widetilde{\mathcal{L}},\widehat{\ell},z)-R(\mathcal{L},\ell,z)\mathbb{J}_{\ell,\widehat{\ell}}\right\|_{\mathcal{B}\left(L^2(\Gamma(\widehat{\ell})),L^2(\Gamma(\ell))\right)}\underset{\ell_0\to 0}{=}o\left(1\right), \tag{5.60}$$

holds uniformly in ℓ_+ satisfying

$$\frac{|\widetilde{\ell}|}{2}\le|\ell_+|\le|\widetilde{\ell}|. \tag{5.61}$$

It suffices to prove that the inequality

$$\left|\left\langle f,\left(\mathbb{J}_{\ell,\widehat{\ell}}R(\widetilde{\mathcal{L}},\widehat{\ell},z)-R(\mathcal{L},\ell,z)\mathbb{J}_{\ell,\widehat{\ell}}\right)g\right\rangle_{L^2(\Gamma(\ell))}\right|\le c(\ell)\|f\|_{L^2(\Gamma(\ell))}\|g\|_{L^2(\Gamma(\widehat{\ell}))}, \tag{5.62}$$

holds for arbitrary $f\in L^2(\Gamma(\ell))$ and $g\in L^2(\Gamma(\widehat{\ell}))$, with

$$\sup_{\ell_+:\ \frac{|\widetilde{\ell}|}{2}\le|\ell_+|\le|\widetilde{\ell}|}c(\ell)=o\left(1\right)\text{ as }\ell_0\to 0. \tag{5.63}$$

Let us denote

$$u:=R(\mathcal{L},\ell,\overline{z})f\text{ and }v:=R(\widetilde{\mathcal{L}},\widehat{\ell},z)g. \tag{5.64}$$

Rewriting the left-hand side of (5.62) we obtain

$$\left\langle f,\mathbb{J}_{\ell,\widehat{\ell}}\left(H(\widetilde{\mathcal{L}},\widehat{\ell})-z\right)^{-1}g\right\rangle_{L^2(\Gamma(\ell))}-\left\langle\left(H(\mathcal{L},\ell)-\overline{z}\right)^{-1}f,\mathbb{J}_{\ell,\widehat{\ell}}g\right\rangle_{L^2(\Gamma(\ell))} \tag{5.65}$$

$$=\left\langle\left(H(\mathcal{L},\ell)-\overline{z}\right)u,\mathbb{J}_{\ell,\widehat{\ell}}v\right\rangle_{L^2(\Gamma(\ell))}-\left\langle u,\mathbb{J}_{\ell,\widehat{\ell}}\left(H(\widetilde{\mathcal{L}},\widehat{\ell})-z\right)v\right\rangle_{L^2(\Gamma(\ell))} \tag{5.66}$$

$$=\left\langle H(\mathcal{L},\ell)u,\mathbb{J}_{\ell,\widehat{\ell}}v\right\rangle_{L^2(\Gamma(\ell))}-\left\langle\overline{z}u,\mathbb{J}_{\ell,\widehat{\ell}}v\right\rangle_{L^2(\Gamma(\ell))} \tag{5.67}$$

$$-\left\langle u,\mathbb{J}_{\ell,\widehat{\ell}}H(\widetilde{\mathcal{L}},\widehat{\ell})v\right\rangle_{L^2(\Gamma(\ell))}+\left\langle u,z\mathbb{J}_{\ell,\widehat{\ell}}v\right\rangle_{L^2(\Gamma(\ell))} \tag{5.68}$$

$$=\left\langle H(\mathcal{L},\ell)u,\mathbb{J}_{\ell,\widehat{\ell}}v\right\rangle_{L^2(\Gamma(\ell))}-\left\langle u,\mathbb{J}_{\ell,\widehat{\ell}}H(\widetilde{\mathcal{L}},\widehat{\ell})v\right\rangle_{L^2(\Gamma(\ell))}. \tag{5.69}$$

