

SURGERY PRINCIPLES FOR THE SPECTRAL ANALYSIS OF QUANTUM GRAPHS

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ABSTRACT. We present a systematic collection of spectral surgery principles for the Laplacian on a compact metric graph with any of the usual vertex conditions (natural, Dirichlet or δ -type), which show how various types of changes of a local or localised nature to a graph impact on the spectrum of the Laplacian. Many of these principles are entirely new; these include “transplantation” of volume within a graph based on the behaviour of its eigenfunctions, as well as “unfolding” of local cycles and pendants. In other cases we establish sharp generalisations, extensions and refinements of known eigenvalue inequalities resulting from graph modification, such as vertex gluing, adjustment of vertex conditions and introducing new pendant subgraphs.

To illustrate our techniques we derive a new eigenvalue estimate which uses the size of the doubly connected part of a compact metric graph to estimate the lowest non-trivial eigenvalue of the Laplacian with natural vertex conditions. This quantitative isoperimetric-type inequality interpolates between two known estimates — one assuming the entire graph is doubly connected and the other making no connectivity assumption (and producing a weaker bound) — and includes them as special cases.

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1. INTRODUCTION

The topic of eigenvalue estimates for various kinds of differential operators – especially for the Laplacian – is a well-established one. The usual goal is to deduce estimates depending only on simple geometric properties of the underlying object, most commonly a domain or a manifold, without having to try to compute the eigenvalues or eigenfunctions explicitly.

In the last decade there has been a pronounced growth of interest in such eigenvalue estimates in the particular case of quantum graph Laplacians; we refer in particular to [ASSW17, Ari16, BaLe17, BKKM17, BD12, Fri05a, KKTK16, KKMM16, KMN13, KN14, Roh17, RoSe18] for the Laplacian and its linear generalisations, and [AST15a, AST15b, AST17] among others for a class of nonlinear Schrödinger operators on graphs. Particular attention has been paid to the lowest non-trivial eigenvalue because, for example, it gives the optimal rate of convergence to the equilibrium of solutions to the corresponding heat equation. In the case of natural vertex conditions (also known as standard and Kirchhoff-continuity in the literature), this eigenvalue equals the spectral gap, and in addition to its role in the heat equation, predicts bifurcation of the ground state from constant in nonlinear Schrödinger equation on the graph [Ada16, MP16], and has a non-trivial relation to graph connectivity: for the discrete Laplacian the spectral gap is also referred to as the *algebraic connectivity* [Fie73]. The corresponding eigenvectors, also called *Fiedler vectors*, are interesting from the point of view of clustering problems [Lux07] and the hot spots conjecture [Eva11], and similar applications are expected in the case of quantum graph Laplacians [KKLM19, KR18]. Higher eigenvalue estimates can play a role in the study of spectral minimal partitions [BBRS12, KKLM19], nodal count statistics [Ber08, BW15], [Ber17, Theorem 7.8] and even quantum chaos [BBK01].

It has become increasingly clear that a central role in estimating eigenvalues is played by what we shall call *surgery operations*: basic changes to the geometry of a graph, such as lengthening an edge or gluing together vertices, that have a predictable effect on one or several eigenvalues. The current work is dedicated to developing and cataloguing these tools in their sharpest form, as well as illustrating their potency with some carefully selected applications (further applications that have been discovered in the course of preparing this manuscript will be published elsewhere). Wherever feasible we treat general eigenvalues (i.e., not just the lowest ones) and more general vertex conditions, in particular Dirichlet and Robin-type couplings. Much care is dedicated to treating the cases of

extremality, i.e. the cases when an inequality becomes an equality. However, we restrict ourselves to the case of *compact* graphs, that is, graphs with a finite number of edges, each of finite length; this guarantees that our Laplacian operators have discrete spectrum. Most techniques presented here should be extendable, with the same proofs, to the eigenvalues below the essential spectrum of Schrödinger-type operators on non-compact graphs.

After introducing our notation and recalling the basic definitions and properties of quantum graphs in Section 2, we will collect all the surgical principles in Section 3, classifying them into three types. Section 3.1 treats operations related to the vertices: cutting and gluing them, or changing the vertex condition. Notably, the main theorem of this section, Theorem 3.4, contains a new, complete characterisation of equality when cutting and gluing vertices, which in turn uses a characterisation of equality in the Courant–Fischer minimax principles which seems to be very little known (see Lemma 4.1). In Section 3.2 we look at operations that increase the total volume (length) of the graph. Here, we study the effect of inserting a graph at a given vertex, of which previously studied operations of attaching a pendant graph and lengthening an edge are special cases. In Section 3.3 we consider operations that transfer edges of the graph from one part to another: in this case, we are primarily interested in the lowest non-trivial eigenvalue. Among others, we introduce the notions of transplantation and unfolding of edges. Theorem 3.18 summarises the spectral consequences of these operations, which for the most part we believe to be entirely new. In Section 3.4 we give a few examples illustrating the necessity of our assumptions and indicating possible further extensions and generalisations.

Some of the surgery operations we consider have appeared elsewhere, but in weaker forms. In [BK12, KMN13, KN14, Roh17], eigenvalue estimates were derived for certain basic surgical operations of quantum graphs, namely gluing vertices and attaching edges; the recent preprint [RoSe18] deals with these operations for more general self-adjoint vertex conditions. However, even with these operations, to date little attention has been paid to characterising the cases of equality. More sophisticated surgery operations where the set of edges is changed were investigated in [BKKM17, KKMM16, BaLe17], often relying on the symmetrisation technique first applied to quantum graphs by L. Friedlander [Fri05a] (see [KKTK16] for a comparison with other techniques). Some estimates of the types not considered here, but which could be derived from the more fundamental results collected in Section 3, appeared as Lemma 4.2 and Lemma 4.5 of [BBW15] and as Theorem 1.3 (edge switching transformation) of [ASSW17].

The proofs of all our surgery results are the subject of Section 4. The remaining sections are devoted to a demonstration of what can be achieved using the new (or significantly improved) surgery principles. Our main goal in this direction is a sharpened isoperimetric-type inequality for the first non-trivial eigenvalue (spectral gap) of the Laplacian with natural vertex conditions. For a general compact metric graph of total length L , this eigenvalue was shown by Nicaise [Nic87] to be no smaller than π^2/L^2 , with equality if and only if $\mathcal{G}$ is a path (i.e., interval); see [Sol02, Fri05a, KN14] for further proofs. Recently, Band and Lévy [BaLe17] obtained a stronger lower bound under the assumption that the graph is doubly connected: the non-trivial eigenvalue is no lower than $4\pi^2/L^2$; see also [BKKM17] for a sharper estimate in the case of higher connectivities.

Here we will obtain results that interpolate between these two inequalities, while containing them as special, limit cases: we will prove lower bounds on the spectral gap in

terms of the size of the *doubly connected part* of the graph $\mathcal{G}$. This is the largest subset of $\mathcal{G}$ each of whose connected components is itself doubly connected (see Definition 6.1 for more details). Our main theorems in this context are Theorem 6.3 and Theorem 6.5. The former bounds the spectral gap of a graph $\mathcal{G}$ from below in terms of a *dumbbell graph* (see Definition 2.3) with the same (or smaller) sized doubly connected component. The latter gives a complementary but equally sharp bound in terms of the length of the longest cycle in $\mathcal{G}$, leading to a comparison with a well-chosen *tadpole graph*. In fact, these results appear to be considerably stronger than the best available analogues for discrete Laplacians, cf. Proposition 6.6 and Corollary 6.7 for more details.

Theorems 6.3 and 6.5 may be viewed as *quantitative isoperimetric inequalities*, which make an appearance in spectral geometry of domains in higher dimensions [BDP17]. Such inequalities give not just a sharp bound on an eigenvalue in terms of the total volume (or in our case length of the graph), but also a correction term which takes into account some measure of the difference of a given domain from the optimising one (the size of the doubly connected component or the length of the longest cycle, in our case). The interested reader may wish to combine our results with the results of [Roh17], in which complementary improved estimates are obtained for tree graphs.

Along the way to our main applications, in Section 5 we will give several smaller, more specialised applications of our techniques to so-called *pumpkin chain* graphs, which give demonstrations of how individual surgery techniques can be used and combined to manipulate special classes of graphs. As a simple example, the “unfolding of edges” principle from Section 3.3 shows that the spectral gap of any tadpole or dumbbell graph is a monotonically decreasing function of the length of the loop(s) for fixed total graph length, without the need for explicit calculations based on the secular equation for the eigenvalues; see Propositions 5.7 and 5.9 for more details. The examples we present in this section have also been chosen because they provide exactly the auxiliary results needed for the proofs of the isoperimetric inequalities in Section 6.

Another application of the techniques developed here will appear in the forthcoming paper [Ken18]; we intend to present further applications elsewhere.

2. PRELIMINARIES AND NOTATION

We shall begin with our notation. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$. We turn it into a metric graph by identifying each edge $e \in \mathcal{E}$ with the interval $[0, |e|]$, where $|e| > 0$ is the length of the edge. We will denote vertices by letters such as v , u and w ; we shall write $e \sim v$ to mean that the vertex v is incident with the edge e . In a slight abuse of notation, we will also write $e \sim vw$ to mean that e is an edge connecting v and w . We will always assume the graph is *compact*, by which we mean that there is a finite number $E = |\mathcal{E}|$ of edges, each edge of finite length; this terminology is in keeping with [BK13]. We denote by $|\mathcal{G}|$ the total length of the graph, i.e. the sum of the lengths of the edges of the graph. $\mathcal{G}$ is allowed to contain loops as well as multiple edges between given pairs of vertices. Often, but not always, we will assume the graph is connected; whenever we do so we will state this assumption explicitly.

We shall be interested in the spectrum of the Laplacian $-\Delta$ on $\mathcal{G}$ equipped at each vertex with one of the following vertex conditions: more precisely, the operator is $-\frac{d^2}{dx^2}$ on

each edge applied to functions which are in the Sobolev space $H^2(e)$ on each edge $e \in \mathcal{E}$, and which satisfy

- *natural*¹ conditions on a subset $\mathcal{V}_N \subset \mathcal{V}$: at $v \in \mathcal{V}_N$, we demand continuity of the functions and that the sum of the normal derivatives at each vertex is zero (the Kirchhoff or “current conservation” condition):

$$\sum_{e \sim v} \partial_\nu f|_e(v) = 0$$

for each $v \in \mathcal{V}_N$, where $\partial_\nu f|_e(v)$ is the normal derivative of f on e at v , with $\nu = \nu_e(v)$ pointing *outward* (away from the edge e , towards the vertex);

- *Dirichlet* conditions on a subset $\mathcal{V}_D \subset \mathcal{V}$: at $v \in \mathcal{V}_D$, any functions in the domain of Δ should take on the value zero.
- δ (or *Kirchhoff–Robin*) conditions on a subset $\mathcal{V}_R \subset \mathcal{V}$: for each $v \in \mathcal{V}_R$ there is a $\gamma = \gamma(v) \in \mathbb{R}$, $\gamma \neq 0$ such that the functions $f \in D(\Delta)$ are continuous at v and the derivatives at v satisfy

$$(2.1) \quad \sum_{e \sim v} \partial_\nu f|_e(v) + \gamma f(v) = 0,$$

where, again, ν is the outer unit normal to the edge. We sometimes refer to γ as the *strength* of the δ -condition, or as the δ -*potential* at v . We will write

$$\mathcal{V}_R^- := \{v \in \mathcal{V}_R : \gamma(v) < 0\}, \quad \mathcal{V}_R^+ := \{v \in \mathcal{V}_R : \gamma(v) > 0\},$$

so that $\mathcal{V}_R = \mathcal{V}_R^- \cup \mathcal{V}_R^+$.

We thus assume that $\mathcal{V} = \mathcal{V}_N \cup \mathcal{V}_D \cup \mathcal{V}_R$, where any of the three sets on the right-hand side may be empty. We see immediately that the δ -condition with $\gamma(v) = 0$ corresponds to the natural condition. Furthermore, Dirichlet conditions correspond formally to δ -conditions of strength $\gamma = \infty$. This correspondence may be made rigorous [BK12], although we will not need it here. We refer to [BK13, Chapter 1] and [Mug14, Chapters 2 and 3] for more details regarding the definitions and elementary properties of function spaces on graphs and the Laplace-type operators defined on them and also to [Ber17] for an elementary introduction to spectral properties of quantum graphs.

Remark 2.1. Any point in the interior of an edge may be declared to be a vertex of degree 2 with natural conditions without affecting the spectral properties of the operator (cf. the discussion just after Assumption 3.1 in [BK17]). We will refer to this as introducing a “dummy” vertex. Conversely, any vertex v of degree 2 with natural conditions may be suppressed. Likewise, the operator is not modified if a subset of the elements of $\mathcal{V}_D$ are identified to form one single Dirichlet vertex.

The corresponding quadratic form is given by the Dirichlet integral

$$(2.2) \quad a(f) = \int_{\mathcal{G}} |f'|^2 dx + \sum_{v \in \mathcal{V}_R} \gamma(v) |f(v)|^2,$$

¹Also known as standard, Neumann, continuity/Kirchhoff, or Neumann–Kirchhoff; observe that on a degree-one vertex natural conditions agree with common Neumann ones.

with the domain formed by all functions from the Sobolev space $H^1(e_j)$ on every edge, which are in addition continuous at the vertices, and zero at all vertices in $\mathcal{V}_D$. If $\mathcal{V}_D = \emptyset$, then this form domain shall be denoted by $H^1(\mathcal{G})$, as is customary; if $\mathcal{V}_D \neq \emptyset$ we shall denote it by $H_0^1(\mathcal{G}; \mathcal{V}_D)$, or just $H_0^1(\mathcal{G})$ if there is no danger of confusion about which set $\mathcal{V}_D$ is to be understood. We will write $C(\mathcal{G})$ for the space of continuous functions on $\mathcal{G}$, i.e., continuous on every edge, and at every vertex, so that $H^1(\mathcal{G}), H_0^1(\mathcal{G}; \mathcal{V}_D) \subset C(\mathcal{G})$ for any $\mathcal{V}_D$.

As is well known, under this set of assumptions the Laplacian described above is self-adjoint and semi-bounded, and has trace class resolvent; in particular, its spectrum consists of a sequence of real eigenvalues of finite multiplicity, which we denote by

$$(2.3) \quad \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots,$$

where each is repeated according to its multiplicity. The corresponding eigenfunctions may be chosen to form an orthonormal basis of $L^2(\mathcal{G})$, and may additionally without loss of generality all be chosen real, as we shall do without further comment throughout the paper. We will tend to use letters such as ψ to denote eigenfunctions, and will refer to an eigenfunction corresponding to eigenvalue λ as a λ -eigenfunction. By standard Kreĭn–Rutman theory, if $\mathcal{G}$ is connected, the first eigenvalue λ_1 is always simple and the corresponding eigenfunction, unique up to scalar multiples, can be chosen strictly positive a.e.; in fact it can be shown to be strictly positive everywhere outside $\mathcal{V}_D$.

The eigenvalues, and their eigenfunctions, depend on both the metric and topological structure of the graph and the δ -coupling parameters; for brevity, in keeping with the custom of considering a quantum graph to be a triple consisting of a metric graph, a differential operator and vertex conditions, we shall write $\lambda_k = \lambda_k(\mathcal{G})$ to reflect this dependence and correspondingly $\sigma(\mathcal{G}) = \{\lambda_k(\mathcal{G}) : k \geq 1\}$ for the spectrum. To distinguish the important case of only natural conditions, $\mathcal{V} = \mathcal{V}_N$, where $\lambda_1 = 0$, we shall often use the superscript “ N ” if we wish to emphasise the presence of only natural conditions at the vertices: in this case, we will write (assuming $\mathcal{G}$ is connected),

$$(2.4) \quad 0 = \lambda_1^N < \lambda_2^N \leq \lambda_3^N \leq \dots$$

Note that the more general notation (2.3) also covers this case. We will likewise use the notation

$$(2.5) \quad 0 < \lambda_1^D < \lambda_2^D \leq \lambda_3^D \leq \dots$$

in place of the λ_k if we want to emphasise that there is at least one Dirichlet vertex. In this case all non-Dirichlet vertices are assumed to be equipped with natural conditions. We mention explicitly that if one or more vertices are equipped with a negative coupling condition, i.e., if $\mathcal{V}_R \neq \emptyset$, then there may be negative eigenvalues; in this case, it is also possible that $\lambda_1(\mathcal{G}) = 0$ but the corresponding eigenfunction is not identically constant. See also [EJ12].

At any rate, these eigenvalues are generally not explicitly computable as the relevant secular equation is transcendental even on graphs as simple as stars, cf. [Bel85, § 4, Thm.] or [Ber17, Sec 5]. Instead, in the tradition of spectral geometry, we can attempt to understand how the eigenvalues change depending on the underlying graph. For example, one may be interested to know if a certain graph minimises or maximises a given eigenvalue among all graphs with certain fixed geometric quantities (length, diameter etc.), or if, for

a given graph with reflection symmetry, the first non-trivial eigenfunction is symmetric or anti-symmetric. To answer such questions one needs to be able to make comparisons. One of the aims of this work is to catalogue the types of alterations to the geometry of a graph that affect the eigenvalue(s) in a predictable way; we shall generally refer to this as “surgery”.

We will often be concerned with the first non-trivial eigenvalue in particular. The starting point is always the variational characterisation of $\lambda_1(\mathcal{G})$ and $\lambda_2(\mathcal{G})$, namely

$$(2.6) \quad \lambda_1(\mathcal{G}) = \inf \left\{ \frac{a(f)}{\int_{\mathcal{G}} |f|^2 dx} : 0 \neq f \in H^1(\mathcal{G}) \text{ or } H_0^1(\mathcal{G}; \mathcal{V}_D) \right\},$$

(H^1 or H_0^1 as appropriate, and where a is given by (2.2)). In the case of a connected graph and pure natural conditions, $\mathcal{V} = \mathcal{V}_N$, where $\lambda_1^N = 0$ and the associated eigenfunction is simply the constant function, it is more natural to consider λ_2^N , which in this case, since $a(f)$ reduces to the integral of the derivatives, is given by

$$(2.7) \quad \lambda_2^N(\mathcal{G}) = \inf \left\{ \frac{\int_{\mathcal{G}} |f'|^2 dx}{\int_{\mathcal{G}} |f|^2 dx} : 0 \neq f \in H^1(\mathcal{G}) \text{ and } \int_{\mathcal{G}} f dx = 0 \right\};$$

the condition $\int_{\mathcal{G}} f dx = 0$ represents the orthogonality in $L^2(\mathcal{G})$ of f to the constant eigenfunctions of $\lambda_1^N(\mathcal{G})$. We therefore introduce the following notation.

Definition 2.2. Given a *connected* graph $\mathcal{G}$ with vertex conditions of the types listed above, we denote by $\mu(\mathcal{G})$ its first eigenvalue $\lambda_1(\mathcal{G})$ if $\mathcal{V}_D \cup \mathcal{V}_R \neq \emptyset$, or its second eigenvalue $\lambda_2^N(\mathcal{G})$ if $\mathcal{V}_D \cup \mathcal{V}_R = \emptyset$ (and thus $\lambda_1^N(\mathcal{G}) = 0$).

The higher eigenvalues may be characterised by corresponding minimax and maximin principles of Courant–Fischer type, cf. (4.1) and (4.2). In all cases, for a given function $f \in H^1(\mathcal{G})$, the quotient appearing in (2.6) or (2.7) is called the *Rayleigh quotient* of f , and equality is achieved if and only if f is an eigenfunction associated with the corresponding eigenvalue.

2.1. Examples of graphs. Here we introduce terminology for several classes of graphs that come up often in applications, in particular, as sharp cases of eigenvalue estimates. In all cases, we assume that any vertices of degree two are suppressed, cf. Remark 2.1.

Definition 2.3. (1) The *path graph* is a graph consisting of two vertices and one edge.

(2) A *loop* is a graph consisting of one edge whose endpoints are the same vertex.

(3) The *star graph* is a graph consisting of E edges all having exactly one vertex in common. We refer to this graph as an *E-star* to emphasise the number of edges.

(4) The *tadpole graph* (also called “lasso graph”) is a graph consisting of a loop attached to a single edge.

(5) The *flower graph* consists of E loops attached to a single vertex. A special case is the *figure-8 graph* which has two loops.

