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Long rainbow cycles and Hamiltonian cycles using many colors in properly edge-colored complete graphs

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ABSTRACT

We prove two results regarding cycles in properly edge-colored graphs. First, we make a small improvement to the recent breakthrough work of Alon, Pokrovskiy and Sudakov who showed that every properly edge-colored complete graph G on n vertices has a rainbow cycle on at least $n-O(n^{3/4})$ vertices, by showing that G has a rainbow cycle on at least $n-O(\log n\sqrt{n})$ vertices. Second, by modifying the argument of Hatami and Shor which gives a lower bound for the length of a partial transversal in a Latin Square, we prove that every properly colored complete graph has a Hamiltonian cycle in which at least $n-O((\log n)^2)$ different colors appear. For large n, this is an improvement of the previous best known lower bound of $n-\sqrt{2n}$ of Andersen.

1. Introduction

Let G be a graph. An edge-coloring is a *proper edge-coloring* if the set of edges incident to a vertex is given distinct colors. If G is edge-colored, we say that $H \subset G$ is *rainbow* if the colors assigned to the edges of H are distinct.

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There has been extensive research on rainbow properties of properly edge-colored graphs. One problem that has seen significant recent interest is to find many edge-disjoint rainbow spanning trees, see Akbari & Alipour [1], Balogh, Liu & Montgomery [5], Pokrovskiy & Sudakov [14], Carraher, Hartke & Horn [6], and Horn & Nelson [13]. A related old conjecture of Andersen [4] is the following, which if true would be best possible.

Conjecture 1. *Every properly edge-colored complete graph on n vertices contains a rainbow path on $n-1$ vertices.*

In the same paper, Andersen proved the following result.

Theorem 2. *Every properly edge-colored complete graph on n vertices contains a Hamiltonian cycle in which at least $n-\sqrt{2n}$ distinct colors appear.*

In this paper, we make the following improvement (for large n) to Theorem 2.

Theorem 3. *There exists a constant C such that for every n the following holds. Every properly edge-colored complete graph on n vertices contains a Hamiltonian cycle in which at least $n-C(\log n)^2$ distinct colors appear.*

Note that our proof of Theorem 3 very closely follows the proof of Hatami & Shor's [12] result on the length of a partial traversal in a Latin square.

Chen & Li [8] consider a similar problem in edge-colored graphs which are not necessarily proper, but in which every vertex is incident to edges of many different colors, and proposed a generalization of Conjecture 1 in this setting.

Instead of asking for a Hamiltonian cycle that uses many colors, another way to approach Conjecture 1, is to attempt to find a long rainbow cycle. This problem has received recent interest. Akbari, Etesami, Mahini & Mahmoody [2] proved that a cycle of length $n/2-1$ exists in every properly colored complete graph G on n vertices. Then Gyárfás & Mhalla [10] proved that a rainbow path of length $(2n+1)/3$ exists in G provided that, for every color α used, the set of edges given the color α forms a perfect matching in G. Not much later Gyárfás, Ruszinkó, Sárközy, & Schelp [11] showed that G contains a rainbow cycle of length $(4/7+o(1))n$ for every proper edge-coloring. Then, independently, both Gebauer & Mousset [9] and Chen & Li [7] showed that a path of length $(3/4-(1))n$ exists when G is properly colored. This was the best known lower bound until very recently Alon, Pokrovskiy & Sudakov [3] established the following breakthrough result.

Theorem 4. *For all sufficiently large n, every properly edge-colored complete graph on n vertices contains a rainbow cycle on at least $n-24n^{3/4}$ vertices.*

Heavily relying on the methods developed in [3], we make the following improvement to Theorem 4.

Theorem 5. *There exists a constant C such that the following holds. If G is a properly edge-colored complete graph on n vertices for n sufficiently large, then there exists a rainbow cycle in G on at least $n-C\log n\sqrt{n}$ vertices.*

To prove Theorem 5, we use the following theorem which is an extension of Theorem 1.3 in [3]. Our improvement, and one of the main observations that drives our proof, is that essentially the same argument as the one given in [3] works when the sizes of the two sets are unbalanced, i.e., only one of the two sets needs to have order $\Omega\left((\log n/p)^2\right)$, the other can be as small as $\Omega\left(\log n/p\right)$.

Theorem 6. *For every sufficiently small $\varepsilon>0$ there exists a constant C such that the following holds. Let G be a properly edge-colored graph on n vertices such that $\delta(G)\geq(1-\xi)n$ for some $\xi=\xi(n)$. Let H be the spanning subgraph obtained by choosing every color class independently at random with probability p. Then the following holds with high probability.*

(a) If $(1-\xi)n \geq C\frac{\log n}{p}$, then all vertices v have degree $(1 \pm \varepsilon)p \cdot d_G(v)$ in H.
(b) For every pair A and B of disjoint vertex sets, if $|A| \geq C\frac{\log n}{p}$ and $|B| \geq \max\{C(\frac{\log n}{p})^2, C\xi n\}$, then $e_H(A,B) \geq (1-\varepsilon)p|A||B|$.

1.1. Definitions and notation

Most of our notation is standard except possibly the following. Let G be a graph and let A and B be disjoint vertex subsets. We let

$$E_G(A,B) = \{xy \in E(G) : x \in A \text{ and } y \in B\},$$

and let $e_G(A,B) = |E_G(A,B)|$. For a vertex subset U, we let

$$N_G(U) = \bigcup_{u \in U} N_G(u).$$

For a path $P = v_1, \ldots, v_m$ we call v_1 and v_m the *endpoints* of P and we call a path Q a *subpath* of P if $Q = v_i, \ldots, v_j$ for some $1 \leq i \leq j \leq m$. For a set X and $0 \leq y \leq |X|$, we let $\binom{X}{y} = \{Y \subseteq X : |Y| = y\}$ be the set of all subsets of X that have order exactly y. All logarithms are base 2 unless otherwise specified.