Henceforth, our objective is to show that

$$\left|\left\langle H(\mathcal{L},\ell)u,\mathbb{J}_{\ell,\widehat{\ell}}v\right\rangle_{L^2(\Gamma(\ell))}-\langle u,\mathbb{J}_{\ell,\widehat{\ell}}H(\widetilde{\mathcal{L}},\widehat{\ell})v\rangle_{L^2(\Gamma(\ell))}\right|=o\left(1\right)\|f\|_{L^2(\Gamma(\ell))}\|g\|_{L^2(\Gamma(\widehat{\ell}))}, \tag{5.70}$$

as $\ell_0\to 0$, uniformly in ℓ_+ satisfying (5.61).

Denoting the left-hand side by Z and integrating by parts one obtains

$$Z:=\langle H(\mathcal{L},\ell)u,\mathbb{J}_{\ell,\widehat{\ell}}v\rangle_{L^2(\Gamma(\ell))}-\langle u,\mathbb{J}_{\ell,\widehat{\ell}}H(\widetilde{\mathcal{L}},\widehat{\ell})v\rangle_{L^2(\Gamma(\ell))} \tag{5.71}$$

$$=\int_{\Gamma_+(\ell)}\overline{u''}\mathbb{J}_{\ell,\widehat{\ell}}v-\overline{u}\mathbb{J}_{\ell,\widehat{\ell}}v''=\int_{\Gamma_+(\ell)}\overline{u''}v-\overline{u}v''=\int_{\partial\Gamma_+}\overline{\partial_\nu u}v-\overline{u}\partial_\nu v, \tag{5.72}$$

where we used

$$(\mathbb{J}_{\ell,\widehat{\ell}}f)(x)=\chi_{\Gamma_+(\ell)}f(x),\ x\in\Gamma_+(\ell)$$

due to the fact that the fact that ℓ_+ is fixed.

By Theorem 3.1, $\operatorname{tr}v\in\widetilde{\mathcal{L}}={}^dP_+\big(\mathcal{L}\cap(D_0\oplus N_0)\big)$. Let $G:\widetilde{\mathcal{L}}\to\mathcal{L}\cap(D_0\oplus N_0)$ be any finite-dimensional linear operator[1] such that ${}^dP_+G\phi=\phi$ for any $\phi\in\mathcal{L}$. We let

$$w=(w_1,w_2)=G\operatorname{tr}v\in\mathcal{L}\cap\left(D_0\oplus N_0\right)\subset{}^dL^2(\partial\Gamma);$$

[1]That is, G is a "generalized inverse" of ${}^dP_+$. It always exist but may no be unique; the choice of G with the least norm is the Moore-Penrose pseudoinverse.

it satisfies

$$ {}^dP_+ w = \operatorname{tr} v, \tag{5.73} $$

$$ \| {}^dP_0 w\|_{{}^dL^2(\partial\Gamma_0)} \le \|w\|_{{}^dL^2(\partial\Gamma)} \lesssim \|\operatorname{tr} v\|_{{}^dL^2(\partial\Gamma_+)}, \tag{5.74} $$

the latter because G, as any finite-dimensional linear operator, is bounded.

Using (5.73) we rewrite the last integral in (5.72),

$$ Z = \int_{\partial\Gamma_+} \overline{\partial_\nu u} v - \overline{u}\partial_\nu v = \omega_\Gamma({}^dP_+ \operatorname{tr}^\ell u,\, {}^dP_+ w). \tag{5.75} $$

Since $\omega_\Gamma(\operatorname{tr}^\ell u, w) = 0$, equation (2.30) yields

$$ Z = \omega_\Gamma({}^dP_+ \operatorname{tr}^\ell u,\, {}^dP_+ w) = \omega_\Gamma({}^dP_0 \operatorname{tr}^\ell u,\, {}^dP_0 w) = \int_{\partial\Gamma_0} \overline{\partial_\nu u} w_1 - \overline{u} w_2. \tag{5.76} $$