(6) The *dumbbell graph* has three edges and two vertices; it consists of an edge joining two loops.

(7) The *pumpkin graph* (also called “mandarin graph”) is a graph consisting of two vertices and E parallel edges of possibly different lengths running between them. We will also write *E-pumpkin* if we wish to emphasise the number of edges.

- (8) In examples (3), (5) and (7), the prefix *equilateral* is applied if all edges have the same length.
- (9) A *pumpkin chain* is built up out of pumpkins glued sequentially at the vertices. More precisely, a $[m_1, \dots, m_n]$ -*pumpkin chain* is a connected graph consisting of $n + 1$ vertices $v_1, \dots, v_{n+1}$ and, for each $k = 1, \dots, n$, m_k parallel edges running between the vertices v_k and v_{k+1} . We will denote the subpumpkin consisting of the vertices v_k and v_{k+1} and the m_k edges joining them by $\mathcal{P}_k$ and refer to it as the k th constituent pumpkin of the chain. Both vertices v_1 and v_{n+1} , which we sometimes denote by v_- and v_+ , and the pumpkins $\mathcal{P}_1$ and $\mathcal{P}_n$ attached to them, shall be called *terminal*. We shall call a constituent pumpkin of a pumpkin chain *trivial* if it only has one edge, and *non-trivial* otherwise. A pumpkin chain is *locally equilateral* if each constituent pumpkin is equilateral (i.e., all edges have the same length), although the lengths of edges in different pumpkins may be different.

We remark that the tadpole graph can be viewed as the $[1, 2]$ - or $[2, 1]$ -pumpkin chain, the figure-8 graph is the $[2, 2]$ -pumpkin chain, and the dumbbell is the $[2, 1, 2]$ -pumpkin chain.

3. A SURGEON'S TOOLKIT

The aim of this section is to catalogue the effects of elementary surgical transformations on the spectrum of the graph. In each case we will first describe how the connectivity and the metric features of $\mathcal{G}$ should be changed in order to produce a new graph $\tilde{\mathcal{G}}$, and then how functions on $\mathcal{G}$ can be lifted to functions on $\tilde{\mathcal{G}}$ by canonically assigning conditions in the vertices of $\tilde{\mathcal{G}}$. Here and throughout we adopt the usual conventions for arithmetic involving ∞ , e.g., $\gamma + \infty = \infty$ if $\gamma \in \mathbb{R}$; $\infty + \infty = \infty$ and so on.

We reiterate that all graphs throughout this section are taken to satisfy the assumptions described at the beginning of Section 2; all results of this section will be proved in Section 4.

3.1. Operations changing vertex conditions. We have already remarked that cutting through/gluing together Dirichlet vertices is a trivial operation (Remark 2.1); let us now consider what happens for more general vertex conditions. Recall that all three kinds of vertex conditions we are considering can be regarded as δ -conditions with parameter $\gamma \in (-\infty, \infty]$.

Definition 3.1 (Gluing vertices). Let $\tilde{\mathcal{G}}$ be obtained from $\mathcal{G}$ by identifying the vertices $v_1, v_2, \dots, v_m$ to obtain a new vertex v_0 . If the δ -conditions with the strengths $\gamma(v_j) \in (-\infty, \infty]$ were imposed at v_j , $j = 1, \dots, m$, then the new vertex v_0 is to be equipped with the δ -condition of strength

$$(3.1) \quad \gamma(v_0) = \gamma(v_1) + \gamma(v_2) + \dots + \gamma(v_m).$$

We will refer to the corresponding surgery transformation as **gluing vertices**.

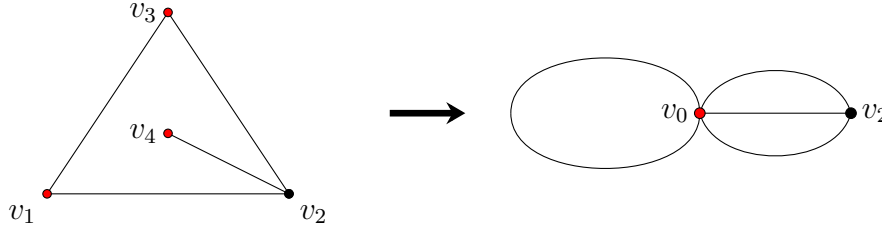


FIGURE 3.1. The graph $\tilde{\mathcal{G}}$ (right) is obtained from $\mathcal{G}$ (left) by gluing the vertices v_1, v_3, v_4 . Conversely, the graph on the left is one of the possible graphs obtainable from the graph on the right upon cutting through v_0 , in this case producing the vertices v_1, v_3, v_4 .

Definition 3.2 (Cutting through vertices). The converse operation to gluing the vertices, i.e. splitting a vertex v_0 into m vertices $v_1, v_2, \dots, v_m$ (called *descendants* of v_0) with δ -type conditions satisfying (3.1) is called **cutting through the vertex** v_0 .

If in addition we are given a certain function ψ satisfying δ -conditions at v_0 we can choose $\gamma(v_1), \dots, \gamma(v_m)$ so that the same function ψ satisfies the conditions at the new vertices. Namely, we let

$$(3.2) \quad \gamma(v_i) = -\frac{1}{\psi(v_0)} \sum_{e \sim v_i} \partial_\nu \psi|_e(v_0), \quad i = 1, \dots, m$$

where the summation is over the edges that are attached to the relevant descendant of v_0 . In particular, when $\psi(v_0) = 0$, we impose Dirichlet conditions at all vertices $v_1, \dots, v_m$. The corresponding surgery transformation will be called **cutting through the vertex** v_0 **along the function** ψ .

In general, when cutting through a vertex v_0 the edges incident with it may be assigned to the new vertices $v_1, \dots, v_m$ in several possible ways. In other words, even if m is fixed, the graph $\tilde{\mathcal{G}}$ created from $\mathcal{G}$ by cutting through a vertex is in general not unique.

Remark 3.3. Suppose $\tilde{\mathcal{G}}$ is created from $\mathcal{G}$ by gluing m vertices $v_1, \dots, v_m$ to form v_0 . Then there is a natural isomorphism $\Phi : L^2(\mathcal{G}) \rightarrow L^2(\tilde{\mathcal{G}})$, that is, we can make the identification

$$L^2(\mathcal{G}) \simeq \bigoplus_{e \in \mathcal{E}} L^2(0, \ell_e) \simeq L^2(\tilde{\mathcal{G}}),$$

where $\mathcal{E}$ is the common set of edges of the two graphs. Moreover, if $f \in C(\mathcal{G})$ (in particular if $f \in H^1(\mathcal{G})$) satisfies $f(v_1) = \dots = f(v_m)$, then also $\Phi(f) \in C(\tilde{\mathcal{G}})$ (corresp. in $H^1(\tilde{\mathcal{G}})$). In this case, we identify $\Phi(f)$ with f and speak of a “canonical identification” of the two; in this way, $C(\tilde{\mathcal{G}})$ and $H^1(\tilde{\mathcal{G}})$ may be regarded as subspaces of $C(\mathcal{G})$ and $H^1(\mathcal{G})$ of codimension $m - 1$, respectively. From now on, we will *always* make this identification, that is, whenever we glue together vertices, we will suppress the notation Φ and identify $C(\tilde{\mathcal{G}})$ and $H^1(\tilde{\mathcal{G}})$ (and its subspaces) with subspaces of $C(\mathcal{G})$ and $H^1(\mathcal{G})$ (and its subspaces), respectively.

By regarding cutting through a vertex as removing continuity conditions from it, if one wishes one may also view this operation as changing the conditions at a single,

“generalised” vertex. Another example of changing conditions is the operation of varying the δ -potential at a vertex v . Both types of operations are finite rank perturbations of the quantum graph operator and result in the interlacing of the eigenvalues of the two graphs.

Theorem 3.4 (Changing vertex conditions). *If the graph $\tilde{\mathcal{G}}$ is obtained from $\mathcal{G}$ by either*

(1) *gluing two vertices,*

or

(2) *increasing the strength of the δ -condition at a single vertex from γ to $\gamma' \in (\gamma, \infty]$,*

then their eigenvalues satisfy the interlacing inequalities

$$(3.3) \quad \lambda_k(\mathcal{G}) \leq \lambda_k(\tilde{\mathcal{G}}) \leq \lambda_{k+1}(\mathcal{G}) \leq \lambda_{k+1}(\tilde{\mathcal{G}}), \quad k \geq 1.$$

If a given value Λ has multiplicities m and $\tilde{m}$ in the spectra of $\mathcal{G}$ and $\tilde{\mathcal{G}}$, respectively, then $|m - \tilde{m}| \leq 1$ and, with the identification in Remark 3.3, the intersection of the respective Λ -eigenspaces has dimension $\min(m, \tilde{m})$.

The inequality of type (3.3) is both well known (cf., e.g., [BK13, Theorems 3.1.8 and 3.1.10], [KMN13], [RoSe18]) and easy to obtain from the variational principles. However, the conclusive treatment of the cases of equality is, to the best of our knowledge, new. It is also of tremendous value for characterising the extremal cases of the inequalities contained in the subsequent sections of the present paper. For example, the following simple observation will be used at least twice.

Remark 3.5. If, in the setting of Theorem 3.4, the eigenvalue arrangement is

$$(3.4) \quad \lambda_k(\mathcal{G}) < \lambda_k(\tilde{\mathcal{G}}) = \lambda_{k+1}(\mathcal{G}) =: \Lambda,$$

then every $\lambda_{k+1}(\mathcal{G})$ -eigenfunction of $\mathcal{G}$ is also an eigenfunction of $\tilde{\mathcal{G}}$. Indeed, it is easy to see that (3.3) will imply that $\min(m, \tilde{m}) = m$ for the eigenvalue Λ and the eigenspace inclusion follows.

We will also often use the following special case of equality.

Corollary 3.6 (Gluing level points). *Suppose $v_1, \dots, v_m \in \mathcal{V}(\mathcal{G})$ and for some $k \geq 1$ there exist eigenfunctions $\psi_1, \dots, \psi_k$ corresponding to $\lambda_1(\mathcal{G}), \dots, \lambda_k(\mathcal{G})$, respectively, such that*

$$\psi_1(v_1) = \dots = \psi_1(v_m), \quad \dots, \quad \psi_k(v_1) = \dots = \psi_k(v_m).$$

Let $\tilde{\mathcal{G}}$ be the graph formed from $\mathcal{G}$ by gluing $v_1, \dots, v_m$. Then

$$\lambda_1(\tilde{\mathcal{G}}) = \lambda_1(\mathcal{G}), \quad \dots, \quad \lambda_k(\tilde{\mathcal{G}}) = \lambda_k(\mathcal{G}).$$

Moreover, $\psi_1, \dots, \psi_k$ are eigenfunctions on $\tilde{\mathcal{G}}$ associated with $\lambda_1(\tilde{\mathcal{G}}), \dots, \lambda_k(\tilde{\mathcal{G}})$, respectively.

Remark 3.7. In [BK13, Proposition 3.1.6] it is additionally shown that if an eigenvalue $\lambda_k(\mathcal{G})$ is simple, with eigenfunction ψ_k normalised to have L^2 -norm one, it can be differentiated with respect to the strength γ of the δ -parameter at a given vertex v . The value of the derivative is

$$(3.5) \quad \frac{d\lambda_k}{d\gamma} = |\psi_k(v)|^2 = \left| \frac{1}{\gamma} \sum_{e \sim v} \partial_\nu \psi_k|_e(v) \right|^2.$$

3.2. Operations increasing the volume. We will now consider operations that expand the graph in some way, either by scaling up a part of it or by attaching a new subgraph to it.

Definition 3.8 (Inserting a graph at a vertex). Let v_0 be a vertex of $\mathcal{G}$ whose set of incident edges is $\{e_1, \dots, e_k\}$ and let $\mathcal{H}$ be another metric graph. Form a new graph $\tilde{\mathcal{G}}$ by removing v_0 from $\mathcal{G}$ and, for each $i = 1, \dots, k$, attaching edge e_i to some vertex $w = w(i)$ of $\mathcal{H}$ instead. Let $w_1, \dots, w_m$, $m \leq k$ be the list of vertices of $\mathcal{H}$ to which an edge has been so attached. If v_0 is equipped with the δ -potential of strength $\gamma(v_0) \in (-\infty, \infty)$, then δ -potentials should be placed at the vertices $w_1, \dots, w_m$ in such a way that they sum to $\gamma(v_0)$. We then say that $\tilde{\mathcal{G}}$ is formed by **inserting $\mathcal{H}$ into $\mathcal{G}$ at v_0** .

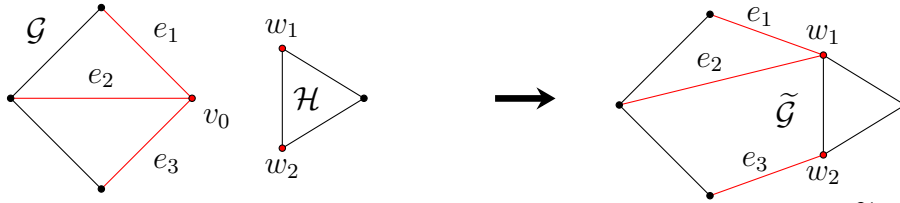


FIGURE 3.2. Inserting $\mathcal{H}$ into $\mathcal{G}$ at v_0 , we obtain the graph $\tilde{\mathcal{G}}$ on the right.

Whenever $w_1 = \dots = w_m$ we have the following special case.

Definition 3.9 (Attaching a pendant graph). Assume that $\mathcal{G}$ and $\mathcal{H}$ are given, with one distinguished vertex in each graph, say $v_1 \in \mathcal{G}$ and $w_1 \in \mathcal{H}$. If $\tilde{\mathcal{G}}$ is formed by gluing together v_1 and w_1 in accordance to Definition 3.1, we speak of **attaching the pendant graph $\mathcal{H}$ to $\mathcal{G}$** .

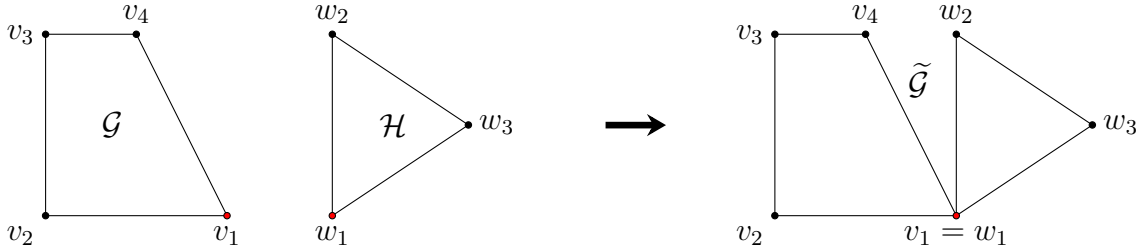


FIGURE 3.3. By gluing together v_1, w_1 we can attach the graph $\mathcal{H}$ to $\mathcal{G}$, thus obtaining the graph $\tilde{\mathcal{G}}$ on the right.

Figure 3.2 shows an example of inserting a graph at v_0 and Figure 3.3 shows an example of attaching a pendant graph.

Theorem 3.10. *The following operations decrease the given eigenvalues.*

- (1) (Attaching a pendant graph) Suppose $\tilde{\mathcal{G}}$ is formed from $\mathcal{G}$ by attaching a pendant metric graph $\mathcal{H}$ at a vertex $v_0 \in \mathcal{V}(\mathcal{G})$. If, for some r and k ,

$$(3.6) \quad \lambda_r(\mathcal{H}) \leq \lambda_k(\mathcal{G}),$$

then

$$(3.7) \quad \lambda_{k+r-1}(\tilde{\mathcal{G}}) \leq \lambda_k(\mathcal{G}).$$

The inequality in (3.7) is strict if the eigenvalue $\lambda_k(\mathcal{G})$ has an eigenfunction which does not vanish at v_0 , $\lambda_k(\mathcal{G}) > \lambda_{k-1}(\mathcal{G})$ and $\lambda_k(\mathcal{G}) > \lambda_r(\mathcal{H})$.

- (2) (Inserting a graph at a vertex) Suppose $\tilde{\mathcal{G}}$ is formed by inserting a graph $\mathcal{H}$ at a vertex v_0 of $\mathcal{G}$. Assume that only natural conditions were imposed at the vertices of $\mathcal{H}$ prior to insertion. Then, for all k such that $\lambda_k(\mathcal{G}) \geq 0$,

$$(3.8) \quad \lambda_k(\tilde{\mathcal{G}}) \leq \lambda_k(\mathcal{G}).$$

The inequality in (3.8) is strict if $\lambda_k(\mathcal{G}) > \max(0, \lambda_{k-1}(\mathcal{G}))$ and the eigenvalue $\lambda_k(\mathcal{G})$ has an eigenfunction which does not vanish at v_0 .

Remark 3.11. An important special case of Theorem 3.10(1) is when the conditions are natural at all vertices of $\mathcal{H}$, while $\mathcal{G}$ has non-negative spectrum (this holds in particular when $\mathcal{V}_R^- = \emptyset$). In this case

$$(3.9) \quad 0 = \lambda_1(\mathcal{H}) \leq \lambda_1(\mathcal{G}) \leq \lambda_k(\mathcal{G})$$

and Theorem 3.10(1) with $r = 1$ shows that attaching the pendant lowers *all* eigenvalues of $\mathcal{G}$:

$$(3.10) \quad \lambda_k(\tilde{\mathcal{G}}) \leq \lambda_k(\mathcal{G}) \quad \text{for all } k \geq 1.$$

The inequality (3.10) was noted in [KMN13, Theorem 2] (for $\mathcal{V} = \mathcal{V}_N$ and $k = 1$) and [Roh17, Proposition 3.1] (for $\mathcal{V} = \mathcal{V}_N$ and general k), and generalised in [RoSe18, Theorem 3.5] to more general self-adjoint vertex conditions.

Several useful inequalities now follow.

Corollary 3.12. (1) (Lengthening an edge) Let $\tilde{\mathcal{G}}$ be obtained from $\mathcal{G}$ by lengthening the edge e . If $\lambda_k(\mathcal{G}) \geq 0$, then

$$(3.11) \quad \lambda_k(\tilde{\mathcal{G}}) \leq \lambda_k(\mathcal{G}) \quad \text{for all } k \geq 1.$$

The inequality is strict if $\lambda_k(\mathcal{G}) > \max(0, \lambda_{k-1}(\mathcal{G}))$ and there is an eigenfunction corresponding to $\lambda_k(\mathcal{G})$ which does not vanish identically on e .

- (2) (Adding an edge between existing vertices) Suppose there exist $v, w \in \mathcal{V}(\mathcal{G})$ and a choice of $n \geq 1$ first eigenfunctions $\psi_1, \dots, \psi_n$ such that

$$(3.12) \quad \psi_k(v) = \psi_k(w)$$

for all $k = 1, \dots, n$. If $\lambda_k(\mathcal{G}) \geq 0$, then the graph $\tilde{\mathcal{G}}$ formed by inserting an edge of arbitrary length between v and w satisfies

$$\lambda_k(\tilde{\mathcal{G}}) \leq \lambda_k(\mathcal{G}), \quad k = 1, \dots, n.$$

- (3) (Adding a long edge between existing vertices) Suppose $\tilde{\mathcal{G}}$ is formed by adding an edge of length ℓ connecting existing vertices v and w of $\mathcal{G}$. Then $(\pi/\ell)^2 \leq \lambda_{k_0}(\mathcal{G})$ implies $\lambda_k(\tilde{\mathcal{G}}) \leq \lambda_k(\mathcal{G})$ for all $k \geq k_0$.

- (4) (*Shrinking a redundant edge*) Let all vertices of the connected graph $\mathcal{G}$ have natural conditions. Suppose there exist an eigenfunction ψ associated with $\lambda_2^N(\mathcal{G})$ and an edge $e \in \mathcal{E}(\mathcal{G})$ such that $\psi|_e \equiv 0$. Then the graph $\tilde{\mathcal{G}}$ formed by shrinking e to a point (i.e., removing e and gluing its incident vertices together) satisfies $\lambda_2^N(\tilde{\mathcal{G}}) = \lambda_2^N(\mathcal{G})$, and $\psi|_{\mathcal{G} \setminus e}$ is an eigenfunction associated with $\lambda_2^N(\tilde{\mathcal{G}})$ (up to the canonical identification described in Remark 3.3).