2. Proof of Theorem 6

We use the following form of the well-known Chernoff bound.

Lemma 7 (*Chernoff Bound*)**.** *Let X be a binomial random variable with parameters (n,p). Then for every $\varepsilon \in (0,1)$ we have that*

$$\mathbb{P}(|X - pn| \geq \varepsilon pn) \leq 2e^{-pn\varepsilon^2/3}.$$

For an $\varepsilon > 0$, call a pair of disjoint vertex subsets A, B of a properly edge-colored graph G $(1-\varepsilon)$-*rainbow* if there are $(1-\varepsilon)|A||B|$ different colors that appear on the edges $E_G(A,B)$.

The following is essentially equivalent to Lemma 2.2 in [3] and is a simple application of the Chernoff bound (Lemma 7).

Lemma 8. *For every $\varepsilon > 0$, there exists a constant C such that the following holds. Let G be a properly edge-colored graph on n vertices, and let H be the spanning subgraph of G obtained by choosing every color class independently at random with probability p. Then, with high probability, for every $(1-\varepsilon)$-rainbow pair S, T such that $|S| = |T| \geq C\frac{\log n}{p}$ we have that*

$$e_H(S,T) \geq (1-2\varepsilon)p|S||T|.$$

The proof of the following lemma is very similar to Lemma 2.3 in [3]. The main differences are that it can be applied to graphs that are not complete and that the sets A and B can be of different sizes.

Lemma 9. *For every sufficiently small $\varepsilon > 0$, there exists C such that when n is sufficiently large the following holds. Let G be a properly edge-colored graph on n vertices and let A and B be two disjoint vertex subsets of size a and b, respectively, with $a \leq b$. Suppose $\delta(G) \geq (1-\xi)n$ for some $\xi = \xi(n) > 0$. If y divides both a and b; $b \geq \max\{Cy^2, C\xi n\}$; and $y \geq C$, then there exist a partition $\{A_i\}$ of A and a partition $\{B_j\}$ of B with all parts having equal size y such that all but at most $\varepsilon \cdot \frac{ab}{y^2}$ of the pairs A_i, B_j are $(1-\varepsilon)$-rainbow.*

Proof. Let $\varepsilon > 0$ be sufficiently small. We choose $C \geq 2\varepsilon^{-2}$. We then have that

$$\frac{(y-1)^2}{2(b-1)} \leq \frac{y^2}{b} \leq \frac{1}{C} \leq \frac{\varepsilon^2}{2} \quad \text{and} \quad \frac{\xi n}{b} \leq \frac{1}{C} \leq \frac{\varepsilon^2}{2}. \tag{1}$$

Let

$$\Omega = \{\alpha : \exists e \in E(A, B) \text{ such that } e \text{ is colored with } \alpha\}$$

be the set of colors used on the edges of $E(A, B)$. Assume S and T are selected uniformly at random from $\binom{A}{y}$ and $\binom{B}{y}$, respectively.

For every distinct $e, e' \in E(S, T)$ that do not share an endpoint

$$\mathbb{P}(e \in E(S, T)) = \frac{y^2}{ab} \qquad \text{and} \qquad \mathbb{P}(e, e' \in E(S, T)) = \frac{y^2(y-1)^2}{ab(a-1)(b-1)}. \tag{2}$$

Let $\alpha \in \Omega$ and $E_\alpha = \{e \in E(A, B) : e \text{ is give color } \alpha\}$. Since the edge-coloring is proper, no two edges in E_α share an endpoint and $|E_\alpha| \le \min\{a, b\} = a$. Therefore, using the Bonferroni inequalities, (2) and (1), we get the following lower bound on the probability that the color α is used on an edge in $E(S, T)$:

$$\begin{aligned} \mathbb{P}(|E_\alpha \cap E(S, T)| \ge 1) &\ge \sum_{e \in E_\alpha} \mathbb{P}(e \in E(S, T)) - \sum_{\{e,e'\} \in \binom{E_\alpha}{2}} \mathbb{P}(e, e' \in E(S, T)) \\ &= \frac{y^2}{ab}|E_\alpha| - \frac{y^2(y-1)^2}{ab(a-1)(b-1)} \binom{|E_\alpha|}{2} \\ &= \left(1 - \frac{(y-1)^2(|E_\alpha| - 1)}{2(a-1)(b-1)}\right) \frac{y^2}{ab}|E_\alpha| \\ &\ge \left(1 - \frac{(y-1)^2}{2(b-1)}\right) \frac{y^2}{ab}|E_\alpha| \ge \left(1 - \frac{\varepsilon^2}{2}\right) \frac{y^2}{ab}|E_\alpha|. \end{aligned} \tag{3}$$

Let Z be the number of different colors used on the edges of $E(S, T)$. Since

$$\sum_{\alpha \in \Omega} |E_\alpha| = |\cup_{\alpha \in \Omega} E_\alpha| = |E(A, B)| \ge a \cdot (b + \delta(G) - n) \ge ab - a\xi n,$$

the linearity of expectation, (3) and (1) together imply that

$$\mathbb{E}(Z) \ge \sum_{\alpha \in \Omega} (1 - \varepsilon^2/2) \frac{y^2}{ab}|E_\alpha| \ge (1 - \varepsilon^2/2) \frac{y^2}{ab}(ab - a\xi n) \ge \left(1 - \frac{\varepsilon^2}{2} - \frac{\xi n}{b}\right) y^2 \ge (1 - \varepsilon^2) y^2.$$