We estimate each term in (5.76) individually. Using $w_1 \in D_0$ and the Cauchy–Schwarz inequality one obtains

$$ \begin{aligned} \Big| \int_{\partial\Gamma_0} \overline{\partial_\nu u} w_1 \Big| &= \Big| \sum_{e\in\mathcal{E}_0} w_1(b_e)u'(b_e) - w_1(a_e)u'(a_e)\Big| = \Big| \sum_{e\in\mathcal{E}_0} w_1(a_e) \int_e u'' \Big| \\ &\le \sum_{e\in\mathcal{E}_0} |w_1(a_e)| \sqrt{\ell_e} \|u''\|_{L^2(e)}. \end{aligned} \tag{5.77} $$

Similarly, using $w_2 \in N_0$ and the Cauchy–Schwarz inequality we get

$$ \begin{aligned} \Big| \int_{\partial\Gamma_0} \overline{u} w_2 \Big| &= \Big| \sum_{e\in\mathcal{E}_0} w_2(b_e)\overline{u}(b_e) + w_2(a_e)\overline{u}(a_e)\Big| = \Big| \sum_{e\in\mathcal{E}_0} w_2(a_e) \int_e \overline{u}' \Big| \\ &\le \sum_{e\in\mathcal{E}_0} |w_2(a_e)| \sqrt{\ell_e} \|u'\|_{L^2(e)}. \end{aligned} \tag{5.78} $$

Therefore, utilizing (5.74), (5.76) – (5.78) we arrive at

$$ \big|Z\big| \lesssim \sqrt{|\ell_0|}\, \| {}^dP_0 w\|_{{}^dL^2(\partial\Gamma_+)} \|u\|_{\widehat{H}^2(\Gamma(\ell))} \lesssim \sqrt{|\ell_0|}\, \|\operatorname{tr} v\|_{{}^dL^2(\partial\Gamma_+)} \|u\|_{\widehat{H}^2(\Gamma(\ell))} \tag{5.79} $$

$$ \le \sqrt{|\ell_0|}\, \|\operatorname{tr}\| \, \|v\|_{\widehat{H}^2(\Gamma(\widetilde{\ell}))} \|u\|_{\widehat{H}^2(\Gamma(\ell))} \tag{5.80} $$

Let us notice that

$$ \sup_{\ell_+:\ \frac{|\widetilde{\ell}|}{2} \le |\ell_+| \le |\widetilde{\ell}|} \|\operatorname{tr}^{\widehat{\ell}}\|_{\mathcal{B}\left(\widehat{H}^2(\Gamma(\ell)), {}^dL^2(\partial\Gamma)\right)} = \mathcal{O}(1) \ \text{ as } \ell_0 \to 0, \tag{5.81} $$

and

$$ \|u\|_{\widehat{H}^2(\Gamma(\ell))} \le \|R(\mathcal{L}, \ell, \overline{z})\|_{\mathcal{B}\left(L^2(\Gamma(\ell)), \widehat{H}^2(\Gamma(\ell))\right)} \|f\|_{L^2(\Gamma(\widetilde{\ell}))} $$

$$ \|v\|_{\widehat{H}^2(\Gamma(\widetilde{\ell}))} \le \|R(\widetilde{\mathcal{L}}, \widehat{\ell}, z)\|_{\mathcal{B}\left(L^2(\Gamma(\widetilde{\ell})), \widehat{H}^2(\Gamma(\ell))\right)} \|g\|_{L^2(\Gamma(\ell))} $$

Combining these with (3.13) we obtain (5.70).