Remark 3.13. Part (1) of Corollary 3.12 in general does not hold for negative eigenvalues, see for example [EJ12]. Note that in a graph with all natural conditions equation (3.12) is trivially always satisfied with $n = 1$, since the eigenfunction is constant; in particular, for $n = 2$, it suffices to check (3.12) for ψ_2 only. Part (2) for $\lambda_2^N(\mathcal{G})$ probably appeared for the first time in [KMN13, Theorem 3], where it was also observed that inserting an edge between two existing vertices may not decrease λ_2^N , unless the edge is sufficiently long. Part (3) gives a quantification of how long this has to be, and also reconciles this observation with the Weyl asymptotics, which requires that all *sufficiently high* eigenvalues must decrease upon the insertion of an additional edge.

Remark 3.14 (Hadamard formula). If an eigenvalue $\lambda = \lambda(\mathcal{G})$ is simple, there is a quantitative expression for its increase if an edge is lengthened, which we refer to as a *Hadamard formula*, by way of analogy with similar formulae on domains:

$$(3.13) \quad \frac{d\lambda}{d|e|} = -\mathcal{E}_e := -(\psi'(x)^2 + \lambda\psi(x)^2), \quad x \in e,$$

where $|e|$ is the length of the edge e , ψ is the λ -eigenfunction, normalised to have $L^2(\mathcal{G})$ -norm one, and the expression $\mathcal{E}_e$, often called the *Prüfer amplitude* of the edge e , can be easily shown to be independent of the location $x \in e$. This has the following immediate consequence, which we will use repeatedly and thus state explicitly: if we lengthen one edge e_1 and shorten another e_2 by the same amount, then the derivative of a simple eigenvalue λ with respect to this operation at the identity exists and has the same sign as

$$(3.14) \quad -\mathcal{E}_{e_1} + \mathcal{E}_{e_2},$$

with $\mathcal{E}_{e_1}, \mathcal{E}_{e_2}$ as in (3.13).

The formula (3.13) is proved in [Fri05b, Proof of the Lemma], [CdV15, Appendix A] and [BaLe17, Lemma 5.2] for natural vertex conditions only, although the same proof also works if $\mathcal{G}$ has some Dirichlet conditions (see also [BK13, Section 3.1.4] for the case of a degree one vertex with Dirichlet conditions), and we will use it without further comment for these conditions. Actually, the same formula turns out to hold in greater generality, for *general* self-adjoint vertex conditions (including δ -type). We intend to return to this point in a later work.

3.3. Operations transferring the volume. We now list some useful new principles based on changing the geometry of the graph by moving edges around, while keeping the total length constant.

Definition 3.15 (Transplantation). Cut through some of the vertices of $\mathcal{G}$ (in the sense of Definition 3.2) to produce two disjoint metric graphs $\mathcal{R}, \mathcal{C}$ (neither of which is required to be connected). Assume that only natural conditions equip the vertices of $\mathcal{C}$. Take

any connected metric graphs $\mathcal{H}_1, \dots, \mathcal{H}_k$ with purely natural vertex conditions and such that $|\mathcal{H}_1| + \dots + |\mathcal{H}_k| = |\mathcal{C}|$. If $\tilde{\mathcal{G}}$ is formed by inserting each $\mathcal{H}_i$ into $\mathcal{R}$ at some vertices $v_1, \dots, v_k \in \mathcal{V} \setminus \mathcal{V}_D$ in accordance with Definition 3.8, then we say that $\tilde{\mathcal{G}}$ is obtained by **transplanting the subgraph $\mathcal{C}$ to the subgraphs $\mathcal{H}_1, \dots, \mathcal{H}_k$ at $v_1, \dots, v_k$** (for short, transplantation of $\mathcal{C}$ to $v_1, \dots, v_k$).

An important special case consists of transplanting one or more edges to a vertex v by either inserting a new pendant edge at v , as depicted in Figure 3.3, or lengthening existing edges incident to v .

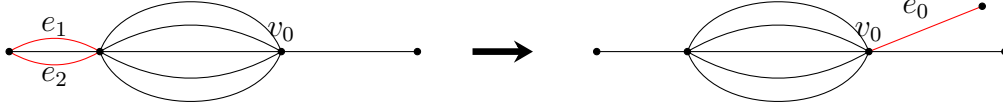


FIGURE 3.4. An example of transplantation: the graph on the right is obtained by transplantation of $\{e_1, e_2\}$ to $\mathcal{H} := \{e_0\}$ at v_0 .

Definition 3.16 (Unfolding edges). Suppose $\mathcal{G}$ has $k \geq 2$ selected parallel edges $e_1, e_2, \dots, e_k$ between the vertices v_1 and v_2 .² Then the operation of replacing the parallel edges by a single edge of length $|e_1| + |e_2| + \dots + |e_k|$ running between v_1 and v_2 (preserving the δ -condition strengths at v_1 and v_2) is called **unfolding parallel edges**.

Let $e_1, e_2, \dots, e_k$ be pendant edges which are attached to the same vertex v and have natural conditions at their endpoints. The operation of replacing $e_1, e_2, \dots, e_k$ by a single pendant edge at v of length $|e_1| + |e_2| + \dots + |e_k|$ (and again preserving the δ -condition strength at v) is called **unfolding pendant edges**.

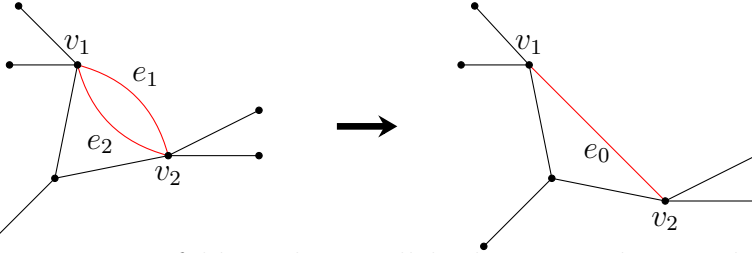


FIGURE 3.5. Unfolding the parallel edges e_1 and e_2 in the graph on the left to obtain the one on the right, that is, replacing the loop e_1, e_2 with a single edge e_0 of equal total length.

A variation on the unfolding of parallel edges is symmetrising them, possibly while reducing their number.

Definition 3.17. Suppose $\mathcal{G}$ has $k \geq 2$ selected parallel edges $e_1, \dots, e_k$ of arbitrary lengths $|e_1|, \dots, |e_k| > 0$ between the vertices v_1 and v_2 .³ We say $\tilde{\mathcal{G}}$ is formed from $\mathcal{G}$

²We explicitly allow the existence of further edges between v_1 and v_2 , as well as the possibility that $v_1 = v_2$.

³Here, as well, we emphasise that we allow both the existence of further parallel edges between v_1 and v_2 , and also $v_1 = v_2$.

by **symmetrisation** of $e_1, \dots, e_k$ if these parallel edges are replaced by $m \leq k$ parallel edges, each of length $(|e_1| + \dots + |e_k|)/m$, and the δ -condition strengths at v_1 and v_2 are preserved.

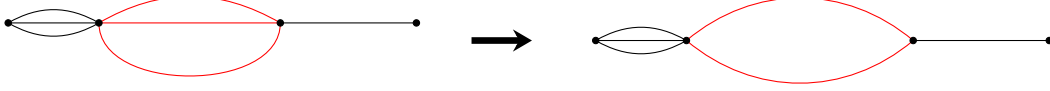


FIGURE 3.6. An example of symmetrisation: The graph on the right is obtained by replacing the 3-pumpkin in the pumpkin chain on the left by an equilateral 2-pumpkin.

The following theorem applies to the first non-trivial eigenvalue $\mu(\mathcal{G})$ of a connected graph $\mathcal{G}$ (see Definition 2.2): in particular, we make no assumptions on the vertex conditions except where explicitly stated.

Theorem 3.18. *The following operations decrease the first non-trivial eigenvalue of a connected graph $\mathcal{G}$, i.e.*

$$(3.15) \quad \mu(\tilde{\mathcal{G}}) \leq \mu(\mathcal{G}).$$

- (1) (Transplantation) Suppose for the subgraph $\mathcal{C} \subset \mathcal{G}$ and $v_1, \dots, v_k \in \mathcal{V}_N(\mathcal{G})$ that there exists a $\mu(\mathcal{G})$ -eigenfunction ψ such that

$$(3.16) \quad 0 \leq \min_{x \in \mathcal{C}} \psi(x) \leq \max_{x \in \mathcal{C}} \psi(x) \leq \min_{i=1, \dots, k} \psi(v_i).$$

If $\mathcal{C}$, together with further graphs $\mathcal{H}_1, \dots, \mathcal{H}_k$, satisfies the requirements of Definition 3.15, then (3.15) holds for the graph $\tilde{\mathcal{G}}$ obtained by transplanting $\mathcal{C}$ to $\mathcal{H}_1, \dots, \mathcal{H}_k$ at $v_1, \dots, v_k$ in accordance with Definition 3.15. The inequality (3.15) is strict if $\min_{x \in \mathcal{C}} \psi(x) < \max_{i=1, \dots, k} \psi(v_i)$.

- (2) (Unfolding parallel edges) Inequality (3.15) holds for the graph $\tilde{\mathcal{G}}$ obtained by unfolding edges $e_1, e_2, \dots, e_k$, $k \geq 2$, in accordance with Definition 3.16. Equality in (3.15) implies that either every eigenfunction of $\mu(\mathcal{G})$ is constant on $e_1 \cup \dots \cup e_k$, or $\mathcal{G}$ is a figure-8 with natural conditions and $\tilde{\mathcal{G}}$ is a loop.⁴
- (3) (Symmetrising parallel edges) Inequality (3.15) holds whenever k edges $e_1, \dots, e_k$ are symmetrised to m parallel edges, $1 \leq m \leq k$, if there is an eigenfunction ψ of $\mu(\mathcal{G})$ which is monotonically increasing along each of the edges $e_1, \dots, e_k$ from v_1 to v_2 . In this case, equality in (3.15) implies that either ψ is constant on $e_1 \cup \dots \cup e_k$, or else $\tilde{\mathcal{G}} = \mathcal{G}$. The same conclusions hold if this principle is applied to several pumpkin subgraphs within $\mathcal{G}$ simultaneously, provided that the conditions are satisfied separately on each pumpkin.
- (4) (Unfolding pendant edges) Inequality (3.15) holds for the graph $\tilde{\mathcal{G}}$ obtained by unfolding pendant edges. Equality in (3.15) implies that either every eigenfunction of μ is constant on the pendant edges, or $\tilde{\mathcal{G}} = \mathcal{G}$.

⁴Since a figure-8 consists of two parallel edges running from the central vertex to itself, unfolding these edges, that is, replacing them with a single edge of the same length, produces a loop. By Corollary 3.6, λ_2^N is unaltered by this operation.

Remark 3.19. Part (3) has already appeared in the literature in a less general form, most recently in [BK KM17]. The other statements are, to the best of our knowledge, completely new. It would be interesting to have a complete characterisation of equality throughout, although this may be difficult as in some cases it seems to depend on the global geometry of $\mathcal{G}$, cf. Example 3.25.

Remark 3.20. The condition for strict inequality in Theorem 3.18(1) is satisfied, for example, if $\psi(x)$ has non-zero variation on $\mathcal{C}$, i.e. if $\min_{x \in \mathcal{C}} \psi(x) < \max_{x \in \mathcal{C}} \psi(x)$. This condition can only fail if ψ is identically 0 on $\mathcal{C}$ or if $\mu = 0$ and ψ is constant on $\mathcal{C}$.

The conclusion of Theorem 3.18(1) also holds, with only trivial modifications of the same proof, if all vertex conditions are natural (so that $\mu = \lambda_2^N$ and ψ changes sign), and we have two subgraphs $\mathcal{C}_1 \subset \{\psi \geq 0\}$ and $\mathcal{C}_2 \subset \{\psi \leq 0\}$, such that $\mathcal{C}_1$ is transplanted to vertices $v_1, \dots, v_{k_1}$ with

$$0 \leq \min_{x \in \mathcal{C}_1} \psi(x) \leq \max_{x \in \mathcal{C}_1} \psi(x) \leq \min_{i=1, \dots, k_1} \psi(v_i)$$

and $\mathcal{C}_2$ is transplanted to vertices $w_1, \dots, w_{k_2}$ such that

$$0 \geq \max_{x \in \mathcal{C}_2} \psi(x) \geq \min_{x \in \mathcal{C}_2} \psi(x) \geq \max_{i=1, \dots, k_2} \psi(w_i).$$

We will not use this, so we do not go into details.

3.4. Effects of surgery: some examples and counterexamples. We now give some basic examples illustrating why some of the assumptions in the above theorems are necessary, and why more can be expected in some cases.

Example 3.21. We start with an example to show that Theorem 3.18(2) (unfolding parallel edges) does not have to apply to the higher eigenvalues. Take $\mathcal{G}$ to be an equal 3-pumpkin each of whose edges e_1, e_2, e_3 has length 1. Then the first few eigenvalues of $\mathcal{G}$ with natural boundary conditions are $0, \pi^2, \pi^2, \pi^2$. If we unfold e_1 and e_2 , we produce a loop $\tilde{\mathcal{G}}$ of length 3, whose first eigenvalues are $0, \frac{4\pi^2}{9}, \frac{4\pi^2}{9}, \frac{16\pi^2}{9}$. In particular, $\lambda_4^N(\tilde{\mathcal{G}}) > \lambda_4^N(\mathcal{G})$.

Example 3.22. Theorem 3.18(4) (unfolding *pendant* edges) also does not apply to the higher eigenvalues. Indeed, by a theorem of Friedlander [Fri05a, Theorem 1], for $k \geq 3$, the unique minimiser of λ_k^N among all graphs of total length $L > 0$ is the equilateral k -star (cf. Definition 2.3). In particular, unfolding any two of its k pendant edges strictly increases λ_k^N .

Example 3.23. On the other hand, Theorem 3.18(2) holds for all eigenvalues if one instead unfolds an *odd* number of edges, i.e., replaces $2k + 1$ parallel edges between two vertices $v_1, v_2 \in \mathcal{V}_N$ by a single edge of the same total length. This is a simple application of Theorem 3.4(1). We do not go into details, but refer to Figure 3.7 to illustrate the principle.

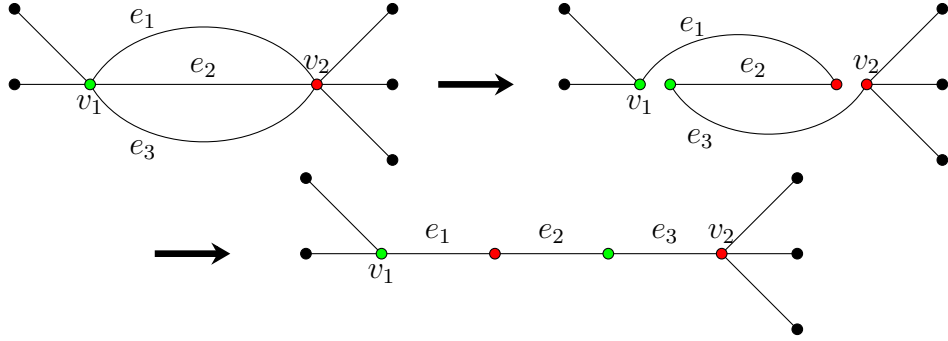


FIGURE 3.7. The three edges between v_1 and v_2 can be unfolded by cutting through the vertices in the right way.

Likewise, Theorem 3.18(2) holds for all $k \geq 1$ if the two edges e_1 and e_2 to be unfolded form a pendant (i.e., together they form a pendant loop); this also follows directly from Theorem 3.4(1). Thus the general question of whether unfolding parallel edges decreases the higher eigenvalues seems to be subtle.

Example 3.24. We now show that the assumption of monotonicity of the eigenfunction in Theorem 3.18(3) (symmetrising parallel edges) cannot be dropped. Choose $\varepsilon > 0$ small and let $\mathcal{G}$ consist of two vertices v_1 and v_2 , joined by edges $e_1, \dots, e_4$ of length $2 - 2\varepsilon$, ε , ε and 1, respectively ($\mathcal{G}$ is thus a 4-pumpkin, of total length 3). Then, for $\varepsilon > 0$ small enough, $\lambda_2^N(\mathcal{G})$ is approximately equal to $4\pi^2/9$, the second eigenvalue of a figure-8 (and of a cycle) of total length 3. If we apply Theorem 3.18(3) to e_1, e_2, e_3 with $m = 2$, we obtain a new graph $\tilde{\mathcal{G}}$ consisting of two vertices and three parallel edges running between them, each of length one (an equilateral 3-pumpkin), so that $\lambda_2^N(\tilde{\mathcal{G}}) = \pi^2$, the second eigenvalue of a cycle of length 2. Obviously, the configuration of $\mathcal{G}$ forces the eigenfunction ψ to satisfy $\psi(v_1) \approx \psi(v_2)$; since e_1 has more than half the total length of $\mathcal{G}$, it is impossible for ψ to be monotonic on it.

Example 3.25. Finally, we give an example to show that inequality in Theorem 3.18(4) can be strict even if $\mu(\mathcal{G})$ is simple and its eigenfunction ψ vanishes identically on the pendants. Take $\mathcal{G}$ to be a 4-star with edge lengths 1, 1, $1 - \varepsilon$ and $1 - \varepsilon$ for some $\varepsilon > 0$ small, connected at a central vertex v_0 . Then $\lambda_2^N(\mathcal{G}) = \pi^2/4$ with eigenfunction ψ supported on the two longer edges and vanishing identically on the shorter ones, with $\psi(v_0) = 0$. Now unfold the shorter edges, so that $\tilde{\mathcal{G}}$ is a 3-star with edge lengths 1, 1 and $2 - 2\varepsilon > 1$. Now since $\tilde{\mathcal{G}}$ can be formed by attaching one of its edges of length 1 as a pendant to a path graph (interval) of length $3 - 2\varepsilon$, whose first non-trivial eigenvalue is $\pi^2/(3 - 2\varepsilon)^2 < \pi^2/4$, Theorem 3.10(1) implies $\lambda_2^N(\tilde{\mathcal{G}}) < \pi^2/4$.

4. PROOFS

4.1. Proof of Theorem 3.4 and its corollaries. For the proof of Theorem 3.4, we will need a sharp form of the Courant–Fischer minimax characterisation of the eigenvalues. If H is a Hilbert space with inner product $(\cdot, \cdot)_H$, $a : D(a) \times D(a) \rightarrow \mathbb{R}$ a closed, symmetric,

sesquilinear form bounded from below and defined on a dense and compactly embedded subspace $D(a) \subset H$, then we have a sequence of associated eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots$, with corresponding eigenvectors $v_1, v_2, \dots$, which satisfy $a(u, v_k) = \lambda_k(u, v_k)_H$ for all $u \in D(a)$ and which can be chosen to form an orthonormal basis of H ; and the eigenvalues can be characterised variationally as

$$(4.1) \quad \lambda_n = \min_{\substack{X \subset D(a) \\ \dim(X)=n}} \max_{0 \neq u \in X} \frac{a(u, u)}{\|u\|_H^2}$$

$$(4.2) \quad = \max_{\substack{Y \subset D(a) \\ \dim(Y)=n-1}} \min_{0 \neq u \in Y^\perp} \frac{a(u, u)}{\|u\|_H^2}.$$

These formulae are well known, cf. [CH53, Chapter 6], [Kat76, Section I.6.10], [ReSi78, Theorem XIII.2]. However, we need additionally the following characterisation of equality, Lemma 4.1. While this is surely not new, it does not seem to be in any of the standard references, including the ones just cited; similar but not identical results are contained in [WS72, Theorem 2.4.3]. For completeness we include a proof.

Lemma 4.1. *With the notation just introduced,*

- (1) *if X realises the minimum in (4.1) (corresp. if Y achieves the maximum in (4.2)), then X (corresp. Y) contains an eigenvector of λ_n ;*
- (2) *if $\lambda_n < \lambda_{n+1}$, then the intersection of all possible minimising n -dimensional subspaces X in (4.1) is the eigenspace of λ_n .*

Proof. (1) We prove the statement only for X . A simple dimension count yields that if $\dim(X) = n$, then we can find a vector $u \in X$ of norm 1 which is orthogonal to the first $n - 1$ eigenvectors $v_1, \dots, v_{n-1}$. We expand u in the eigenvector basis,

$$(4.3) \quad u = \sum_{k=n}^{\infty} \alpha_k v_k,$$

for some coefficients $\alpha_k = (u, v_k)_H \in \mathbb{R}$; the normalisation condition reads $\sum_{k=n}^{\infty} \alpha_k^2 = 1$. Moreover,

$$(4.4) \quad a(u, u) = \sum_{k=n}^{\infty} \alpha_k^2 \lambda_k \leq \lambda_n,$$

the inequality following from the minimality of X . This is only possible if $\alpha_k = 0$ whenever $\lambda_k > \lambda_n$. Thus u is a linear combination of eigenvectors having eigenvalue equal to λ_n .