Clearly, $Z \le |E(S, T)| \le y^2$, so $y^2 - Z \ge 0$, and $\mathbb{E}(y^2 - Z) \le \varepsilon^2 y^2$. Markov's inequality then implies that $\mathbb{P}(y^2 - Z \ge \varepsilon y^2) \le \varepsilon$, so

$$\mathbb{P}\,(S, T \text{ is } (1-\varepsilon)\text{-rainbow}) \ge 1 - \varepsilon. \tag{4}$$

Select an ordered partition $\{A_i\}$ of A and an ordered partition $\{B_j\}$ of B so that each part has equal size y uniformly at random from all such ordered partitions. For every $i \in [a/y]$ and $j \in [b/y]$, by symmetry and (4), the probability that A_i, B_j is $(1-\varepsilon)$-rainbow is at least $1-\varepsilon$. This with the linearity of expectation implies that the expected number of pairs of sets A_i and B_j that are $(1-\varepsilon)$-rainbow is at least $(1-\varepsilon) \cdot \frac{a}{y} \cdot \frac{b}{y}$, so there exists such a partition such that all but at most $\varepsilon \frac{ab}{y^2}$ pairs A_i, B_j are $(1-\varepsilon)$-rainbow. $\square$

Proof of Theorem 6. Assume $\varepsilon > 0$ is sufficiently small and that $C' \ge \varepsilon^{-1}$ is large enough so that both Lemmas 8 and 9 simultaneous apply with C' and $\varepsilon' = \varepsilon/5$ playing the roles of C and ε, respectively. We will show that the conclusions of Theorem 6 hold with $C = 3C'\varepsilon^{-1}$.

To prove part (a) of the lemma, assume that $\delta(G) \ge (1 - \xi)n \ge C \frac{\log n}{p}$, so

$$p \cdot d_G(v) \ge p(1 - \xi)n \ge C \log n. \tag{5}$$

For every $v \in V(G)$, the edges incident to v are rainbow, so the number of edges incident to v that are in H is binomial distributed with parameters $(d_G(v), p)$. The fact that $C \ge 3C'\varepsilon^{-1} \ge 3\varepsilon^{-2}$

and (5) imply that

$$n \cdot 2e^{-p \cdot d_G(v)\varepsilon^2/3} \leq 2n^{-1},$$

so the Chernoff bound (Lemma 7) and the union bound imply that with high probability, for every $v \in V(G)$, we have

$$d_H(v) = (1 \pm \varepsilon)p \cdot d_G(v).$$

This proves (a).

We will now prove (b). Fix $y = \lceil C' \log n/p \rceil$. By Lemma 8, with high probability,

$$e_H(S, T) \geq (1 - 2\varepsilon')y^2, \text{ for every } (1 - \varepsilon')\text{-rainbow pair } S, T \text{ such that } |S| = |Y| = y. \tag{6}$$

We will now assume that (6) holds and show that this implies that (b) holds. Since (6) holds with high probability, this will complete the proof of the lemma.

Let b be the smallest number larger than $\max\{C'y^2, C'\xi n\}$ that is divisible by y. Let A' and B' be vertex disjoint subsets of orders y and b, respectively. Lemma 9 implies that there exists a partition $\{B'_j\}$ of B' into parts of size y such that all but an ε' fraction of the pairs A', B'_j are $(1 - \varepsilon')$-rainbow, i.e., if we let $J = \{j : A', B'_j \text{ is } (1 - \varepsilon')\text{-rainbow}\}$, then $|J| \geq (1 - \varepsilon')yb/y^2$. Therefore, with (6),

$$e_H(A', B') \geq \sum_{j \in J} e_H(A', B'_j) \geq (1 - \varepsilon')yb/y^2 \cdot (1 - 2\varepsilon')y^2 \geq (1 - 3\varepsilon')yb. \tag{7}$$

Finally, to complete the proof of (b), let A and B be disjoint vertex subsets such that $|A| \geq C \log n/p$ and $|B| \geq \max\{C(\log n/p)^2, C\xi n\}$. Recall that $\varepsilon' = 3C'/C$, so $\varepsilon'|A| \geq y$ and $\varepsilon'|B| \geq b$. Therefore, there exists a collection of at least $(1 - \varepsilon')|A|/y$ disjoint subsets of A each of size y and a collection of at least $(1 - \varepsilon')|B|/b$ disjoint subsets of B each of size b. With (7) and the fact that $\varepsilon = 5\varepsilon'$, we have that

$$e_H(A, B) \geq (1 - \varepsilon')|A|/y \cdot (1 - \varepsilon')|B|/b \cdot (1 - 3\varepsilon')yb \geq (1 - \varepsilon)|A||B|. \quad \square$$

3. Long rainbow cycles

This appears as Lemma 3.1 in [3].

Lemma 10. *For all γ, ξ, n with $\xi \geq \gamma$ and $3\gamma\xi - \gamma^2/2 > n^{-1}$ the following holds. Let G be a properly edge-colored graph on n vertices such that $\delta(G) \geq (1 - \xi)n$. Then G contains a rainbow path forest $\mathcal{P}$ with at most γn paths and $|E(\mathcal{P})| \geq (1 - 4\xi)n$.*

For $0 < a \leq b$, call a graph H an (a, b)-*expander*, if the following holds:

(E1) $\delta(H) \geq a$;
(E2) if $A \subseteq V(H)$ such that $|A| \geq a$, then $|N_H(A)| \geq n - a - b$; and
(E3) if A and B are disjoint subsets of order a and b, respectively, then $E_H(A, B) \neq \emptyset$.

Note that (E3) implies (E2), because (E3) implies $|V(G) \setminus (N_H(A') \cup A')| \leq b$ for every $A' \in \binom{A}{a}$, but it is more convenient to state (E2) separately.