Step 2. Let us denote $\widehat{\ell} := (\ell_+, 0)$, $\widetilde{\ell} = (\widetilde{\ell}_+, 0)$ and let $\mathcal{J}_{\widehat{\ell}} : L^2(\Gamma(\widetilde{\ell})) \to L^2(\Gamma(\widehat{\ell}))$ be defined as

$$ \left(\mathcal{J}_{\widehat{\ell}} f\right)(x) = \sqrt{\frac{\widetilde{\ell}_e}{\ell_e}} f\left(\frac{x\widetilde{\ell}_e}{\ell_e}\right), \quad x \in e \in \mathcal{E}_+. $$

We remark that in this case, the operators $\mathcal{J}_{\widehat{\ell}}$ are unitary. We need to prove

$$ \left\| \mathcal{J}_{\widehat{\ell}} R(\widetilde{\mathcal{L}}, \widetilde{\ell}, z) - R(\widetilde{\mathcal{L}}, \widehat{\ell}, z)\mathcal{J}_{\widehat{\ell}} \right\|_{\mathcal{B}\left(L^2(\Gamma(\widetilde{\ell})), L^2(\Gamma(\widehat{\ell}))\right)} \underset{\ell_+ \to \widetilde{\ell}_+}{=} o(1), \tag{5.82} $$

where $R(\widetilde{\mathcal{L}},\widehat{\ell},z)$ denotes the resolvent of $H(\widetilde{\mathcal{L}},\widehat{\ell})$, the Laplace operator acting in $L^2(\Gamma(\widehat{\ell}))$ and associated with the Lagrangian plane $\widetilde{\mathcal{L}}$ as in Theorem 3.1.

This case has been considered in [BK12, Theorem 3.7]. In particular, it is proved there that the operator valued function

$$\widehat{\ell}\mapsto \mathcal{J}_{\widehat{\ell}}R(\widetilde{\mathcal{L}},\widehat{\ell},z)\mathcal{J}_{\widehat{\ell}}^{-1}, \tag{5.83}$$

is continuous. This together with the fact that $\mathcal{J}_{\widehat{\ell}}$ is unitary implies (5.82).

Step 3. In this step we show how to combine the results from previous steps to derive (5.58). To this end we use (5.51) and $\mathcal{J}_\ell=\mathbb{J}_{\ell,\widehat{\ell}}\mathcal{J}_{\widehat{\ell}}$ to notice the following:

$$\begin{aligned}\mathcal{J}_\ell\, R(\widetilde{\mathcal{L}},\widetilde{\ell},z)-R(\mathcal{L},\ell,z)\mathcal{J}_\ell &= \mathbb{J}_{\ell,\widehat{\ell}}\mathcal{J}_{\widehat{\ell}}R(\widetilde{\mathcal{L}},\widetilde{\ell},z)-R(\mathcal{L},\ell,z)\mathbb{J}_{\ell,\widehat{\ell}}\mathcal{J}_{\widehat{\ell}}\\ &= \big(\mathbb{J}_{\ell,\widehat{\ell}}R(\widetilde{\mathcal{L}},\ell,z)-R(\mathcal{L},\ell,z)\mathbb{J}_{\ell,\widehat{\ell}}\big)\mathcal{J}_{\widehat{\ell}}+\mathbb{J}_{\ell,\widehat{\ell}}\big(\mathcal{J}_{\widehat{\ell}}R(\widetilde{\mathcal{L}},\widetilde{\ell},z)-R(\widetilde{\mathcal{L}},\ell,z)\,\mathcal{J}_{\widehat{\ell}}\big).\end{aligned} \tag{5.84}$$

For all $\ell,\widehat{\ell}$ one has

$$\|\mathcal{J}_{\widehat{\ell}}\|_{\mathcal{B}(L^2(\Gamma(\widetilde{\ell}),\Gamma(\widehat{\ell})))}=1,\ \ \|\mathbb{J}_{\ell,\widehat{\ell}}\|_{\mathcal{B}(L^2(\Gamma(\widehat{\ell}),\Gamma(\ell)))}=1. \tag{5.85}$$

Moreover,

$$\ell=(\ell_+,\ell_0)\to\widetilde{\ell}=(\widetilde{\ell}_+,0)\iff \ell_+\to\widetilde{\ell}_+,\ \ell_0\to 0. \tag{5.86}$$

Therefore, using (5.60) (uniformly in ℓ_+ satisfying (5.61)), (5.82), (5.84) and the triangle inequality, we obtain (5.58). □