(2) We first show that if X is an arbitrary minimising subspace, then every eigenvector of λ_n is in X ; after a re-numbering of all eigenvalues equal to λ_n if necessary, it suffices to prove $v_n \in X$. As in (1), we obtain a vector $u \in X$ of norm one having the form (4.3), such that (4.4) also holds. But since $\lambda_n < \lambda_{n+1}$, we must have $\alpha_k = 0$ for all $k \geq n + 1$. That is, $u = v_n$.

Finally, let $\lambda_{n-m} < \lambda_{n-m+1} = \dots = \lambda_n < \lambda_{n+1}$, i.e., suppose λ_n has multiplicity m . We need to show that a vector u that belongs to every minimising X must in fact be a

superposition of $v_{n-m+1}, \dots, v_n$. We expand u ,

$$(4.5) \quad u = \sum_{k=1}^{\infty} \alpha_k v_k.$$

Taking $X := \text{span}\{v_1, \dots, v_n\}$, we conclude that $\alpha_k = 0$ for all $k > n$. Let now $n > m$ and suppose without loss of generality that $\alpha_1 \neq 0$. Then we take

$$X := \text{span}\{v_1 + \varepsilon v_{n+1}, v_2, v_3, \dots, v_n\}.$$

For small enough ε this subspace is minimising for (4.1) and cannot contain u since the expansion of the latter contains v_1 but does not contain v_{n+1} . Similarly, $\alpha_k = 0$ for all $k \leq n - m$, which completes the proof. $\square$

Theorem 3.4 is true as a special case of a more general theorem about rank-1 perturbations of quadratic forms. Since the vertex conditions can enter into a form in two different ways, we need to consider perturbations of two different types.

Definition 4.2. Let X be a normed space and Z a closed subspace. We say that Z is a *co-dimension 1 subspace* of X and write $Z \subset_1 X$, if the quotient space X/Z is 1-dimensional.

Let a and $\tilde{a}$ be closed semi-bounded Hermitian forms with domains $D(a)$ and $D(\tilde{a})$. Then we say that $\tilde{a}$ is a *positive rank-1 perturbation* of the form a if $\tilde{a} = a$ on some $Z \subset_1 D(a)$ and

- either $Z = D(\tilde{a}) \subset_1 D(a)$,
- or $D(\tilde{a}) = D(a)$ and $\tilde{a} \geq a$.

We shall call the former case perturbation of type (R) (for “restriction”) and the latter perturbation of type (V) (for “variation”).

This definition is, in particular, applicable to the form given by (2.2) for any of the vertex conditions we are considering; moreover, it is easy to see that the operations in Theorem 3.4 satisfy the definition of rank-1 perturbation of the forms, where as always we make the identification of Remark 3.3. Indeed, gluing two vertices introduces a single constraint and is therefore of type (R); changing the strength of the δ -condition corresponds to variation of the form except in the case when $\gamma' = \infty$, in which case the perturbation is again of type (R).

Theorem 4.3 (Interlacing with equality characterisation). *Let $\tilde{a}$ be a positive rank-1 perturbation of the form a . Then the eigenvalues of the two forms satisfy the interlacing inequalities*

$$(4.6) \quad \lambda_k(a) \leq \lambda_k(\tilde{a}) \leq \lambda_{k+1}(a) \leq \lambda_{k+1}(\tilde{a}), \quad k \geq 1.$$

If a given value Λ has multiplicities m and $\tilde{m}$ in the spectra of a and $\tilde{a}$, respectively, then $|m - \tilde{m}| \leq 1$ and the intersection of the respective Λ -eigenspaces of the two forms has dimension $\min(m, \tilde{m})$.

We remark that, in particular, the common eigenfunction(s) must belong to the domain $D(\tilde{a})$ which may be smaller than $D(a)$.

Proof of Theorem 4.3 and hence of Theorem 3.4. Introduce, for convenience, the notation

$$(4.7) \quad M(a, X) := \max_{0 \neq u \in X} \frac{a(u, u)}{\|u\|^2}.$$

The inequalities (4.6) are standard; the inequality $\lambda_k(a) \leq \lambda_k(\tilde{a})$ follows from minimising over a smaller set $D(\tilde{a})$ in (4.1) in the type (R) case or from inequality $a \leq \tilde{a}$ in the type (V) case. The inequality $\lambda_k(\tilde{a}) \leq \lambda_{k+1}(a)$ follows from the rank of the perturbation. Indeed, let X be any $(k+1)$ -dimensional subspace of $D(a)$ such that $\lambda_{k+1}(a) = M(a, X)$. Then the subspace $X \cap Z$ (with Z as in Definition 4.2) has dimension at least k . Choosing an arbitrary subset $\tilde{X}$ of $X \cap Z$ of dimension k , we have $\lambda_k(\tilde{a}) \leq M(\tilde{a}, \tilde{X}) = M(a, \tilde{X}) \leq M(a, X) = \lambda_{k+1}(a)$.

The case of equality is more interesting. The fact that the multiplicities m and $\tilde{m}$ differ by at most 1 is a simple consequence of the interlacing. Let $\Lambda \in \mathbb{R}$ be an eigenvalue of a and denote by $E(a, \Lambda) \subset H$ the corresponding eigenspace. With $r := m - 1$ and an appropriate k , we can write

$$(4.8) \quad \lambda_{k-1}(a) < \lambda_k(a) = \dots = \Lambda = \dots = \lambda_{k+r}(a) < \lambda_{k+r+1}(a).$$

We have to consider four cases of arrangements of eigenvalues of $\tilde{a}$ among (4.8). We will deal with these arrangements two at a time.

Suppose the equalities line up in one of the following two ways:

$$(4.9) \quad \dots < \lambda_k(a) = \lambda_k(\tilde{a}) = \dots = \lambda_{k+r}(a) = \lambda_{k+r}(\tilde{a}) < \dots$$

$$(4.10) \quad \dots < \lambda_{k-1}(\tilde{a}) = \lambda_k(a) = \lambda_k(\tilde{a}) = \dots = \lambda_{k+r}(a) = \lambda_{k+r}(\tilde{a}) < \dots$$

In these two cases $m \leq \tilde{m}$ and we will show that $E(a, \Lambda) \subset E(\tilde{a}, \Lambda)$; the claim then follows from a dimension count argument. Let $\tilde{X}$ be any subspace of dimension $k+r$ such that $M(\tilde{a}, \tilde{X}) = \Lambda$. Since $\tilde{X} \subset D(a)$, we can also consider a on $\tilde{X}$. We have

$$(4.11) \quad \Lambda = \lambda_{k+r}(a) \leq M(a, \tilde{X}) \leq M(\tilde{a}, \tilde{X}) = \Lambda,$$

and therefore $M(a, \tilde{X}) = \Lambda$. Thus $\tilde{X}$ is a minimising subspace for $\lambda_{k+r}(a)$ and by Lemma 4.1(2) we get $E(a, \Lambda) \subset \tilde{X}$. Since the intersection of all such $\tilde{X}$ coincides with $E(\tilde{a}, \Lambda)$, we are done.

Now suppose the equalities line up in one of the following ways:

$$(4.12) \quad \dots < \lambda_{k-1}(\tilde{a}) = \lambda_k(a) = \lambda_k(\tilde{a}) = \dots = \lambda_{k+r-1}(\tilde{a}) = \lambda_{k+r}(a) < \dots$$

$$(4.13) \quad \dots < \lambda_k(a) = \lambda_k(\tilde{a}) = \dots = \lambda_{k+r-1}(\tilde{a}) = \lambda_{k+r}(a) < \dots$$

Here we need to show that $E(\tilde{a}, \Lambda) \subset E(a, \Lambda)$. Consider any minimising subspace X of dimension $k+r$ for a , i.e. $M(a, X) = \Lambda$. Let $\tilde{X} \subset Z \cap X$, where Z is the subspace from Definition 4.2 of co-dimension 1 and therefore $\tilde{X}$ can be chosen to have dimension $k+r-1$. On $\tilde{X}$ the two forms agree, therefore we have

$$(4.14) \quad \Lambda = \lambda_{k+r-1}(\tilde{a}) \leq M(\tilde{a}, \tilde{X}) = M(a, \tilde{X}) \leq M(a, X) = \Lambda.$$

Thus $M(\tilde{a}, \tilde{X}) = \Lambda$, $\tilde{X}$ is a minimising subspace for $\lambda_{k+r-1}(\tilde{a})$ and by Lemma 4.1(2) we conclude $E(\tilde{a}, \Lambda) \subset \tilde{X} \subset X$. As above, we now take the intersection over all possible X and use Lemma 4.1(2) again to conclude $E(\tilde{a}, \Lambda) \subset E(a, \Lambda)$. $\square$

To continue in the spirit of this section, we will establish Corollary 3.6 by proving a slightly more general claim. For the rest of this subsection, a continues to be any closed semi-bounded Hermitian form, with domain $D(a)$ in a normed space X .

Definition 4.4. A vector $v \in D(a)$ is *invariant with respect to perturbation of a to $\tilde{a}$* if $v \in D(\tilde{a})$ and $a(v, v) = \tilde{a}(v, v)$.

Note that if $\tilde{a}$ is a rank-1 perturbation of a of type (R), the condition $a(v, v) = \tilde{a}(v, v)$ is satisfied automatically and if it is a rank-1 perturbation of type (V), the condition $v \in D(\tilde{a})$ is satisfied automatically.

Lemma 4.5. *Let $\tilde{a}$ be a positive rank-1 perturbation of the form a . If the first k eigenvectors of a are invariant with respect to the perturbation, then*

$$(4.15) \quad \lambda_i(a) = \lambda_i(\tilde{a}), \quad 1 \leq i \leq k.$$

Proof of Lemma 4.5 and hence of Corollary 3.6. By positivity of the perturbation (see Theorem 4.3),

$$\lambda_i(a) \leq \lambda_i(\tilde{a})$$

for $i = 1, \dots, k$. Denote the eigenvectors of a by $v_1, \dots, v_k$ and form $X_i = \text{span}\{v_1, \dots, v_i\}$ with $i \leq k$. By assumption, X_i is contained in the form domain of $\tilde{a}$ and

$$\lambda_i(\tilde{a}) \leq \max_{0 \neq u \in X_i} \frac{\tilde{a}(u, u)}{\|u\|_H^2} = \max_{0 \neq u \in X_i} \frac{a(u, u)}{\|u\|_H^2} = \lambda_i(a).$$

Thus we have equality of the eigenvalues. $\square$

Remark 4.6. The rank of the perturbation plays no role in the proof, but we have not defined general positive perturbations of the form a and will not require them.

4.2. Proof of Theorem 3.10 and its corollaries.

Proof of Theorem 3.10. (1) Suppose that

$$(4.16) \quad \lambda_r(\mathcal{H}) \leq \lambda_k(\mathcal{G}).$$

The spectrum of the union of the two graphs (considered as a single, disconnected graph $\mathcal{G} \dot{\cup} \mathcal{H}$) is the union of the two spectra. Therefore,

$$(4.17) \quad \lambda_k(\mathcal{G}) = \lambda_n(\mathcal{G} \dot{\cup} \mathcal{H}),$$

for some $n = n(k, r) \geq k + r$. Now attaching $\mathcal{H}$ to $\mathcal{G}$ to form $\tilde{\mathcal{G}}$ is an operation covered by Theorem 3.4(1) (gluing vertices) and we get

$$(4.18) \quad \lambda_{n-1}(\tilde{\mathcal{G}}) \leq \lambda_n(\mathcal{G} \dot{\cup} \mathcal{H}).$$

Combining this with the estimate $n \geq k + r$, we thus obtain

$$(4.19) \quad \lambda_{k+r-1}(\tilde{\mathcal{G}}) \leq \lambda_k(\mathcal{G}).$$

To obtain strict inequality under the additional conditions stipulated in the theorem, let $\lambda_n(\mathcal{G} \dot{\cup} \mathcal{H})$ be the first occurrence of $\Lambda := \lambda_k(\mathcal{G})$ in the spectrum of $\mathcal{G} \dot{\cup} \mathcal{H}$, namely $\lambda_{n-1}(\mathcal{G} \dot{\cup} \mathcal{H}) < \lambda_n(\mathcal{G} \dot{\cup} \mathcal{H}) = \Lambda$. From $\Lambda > \lambda_{k-1}(\mathcal{G})$ and $\Lambda > \lambda_r(\mathcal{H})$ we still have $n \geq k + r$. Assume we do not have strict inequality in (4.18), i.e.

$$\lambda_{n-1}(\mathcal{G} \dot{\cup} \mathcal{H}) < \lambda_{n-1}(\tilde{\mathcal{G}}) = \lambda_n(\mathcal{G} \dot{\cup} \mathcal{H}),$$

which fits the circumstances described in Remark 3.5. But the eigenspace of $\lambda_n(\mathcal{G} \dot{\cup} \mathcal{H})$ contains a function which vanishes identically on $\mathcal{H}$ and is non-zero at v_0 ; this function cannot be contained in the eigenspace of $\tilde{\mathcal{G}}$ because it does not belong to its form domain. Therefore $\lambda_{n-1}(\tilde{\mathcal{G}}) < \lambda_n(\mathcal{G} \dot{\cup} \mathcal{H}) = \Lambda$ and thus $\lambda_{k+r-1}(\tilde{\mathcal{G}}) \leq \lambda_{n-1}(\tilde{\mathcal{G}}) < \Lambda$.

(2) Denote by $\hat{\mathcal{H}}$ the graph obtained from $\mathcal{H}$ by gluing the vertices $w_1, \dots, w_m$ to form a single vertex w^* . Attach $\hat{\mathcal{H}}$ to $\mathcal{G}$ by gluing w^* and v_0 , keeping the δ -condition γ_0 at v_0 . Call the new graph $\hat{\mathcal{G}}$. Since all vertex conditions on $\hat{\mathcal{H}}$ are natural, $\lambda_1(\hat{\mathcal{H}}) = 0$ and we can use part (1) of the theorem with $r = 1$, as long as $\lambda_k(\mathcal{G}) \geq 0$. We conclude that $\lambda_k(\hat{\mathcal{G}}) \leq \lambda_k(\mathcal{G})$ and then cut through the vertex v_0 of $\hat{\mathcal{G}}$ to restore the vertices $w_1, \dots, w_m$ and thus create the graph $\tilde{\mathcal{G}}$. Since the δ -conditions at $w_1, \dots, w_m$ sum to γ_0 , by Theorem 3.4(1) we have $\lambda_k(\tilde{\mathcal{G}}) \leq \lambda_k(\hat{\mathcal{G}})$ for all $k \geq 1$. To obtain strict inequality we use the strict version of part (1). $\square$

Proof of Corollary 3.12. (1) follows directly from Theorem 3.10(2) by picking an arbitrary internal point of e , making this a dummy vertex, and inserting a one-edge graph $\mathcal{H}$ at this point in accordance with Definition 3.8.

(2) Form a new graph $\hat{\mathcal{G}}$ from $\mathcal{G}$ by gluing the vertices v and w (remembering to add the δ -potentials if present); call the new vertex v_0 . Then by Corollary 3.6,

$$\lambda_1(\hat{\mathcal{G}}) = \lambda_1(\mathcal{G}), \dots, \lambda_n(\hat{\mathcal{G}}) = \lambda_n(\mathcal{G}).$$

Now insert a one-edge graph $\mathcal{H}$ at v_0 separating the vertices v and w again.

(3) Denote by $\mathcal{H}$ the graph consisting of the long edge, with natural endpoints. Then the assumption on ℓ implies that $\lambda_2^N(\mathcal{H}) \leq \lambda_{k_0}(\mathcal{G})$. We now glue one endpoint of $\mathcal{H}$ and v to form a new graph $\hat{\mathcal{G}}$ and apply Theorem 3.10(1) with $r = 2$ to obtain

$$\lambda_{k+1}(\hat{\mathcal{G}}) \leq \lambda_k(\mathcal{G})$$

for all $k \geq k_0$. We now glue w and the other endpoint of $\mathcal{H}$ to form $\tilde{\mathcal{G}}$; then Theorem 3.4(1) implies

$$\lambda_k(\tilde{\mathcal{G}}) \leq \lambda_{k+1}(\hat{\mathcal{G}})$$

for all k , whence the claim.

(4) By Corollary 3.6, we may assume that the edge e on which ψ vanishes is pendant (possibly a loop) by gluing its incident vertices if necessary. Now $\tilde{\mathcal{G}}$ is formed from $\mathcal{G}$ by removing the pendant graph consisting of e ; hence $\lambda_2^N(\mathcal{G}) \leq \lambda_2^N(\tilde{\mathcal{G}})$ by Theorem 3.10(1). But it is easy to check directly that the restriction of ψ to $\tilde{\mathcal{G}}$ satisfies the eigenvalue equation and the vertex conditions. Therefore $\lambda_2^N(\mathcal{G}) > 0$ is still an eigenvalue of the graph $\tilde{\mathcal{G}}$ and it must be the second one since the first one is zero. $\square$

4.3. Proof of Theorem 3.18. The following lemma will be useful for the proof.

Lemma 4.7. *Let F and G be two real-valued functions defined on a probability space X . If F has zero mean and for almost all $x \in X$ either*

$$(4.20) \quad 0 \leq F(x) \leq G(x) \quad \text{or} \quad G(x) \leq F(x) \leq 0,$$

then

$$(4.21) \quad \text{var}(F) \leq \text{var}(G).$$

If $0 < |F(x)| < |G(x)|$ on a set of non-zero measure, then (4.21) is strict.

Proof. The conditions on F and G can be rewritten as $F(G - F) \geq 0$ a.e. We now estimate

$$\begin{aligned} \text{var}(G) &= \mathbb{E}(G - \mathbb{E}G)^2 = \mathbb{E}(F + (G - F - \mathbb{E}G))^2 \\ &= \mathbb{E}(F^2 + 2F(G - F) - 2F\mathbb{E}G + (G - F - \mathbb{E}G)^2) \\ &= \mathbb{E}F^2 + 2\mathbb{E}F(G - F) + \mathbb{E}(G - F - \mathbb{E}G)^2 \geq \mathbb{E}F^2 = \text{var}(F), \end{aligned}$$

where we used $\mathbb{E}F = 0$ to get to the last line. If $0 < |F| < |G|$ on a set of positive measure, then also $F(G - F) > 0$ on the same set, implying that the last inequality in the above calculation is strict. $\square$

Proof of Theorem 3.18. (1) Denote by ψ the eigenfunction on $\mathcal{G}$ satisfying the assumptions of the theorem. Construct a test function φ on $\tilde{\mathcal{G}}$ by setting

$$\varphi(x) = \begin{cases} \psi(x) & \text{if } x \in \mathcal{G} \setminus \{v_1, \dots, v_k\} = \tilde{\mathcal{G}} \setminus \bigcup_{i=1}^k \mathcal{H}_i, \\ \psi(v_i) & \text{if } x \in \mathcal{H}_i, i = 1, \dots, k. \end{cases}$$

Then $\varphi \in H^1(\tilde{\mathcal{G}})$ since it is continuous and piecewise- H^1 ; moreover, it is still zero at any Dirichlet vertices, with

$$(4.22) \quad \int_{\tilde{\mathcal{G}}} |\varphi|^2 dx \geq \int_{\mathcal{G}} |\psi|^2 dx, \quad \int_{\tilde{\mathcal{G}}} |\varphi'|^2 dx \leq \int_{\mathcal{G}} |\psi'|^2 dx.$$

In particular, if $\mu = \lambda_1$, we see φ is a valid test function with smaller Rayleigh quotient than ψ ; thus $\lambda_1(\tilde{\mathcal{G}}) \leq \lambda_1(\mathcal{G})$.