Lemma 11. *Let $0 < a \leq b \leq n/4$, $r > 0$, and let G, H_1, H_2, and H_3 be edge-disjoint spanning subgraphs of the complete graph on n vertices whose edges are edge-colored by pairwise disjoint sets of colors such that H_1, H_2 and H_3 are each (a, b)-expanders. Suppose $\mathcal{P} = \{P_1, \ldots, P_r\}$ is a rainbow path forest in G such that for $U = V(G) \setminus V(\mathcal{P})$ we have $|U| \leq b$. If $|P_1| \leq n - a - |U|$, then there exists $e_j \in E(H_j)$ for $j \in [3]$, and $i \in [r]$, such that there are two disjoint paths P'_1 and P'_i in the graph induced in G by $V(P_1) \cup V(P_i) \cup U$ with the additional edges $\{e_1, e_2, e_3\}$ where*

(1) P'_i is a subpath of P_i on less than $|P_i|/2$ vertices (we allow P'_i to be the path without vertices here),
(2) $|P'_1| \geq |P_1| + |P_i| - |P'_i|$, and
(3) $\mathcal{P}' = \mathcal{P} - P_1 - P_i \cup \{P'_1, P'_i\}$ is a rainbow path forest.

Proof. Assume $|P_1| \leq n - a - |U|$ and that the conclusions of the lemma do not hold. Let $v_1, \dots, v_m$ be the vertices of P_1 in the order they appear on the path, let T be the set of vertices on the paths $P_2, \dots, P_r$, and recall that $U = V(G) \setminus V(\mathcal{P})$. We have that

$$|T| = n - |P_1| - |U| \geq a. \tag{8}$$

Claim 11.1. *Let σ be a permutation of $\{1, 2, 3\}$. For every path P such that*

- $|P| \geq |P_1|$,
- $V(P) \subseteq V(P_1) \cup U$, *and*
- $E(P) \subseteq E(P_1) \cup \{e_{\sigma(1)}, e_{\sigma(2)}\}$ *where* $e_\sigma(j) \in E(H_{\sigma(j)})$ *for* $j \in [2]$,

there are no edges in $E(H_{\sigma(3)})$ incident to an endpoint of P and a vertex in T.

Proof. Suppose, for a contradiction, that v is an endpoint of such a path P and there exists $x \in N_{H_{\sigma(3)}}(v) \cap T$. Let $P_i \in \{P_2, \dots, P_r\}$ be the path containing x. We can construct P_1' by combining P with the longer of the two subpaths in P_i that have x as an endpoint. By letting P_i' be the subpath of P_i with the vertex set $V(P_i) \setminus V(P_1')$ (recall that the statement of the theorem allows P_i' to be the path without vertices), we have two paths P_1' and P_i' that satisfy the conclusions of the lemma. $\square$

Let $\mathcal{P}$ be the set of all paths that do not contain an edge from $E(H_3)$ and that satisfy the conditions of Claim 11.1. Let

$$X = \{x \in V(G) : \text{there exists } P \in \mathcal{P} \text{ such that } x \text{ is an endpoint of } P\}.$$

Claim 11.1 implies that $E_{H_3}(X, T) = \emptyset$, so, with (8), we will contradict (E3) and prove the lemma if we can show that $|X| \geq b$.

Claim 11.1 implies that, in H_1, v_1 does not have a neighbor in T, so if we let,

$$Y = (N_{H_1}(v_1) \cap U) \cup \{v_j : v_{j+1} \in N_{H_1}(v_1) \cap V(P_1)\},$$

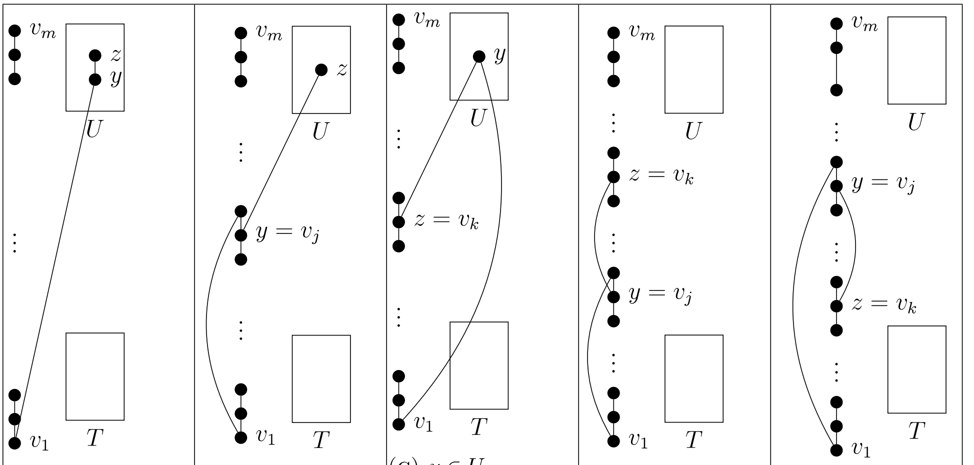
(E1) implies that $|Y| = |N_{H_1}(v_1)| \geq a$. Therefore, by (E2) and the fact that $a \leq b \leq n/4$,

$$|N_{H_2}(Y)| \geq n - a - b \geq 2b. \tag{9}$$

We also have that $Y \subseteq X$. To see this, observe that if $y \in Y \cap U$, then $y, v_1, \dots, v_m$ is in $\mathcal{P}$, and if $y \in Y \setminus U$, then y is on P_1, so $y = v_j$ for some $j \in [m]$ and the path $v_j, \dots, v_1, v_{j+1}, \dots, v_m$ is in $\mathcal{P}$.