Our next goal is to show that the generalized resolvent convergence of the Schrödinger operators implies convergence of spectral projections and thus convergence of spectra in the Hausdorff sense. In case of non-negative operators this result was established in [P12, Theorem 4.3.3]. In the present setting [P12, Theorem 4.3.3] is not directly applicable since the bottom of the spectrum of $H(\mathcal{L},\ell)$ may tend to negative infinity as $\ell\to\widetilde{\ell}$ (cf., [KS06, Section 3.3]). Nevertheless, the convergence of spectra still holds. We carry out the proof following the standard line of arguments from [P12], [RS, Theorem VIII.20, VIII.23, VIII.24].

Proof of Theorem 3.6. The convergence of spectra follows from Theorem 3.5 and [P12, Proposition 4.3.1]. First of all, by Theorem 3.9, conditions *(i)-(iii)* of Theorem 3.8 hold. Hence Theorem 5.4 and Theorem 5.5 are applicable.

Next, in order to simplify notation let us denote

$$\begin{aligned}&\widetilde{R}_\pm:=R(\widetilde{\mathcal{L}},\widetilde{\ell},\pm\mathbf{i}),\quad R_\pm:=R(\mathcal{L},\ell,\pm\mathbf{i}),\\ &\widetilde{H}:=H(\widetilde{\mathcal{L}},\widetilde{\ell}),\quad H:=H(\mathcal{L},\ell).\end{aligned} \tag{5.87}$$

Let us prove the first assertion in (3.9). Proceeding as in [P12, Theorem 4.2.9] and using (5.58) we get

$$\Big\|\mathcal{J}_\ell\widetilde{R}_\pm^p-R_\pm^p\mathcal{J}_\ell\Big\|_{\mathcal{B}\big(L^2(\Gamma(\widetilde{\ell})),L^2(\Gamma(\ell))\big)}\underset{\ell\to\widetilde{\ell}}{=}o(1),\ p\in\mathbb{N}. \tag{5.88}$$

Next, for arbitrary $p,q\in\mathbb{N}$ one has

$$\mathcal{J}_\ell\widetilde{R}_+^p\widetilde{R}_-^q-R_+^pR_-^q\mathcal{J}_\ell=\big(\mathcal{J}_\ell\widetilde{R}_+^p-R_+^p\mathcal{J}_\ell\big)\widetilde{R}_-^q \tag{5.89}$$

$$+R_+^p\big(\mathcal{J}_\ell\widetilde{R}_-^q-R_-^q\mathcal{J}_\ell\big). \tag{5.90}$$

Let us notice that

$$\|R_+^p\|_{\mathcal{B}(L^2(\Gamma(\ell)))}\le 1,\ \ \|\widetilde{R}_+^p\|_{\mathcal{B}(L^2(\Gamma(\widetilde{\ell})))}\le 1. \tag{5.91}$$

Therefore (5.88)–(5.90) yield

$$\Big\|\mathcal{J}_\ell\widetilde{R}_+^p\widetilde{R}_-^q-R_+^pR_-^q\mathcal{J}_\ell\Big\|_{\mathcal{B}\big(L^2(\Gamma(\widetilde{\ell})),L^2(\Gamma(\ell))\big)}\underset{\ell\to\widetilde{\ell}}{=}o(1),\ p,q\in\mathbb{N}. \tag{5.92}$$

By the Stone–Weierstrass theorem polynomials in $(x+\mathbf{i})^{-1}$ and $(x-\mathbf{i})^{-1}$ are dense in $C(\overline{\mathbb{R}})$, the space of continuous functions for which the limits at both $+\infty$ and $-\infty$ exist and are equal. That is, given any $f\in C(\overline{\mathbb{R}})$ and arbitrary $\varepsilon>0$ there exits a polynomial $P(u,v)$ such that

$$\text{ess sup}_{x\in\mathbb{R}}|f(x)-P((x+\mathbf{i})^{-1},(x-\mathbf{i})^{-1})|<\varepsilon. \tag{5.93}$$