If $\mu = \lambda_2^N$, then φ will not in general be a valid test function since $\int_{\tilde{\mathcal{G}}} \varphi dx \neq 0$. Assume without loss of generality that $|\mathcal{G}| = |\tilde{\mathcal{G}}| = 1$ (otherwise just divide everything by the total length). Noting that $\int_{\mathcal{G}} \psi dx = 0$, we apply Lemma 4.7 to $F = \psi$, $G = \varphi$ and $X = L^2(\mathcal{G}) \simeq L^2(\tilde{\mathcal{G}})$ (here we identify $\mathcal{C}$ with $\bigcup_{i=1}^k \mathcal{H}_i$, and spaces of L^2 -functions on them, arbitrarily, noting that the sets have the same measure). This yields

$$(4.23) \quad \int_{\mathcal{G}} |\psi|^2 dx \leq \int_{\tilde{\mathcal{G}}} \left[\varphi - \int_{\tilde{\mathcal{G}}} \varphi dx \right]^2 dx.$$

In particular, if we set $\tilde{\varphi} := \varphi - \int_{\tilde{\mathcal{G}}} \varphi dx$, then since also $\tilde{\varphi}' = \varphi'$, we see that (4.22) holds with $\tilde{\varphi}$ in place of φ . Since $\tilde{\varphi}$ is now a valid test function, as above we conclude that $\lambda_2^N(\tilde{\mathcal{G}}) \leq \lambda_2^N(\mathcal{G})$.

If, in addition to (3.16) we have $\min_{x \in \mathcal{C}} \psi(x) < \max_{j=1, \dots, k} \psi(v_j)$, then either $\min_{x \in \mathcal{C}} \psi(x) < \min_i \psi(v_i)$ or $\max_{x \in \mathcal{C}} \psi(x) < \max_i \psi(v_i)$. From continuity of ψ we conclude that there is a set of non-zero measure on which $\psi(x) < \varphi(x)$ and we obtain strict inequality in (4.22) or in (4.23), correspondingly (in the latter case through the strict version of Lemma 4.7).

(2) Let ψ be an eigenfunction corresponding to $\mu(\mathcal{G})$. Denote $\psi(v_1) = a$, $\psi(v_2) = b$ and assume without loss of generality that $0 \leq |a| \leq b$. On the edge e_1 locate an interval $[x_1, x_2]$ such that ψ goes from a to b on this interval, while remaining in the range $[a, b]$.

To be more specific, one can take

$$\begin{aligned} x_1 &:= \max\{x \in [0, |e_1|] : \psi(x) \leq a\} \\ x_2 &:= \min\{x \in [x_1, |e_1|] : \psi(x) \geq b\}. \end{aligned}$$

Change the value of the function on the interval $[x_1, x_2]$ to b , thus increasing its variance, see Lemma 4.7, but creating a discontinuity at x_1 . Insert the edge e_2 at the point x_1 , restoring the continuity. This process creates a test function φ_1 with greater variance. We can repeat this process, further absorbing the edges $e_3, \dots, e_k$ and eventually obtaining the test function $\varphi := \varphi_{k-1}$ on $\tilde{\mathcal{G}}$. As in the proof of part (1), inequalities (4.22)-(4.23) are satisfied leading to non-increase of $\mu(\mathcal{G})$.

It is easy to see that the variance is strictly increased by this process as long as $a \neq b$. To prove strict inequality in the case when $\varphi(v_1) = \varphi(v_2)$ for every eigenfunction φ associated with $\mu(\mathcal{G})$, suppose that there is at least one eigenfunction, call it ψ , which is non-constant on $e_1 \cup \dots \cup e_k$. By working recursively, we may assume that $k = 2$, and in fact, by Corollary 3.6 we may also assume without loss of generality that $v_1 = v_2 =: v$, and e_1 and e_2 are loops at v .

Then $\tilde{\mathcal{G}}$ is formed from $\mathcal{G}$ by cutting through v to create a longer loop e_0 out of e_1 and e_2 , cf. Figure 4.1. By Theorem 3.4, the only way we can have $\mu(\tilde{\mathcal{G}}) = \mu(\mathcal{G})$ is if the image of ψ under this cut, which we will still call ψ , is also an eigenfunction for $\mu(\tilde{\mathcal{G}})$ on $\tilde{\mathcal{G}}$; in particular, at the point v' on e_0 which is the image of v under the cut, it takes the value $\psi(v') = \psi(v)$.

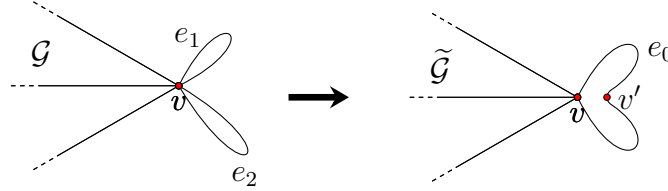


FIGURE 4.1. The graph $\mathcal{G}$ in which $v_1 = v_2 = v$ (left) and the graph $\tilde{\mathcal{G}}$ obtained by cutting through v (right).

Now since ψ is non-constant on the loop e_0 and it takes the same value at least twice on this loop, we can conclude that $\mu \geq 4\pi^2/|e_0|^2 = \lambda_2^N(e_0)$. Moreover, as ψ will change sign on e_0 , it cannot correspond to the first eigenvalue and we may assume we are dealing with $\mu(\mathcal{G}) = \lambda_2^N(\mathcal{G})$.

However, $\tilde{\mathcal{G}}$ can be viewed as being formed by attaching a pendant to the loop e_0 at v . Since $\lambda_2^N(e_0)$ has an eigenfunction not vanishing at v , Theorem 3.10(1) yields $\lambda_2^N(\tilde{\mathcal{G}}) < \lambda_2^N(e_0) \leq \lambda_2^N(\mathcal{G})$. The only exception is when the pendant is empty, that is, $\tilde{\mathcal{G}} = e_0$.

(3) The proof uses a symmetrisation argument, which is an easy variant of one that has appeared several times in the literature [Fri05a], [BaLe17, Theorem 2.1], [BKKM17, Theorem 3.4].

We denote by $\mathcal{P}_1, \dots, \mathcal{P}_n$ the pumpkin subgraphs to be symmetrised, by $e_{1i}, \dots, e_{k_i i}$ the edges of $\mathcal{P}_i$ and by $\tilde{e}_{1i}, \dots, \tilde{e}_{m_i i}$ the edges of the symmetrised pumpkin $\tilde{\mathcal{P}}_i$, $i = 1, \dots, n$.

Suppose the two vertices of $\mathcal{P}_i$ are v_i^- and v_i^+ (where $\psi(v_i^-) \leq \psi(v_i^+)$ and it is possible that $v_i^\pm = v_j^\pm$ for some $i \neq j$).

We now construct a test function $\psi^* \in H^1(\tilde{\mathcal{G}})$ out of ψ by symmetrising ψ on each pumpkin separately: we set $\psi^*(x) = \psi(x)$ if $x \notin \mathcal{P}_1 \cup \dots \cup \mathcal{P}_n$ and, on $\mathcal{P}_i$, if $\psi \geq 0$ on $\mathcal{P}_i$, we define $\psi^*(x)$ to be the continuous function such that $\psi^*(v_i^-) = \psi(v_i^-)$ and

$$|\{x \in \tilde{e}_{ji} : \psi^*(x) < t\}| = \frac{1}{m_i} |\{x \in \mathcal{P}_i : \psi(x) < t\}|$$

for $t \in \mathbb{R}$ and $j = 1, \dots, m_i$.

The following facts are standard and may be easily checked (cf. also the references given above): ψ^* is in fact an H^1 -function such that, by construction, $\psi^*(v) = \psi(v)$ for all $v \in \mathcal{V}(\tilde{\mathcal{G}}) \simeq \mathcal{V}(\mathcal{G})$; in particular, if ψ satisfies a Dirichlet condition at some vertex, then so too does ψ^* ; $\|\psi^*\|_{L^2(\tilde{\mathcal{G}})} = \|\psi\|_{L^2(\mathcal{G})}$,

$$\int_{\tilde{\mathcal{G}}} \psi^* dx = \int_{\mathcal{G}} \psi dx,$$

and $\|(\psi^*)'\|_{L^2(\tilde{\mathcal{G}})} \leq \|\psi'\|_{L^2(\mathcal{G})}$. Thus ψ^* is a valid test function for $\mu(\tilde{\mathcal{G}})$ with a Rayleigh quotient no larger than the one of ψ , and so we conclude $\mu(\tilde{\mathcal{G}}) \leq \mu(\mathcal{G})$.

If there is at least one pumpkin $\mathcal{P}_i$ on which ψ is not constant, then the inequality $\|(\psi^*)'\|_{L^2(\tilde{\mathcal{G}})} \leq \|\psi'\|_{L^2(\mathcal{G})}$ is strict unless $k_i = m_i$ and all edges $e_1, \dots, e_{k_i}$ have the same length (i.e., strict unless the symmetrisation is trivial).

(4) Since unfolding pendant edges can be done recursively, it is enough to prove the statement in the case of two pendant edges e_1 and e_2 . Denote their attachment vertex by v_0 , and their other vertices by v_1 and v_2 , respectively, and suppose the edge to be created is e_0 , $|e_0| = |e_1| + |e_2|$. If $\mu(\mathcal{G}) = \lambda_1(\mathcal{G})$ (or else λ_2^N has an eigenfunction which does not change sign on $e_1 \cup e_2$), then this follows by transplanting an edge. More precisely, suppose there is an eigenfunction ψ which is non-negative on $e_1 \cup e_2$, and suppose it reaches a maximum over $e_1 \cup e_2$ at $x_0 \in e_1$. We then transplant e_2 to x_0 . The resulting graph is $\tilde{\mathcal{G}}$, and by part (1), we have $\mu(\tilde{\mathcal{G}}) \leq \mu(\mathcal{G})$. Moreover, if ψ is not constant on $e_1 \cup e_2$, then obviously $0 \leq \min_{x \in e_2} \psi(x) < \psi(x_0)$, so (1) yields strict inequality.

Now suppose $\mu = \lambda_2^N(\mathcal{G})$, $\mathcal{G} \supsetneq e_1 \cup e_2$, and ψ is an eigenfunction which changes sign on $e_1 \cup e_2$. First we observe the *a priori* bound

$$(4.24) \quad \lambda_2^N(\tilde{\mathcal{G}}) < \lambda_2^N(e_1 \cup e_2),$$

which is obtained from Theorem 3.10(1) by attaching the pendant $\mathcal{R} := \mathcal{G} \setminus (e_1 \cup e_2)$ to the interval $e_1 \cup e_2$ at v_1 .

First consider the case $\psi(v_0) = 0$. If $\psi \equiv 0$ on e_2 , then ψ changes sign on e_1 and takes zero value at v_0 , and we get $\mu > \lambda_2^N(e_1) > \lambda_2^N(e_1 \cup e_2)$. By (4.24), we are done. Similarly we treat the case of ψ vanishing on e_1 . If $\psi(v_0) = 0$, but ψ is not identically zero on either e_1 or e_2 , then we can create an eigenfunction of $e_1 \cup e_2$ by possibly multiplying ψ by a constant on e_1 or e_2 . Since this does not affect the eigenvalue, we obtain $\mu \geq \lambda_2^N(e_1 \cup e_2)$ and we can again invoke (4.24).

So suppose $\psi(v_0) \neq 0$. Our strategy of proof is as follows: we cut $\mathcal{G}$ at v_0 along ψ to create a path graph (interval) out of $e_1 \cup e_2$ with a δ -condition at v_0 (cf. Definition 3.2).

We either remove this δ -condition or shift it to an endpoint of the path, which will lower the eigenvalue. We then re-glue the path to the rest of $\mathcal{G}$ to create $\tilde{\mathcal{G}}$.

We denote by $\mathcal{I}_{v_0, \gamma}$ the (quantum) graph consisting of $e_1 \cup e_2$ with a δ -potential of strength γ at v_0 and natural conditions at v_1 and v_2 . We obtain the value of γ from the eigenfunction ψ as described in Definition 3.2. By $\mathcal{R}_{v_0, -\gamma}$ we denote the graph $\mathcal{G}$ with e_1 and e_2 removed, and with a δ -potential $-\gamma$ at v_0 and natural conditions elsewhere; see Figure 4.2.

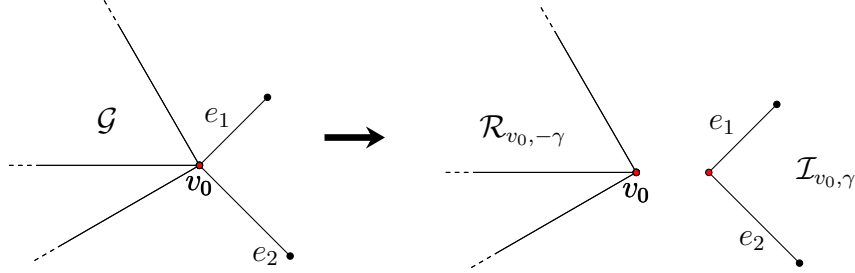


FIGURE 4.2. The graph $\mathcal{G}$ (left), and the graphs $\mathcal{R}_{v_0, -\gamma}$ and $\mathcal{I}_{v_0, \gamma}$ created by cutting through v_0 (right).

First suppose that $\gamma \geq 0$. We have $\mu \geq \lambda_2(\mathcal{I}_{v_0, \gamma})$ because ψ is an eigenfunction for $\mathcal{I}_{v_0, \gamma}$ which changes sign on $e_1 \cup e_2$. But by Theorem (3.3), we have $\lambda_2(\mathcal{I}_{v_0, \gamma}) \geq \lambda_2(\mathcal{I}_{v_0, 0}) = \lambda_2^N(e_1 \cup e_2)$ and we invoke (4.24) to complete this case.⁵

Finally consider the case $\gamma < 0$. We will need the following auxiliary result, which we state and prove in greater generality than we require for the proof. Denote by ℓ the length $|e_1 \cup e_2|$ of the interval graph $\mathcal{I}$.

Lemma 4.8. *With the above notation and with $\gamma < 0$, the function $x \mapsto \lambda_2(\mathcal{I}_{x, \gamma})$ is strictly monotonically increasing in $x \in [0, \ell/2]$. In particular, it reaches its unique minimum at $x = 0$.*

Proof. Firstly, by standard Sturm–Liouville theory the eigenvalue is simple, and its eigenfunction ψ has exactly one zero in $[0, \ell]$, say at z . Moreover, an argument using strict monotonicity with respect to domain inclusion and changes of γ shows that ψ cannot have an interior extremum. We will show using the Hadamard formula of Remark 3.14 that

$$(4.25) \quad \frac{d}{dx} \lambda_2(\mathcal{I}_{x, \gamma}) > 0$$

for all $x \in (0, \ell/2)$, which will prove the lemma.⁶ Now since increasing x is equivalent to lengthening the edge $[0, x]$ and shortening $[x, \ell]$ by the same amount, invoking (3.13) and (3.14) and using the continuity of ψ at x , we see (4.25) is equivalent to

$$(4.26) \quad (\psi_-)'(x)^2 < (\psi_+)'(x)^2,$$

⁵One can in fact show that the case $\gamma \geq 0$ is impossible under the assumption that ψ is a second eigenfunction that changes sign on $e_1 \cup e_2$.

⁶Similar arguments, albeit with a different goal, will be used repeatedly in Section 5.

where $(\psi_{\pm})'(x)$ denote the left $(-)$ and right $(+)$ derivatives of ψ at x . We claim that the zero $z \in (x, \ell)$. Indeed, if not, then, since $\gamma < 0$,

$$\lambda_2(\mathcal{I}_{x,\gamma}) = \lambda_1^D(0, z) = \lambda_2^N(0, 2z) \geq \lambda_2^N(0, \ell) > \lambda_2(\mathcal{I}_{x,\gamma}).$$

So suppose that $\psi(x) > 0$; then clearly $(\psi_-)'(x), (\psi_+)'(x) < 0$. The vertex condition at x together with $\gamma < 0$ now yields (4.26) and hence (4.25). $\square$

We now return to the proof of Theorem 3.18(4). It follows using Lemma 4.8 that

$$\mu = \lambda_2(\mathcal{I}_{v_0,\gamma}) > \lambda_2(\mathcal{I}_{v_1,\gamma}),$$

where in $\mathcal{I}_{v_1,\gamma}$ the δ -potential has been shifted from v_0 to v_1 , the degree one vertex of e_1 . We now glue $\mathcal{I}_{v_1,\gamma}$ to $\mathcal{R}_{v_0,-\gamma}$ to create $\tilde{\mathcal{G}}$. By Theorem 3.4, we have

$$(4.27) \quad \lambda_2^N(\tilde{\mathcal{G}}) \leq \lambda_3(\mathcal{R}_{v_0,-\gamma} \dot{\cup} \mathcal{I}_{v_1,\gamma}) \leq \mu = \lambda_2^N(\mathcal{G}),$$

where the second inequality follows from $\mu > \lambda_2(\mathcal{I}_{v_1,\gamma})$ and $\mu \in \sigma(\mathcal{R}_{v_0,-\gamma})$. This establishes the desired inequality in this case. To show that it is actually strict, assume the contrary: there is equality throughout (4.27). Then we must also have

$$(4.28) \quad \lambda_2(\mathcal{I}_{v_1,\gamma}) = \lambda_2(\mathcal{R}_{v_0,-\gamma} \dot{\cup} \mathcal{I}_{v_1,\gamma}) < \lambda_2^N(\tilde{\mathcal{G}}) = \lambda_3(\mathcal{R}_{v_0,-\gamma} \dot{\cup} \mathcal{I}_{v_1,\gamma}) = \mu.$$

This puts us in the circumstances of Remark 3.5 and we conclude that every eigenfunction of $\lambda_3(\mathcal{R}_{v_0,-\gamma} \dot{\cup} \mathcal{I}_{v_1,\gamma})$ is an eigenfunction of $\tilde{\mathcal{G}}$. But the former eigenspace contains a function which is equal to ψ on $\mathcal{R}$ and to zero on $\mathcal{I}$ and cannot be glued continuously since $\psi(v_0) \neq 0$. This final contradiction concludes the proof. $\square$

5. PUMPKINS EVERYWHERE

We are now ready to show how various surgery principles can be combined to allow a fine spectral analysis of one's graph. In this section, we will concentrate on the particular classes of *pumpkins* and *pumpkin chains* (see Definition 2.3), for three reasons: firstly, they are good examples on which to illustrate the principles; secondly, they will be used in a central way in our principal application in Section 6; and thirdly (which also largely explains the first two), at least when considering the first non-trivial eigenvalue $\mu(\mathcal{G})$, these graphs are in a certain sense generic, as we shall now demonstrate using our surgery principles.

5.1. Pumpkin chains. We recall that a $[m_1, \dots, m_n]$ -pumpkin chain consists of vertices $v_1, \dots, v_{n+1}$ together with, for each $k = 1, \dots, n$, some number m_k of parallel edges running between v_k and v_{k+1} . We will say that a (continuous) function ψ defined on the given pumpkin chain is *monotonically increasing along the chain* (or *monotonically increasing* for short) if it satisfies $\psi(v_1) \leq \psi(v_2) \leq \dots \leq \psi(v_{n+1})$, and on each edge connecting v_k to v_{k+1} , ψ is monotonically increasing from v_k to v_{k+1} , $k = 1, \dots, n$. We call ψ *monotonically decreasing* (along the chain) if it satisfies the reverse inequalities, and *monotonic* if it is monotonically increasing or decreasing.

We start out by showing that for any graph $\mathcal{G}$ there is a naturally associated pumpkin chain with the same first non-trivial eigenvalue.

Lemma 5.1. *Let $\mathcal{G}$ be compact and connected, and suppose $\mu(\mathcal{G}) \neq 0$. Then there is a pumpkin chain $\mathcal{P}_1$ with the same or smaller total length such that*

$$\mu(\mathcal{G}) = \mu(\mathcal{P}_1);$$

moreover, $\mu(\mathcal{P}_1)$ has an eigenfunction which is monotonically increasing along the chain. Finally, if $\mathcal{P}_1$ is a path, then either it has shorter total length than $\mathcal{G}$, or $\mathcal{G}$ is a path itself, in which case they coincide.

In fact, if there is an eigenfunction associated with $\mu(\mathcal{G})$ which does not vanish identically on any edge of $\mathcal{G}$, then the $\mathcal{P}_1$ we construct has the same length as $\mathcal{G}$. For the proof of the lemma, we first introduce what we shall call *critical values* or *critical levels* of a function.