We will now describe a mapping from $N_{H_2}(Y)$ to X such that, for every $x \in X$, at most two vertices in $N_{H_2}(Y)$ are mapped to x. By (9), this will imply that $|X| \geq b$, which, as was previously mentioned, will prove the claim. To this end, let $z \in N_{H_2}(Y)$, and arbitrarily select some $y \in N_{H_2}(v) \cap Y$. Recall that there exists $P_y \in \mathcal{P}$ that has y as an endpoint and that does not contain edges from either H_2 or H_3. First assume that z is not on P_1. Then, by using $yz \in E(H_2)$, we can append z to P_y to create an element of $\mathcal{P}$ with z as an endpoint. Therefore, $z \in X$, so, in this case, we map z to itself (see Figs. 1a and 1b). Now assume that, $z = v_k$ for some $k \in [m]$, and recall that Y does not intersect T, so $y \in V(P_1) \cup U$. If $y \in U$, since $yv_1 \in E(H_1)$ and $yv_k \in E(H_2)$ we have that $k \neq 1$, so we can map v_k to v_{k-1}, because $v_{k-1} \dots v_1 y v_k \dots v_m$ is in $\mathcal{P}$ (see Fig. 1c). If $y \in V(P_1)$, then $y = v_j$ for some $2 \leq j \leq m - 1$. Recall that by the definition of Y, $v_1 v_{j+1}$ is an edge in H_1. Since $v_j v_{j+1}$ and $v_j v_{j-1}$ are both in $E(G)$ and $v_j v_k \in E(H_2)$, we either have that $k \geq j + 2$ or $k \leq j - 2$. If $k \geq j + 2$, the path $v_{k-1}, \dots, v_{j+1}, v_1, \dots, v_j, v_k, \dots, v_m$ is in $\mathcal{P}$, so we map v_k to v_{k-1} (see Fig. 1d). Similarly, if $k \leq j - 2$, the path $v_{k+1}, \dots, v_j, v_k, \dots, v_1, v_{j+1}, \dots, v_m$ is in $\mathcal{P}$, so we map v_k to v_{k+1} (see Fig. 1e). Note that we have now proved the lemma, because for every $x \in X$, at most two vertices in $N_{H_2}(Y)$ are mapped to x; if $x \in X \cap U$, then the only vertex that can be mapped to x is x itself, and if $x = v_k$ for some $v_k \in X \cap V(P_1)$ then v_{k-1} and v_{k+1} are the only vertices that can be mapped to x. $\square$

Proof of Theorem 5. Assume $\varepsilon > 0$ is sufficiently small, and pick C large enough so that, provided n is sufficiently large, Theorem 6 applies. Let $p = C\frac{\log n}{\sqrt{n}}$, $a = \sqrt{n}$, $b = n/4$, $\xi = 4p$, and $\gamma = \frac{1}{C \log n \sqrt{n}}$. Let G_1 be a properly edge-colored complete graph on n vertices.



(A) $y, v \in U$, and $z \mapsto z$ (B) $y \in V(P_1)$, $v \in U$, and $z \mapsto z$ (C) $y \in U$, $z \in V(P_1)$, and $z \mapsto v_{k-1}$ (D) $y, z \in V(P_1)$, and $z \mapsto v_{k-1}$ (E) $y, v \in V(P_1)$ and $z \mapsto v_{k+1}$

Fig. 1. The cases considered at the end of the proof of Lemma 11.

Claim 5.1. *There exist edge-disjoint spanning subgraphs, G, H_1, H_2 and H_3, of G_1 that are properly edge-colored with pairwise disjoint sets of colors such that*

(a) $\delta(G) \geq (1 - \xi)n$, and,
(b) for every $m \leq \sqrt{n}$ and $i \in [3]$, if m color classes are removed from H_i, then the resulting graph is an (a, b)-expander.

Proof. We form H_1 by selecting the color classes of G_1 randomly and independently with probability p. With high probability, the conclusions of Theorem 6 hold. We fix such a subgraph H_1.

Note that every vertex has degree $(1 \pm \varepsilon)p(n-1)$ in H_1, so if we let G_2 be the graph formed by removing the edges of H_1 from G_1, we have that every vertex has degree $(1 - p(1 \pm \varepsilon))(n-1)$ in G_2. Therefore, if we form H_2, by selecting the color classes of G_2 randomly and independently with probability p, the conclusions of Theorem 6 hold in H_2 with high probability. We fix such a graph H_2 and note that every vertex has degree $(1 \pm 1.1\varepsilon)p(n-1)$ in H_2. We then let G_3 be the graph formed by removing the edges of H_2 from G_2. Every vertex has degree $(1 - 2(p \pm 1.1\varepsilon))(n-1)$ in G_3, so if we form H_3 by selecting the color classes of G_3 randomly and independently with probability p, H_3 satisfies the conclusions of Theorem 6 with high probability, so we can fix such an H_3. We now have that for every $j \in [3]$, and every vertex v,

$$d_{H_j}(v) \geq (1 \pm 1.2\varepsilon)p(n-1) > 2\sqrt{n}. \tag{10}$$

Let G be the graph formed by removing the edges of H_3 from G_3, and note that, with (10), for every vertex v we have that

$$d_G(v) = (n-1) - \sum_{j \in [3]} d_{H_j}(v) \geq (n-1) - 3(1 + 1.2\varepsilon)np \geq (1 - \xi)n.$$

Because $a \geq C\frac{\log n}{p}$, $b \geq C\left(\frac{\log n}{p}\right)^2$ and $b \geq C\xi n$, the conclusions of Theorem 6, imply that, for $j \in [3]$ and every pair of disjoint vertex sets A and B with sizes at least a and b, respectively, $e_{H_j}(A, B) \geq (1 - \varepsilon)p|A||B|$.