Combining (5.92) and (5.93) we arrive at

$$\left\|\mathcal{J}_\ell f(\widetilde{H})-f(H)\mathcal{J}_\ell\right\|_{\mathcal{B}\left(L^2(\Gamma(\widetilde{\ell})),L^2(\Gamma(\ell))\right)}\underset{\ell\to\widetilde{\ell}}{=}o(1), \tag{5.94}$$

for all $f\in C(\overline{\mathbb{R}})$. As in the case of positive operators, (5.94) gives rise to a similar identity for the spectral projections corresponding to bounded open sets. Namely, in the present context the analogue of [P12, Corollary 4.2.12] reads as

$$\left\|\mathcal{J}_\ell\chi_{(a,b)}(\widetilde{H})-\chi_{(a,b)}(H)\mathcal{J}_\ell\right\|_{\mathcal{B}\left(L^2(\Gamma(\widetilde{\ell})),L^2(\Gamma(\ell))\right)}\underset{\ell\to\widetilde{\ell}}{=}o(1), \tag{5.95}$$

where $-\infty<a<b<\infty$ and $a,b\notin\operatorname{Spec}(\widetilde{H})$. In order to show (5.95), let us pick any $\psi\in C(\overline{\mathbb{R}})$ satisfying

$$0\le\psi\le1,\ \operatorname{supp}(\psi)\subset\mathbb{R}\setminus\{a,b\}\quad\text{and}\quad\psi(x)=1\text{ whenever }x\in\operatorname{Spec}(\widetilde{H}). \tag{5.96}$$

Then

$$\begin{aligned}\left\|\mathcal{J}_\ell\chi_{(a,b)}(\widetilde{H})-\chi_{(a,b)}(H)\mathcal{J}_\ell\right\|&\le\left\|\mathcal{J}_\ell\psi(\widetilde{H})\chi_{(a,b)}(\widetilde{H})-\psi(H)\chi_{(a,b)}(H)\mathcal{J}_\ell\right\|\\&\quad+\left\|\mathcal{J}_\ell(1-\psi)(\widetilde{H})\chi_{(a,b)}(\widetilde{H})-(1-\psi)(H)\chi_{(a,b)}(H)\mathcal{J}_\ell\right\|,\end{aligned} \tag{5.97}$$

where the norms are taken in $\mathcal{B}\left(L^2(\Gamma(\widetilde{\ell})),L^2(\Gamma(\ell))\right)$. Since $\psi\chi_{(a,b)}\in C(\overline{\mathbb{R}})$, the expression in (5.97) is $o(1)$ as $\ell\to\widetilde{\ell}$. Using $(1-\psi)(\widetilde{H})=0$, we rewrite and estimate (5.97) as follows

$$\|(1-\psi)(H)\chi_{(a,b)}(H)\mathcal{J}_\ell\|\le\left\|(1-\psi)(H)\mathcal{J}_\ell\right\|\le\left\|\mathcal{J}_\ell(1-\psi)(\widetilde{H})-(1-\psi)(H)\mathcal{J}_\ell\right\|. \tag{5.98}$$

Since $1-\psi\in C(\overline{\mathbb{R}})$, the expression in (5.98) is $o(1)$ as $\ell\to\widetilde{\ell}$. Hence, (5.95) and the first part of (3.9) hold as asserted. Analogously, the second part of (3.9) can be derived from the second part of (3.6). □

Proof of Theorem 3.7. Due to Theorem 3.6 it is enough to show that (3.8) implies Condition 3.2. Seeking a contradiction we assume that Condition 3.2 is not fulfilled and will show that (3.8) does not hold. In fact we will prove a slightly stronger statement,

$$\dim(\ker(H(\mathcal{L},\ell)))>\dim(\ker(H(\widetilde{\mathcal{L}},\widetilde{\ell}))),\quad\ell\in\mathbb{R}^{|\mathcal{E}|}_{>0}. \tag{5.99}$$

In particular, the multiplicity of zero eigenvalues of the limiting and the approximating operators do not match.