Definition 5.2. Suppose $f \in C(\mathcal{G})$. A point $x \in \mathcal{G}$ shall be called a *critical point* (of the function f) if $x \in \mathcal{V}(\mathcal{G})$ or if f attains a local maximum or minimum at x . A value $t \in \mathbb{R}$ shall be called a *critical value* or *critical level* (of f) if there exists a critical point x of f such that $f(x) = t$. The preimage $\{x \in \mathcal{G} : f(x) = t\}$ of a critical value t will also be called the *critical set* (of f at the critical level t).

Note that since $\mathcal{G}$ is compact any *eigenfunction* has only finitely many critical levels; and, as long as $\mu(\mathcal{G}) \neq 0$, apart possibly from $t = 0$ every corresponding critical set is also finite.

Proof of Lemma 5.1. Fix any eigenfunction ψ associated with $\mu := \mu(\mathcal{G})$. We may assume ψ does not vanish identically on any edge of the remaining connected component by shrinking the edge to zero if necessary, which decreases the total length without affecting μ or ψ by Corollary 3.12(4). Thus each critical set of ψ is finite; we already know that ψ has finitely many critical sets. By inserting dummy vertices as necessary we may assume that every point in every critical set is a vertex. For each critical set we glue together all the vertices belonging to it; Corollary 3.6 implies this leaves μ and ψ unchanged. The new graph $\tilde{\mathcal{G}}$ is, by construction, a pumpkin chain, and ψ is now (strictly) monotonic along the chain. Indeed, ψ does not take on the same value at any two distinct vertices, and if v_1, v_2, v_3 are any vertices such that $\psi(v_1) < \psi(v_2) < \psi(v_3)$, then there is no edge from v_1 to v_3 , since otherwise ψ would take on a value on that edge equal to $\psi(v_2)$. It follows in particular that each vertex is connected at most to a predecessor and a successor, that is, $\tilde{\mathcal{G}}$ is a pumpkin chain.

Next observe that $\mu = \mu(\tilde{\mathcal{G}})$, that is, μ is the smallest eigenvalue of $\tilde{\mathcal{G}}$ having a non-trivial eigenfunction. In the case $\mu = \lambda_2^N(\mathcal{G})$, then $\lambda_2^N(\mathcal{G}) \leq \lambda_2^N(\tilde{\mathcal{G}}) = \mu(\tilde{\mathcal{G}})$ by Theorem 3.4(1), but ψ is a non-constant eigenfunction of μ on $\tilde{\mathcal{G}}$, so there is equality. The argument if $\mu = \lambda_1(\mathcal{G})$ and $\mu(\tilde{\mathcal{G}}) = \lambda_1(\tilde{\mathcal{G}})$ is similar. It is impossible for $\mu = \lambda_1(\mathcal{G})$ and $\mu(\tilde{\mathcal{G}}) = \lambda_2^N(\tilde{\mathcal{G}})$, since then ψ would be a non-constant, non-sign-changing eigenfunction of $\tilde{\mathcal{G}}$, a contradiction to the theorem of Kreĭn–Rutman and $\lambda_1^N(\tilde{\mathcal{G}}) = 0$ with only constants as eigenfunctions.

Finally, if $\mathcal{G}$ is not a path graph after the initial “shrinking” procedure, then $\mathcal{P}_1$ is also not one, since the above procedure cannot decrease the degree of any vertex; if ψ did not vanish on any edge, then no edge has been shrunk and thus $|\mathcal{G}| = |\mathcal{P}_1|$. $\square$

The monotonicity of the eigenfunction of $\mathcal{P}_1$ means we are now in a position to symmetrise each pumpkin using Theorem 3.18(3) (or (2)). Note the contrast to [KKMM16,

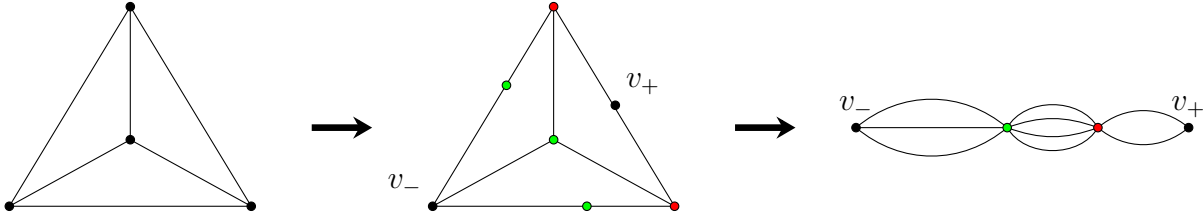


FIGURE 5.1. Turning the quantum graph constructed upon the complete graph K_4 into a pumpkin chain. The new vertices appearing in the middle graph are those points where the eigenfunction attains the same values as in the central vertex together with the point v_+ where it attains its maximum.

Lemma 5.4], which shows that λ_2^N can be bounded from *above* by the corresponding eigenvalue of a locally equilateral pumpkin chain with the same diameter (but generally smaller total length).

Lemma 5.3. *Let $\mathcal{G}$ be compact and connected, and suppose $\mu(\mathcal{G}) \neq 0$. Then there is a locally equilateral pumpkin chain $\mathcal{P}_2$ with the same or smaller total length such that*

$$\mu(\mathcal{G}) \geq \mu(\mathcal{P}_2).$$

Proof. By Lemma 5.1 we may assume that $\mathcal{G}$ is already a pumpkin chain and $\mu(\mathcal{G})$ has an eigenfunction which is monotonic along the chain. Now apply Theorem 3.18(3) to all constituent pumpkins simultaneously with $k = m$ in each case. $\square$

Before proceeding, we wish to give some basic properties of locally equilateral pumpkin chains. First, we look at a decomposition of the corresponding eigenspaces. Given a locally equilateral pumpkin chain $\mathcal{P}$ with terminal vertices v_-, v_+ , we stipulate the following:

- (1) a function on $\mathcal{P}$ is called *longitudinal* if it depends only on $\text{dist}(\cdot, v_-)$;
- (2) a function on $\mathcal{P}$ is called *transversal* if it is supported on exactly one pumpkin.

Lemma 5.4. *Let $\mathcal{G}$ be a locally equilateral pumpkin chain, and assume that no vertices apart possibly from the terminal vertices are equipped with a Dirichlet condition. Then*

- (1) $L^2(\mathcal{G})$ has an orthonormal basis consisting of longitudinal and transversal eigenfunctions, such that each transversal eigenfunction is supported on exactly one pair of parallel edges; and
- (2) there is an infinite sequence of eigenvalues having longitudinal eigenfunctions, and for each such eigenvalue the span of the longitudinal eigenfunctions in the corresponding eigenspace is one-dimensional.

Proof. (1) Suppose ψ is any eigenfunction. Define its *longitudinal part* ψ^{lon} by averaging the value of ψ over all parallel edges, i.e., if $\mathcal{P}_i$ is any constituent pumpkin of $\mathcal{G}$, which itself consists of the edges e_j , $j = 1, \dots, k_i$, then we set

$$\psi^{\text{lon}}|_{e_j}(x) := \frac{1}{k_i} \sum_{e \in \mathcal{P}_i} \psi|_e(x), \quad x \in \mathcal{P}_i.$$

Then ψ^{lon} , if it is non-zero, is still an eigenfunction with the same eigenvalue as ψ since it still satisfies the eigenvalue equation pointwise, $\psi^{\text{lon}}(v) = \psi(v)$ at every vertex v of $\mathcal{G}$, and all three vertex conditions, Dirichlet, natural and δ , are preserved by the averaging process. Indeed, it follows immediately from the definition that at any vertex v , supposing that $\mathcal{P}_i$ is an incident pumpkin with k_i edges, then

$$\sum_{e \in \mathcal{P}_i} \partial_\nu \psi^{\text{lon}}|_e(v) = \sum_{e \in \mathcal{P}_i} \partial_\nu \left(\frac{1}{k_i} \sum_{e \in \mathcal{P}_i} \psi|_e \right) (v) = \frac{1}{k_i} \sum_{e \in \mathcal{P}_i} \sum_{e \in \mathcal{P}_i} \partial_\nu \psi|_e(v) = \sum_{e \in \mathcal{P}_i} \partial_\nu \psi|_e(v),$$

and thus Kirchhoff and δ conditions remain satisfied in the strong sense. Moreover, by construction, ψ^{lon} is longitudinal. Since the function $\psi - \psi^{\text{lon}}$ vanishes at all vertices of $\mathcal{G}$, it is within the span of transversal eigenfunctions each of which is supported on just one pair of parallel edges. A Gram–Schmidt process completes the proof of (1).

(2) The existence of infinitely many such eigenvalues follows since the problem corresponds to a (one-dimensional) Sturm–Liouville problem with non-smooth but piecewise constant weight function, possibly with a finite number of δ potentials, cf. [KKMM16, Section 5.2]. In particular, the simplicity of each eigenvalue within the space of longitudinal functions follows from basic Sturm–Liouville theory. $\square$

Similar ideas have been developed, for example, in [Mug14, Theorem 8.30 and § 8.3.1] in a more general context that may lack a longitudinal direction and therefore not allow for a one-dimensional reduction. Longitudinal eigenfunctions correspond to the trivial representation of the symmetry of exchanging edges within each pumpkin, cf. [BBJL17].

Lemma 5.5. *Let $\mathcal{G}$ be a locally equilateral pumpkin chain, such that no vertices other than the terminal ones may be equipped with Dirichlet conditions. Additionally, assume that $\mathcal{G}$ is not a pumpkin with all natural conditions. Then the first non-trivial eigenvalue $\mu(\mathcal{G})$ is simple. The corresponding eigenfunction ψ is the first non-constant longitudinal eigenfunction. In particular, if all vertices of $\mathcal{G}$ are equipped with natural conditions except possibly one of the terminal vertices, which may be equipped with an arbitrary δ -potential $\gamma \in (-\infty, \infty]$, then ψ may be chosen to be monotonically increasing along the chain.*

Proof. Let $\lambda^* = \lambda^*(\mathcal{G})$ be the smallest eigenvalue having a non-constant longitudinal eigenfunction, call it ψ . If not all vertex conditions are natural, i.e., we are considering $\mu = \lambda_1$, then ψ does not change sign, meaning $\lambda^* = \lambda_1$, and this eigenvalue is simple in the spectrum of $\mathcal{G}$.

Now consider the case of only natural conditions, i.e., $\mu = \lambda_2^N$. In this case, by invoking the decomposition of the spectrum and simplicity of longitudinal eigenvalues established in Lemma 5.4, it suffices to show that $\lambda^* = \lambda_2^N$ and that no transversal eigenfunction has the same eigenvalue.

To this end, first observe that the eigenvalue of any transversal eigenfunction is always a non-zero eigenvalue of one of the constituent pumpkins of $\mathcal{G}$. If we denote the longest edge length in $\mathcal{G}$ by $|e_{\max}|$, then the smallest of these is $\pi^2/|e_{\max}|^2$. Hence, to complete the proof, we merely have to show that $\lambda_2^N(\mathcal{G}) < \pi^2/|e_{\max}|^2$.

Denote by $\mathcal{P} \subsetneq \mathcal{G}$ any constituent pumpkin whose smallest non-trivial eigenvalue equals $\pi^2/|e_{\max}|^2$; call its vertices v_1 and v_2 . Then $\mathcal{G}$ may be formed from $\mathcal{P}$ by attaching the pendant(s) $\mathcal{G} \setminus \mathcal{P}$ to $\mathcal{P}$ at v_1 and/or v_2 as appropriate. Since $\lambda_2^N(\mathcal{P})$ has an eigenfunction

which is non-zero at v_1 and at v_2 , applying Theorem 3.10(1) with $r = 1$ yields

$$\lambda_2^N(\mathcal{G}) < \lambda_2^N(\mathcal{P}) = \frac{\pi^2}{|e_{\max}|^2}$$

Now since ψ corresponds to the first non-trivial eigenfunction of a one-dimensional Sturm–Liouville problem with L^∞ - (indeed, piecewise constant) weights (cf., e.g., [KKMM16, Section 5.2]), its monotonicity with respect to $\text{dist}(\cdot, v_-)$ is a routine statement from Sturm–Liouville theory. $\square$

With this background, we will now give three particular examples of special classes of pumpkin chains. On the one hand, this is a further illustration of what results can be obtained using the tools presented in Section 3, in particular both the unfolding principles and the Hadamard principle (Remark 3.14). At the same time, the examples show how the spectral gap λ_2^N is reduced if more “mass” is concentrated symmetrically at the periphery of the graph, or as the pumpkin chain becomes “thinner” (more path-like). On the other hand, these examples will be needed for our principal application, in Section 6.

We remark that the Hadamard-type formula for quantum graphs was used for a comparable but complementary analysis in [BaLe17, Section 5], where the goal was to study properties of graphs which represented “critical points” with respect to this formula for a given graph topology, i.e., given an underlying discrete graph, to study those graphs whose every edge length was a critical point for λ_2^N . Here, the goal is to see how monotonic behaviour of the eigenfunction can be used to show that a continuous change in edge lengths can transform a graph into another one, such that λ_2^N always increases or decreases under this transformation.

5.2. Pumpkin-on-a-stick graphs. Here we assume $\mathcal{V} = \mathcal{V}_N$ and consider $\mu = \lambda_2^N$. We will consider the following class of graphs.

Definition 5.6. A pumpkin chain $\mathcal{G}$ shall be called a *pumpkin-on-a-stick* if it is a $[1, m, 1]$ -pumpkin chain for some $m \geq 1$, and the m -pumpkin is equilateral. Here we allow the terminal one-pumpkins to be degenerate, i.e. have zero length.

In any case, we refer to the m -pumpkin as the (non-trivial) pumpkin, and the union of the 1-pumpkins as the stick (cf. Figure 5.2). Note that equilateral pumpkins, tadpoles (lassos) and even path graphs are all special cases. We are interested in the following parameters, and the behaviour of λ_2^N with respect to them:

- (1) the total length L ;
- (2) the number of edges m of the pumpkin;
- (3) the lengths ℓ_1 and ℓ_2 of the 1-pumpkins (which we think of as the “left” and the “right” ones, respectively), as well as the length $\ell := \ell_1 + \ell_2$ of the stick.

For a given L , which will be fixed throughout, we will denote by $\mathcal{P}[\ell_1, m, \ell_2]$ the pumpkin-on-a-stick whose 1-pumpkins have length ℓ_1 and ℓ_2 , and whose central pumpkin has m edges. If we denote by $T \subset \mathbb{R}^2$ the closed triangle whose vertices are $(0, 0)$, $(L, 0)$ and $(0, L)$, our condition on the 1-pumpkins reads $(\ell_1, \ell_2) \in T$. Up to rigid transformations, any pumpkin-on-a-stick of total length L is determined uniquely by the parameters $m \geq 1$ and $(\ell_1, \ell_2) \in T$.

The following proposition gives a complete description of how λ_2^N depends on these parameters. The proof is based principally on Theorem 3.18(3) (local symmetrisation) and (3.13) (the Hadamard-type formula). We emphasise that the proof does *not* involve any explicit calculations; in particular, we do not use any properties of the corresponding secular equations for the eigenvalues. We exclude the trivial case $m = 1$ from our considerations.

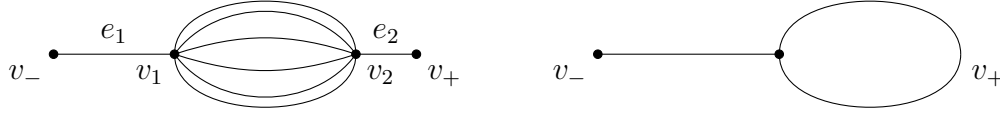


FIGURE 5.2. A “generic” pumpkin-on-a-stick (left), with $m = 6$ and stick $e_1 \cup e_2$, where $|e_1| = \ell_1$ and $|e_2| = \ell_2$; and a tadpole (right), with $m = 2$ and $\ell_2 = 0$.

Proposition 5.7. *Let L be fixed and suppose $m \geq 2$ and $(\ell_1, \ell_2) \in T$. Then $\lambda_2^N(\mathcal{P}[\ell_1, m, \ell_2])$ is simple with corresponding eigenfunction monotonic from v_- to v_+ , unless the graph is a pumpkin. For each fixed $m \geq 2$, the function $(\ell_1, \ell_2) \mapsto \lambda_2^N(\mathcal{P}[\ell_1, m, \ell_2])$ is continuous on the closed triangle T . Moreover,*

(1) *for fixed $m \geq 2$ we have*

$$\lambda_2^N(\mathcal{P}[\ell_1, m, \ell_2]) > \lambda_2^N(\mathcal{P}[\ell'_1, m, \ell'_2])$$

whenever $\ell_1 \leq \ell'_1$ and $\ell_2 \leq \ell'_2$ with at least one inequality strict;

(2) *for fixed $(\ell_1, \ell_2) \in T$ the function $m \mapsto \lambda_2^N(\mathcal{P}[\ell_1, m, \ell_2])$ is strictly monotonically decreasing in $m \geq 2$;*

(3) *for fixed $m \geq 2$ and fixed $\ell = \ell_1 + \ell_2$, the function $\ell_1 \mapsto \lambda_2^N(\mathcal{P}[\ell_1, m, \ell - \ell_1])$ is strictly monotonically increasing in $\ell_1 \in [0, \ell/2]$;*

(4) *in particular, among all pumpkin-on-a-stick graphs with fixed ℓ and m , the minimum of λ_2^N is achieved at $\ell_1 = 0$, and among all pumpkin-on-a-stick graphs with fixed ℓ , the minimum is achieved at $\ell_1 = 0$ and $m = 2$, i.e. at the tadpole whose tail has length ℓ .*

Finally, for given $m \geq 2$, $\lambda_2^N(\mathcal{P}[\ell_1, m, \ell_2])$ satisfies the bound

$$(5.1) \quad \frac{\pi^2}{L^2} \leq \lambda_2^N(\mathcal{P}[\ell_1, m, \ell_2]) \leq \frac{\pi^2 m^2}{L^2},$$

with equality in the lower estimate if and only if $\mathcal{P}[\ell_1, m, \ell_2]$ is a path (i.e. $\ell = L$) and in the upper one if and only if $\mathcal{P}[\ell_1, m, \ell_2]$ is a pumpkin (i.e. $\ell = 0$).

We observe that (1) contains the statement that the spectral gap of a tadpole is a strictly increasing function of the length of its loop (if the total length is fixed); in particular, it runs from π^2/L^2 if the loop has length 0 to $4\pi^2/L^2$ if the loop has length L .

Proof of Proposition 5.7. The statements about simplicity and the corresponding eigenfunction were proved in Lemma 5.5. The statements about continuity follow from general results about the stability of the spectrum with respect to changes in the edge lengths, including in the degenerate case when an edge contracts to zero; see for example [BK12]

or [BaLe17, Appendix A] (or [BLS18] for general vertex conditions including δ -type). For the rest, we will rely primarily on the unfolding principles from Theorem 3.18 and the Hadamard-type formula in the form (3.13).

(1) Consider two pumpkin-on-a-chain graphs $\mathcal{P} = \mathcal{P}[\ell_1, m, \ell_2]$ and $\mathcal{P}' = \mathcal{P}[\ell'_1, m, \ell'_2]$ with, say, $\ell_1 < \ell'_1$ and $\ell_2 \leq \ell'_2$. In the graph $\mathcal{P}$, on each parallel edge insert dummy vertices at the distance $(\ell'_1 - \ell_1)/m$ from the left terminal vertex and at the distance $(\ell'_2 - \ell_2)/m$ from the right terminal vertex, cf. Figure 5.3.

We glue the dummy vertices in such a way as to obtain a locally equilateral pumpkin chain with three m -pumpkins. This leaves λ_2^N unchanged by Corollary 3.6. The two side pumpkins can be unfolded, leading to the graph $\mathcal{P}'$; Theorem 3.18(2) yields the inequality $\lambda_2^N(\mathcal{P}) > \lambda_2^N(\mathcal{P}')$, which is strict because the longitudinal eigenfunction corresponding to $\lambda_2^N(\mathcal{P})$ is strictly monotonic in the longitudinal direction, and $\ell_1 < \ell'_1$.