For each $j \in [3]$, form H_j' by removing an arbitrary set of $m \leq \sqrt{n}$ color classes from H_j. By (10), and the fact that H_j is properly edge-colored, we have that $d_{H_j'}(v) \geq d_{H_j}(v) - m \geq a$, so (E1) holds. For every pair of disjoint sets A and B with orders at least a and b, respectively, since $pb = C\sqrt{n}\log n/3 > 2\sqrt{n} \geq 2m$,

$$e_{H'_j}(A, B) \geq e_{H_j}(A, B) - m|A| \geq (1-\varepsilon)p|A||B| - m|A| = |A|((1-\varepsilon)p|B| - m) > 0.$$

Hence, we have established that (E1) and (E3) from the definition of an (a, b)-expander hold in H'_j. As was mentioned in the definition of an (a, b)-expander, (E3) implies (E2), so this completes the proof of the claim. □

Because $\gamma\xi = 4/n$ and $\gamma^2 = o(1/n)$, we have that $3\gamma\xi - \gamma^2/2 \geq 1/n$, so we can apply Lemma 11 to form a rainbow path forest $\mathcal{P} = \{P_1, \ldots, P_r\}$ such that $|V(\mathcal{P})| \geq (1-4\xi)n$ and $r \leq \gamma n$.

We now apply the following algorithm to $\mathcal{P}$.

- If $|P_1| \geq n - 4\xi n - a$ or one of H_1, H_2, or H_3 is not an (a, b)-expander, then terminate.
- Otherwise, we let $\mathcal{P}'$ and e_1, e_2 and e_3 be as in the statement of Lemma 11.
- For each $j \in [3]$, remove the color class corresponding to e_j from H_j and then repeat with $\mathcal{P} = \mathcal{P}'$.

Note that at most $(r-1)\log n < \sqrt{n}$ iterations of the algorithm will execute, since each of the $r-1$ paths $\{P_2, \ldots, P_r\}$ can be used to extend P_1 at most $\log n$ times. To see this, observe that every time such a path P_i is used to extend P_1, at least half of the remaining vertices in P_i are removed. Therefore, by Claim 5.1, the algorithm must terminate with $|P_1| \geq n - 4\xi n - a$.

After the algorithm terminates, we have (a, b)-expanders H_1, H_2 and H_3 and a rainbow path P_1 on at least $n - 4\xi n - a$ vertices such that the edges of H_1, H_2, H_3 and P_1 are colored with disjoint sets of colors. We can now use a procedure similar to the one in the proof of Lemma 11 to form a rainbow cycle of length at least $n - 4\xi n - 3a$ and this will complete the proof. Let $v_1, \ldots, v_m$ be the vertices of P_1 in the order they appear on the path. Let $A_1 = \{v_1, \ldots, v_a\}$ be the first a vertices on P_1 and let $A_2 = \{v_{m-(a-1)}, \ldots, v_m\}$ be the last a vertices on P_1. Assume $E_{H_1}(A_1, A_2) = \emptyset$, since otherwise we have the desired cycle. Let

$$B = \{v_j : v_{j+1} \in N_{H_1}(A_1) \cap (V(P_1) \setminus A_1)\},$$

and note that A_2 and B are disjoint, and, because H_1 is an (a, b)-expander,

$$|B| \geq |N_{H_1}(A_1)| - |V(G) \setminus V(P_1)| - |A_1| \geq (n-a-b) - (4\xi n + a) - a \geq b.$$

Therefore, there exist $v_j \in B$ and $v_k \in A_2$ such that $v_k v_j \in E_{H_2}(A_2, B)$. Recall that there exists $v_i \in A_1$ such that $v_i v_{j+1} \in E(H_1)$ and note that $i < j < k$. Because $v_i, v_{j+1}, \ldots, v_k, v_j, v_{j-1}, \ldots, v_i$ is a cycle that contains all of the vertices $v_i, \ldots, v_k$, we have the desired cycle. □

4. Spanning rainbow path forest with few paths

In this section we prove Theorem 3. In fact, we prove the following more general result which implies the theorem.

Theorem 12. *There exists a constant C such that for every n and for all $\xi = \xi(n) > 0$ the following holds. If G is a properly edge-colored graph on n vertices and $\delta(G) \geq (1-\xi)n$, then G contains a spanning rainbow path forest with at most $C(\log n)^2 + 3\xi n$ paths.*

We will need the following technical lemma. We defer its proof until after the proof of Theorem 12.

Lemma 13. *Suppose that $0 < c \leq 1$ and that $n_1, \ldots, n_k$ is a sequence of strictly increasing positive integers such that for all $m \leq j < \ell \leq k$*

$$n_j - n_{j-1} \geq \frac{n_\ell - n_j}{n_j}\left((1+c)\,n_j - (2n_\ell - n_{\ell-1})\right). \tag{11}$$

Then $k \leq (\log_r n_k)^2 + 2\log_r n_k + m + 1$ where $r = 1 + c/3$.

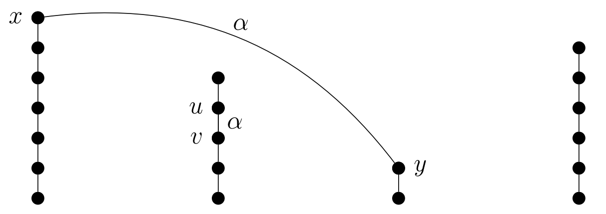


Fig. 2. Because $e = xy$ and $e' = uv$ are both given the color α, swapping e' for e creates a different rainbow path forest. The color α is associated with this swap.