Our first objective is to prove that any $\varphi\in\ker(H(\widetilde{\mathcal{L}},\widetilde{\ell}))$ is constant on each edge. By Proposition 4.5 there exist subspaces $\mathcal{L}_D,\mathcal{L}_N$ such that

$$\mathcal{L}=\{(\phi_1,\phi_2)\in{}^dL^2(\partial\Gamma):\phi_1\in\mathcal{L}_D,\phi_2\in\mathcal{L}_N\}.$$

By Theorem 3.1, one has

$$\widetilde{\mathcal{L}}:=\{{}^dP_+(\phi_1,\phi_2):\phi_1\in\mathcal{L}_D\cap D_0,\phi_2\in\mathcal{L}_N\cap N_0\}. \tag{5.100}$$

Then by Proposition 4.5 the vertex conditions of $H(\widetilde{\mathcal{L}},\widetilde{\ell})$ are scale invariant. From (2.17) one has

$$0=\langle\varphi,H(\widetilde{\mathcal{L}},\widetilde{\ell})\varphi\rangle_{L^2(\Gamma(\widetilde{\ell}))}=\|\varphi'\|_{L^2(\Gamma(\widetilde{\ell}))}. \tag{5.101}$$

Thus φ is constant on each edge, in particular $\varphi(a_e) = \varphi(b_e)$ for every $e \in \mathcal{E}_+$.

Next, for each $\varphi \in \ker(H(\widetilde{\mathcal{L}}, \widetilde{\ell}))$ we construct $f_\varphi \in \ker(H(\mathcal{L}, \ell))$ as follows. Since $\operatorname{tr}\varphi \in \widetilde{\mathcal{L}}$, by Theorem 3.1 there exists

$$(\phi_1, \phi_2) \in \mathcal{L} \cap (D_0 \oplus N_0) \tag{5.102}$$

such that $\operatorname{tr}\varphi = {}^dP_+(\phi_1, \phi_2)$. Note that since φ was constant on edges from $\mathcal{E}_+$ and $\phi_1 \in D_0$,

$$\phi_1(a_e) = \phi_1(b_e) \quad \text{on every edge } e. \tag{5.103}$$

Let us define a function f_φ, constant on each edge, by the formula

$$f_\varphi := \sum_{e\in\mathcal{E}} \phi_1(a_e)\chi_e, \tag{5.104}$$

We claim that $\operatorname{tr} f_\varphi \in \mathcal{L}$. By construction and property (5.103), $\gamma_D f_\varphi = \phi_1$. Since f_φ is constant on edges, $\gamma_N f_\varphi = 0$. Finally, by Proposition 4.5 $(\phi_1, \phi_2) \in \mathcal{L}$ implies $(\phi_1, 0) \in \mathcal{L}$. Therefore, $f_\varphi \in \ker(H(\mathcal{L}, \ell))$ and $f_\varphi|_{\Gamma_+} = \varphi$.

We have now produced a function $f_\varphi \in \ker(H(\mathcal{L}, \ell))$ for every $\varphi \in \ker(H(\widetilde{\mathcal{L}}, \widetilde{\ell}))$. It it easy to see that f_φ are linearly independent if the corresponding φ are. Furthermore, no non-trivial linear combination of f_φ can be zero on Γ_+.

Let us now utilize Lemma 3.3 to produce a nonzero $f \in \ker(H(\mathcal{L}, \ell))$ such that $f|_{\Gamma_+} = 0$. It is clearly linearly independent of all f_φ, leading to

$$\dim(\ker(H(\widetilde{\mathcal{L}}, \widetilde{\ell}))) < \dim(\operatorname{span}\{f_\varphi, f : \varphi \in \ker(H(\widetilde{\mathcal{L}}, \widetilde{\ell}))\}) \leq \dim(\ker(H(\mathcal{L}, \ell))) \tag{5.105}$$

as required. □

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