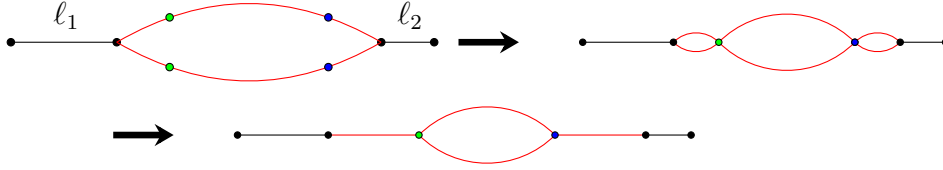


FIGURE 5.3. The first graph $\mathcal{P} = \mathcal{P}[\ell_1, m, \ell_2]$ is turned into an auxiliary graph and finally into $\mathcal{P}' = \mathcal{P}[\ell'_1, m, \ell'_2]$.

(2) This follows directly from an application of Theorem 3.18(3) to the non-trivial pumpkin, again noting that the eigenfunction does not vanish on the pumpkin in question.

(3) Here we use our Hadamard formula (3.14). First some notation: we label the internal vertices as v_1 and v_2 as depicted in Figure 5.2, so that v_1 is closer to v_- and v_2 is closer to v_+ . We also write $e_1 \sim v_-v_1$ and $e_2 \sim v_2v_+$, and denote by e_p any of the m parallel edges of the central pumpkin.

Fix a pumpkin-on-a-stick and assume that for this graph $\ell_1 > \ell_2$. We claim that to prove (3) it is sufficient to show that

$$(5.2) \quad \mathcal{E}_{e_1} > \mathcal{E}_{e_2},$$

where $\mathcal{E}_e$ is the Prüfer amplitude defined in (3.13). Indeed, if this holds *whenever* $\ell_1 > \ell_2$, then by (3.14), further lengthening e_1 and shortening e_2 *always* decreases λ_2^N . Since λ_2^N is continuous also at $\ell_2 = 0$, this yields strict monotonicity on the entire range of possible values of ℓ_1 .

To prove (5.2), we first observe that the set $\{\psi = 0\}$ of zeros of the eigenfunction (which we recall consists of all points of the form $\{x : \text{dist}(x, v_-) = c\}$ for some $c > 0$ depending on the graph) is closer to v_1 than v_2 (we do not rule out the possibility that it is on the edge e_1). Indeed, if it were not, then by reflecting $\{\psi \geq 0\}$ across the set $\{\psi = 0\}$ (i.e. creating a new graph which is reflection symmetric across $\{\psi = 0\}$, such that each half is a copy of $\{\psi \geq 0\}$) we would obtain a new pumpkin-on-a-stick $\tilde{\mathcal{P}}$ with a shorter pumpkin and a shorter edge replacing e_1 (since v_2 is closer to v_+ than v_1 is to v_-). But then the odd extension of ψ from $\{\psi \geq 0\}$ to $\tilde{\mathcal{P}}$ is still an eigenfunction but with the same eigenvalue. Since at least one edge of $\tilde{\mathcal{P}}$ is strictly shorter than its original counterpart,

we have created a (possibly degenerate) pumpkin chain with shorter edges and the same λ_2^N : this is an obvious contradiction to Corollary 3.12(1) (lengthening edges).

In particular, since $\psi|_{e_p}$ is sinusoidal and monotonic, the fact that the zero is closer to v_1 than v_2 means that $|\psi(v_1)| < |\psi(v_2)|$.

Armed with this, we now consider the Prüfer amplitudes: since $\mathcal{E}_e$ depends only on the edge e in question, writing λ for $\lambda_2^N(\mathcal{P}[\ell_1, m, \ell_2])$,

$$(5.3) \quad [\partial_\nu \psi|_{e_p}(v_1)]^2 + \lambda \psi|_{e_p}(v_1)^2 = [\partial_\nu \psi|_{e_p}(v_2)]^2 + \lambda \psi|_{e_p}(v_2)^2.$$

Now we use the vertex conditions to translate this into a comparison between $\mathcal{E}_{e_1}$ and $\mathcal{E}_{e_2}$: continuity implies $\psi|_{e_p}(v_i) = \psi|_{e_i}(v_i) \equiv \psi(v_i)$ for $i = 1, 2$, while the Kirchhoff condition and the fact that ψ is identical on each of the m parallel edges of the pumpkin mean that $\partial_\nu \psi|_{e_i}(v_i) = m \partial_\nu \psi|_{e_p}(v_i)$ for $i = 1, 2$. Inserting these into (5.3) yields

$$(5.4) \quad \frac{1}{m} [\partial_\nu \psi|_{e_1}(v_1)]^2 + \lambda \psi|_{e_1}(v_1)^2 = \frac{1}{m} [\partial_\nu \psi|_{e_2}(v_2)]^2 + \lambda \psi|_{e_2}(v_2)^2.$$

Combining this with what we showed earlier, viz. $|\psi|_{e_1}(v_1)| < |\psi|_{e_2}(v_2)|$, we deduce that $|\partial_\nu \psi|_{e_1}(v_1)| > |\partial_\nu \psi|_{e_2}(v_2)|$. Multiplying this latter inequality by $(1 - 1/m)$ and adding it to (5.4) now gives

$$\mathcal{E}_{e_1} = [\partial_\nu \psi|_{e_1}(v_1)]^2 + \lambda \psi|_{e_1}(v_1)^2 > [\partial_\nu \psi|_{e_2}(v_2)]^2 + \lambda \psi|_{e_2}(v_2)^2 = \mathcal{E}_{e_2},$$

which was to be proved.

(4) This follows immediately from (2) and (3).

Finally, the bounds (5.1) follow directly from the (strict) monotonicity results of (1), since decreasing ℓ to L produces a pumpkin corresponding to the upper bound, increasing ℓ to 0 yields a path, and the behaviour of λ_2^N is (strictly) monotonic between the two extremities. $\square$

5.3. Pumpkin dumbbells. We again assume $\mathcal{V} = \mathcal{V}_N$ and $\mu = \lambda_2^N$. By a *dumbbell*, we understand a graph consisting of an edge, or *handle*, e_0 , with a loop attached to each end; in other words, it is a locally equilateral $[2, 1, 2]$ -pumpkin chain. Here we consider a slightly more general class:

Definition 5.8. Fix $m \geq 1$. A locally equilateral $[m, 1, m]$ -pumpkin chain will be called a *pumpkin dumbbell*. Any constituent pumpkin is allowed to be degenerate. The middle 1-pumpkin will be called the *handle*.

We will again fix the total length L and denote by $\mathcal{D}[\ell_1, \ell_2, m]$ the pumpkin dumbbell of length L , unique up to symmetries, such that

- (1) the (“left”) pumpkin adjacent to v_- has total length $\ell_1 \in [0, L]$;
- (2) the (“right”) pumpkin adjacent to v_+ has total length $\ell_2 \in [0, L]$, with $\ell_1 + \ell_2 \leq L$;
- (3) the two outer pumpkins both have $m \geq 1$ edges of length ℓ_1/m and ℓ_2/m , respectively.

See Figure 5.4. As before, we will also write $T \subset \mathbb{R}^2$ for the closed triangle whose vertices are $(0, 0)$, $(L, 0)$ and $(0, L)$, so that $(\ell_1, \ell_2) \in T$. The pumpkin dumbbell coincides with a pumpkin-on-a-stick if $\ell_1 = 0$ or $\ell_2 = 0$, and also covers the special cases of a path if $\ell_1 = \ell_2 = 0$, and a figure-8 if $m = 2$ and $\ell_1 + \ell_2 = L$. If $m = 2$, then we will also write

$$\mathcal{D}[\ell_1, \ell_2] := \mathcal{D}[\ell_1, \ell_2, 2]$$

for a (conventional) dumbbell. If in addition $\ell_2 = 0$, then we shall write

$$(5.5) \quad \mathcal{L}[\ell_1] := \mathcal{D}[\ell_1, 0] \equiv \mathcal{D}[\ell_1, 0, 2] \equiv \mathcal{P}[L - \ell_1, 2, 0]$$

for the tadpole (or lasso) of total length L , whose loop is of length ℓ_1 .

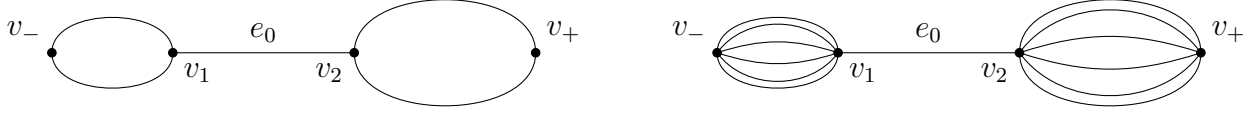


FIGURE 5.4. A dumbbell consisting of a handle e_0 joining two loops, which may (as here) be imagined as being 2-pumpkins (left); a more general pumpkin dumbbell with $m = 6$ (right), for which the pumpkin on the left has total length ℓ_1 and the one on the right total length ℓ_2 .

Our result, in addition to monotonicity statements analogous to those in Proposition 5.7, states that “balancing” the two pumpkins, i.e., making them more equal in size, lowers the spectral gap λ_2^N . The tools used will be essentially the same.

Proposition 5.9. *Suppose that $\mathcal{D}[\ell_1, \ell_2, m]$ is a pumpkin dumbbell of fixed total length $L \geq \ell_1 + \ell_2$ with $m \geq 2$. Then $\lambda_2^N(\mathcal{D}[\ell_1, \ell_2, m])$ is simple with corresponding eigenfunction monotonic from v_- to v_+ , unless $\ell_1 = L$ or $\ell_2 = L$, i.e., unless it is a pumpkin. The function $(\ell_1, \ell_2) \mapsto \lambda_2^N(\mathcal{D}[\ell_1, \ell_2, m])$ is continuous on the closed triangle T . Moreover,*

- (1) *for each fixed $m \geq 2$ and $\ell_2 \in [0, L]$, the function $\ell_1 \mapsto \lambda_2^N(\mathcal{D}[\ell_1, \ell_2, m])$ is a strictly monotonically increasing function of $\ell_1 \in [0, L - \ell_2]$. A corresponding statement holds mutatis mutandis if the roles of ℓ_1 and ℓ_2 are interchanged;*
- (2) *if $\ell := \ell_1 + \ell_2$ and $m \geq 2$ are fixed, then $\ell_1 \mapsto \lambda_2^N(\mathcal{D}[\ell_1, \ell - \ell_1, m])$ is strictly monotonically increasing in $\ell_1 \in (0, \ell/2)$;*
- (3) *if $\ell_1, \ell_2 \in (0, L)$ are fixed, then $m \mapsto \lambda_2^N(\mathcal{D}[\ell_1, \ell_2, m])$ is strictly monotonically increasing in $m \geq 1$;*
- (4) *in particular, among all pumpkin dumbbells $\mathcal{D}[\ell_1, \ell_2, m]$ for which $\ell = \ell_1 + \ell_2$ is fixed, $\lambda_2^N(\mathcal{D}[\ell_1, \ell_2, m])$ is uniquely minimised when $\ell_1 = \ell_2 = \ell/2$; among all such pumpkin dumbbells where $m \geq 2$ is also allowed to vary, the minimum is achieved only by the regular dumbbell ($m = 2, \ell_1 = \ell_2 = \ell/2$).*

Finally, for given $m \geq 2$, $\lambda_2^N(\mathcal{D}[\ell_1, \ell_2, m])$ satisfies the bound

$$(5.6) \quad \frac{\pi^2}{L^2} \leq \lambda_2^N(\mathcal{D}[\ell_1, \ell_2, m]) \leq \frac{\pi^2 m^2}{L^2},$$

with equality in the lower estimate if and only if $\mathcal{D}[\ell_1, \ell_2, m]$ is a path (i.e. $\ell_1 = \ell_2 = 0$) and in the upper one if and only if $\ell_1 + \ell_2 = L$.

Proof. As in the proof of Proposition 5.7, the statements about simplicity and the properties of the eigenfunction were proved in Lemma 5.5, while the statements about continuity are standard.

(1) The proof is essentially the same as the proof of monotonicity in Proposition 5.7(1). Suppose $0 \leq \ell_1 < \ell'_1 \leq L - \ell_2$. We consider $\mathcal{D}[\ell'_1, \ell_2, m]$ and glue the m points (treated

as dummy vertices) on the pumpkin of length ℓ'_1 at distance $(\ell'_1 - \ell_1)/m > 0$ from its vertex. Since the eigenfunction is longitudinal, this does not change λ_2^N . An application of Theorem 3.18(2) or (3) to the m obtained parallel edges transforms the graph into $\mathcal{D}[\ell_1, \ell_2, m]$ and decreases the eigenvalue strictly since the eigenfunction only has isolated zeros. This proves the statement. Obviously, we may interchange the roles of ℓ_1 and ℓ_2 if we wish.

(2) Here we will use the Hadamard formula in the form of (3.14) as in the proof of Proposition 5.7(3). Suppose that m and $\ell_1 + \ell_2$ are fixed with $\ell_1 < \ell_2$ and denote by e_1 and e_2 any of the m edges of the pumpkins adjacent to v_- and v_+ , respectively, so that $|e_1| = \ell_1/m$ and $|e_2| = \ell_2/m$. As depicted in Figure 5.4, we will write e_0 for the handle joining the two pumpkins and v_1, v_2 for its incident vertices, where v_1 is closer to v_- and v_2 is closer to v_+ .

Exactly as in the proof of Proposition 5.7, by the Hadamard formula, it suffices to prove

$$(5.7) \quad \mathcal{E}_{e_1} > \mathcal{E}_{e_2}$$

for the graph $\mathcal{D}[\ell_1, \ell_2, m]$ if $0 < \ell_1 < \ell_2$, where, again, $\mathcal{E}_{e_i}$ is defined in (3.13). Indeed, this implies that shortening *all* edges of the longer pumpkin and lengthening *all* the edges of the shorter pumpkin by the same amount will always (strictly) lower λ_2^N .

Now a similar symmetry argument to the one given in the proof of Proposition 5.7(3) shows that the zero set $\{\psi = 0\}$ is closer to v_2 than to v_1 . To this end, we first claim that $\text{dist}(\{\psi = 0\}, v_2) < \text{dist}(\{\psi = 0\}, v_1)$: since $\lambda_2^N(\mathcal{D}[\ell_1, \ell_2, m]) = \lambda_1(\{\psi \leq 0\})$, if this were not true we could reflect $\{\psi \leq 0\}$ across $\{\psi = 0\}$ to create a dumbbell with strictly shorter handle and/or second loop e_2 but the same eigenvalue λ_2^N . This is then a contradiction to the fact that lengthening an edge strictly decreases λ_2^N .

As before, it follows from the fact that ψ is a monotonic sinusoidal curve on each edge that $|\psi(v_1)| > |\psi(v_2)|$. Using the definition of the Prüfer amplitude $\mathcal{E}_e$, the vertex conditions and the independence of ψ from the parallel edges in question as before, we then obtain, writing λ for the eigenvalue,

$$\begin{aligned} m [\partial_\nu \psi|_{e_1}(v_1)]^2 + \lambda \psi|_{e_1}(v_1)^2 &= [\partial_\nu \psi|_{e_0}(v_1)]^2 + \lambda \psi|_{e_0}(v_1)^2 \\ &= [\partial_\nu \psi|_{e_0}(v_2)]^2 + \lambda \psi|_{e_0}(v_2)^2 = m [\partial_\nu \psi|_{e_2}(v_2)]^2 + \lambda \psi|_{e_2}(v_2)^2 \end{aligned}$$

which, when combined with $\psi|_{e_1}(v_1)^2 > \psi|_{e_2}(v_2)^2$ as shown earlier, implies (5.7).

(3) This is, again, a direct consequence of Theorem 3.18(3) applied to each of the pumpkins.

(4) This follows immediately from (2) together with the observations that, for fixed $\ell \in (0, L)$, $\lambda_2^N(\mathcal{D}[\ell, \ell - \ell_1, m]) = \lambda_2^N(\mathcal{D}[\ell - \ell_1, \ell, m])$, and that $(\ell_1, \ell - \ell_1) \mapsto \lambda_2^N(\mathcal{D}[\ell, \ell - \ell_1, m])$ is continuous as $\ell_1 \rightarrow 0$ or $\ell_1 \rightarrow \ell/2$.

Finally, the bounds (5.6) follow, analogously to the proof of Proposition 5.7, from the (strict) monotonicity in (1): starting from an arbitrary given dumbbell, unfolding to produce a path yields the first inequality, while expanding the pumpkins until the handle disappears yields the second. $\square$

5.4. Pumpkin chains with a Dirichlet vertex. Our third result, another application of the Hadamard-type formula, is for a slightly larger class of graphs. We will consider locally equilateral pumpkin chains $\mathcal{P}$ having constituent pumpkins $\mathcal{P}_1, \dots, \mathcal{P}_n$ (given in

sequence, i.e., such that $\mathcal{P}_i$ has $\mathcal{P}_{i-1}$ and $\mathcal{P}_{i+1}$ as its neighbours). We will assume that one of the terminal vertices, say v_- , the one adjacent to $\mathcal{P}_1$, has a Dirichlet condition, while at all the others we have the usual natural condition; see Figures 5.5 and 5.6. Here we consider the associated smallest eigenvalue $\lambda_1^D(\mathcal{P}) > 0$.⁷ The following result states that swapping the order of pumpkins to move the fatter ones further away from the Dirichlet point decreases λ_1^D .

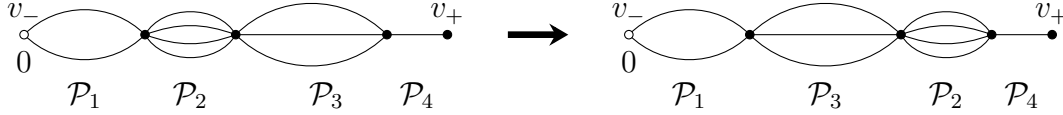


FIGURE 5.5. Swapping the pumpkins $\mathcal{P}_2$ on four edges and $\mathcal{P}_3$ on three edges decreases $\lambda_1^D \dots$

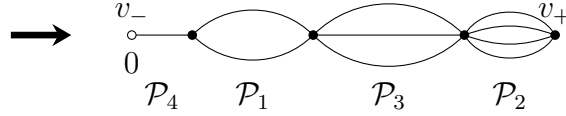


FIGURE 5.6. ...and so on. The 0 indicates the unique Dirichlet vertex, v_- , drawn in white.

Proposition 5.10. *Suppose that the locally equilateral pumpkin chain $\mathcal{P}$ consists of pumpkins $\mathcal{P}_1, \dots, \mathcal{P}_n$, $n \geq 2$, such that $\mathcal{P}_1$ and $\mathcal{P}_n$ are the terminal pumpkins with a Dirichlet condition being imposed at the terminal vertex v_- of $\mathcal{P}_1$, natural conditions at all the other vertices, and the $\mathcal{P}_i$ are ordered by increasing distance from v_- (as depicted in Figure 5.5). Suppose further that for some $i = 1, \dots, n-1$, $\mathcal{P}_i$ has $m_i \geq 2$ parallel edges and $\mathcal{P}_{i+1}$ has $1 \leq m_{i+1} < m_i$ parallel edges. Denote by $\tilde{\mathcal{P}}$ the corresponding pumpkin chain obtained from $\mathcal{P}$ by exchanging $\mathcal{P}_i$ and $\mathcal{P}_{i+1}$, i.e., consisting of the pumpkins $\mathcal{P}_1, \dots, \mathcal{P}_{i-1}, \mathcal{P}_{i+1}, \mathcal{P}_i, \mathcal{P}_{i+2}, \dots, \mathcal{P}_n$ listed in order of increasing distance from v_- . Then we have*

$$\lambda_1^D(\mathcal{P}) > \lambda_1^D(\tilde{\mathcal{P}}).$$

Thus if the set of constituent pumpkins is fixed, then the pumpkin chain minimising λ_1^D orders them by increasing thickness away from the Dirichlet vertex, cf. Figure 5.6.

Remark 5.11. (1) The assertion of Proposition 5.10 still holds if we replace the Dirichlet condition at v_- by a δ condition with positive strength γ . The proof is identical and we do not go into details.

(2) If natural conditions are imposed on all vertices of $\mathcal{P}$, then one can apply Proposition 5.10 to either of the nodal domains $\mathcal{P}^\pm$ of the eigenfunction associated with $\lambda_2^N(\mathcal{P})$ and deduce an analogous result as in Proposition 5.10, namely comparison with a pumpkin chain $\mathcal{P}'$ where fatter pumpkins have been moved towards the endpoints v_-, v_+ .