Proof of Theorem 12. Let G be a properly edge-colored graph on n vertices such that $\delta(G) \geq (1-\xi)n$. For a rainbow path forest F, let $p(F)$ be the number of paths in F and let $A(F)$ be the set of endpoints of paths in F. For two rainbow path forests, F and F', we say that F' is obtained from F by a *swap* if there exists an edge e in G that is incident to endpoints of distinct paths in F such that either $F' = F + e$ and there are no edges in $E(F)$ given the same color as e, or $F' = F + e - e'$ where $e' \in E(F)$ such that e' and e are given the same color (see Fig. 2). Note that when F' is obtained from F by a swap, there is a unique color, say α, that is used on the edges in $E(F')\triangle E(F)$. We call α the color *associated* with the swap.

Let $\mathcal{S}(F)$ be the set of rainbow path forests F' that can be obtained from F by a sequence of swaps, i.e., $F' \in \mathcal{S}(F)$ if there exists a sequence of rainbow path forests $F = F_1, F_2, \ldots, F_m = F'$ such that, for every $i \in [m-1]$, F_{i+1} is obtained from F_i by a swap. Note that $p(F) \geq p(F')$ for all $F' \in \mathcal{S}(F)$. If, for all $F' \in \mathcal{S}(F)$, we have that $p(F') = p(F)$ then we say that F is *swap-maximal*. (Spanning path forests with fewer paths have more edges.) Let $C(\mathcal{S}(F))$ contain the set of colors α such that α is the color associated with a swap between two forests in $\mathcal{S}(F)$. For every collection of path forests $\mathcal{F}$, let $A(\mathcal{F}) = \bigcup_{F\in\mathcal{F}} A(F)$ be the set of vertices x for which there exists at least one path forest in $\mathcal{F}$ in which x is an endpoint of a path.

Let k be the minimum number of paths in a spanning rainbow path forest of G and fix F_k a rainbow spanning path forest with k paths. Note that F_k is swap-maximal. We use the following iterative procedure to select swap-maximal forests $F_k \supseteq F_{k-1} \supseteq \cdots \supseteq F_1$. ($F_{k-1}, \ldots, F_1$ will be swap-maximal rainbow forest, but they will not span G.) Suppose that, for $j \geq 2$, $F_k, F_{k-1}, \ldots, F_j$, have been selected so that for all $j \leq \ell \leq k$, F_ℓ is swap-maximal and $p(F_\ell) = \ell$. To select the forest F_{j-1}, we first define a forest F^x_{j-1} and path P^x_{j-1} for every $x \in A(\mathcal{S}(F_j))$. To this end, let $x \in A(\mathcal{S}(F_j))$, and

(i) pick $F \in \mathcal{S}(F_j)$ such that x is an endpoint of a path P in F, and,
(ii) subject to (i), the path P in F containing x is as short as possible.

Note that for every P and F selected in this way, $F - P$ is swap-maximal. To see this, first note that (ii) implies that the colors in $C(\mathcal{S}(F-P))$ are not used on the edges of P. But then, for every $F' \in \mathcal{S}(F-P)$, the forest $P + F'$ is rainbow, so $P + F' \in \mathcal{S}(F)$. This implies that $p(P+F') = p(F) = j$, so $p(F') = p(F-P) = j-1$, which further implies that $F - P$ is swap-maximal. Define $P^x_{j-1} = P$ and $F^x_{j-1} = F - P^x_{j-1}$. To complete the procedure for constructing F_{j-1}, pick $x \in A(\mathcal{S}(F_j))$ so that $|A(\mathcal{S}(F^x_{j-1}))|$ is as small as possible, and then let $F_{j-1} = F^x_{j-1}$.

For every $j \in [k]$, define $A_j = A(\mathcal{S}(F_j))$, $n_j = |A_j|$, $C_j = C(\mathcal{S}(F_j))$ and G_j to be the graph with vertex set $V(G)$ that contains only the edges of G that are assigned a color from C_j. Define $d_j(x) = d_{G_j}(x)$ for $x \in V(G)$, and, for $U \subseteq V(G)$, let $d_j(x, U) = |N_{G_j}(x) \cap U|$. Similarly, for disjoint vertex subsets A and B, we let $E_j(A, B) = E_{G_j}(A, B)$ and $e_j(A, B) = e_{G_j}(A, B)$.

The following claim summarizes some of the important facts implied by this construction.

Claim 12.1. *For every $1 \leq j \leq k$, we have that F_j is swap-maximal. For every $2 \leq j \leq k$, every $x \in A_j$ and every $F \in \mathcal{S}(F^x_{j-1})$, we have that $P^x_{j-1} + F \in \mathcal{S}(F_j)$, so $A(\mathcal{S}(F^x_{j-1}))$ is proper subset of A_j. This and the selection of F_j imply that $n_j > |A(\mathcal{S}(F^x_{j-1}))| \geq n_{j-1}$.*

Claim 12.2. *For every $1 \leq j \leq k$, we have that $\frac{1}{2}n_j - j \leq |C_j| \leq n_j - j$.*

Proof. Let H be the subforest of F_j created by first removing from F_j all edges that were not assigned a color from C_j and then removing all isolated vertices that are not an endpoint of a path in F_j. Recall that for $v \in V(F_j)$, we have that $v \in A_j$ if and only if there exists a path forest in $\mathcal{S}(F_j)$ in which v is an endpoint of a path. Therefore, A_j contains all of the endpoints of paths in F_j and these endpoints are also in $V(H)$. Now consider a vertex v in $V(F_j)$ that is not the endpoint of a path in F_j. Then $v \in A_j$ if and only if at least one of the two edges incident to v in F_j is colored with a color from C_j. Note that we have established that $A_j = V(H)$.