⁷Recall our notational convention given in (2.5).

Proof of Proposition 5.10. The eigenvalue $\lambda_1^D(\mathcal{P})$ is simple, and by Lemma 5.5, if its associated eigenfunction ψ is chosen positive, then it is monotonically increasing from v_- to v_+ and invariant under permutations of the edges within any pumpkin, i.e., $\psi(x)$ depends only on $\text{dist}(x, v_-)$.

We claim that it is sufficient to show that if $\mathcal{P}_{i-1}, \mathcal{P}_i, \mathcal{P}_{i+1}$ are consecutive pumpkins in such a pumpkin chain with m_{i-1}, m_i, m_{i+1} edges, respectively, such that $m_{i-1} = m_{i+1} < m_i$, then shortening the edges of $\mathcal{P}_{i-1}$ and lengthening those of $\mathcal{P}_{i+1}$ for a fixed total length always (strictly) decreases λ_1^D (and vice versa). Indeed, we may view the graphs $\mathcal{P}$ and $\tilde{\mathcal{P}}$ as those obtained respectively by passing to the limit as $\mathcal{P}_{i+1}$ shrinks to a point, and as $\mathcal{P}_{i-1}$ shrinks to a point. Since the eigenvalue is continuous with respect to passing to these limits (we again refer to [BaLe17, Appendix A] or [BLS18]), it follows that $\lambda_1^D(\mathcal{P}) > \lambda_1^D(\tilde{\mathcal{P}})$.

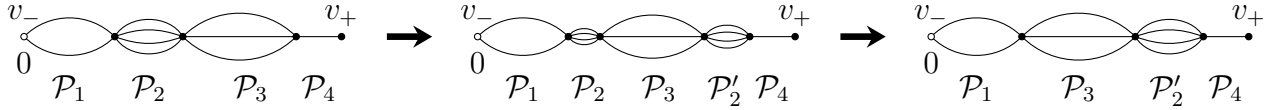


FIGURE 5.7. The pumpkin $\mathcal{P}'_2$ is a clone of $\mathcal{P}_2$.

Denote by e_{i-1}, e_i, e_{i+1} any of the parallel edges of $\mathcal{P}_{i-1}, \mathcal{P}_i, \mathcal{P}_{i+1}$, respectively, and suppose that v_i is the vertex between $\mathcal{P}_{i-1}$ and $\mathcal{P}_i$, and v_{i+1} is the vertex between $\mathcal{P}_i$ and $\mathcal{P}_{i+1}$. Then by (3.14), to prove the proposition we only need to show that, in the situation just described,

$$\mathcal{E}_{e_{i-1}} > \mathcal{E}_{e_{i+1}}.$$

The argument is thus reduced to one similar to those given in the proofs of Proposition 5.7(3) and Proposition 5.9(2): we note that, firstly, $0 < \psi(v_{i-1}) < \psi(v_{i+1})$ since ψ is monotonically increasing and non-constant along the pumpkin chain, and secondly, writing λ for the eigenvalue λ_1^D of the pumpkin chain in question,

$$(5.8) \quad [\partial_\nu \psi|_{e_i}(v_i)]^2 + \lambda \psi(v_i)^2 = \mathcal{E}_{e_i} = [\partial_\nu \psi|_{e_i}(v_{i+1})]^2 + \lambda \psi(v_{i+1})^2.$$

The Kirchhoff condition at v_i and v_{i+1} , together with the fact that ψ is equal on all edges of a given pumpkin, then reads

$$m_{i-1} |\partial_\nu \psi|_{e_{i-1}}(v_i)| = m_i |\partial_\nu \psi|_{e_i}(v_i)|, \quad m_i |\partial_\nu \psi|_{e_i}(v_{i+1})| = m_{i+1} |\partial_\nu \psi|_{e_{i+1}}(v_{i+1})|;$$

inserting this into (5.8) implies

$$\frac{m_{i-1}}{m_i} [\partial_\nu \psi|_{e_{i-1}}(v_i)]^2 + \lambda \psi(v_i)^2 = \frac{m_{i+1}}{m_i} [\partial_\nu \psi|_{e_{i+1}}(v_{i+1})]^2 + \lambda \psi(v_{i+1})^2.$$

Recalling that $0 < m_{i-1} = m_{i+1} < m_i$ and $\psi(v_i)^2 < \psi(v_{i+1})^2$, this implies that $[\partial_\nu \psi|_{e_{i-1}}(v_i)]^2 > [\partial_\nu \psi|_{e_{i+1}}(v_{i+1})]^2$ and

$$\mathcal{E}_{e_{i-1}} = [\partial_\nu \psi|_{e_{i-1}}(v_i)]^2 + \lambda \psi(v_i)^2 > [\partial_\nu \psi|_{e_{i+1}}(v_{i+1})]^2 + \lambda \psi(v_{i+1})^2 = \mathcal{E}_{e_{i+1}},$$

as required. $\square$

6. THE SIZE OF THE DOUBLY CONNECTED PART

We are finally in a position to give the principal application of the paper, namely the quantitative lower bound on λ_2^N . It involves the following quantity.

Definition 6.1. The *doubly connected part* $\mathcal{D}_{\mathcal{G}}$ of a graph $\mathcal{G}$ is the closed subgraph consisting of all $x \in \mathcal{G}$ for which there is a non-self-intersecting path in $\mathcal{G}$ starting and ending at x .

The doubly connected part $\mathcal{D}_{\mathcal{G}}$ can be obtained by deleting every bridge (including pendant edges), followed by the removal of any isolated vertices. It may also be characterised as being the largest subgraph of $\mathcal{G}$ whose every connected component is itself doubly edge connected.

Example 6.2. Suppose $\mathcal{D} = \mathcal{D}[\ell_1, \ell_2]$ is a dumbbell having loops e_1 and e_2 of length ℓ_1 and ℓ_2 , respectively. Then $\mathcal{D}_{\mathcal{D}} = e_1 \cup e_2$. More generally, if $\mathcal{G}$ is a pumpkin chain, then $\mathcal{D}_{\mathcal{G}}$ consists of the (possibly disjoint) union of the non-trivial constituent pumpkins of $\mathcal{G}$.

Our goal is to use the tools of Section 3 and the results of Section 5 to derive a lower bound on $\lambda_2^N(\mathcal{G})$ in terms of the total length $|\mathcal{D}_{\mathcal{G}}|$ of $\mathcal{D}_{\mathcal{G}}$ (as well as the length $L := |\mathcal{G}|$ of $\mathcal{G}$). This bound will interpolate between the inequalities of Nicaise [Nic87, Théorème 3.1] and Band–Lévy [BaLe17, Theorem 2.1]. We recall that the former states that for any compact graph $\mathcal{G}$ of total length L , we have

$$(6.1) \quad \lambda_2^N(\mathcal{G}) \geq \lambda_2^N(\mathcal{I}) = \frac{\pi^2}{L^2},$$

where $\mathcal{I}$ is a path graph (interval) of length L ; the latter is for *doubly connected* compact graphs $\mathcal{G}$ of total length L , i.e., graphs $\mathcal{G}$ for which $\mathcal{D}_{\mathcal{G}} = \mathcal{G}$, and reads

$$(6.2) \quad \lambda_2^N(\mathcal{G}) \geq \lambda_2^N(\mathcal{C}) = \frac{4\pi^2}{L^2},$$

where $\mathcal{C}$ is a loop of the same total length (the same inequality was proved earlier in [KN14] for Eulerian graphs). Our theorem is as follows.

Theorem 6.3. *Suppose that the compact and connected graph $\mathcal{G}$ has total length L , and its doubly connected part has total length $V := |\mathcal{D}_{\mathcal{G}}| \in [0, L]$. Let $\mathcal{D} = \mathcal{D}[\frac{V}{2}, \frac{V}{2}]$ be the dumbbell of length L having both loops of length $V/2$. Then*

$$\lambda_2^N(\mathcal{G}) \geq \lambda_2^N(\mathcal{D}).$$

Variants are possible: see Theorem 6.5 and Corollary 6.7. Also note that since $\lambda_2^N(\mathcal{D}[\frac{V}{2}, \frac{V}{2}])$ is monotonically increasing with respect to V (see Proposition 5.9(1)), for any non-tree Theorem 6.3 yields a strictly better estimate than (6.1). Moreover, it contains (6.1) and (6.2) as special cases (the former corresponds to $V = 0$, the latter to $V = L$), and thus interpolates smoothly and monotonically between them as V ranges from 0 to L .

Remark 6.4. Actually, using Lemma 5.1 and Theorem 3.18(2), we can immediately sketch simpler short proofs of (6.1) and (6.2): to prove (6.1), assume without loss of generality that $\mathcal{G}$ is a pumpkin chain, and apply Theorem 3.18(2) repeatedly until every pumpkin is turned into a path. To prove (6.2), note that after applying Lemma 5.1 every constituent pumpkin has at least two edges. Hence apply Theorem 3.18(2) to reduce each

to a pumpkin on two edges. The resulting graph is a $[2, 2, \dots, 2]$ -pumpkin chain. Cutting through all the vertices of degree four lowers λ_2^N and produces $\mathcal{C}$.

Proof of Theorem 6.3. Fix any eigenfunction ψ of $\mathcal{G}$ associated with $\lambda_2^N(\mathcal{G})$. We may assume without loss of generality that ψ does not vanish identically on any edge of $\mathcal{G}$. Indeed, if it did, say on the edge $e \sim v_1 v_2$, we could form a new graph $\tilde{\mathcal{G}}$ by deleting e and gluing v_1 and v_2 ; then $\lambda_2^N(\tilde{\mathcal{G}}) = \lambda_2^N(\mathcal{G})$ by Corollary 3.12(4). Moreover, $|\tilde{\mathcal{G}}| \leq L$, and the bounds in the theorem are decreasing functions of L .

Step 1. We start by creating a pumpkin chain $\mathcal{P}$ out of $\mathcal{G}$ in accordance with Lemma 5.1: then $\mathcal{P}$ still has length L , $\lambda_2^N(\mathcal{P}) = \lambda_2^N(\mathcal{G})$, and up to the usual identification ψ is still an eigenfunction on $\mathcal{P}$, monotonically increasing on each pumpkin. Now gluing together vertices can only shorten the paths used in Definition 6.1, while new paths could be created; hence the size of the doubly connected part can only increase, i.e., $|\mathcal{D}_{\mathcal{P}}| \geq V$.

Step 2. We now apply Theorem 3.18(3) with $m = 2$ to each constituent pumpkin of $\mathcal{P}$, using ψ as the eigenfunction with the necessary properties, to obtain a locally equilateral pumpkin chain $\tilde{\mathcal{P}}$ with $\lambda_2^N(\tilde{\mathcal{P}}) \leq \lambda_2^N(\mathcal{P})$, each of whose pumpkins has either one or two edges, and the sum of the lengths of the two-pumpkins is still $|\mathcal{D}_{\mathcal{P}}|$.

Step 3. Let $\tilde{\psi}$ be an eigenfunction corresponding to $\lambda_2^N(\tilde{\mathcal{P}})$: then, as established in Lemma 5.5, $\tilde{\psi}$ is longitudinal (in particular invariant with respect to permutations of the edges of each two-pumpkin), monotonic between the two terminal vertices of $\tilde{\mathcal{P}}$, and does not vanish on any edge. In particular, the two sets $\tilde{\mathcal{P}}^+ := \{x \in \tilde{\mathcal{P}} : \tilde{\psi}(x) \geq 0\}$ and $\tilde{\mathcal{P}}^- := \{x \in \tilde{\mathcal{P}} : \tilde{\psi}(x) \leq 0\}$ are connected and, up to identifying the (at most two) points where $\tilde{\psi} = 0$, creating the vertex v_0 , are themselves locally equilateral pumpkin chains as in Figure 6.1.



FIGURE 6.1. A depiction of the set $\tilde{\mathcal{P}}^+$; the point 0 refers to the set $\{\tilde{\psi} = 0\}$, which is taken as a Dirichlet condition at the vertex.

We note that $\tilde{\psi}$ continues to be an eigenfunction on $\tilde{\mathcal{P}}^\pm$ and, since it does not change sign, we have $\lambda_2^N(\tilde{\mathcal{P}}) = \lambda_1^D(\tilde{\mathcal{P}}^\pm)$, the latter graphs being equipped with a Dirichlet condition at the vertex corresponding to $\{\tilde{\psi} = 0\}$. We suppose that the sum of the lengths of the two-pumpkins of $\tilde{\mathcal{P}}^\pm$ is $\ell_\pm$; then $\ell_- + \ell_+ = |\mathcal{D}_{\mathcal{P}}|$.

We may thus apply Proposition 5.10 to each of $\tilde{\mathcal{P}}^\pm$ separately to shift the two-pumpkins away from the vertex v_0 : suppose that $\tilde{\mathcal{P}}^+$ consists of pumpkins $\mathcal{P}_1, \dots, \mathcal{P}_m$, each having either one or two edges, and denote by $\hat{\mathcal{P}}^+$ the pumpkin chain with a Dirichlet condition in which all the one-pumpkins have been lined up after the vertex v_0 and the two-pumpkins are in a row after them. Cutting through the vertices of degree four (cf. Figure 6.2), which each separate two of the neighbouring two-pumpkins of $\hat{\mathcal{P}}^+$, we are left with a tadpole graph (which we will still denote by $\hat{\mathcal{P}}^+$) having a loop of length ℓ_+ and a pendant edge of length $|\hat{\mathcal{P}}^+| - \ell_+$, such that $\lambda_1^D(\tilde{\mathcal{P}}^+) \geq \lambda_1^D(\hat{\mathcal{P}}^+)$.

Similarly, we obtain a tadpole graph $\widehat{\mathcal{P}}^-$ having the same total length as $\widetilde{\mathcal{P}}^-$, a loop of length ℓ_- , a pendant edge with a Dirichlet vertex, and an eigenvalue $\lambda_1(\widehat{\mathcal{P}}^-) \leq \lambda_1(\widetilde{\mathcal{P}}^-)$.

Gluing $\widehat{\mathcal{P}}^\pm$ together at their Dirichlet vertices, we obtain a dumbbell $\mathcal{D}[\ell_-, \ell_+]$ with the same total length as $\mathcal{G}$, and such that $\lambda_2^N(\mathcal{D}[\ell_-, \ell_+]) \leq \max \lambda_1(\widehat{\mathcal{P}}^\pm)$ (since the eigenfunctions on the latter two graphs may be glued to create a valid non-trivial eigenfunction on $\mathcal{D}[\ell_-, \ell_+]$).

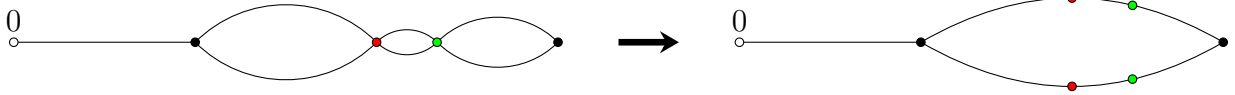


FIGURE 6.2. The graph created from $\widetilde{\mathcal{P}}^+$ by shifting all the two-pumpkins away from the Dirichlet vertex denoted by 0 (left); the graph created by cutting through all the vertices of degree four, to produce the tadpole $\widehat{\mathcal{P}}^+$ (right).

Step 4. We have thus found a dumbbell $\mathcal{D} = \mathcal{D}[\ell_-, \ell_+]$ with $\ell_- + \ell_+ = |\mathcal{D}_{\mathcal{P}}| \geq V$ such that $\lambda_2^N(\mathcal{D}) \leq \lambda_2^N(\widetilde{\mathcal{P}})$. By Proposition 5.9(2), we finally have $\lambda_2^N(\mathcal{D}[\frac{V}{2}, \frac{V}{2}]) \leq \lambda_2^N(\mathcal{D}) \leq \lambda_2^N(\widetilde{\mathcal{P}}) \leq \lambda_2^N(\mathcal{P}) = \lambda_2^N(\mathcal{G})$, proving Theorem 6.3. $\square$

We shall now give a variant of Theorem 6.3 which is stronger if the doubly connected part $\mathcal{D}_{\mathcal{G}}$ is connected, or even if one of its connected components is sufficiently large compared with the rest: here, our object of comparison will be a tadpole graph rather than a dumbbell. We recall our notation from (5.5): for fixed L , $\mathcal{L}[V]$ is the tadpole with total length L and loop length $V \in [0, L]$.

Theorem 6.5. *Suppose the compact and connected graph $\mathcal{G}$ has total length L , and its doubly connected part has a connected component of length $V \in [0, L]$. Then*

$$\lambda_2^N(\mathcal{G}) \geq \lambda_2^N(\mathcal{L}[V]).$$

Note that, for given $V > 0$, $\lambda_2^N(\mathcal{L}[V]) > \lambda_2^N(\mathcal{D}[\frac{V}{2}, \frac{V}{2}])$, as follows from Proposition 5.9(3). Thus if for example $\mathcal{D}_{\mathcal{G}}$ is connected, or even if $\mathcal{D}_{\mathcal{G}}$ has a connected component whose total length is sufficiently close to V , then Theorem 6.5 provides a better estimate than Theorem 6.3.

Sketch of proof of Theorem 6.5. The proof is a simple modification of the proof of Theorem 6.3, so we do not go into much detail. Assuming without loss of generality that the eigenfunction does not vanish identically on any edge, the largest doubly connected component of the pumpkin chain $\mathcal{P}$ formed from $\mathcal{G}$ is at least as large as the largest doubly connected component in $\mathcal{G}$. Locally symmetrising, unfolding as necessary to create a pumpkin-on-a-stick with $m = 2$ and finally invoking Proposition 5.7(4), we get that $\lambda_2^N(\mathcal{G})$ can be estimated from below by the second eigenvalue of the tadpole whose loop is equal to the size of the largest doubly connected component in $\mathcal{G}$. $\square$

Finally, to illustrate the strength of the above theorems, we give a comparison with what is known for *discrete (combinatorial) graph Laplacians*. We recall one of the principal results in this direction, which is in terms of the (discrete) *girth* s of a combinatorial graph G , defined as the shortest cycle length in the graph.

Proposition 6.6. *Among all connected combinatorial graphs G on n vertices with girth $s \geq 3$, the algebraic connectivity (smallest non-trivial eigenvalue of the combinatorial Laplacian) is minimised by the tadpole graph consisting of a cycle of length s attached at one vertex to a path of length $n - s$.*

This is the principal result of [Guo08], following a conjecture of [FK98]. By way of comparison, Theorem 6.5 implies a corresponding statement in terms of the *circumference* $\mathfrak{c} = \mathfrak{c}(\mathcal{G})$ of the metric graph $\mathcal{G}$, which we define to be the *maximum cycle length within $\mathcal{G}$* . (By *cycle* of a metric graph, we mean any closed path within $\mathcal{G}$ in which no edge appears twice, although vertices may be crossed multiple times. The assumption that $\mathcal{G}$ has a finite number of edges of finite length guarantees that this maximum is well defined.) If $\mathcal{G}$ is a tree, we define $\mathfrak{c}(\mathcal{G}) := 0$. We will also use $s = s(\mathcal{G})$ to denote the *metric girth*, i.e., shortest cycle length in $\mathcal{G}$.

Corollary 6.7. *Suppose the compact and connected graph $\mathcal{G}$ has total length L , its circumference is $\mathfrak{c} \in [0, L]$ and its girth is $s \in [0, \mathfrak{c}]$. Then*

$$\lambda_2^N(\mathcal{G}) \geq \lambda_2^N(\mathcal{L}[\mathfrak{c}]) \geq \lambda_2^N(\mathcal{L}[s]).$$

Here $\mathcal{L}[V]$ is, as before, the tadpole graph of total length L and loop length $V \in [0, L]$. Actually, the discrete equivalent of Corollary 6.7 has just been proved; see [XLS18].

Proof of Corollary 6.7. Let $\mathcal{C} \subset \mathcal{G}$ be a cycle of length $\mathfrak{c}$. Then it is immediate from the definition that $\mathcal{C}$ is contained in a connected component of $\mathcal{D}_{\mathcal{G}}$, meaning that this component has length at least as large as $\mathfrak{c}$. Now apply Theorem 6.5 and, if necessary, use that the mapping $V \mapsto \lambda_2^N(\mathcal{L}[V])$ is an increasing function of V for fixed L by Proposition 5.7(1). $\square$

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