Since H is rainbow and C_j is exactly the set of colors used on the edges of H, $|E(H)| = |C_j|$. Because H is a path forest and $|V(H)| = |A_j| = n_j$, we have that $|C_j| = |E(H)| = n_j - p(H)$. Therefore, to complete the proof, we only need to show that the number of paths in H, $p(H)$, is between j and $\frac{1}{2}n_j + j$. The lower bound on $p(H)$ follows because H contains the endpoints of the $p(F_j) = j$ paths in F_j, and $H \subseteq F_j$. The upper bound on $p(H)$ comes from the fact that the isolated vertices in H must be endpoints of paths in F_j. Therefore, there are at most $2j$ isolated vertices in H. Since all of the paths in H that are not isolated vertices must contain at least 2 vertices, $p(H) \leq \frac{1}{2}(n_j - 2j) + 2j = \frac{1}{2}n_j + j$. □

Claim 12.3. *For every $j \in [k]$ and $x \in A_j$, we have that $d_j(x) \leq |C_j|$ and $d_j(x, A_j) \geq n_{j-1} - \xi n$.*

Proof. The first inequality, $d_j(x) \leq |C_j|$, follows from the fact that G_j is properly edge-colored and only uses colors from C_j. To establish the second inequality, $d_j(x, A_j) \geq n_{j-1} - \xi n$, we first note that, by Claim 12.1, $A(\mathcal{S}(F_{j-1}^x))$ is a subset of A_j, and that $|A(\mathcal{S}(F_{j-1}^x))| \geq n_{j-1}$. Therefore, since $\delta(G) \geq n - \xi n$, we will prove the second inequality by showing that if $y \in A(\mathcal{S}(F_{j-1}^x))$ such that $xy \in E(G)$, then the edge xy is assigned a color from C_j.

To this end, let $y \in A(\mathcal{S}(F_{j-1}^x)) \cap N_G(x)$ and let α be the color assigned to xy. Recall that x is one of the endpoints of P_{j-1}^x and that, by the definition of $A(\mathcal{S}(F_{j-1}^x))$, there exists $F \in \mathcal{S}(F_{j-1}^x)$ in which y is the endpoint of a path. By Claim 12.1, if $F' = P_{j-1}^x + F$, then $F' \in \mathcal{S}(F_j)$. Furthermore, because F_j is swap-maximal, there exists $e \in E(F')$ such that e is assigned the color α. Therefore, α is the color associated with the swap in which $F' + xy - e$ is obtained from F'. This implies that $F' + xy - e \in \mathcal{S}(F_j)$ and $\alpha \in C_j$. This proves the claim. □

Claim 12.4. *For every $2 \leq j < \ell \leq k$,*

$$(n_\ell - n_j)\left(\frac{3}{2}n_j - 2n_\ell + n_{\ell-1} - \xi n\right) \leq e_j(A_\ell \setminus A_j, A_j) \leq n_j(n_j - n_{j-1} - j + \xi n).$$

Proof. Let $x \in A_\ell \setminus A_j$. First note that $d_j(x, A_j) \geq d_\ell(x, A_j) - |C_\ell \setminus C_j|$ because the edges of G and hence G_ℓ are properly colored. Because Claim 12.2 implies that

$$|C_\ell \setminus C_j| \leq (n_\ell - \ell) - \left(\frac{1}{2}n_j - j\right),$$

we have that

$$d_j(x, A_j) \geq d_\ell(x, A_\ell) - |C_\ell \setminus C_j| \geq d_\ell(x, A_j) - (n_\ell - \ell) + \left(\frac{1}{2}n_j - j\right). \tag{12}$$

We also have that,

$$d_\ell(x, A_j) \geq d_\ell(x, A_\ell) - |A_\ell \setminus A_j| = d_\ell(x, A_\ell) - (n_\ell - n_j),$$

and, by Claim 12.3,

$$d_\ell(x, A_\ell) \geq n_{\ell-1} - \xi n,$$

together with (12) we have

$$d_j(x, A_j) \geq \big((n_{\ell-1} - \xi n) - (n_\ell - n_j)\big) - (n_\ell - \ell) + \left(\frac{1}{2}n_j - j\right)$$
$$= \frac{3}{2}n_j + n_{\ell-1} - 2n_\ell + \ell - j - \xi n \geq \frac{3}{2}n_j - 2n_\ell + n_{\ell-1} - \xi n.$$

Summing over all vertices in $A_\ell \setminus A_j$ gives the lower bound.

For every $x \in A_j$, by Claim 12.2, $d_j(x, A_\ell) \leq d_j(x) \leq |C_j| \leq n_j - j$. By Claim 12.3, we also have that $d_j(x, A_j) \geq n_{j-1} - \xi n$. Therefore,

$$d_j(x, A_\ell \setminus A_j) = d_j(x, A_\ell) - d_j(x, A_j) \leq (n_j - j) - (n_{j-1} - \xi n).$$

Summing over all vertices in A_j gives the upper bound. □

Let $m = \lceil 3\xi n \rceil$. Note that we can assume that $k > m$, as otherwise F_k has at most $\lceil 3\xi n \rceil$ paths, and F_k would satisfy the conclusion of the theorem. For every j such that $m \leq j \leq k$, using only the fact that $n_j \geq j \geq m \geq 3\xi n$, we can deduce that

$$\frac{3}{2}n_j - 2n_\ell + n_{\ell-1} - \xi n \geq \frac{7}{6}n_j - (2n_\ell - n_{\ell-1})$$

and

$$n_j - n_{j-1} - j + \xi n \leq n_j - n_{j-1}.$$

Therefore, Claim 12.4 implies that, for every $m \leq j < \ell \leq k$,

$$n_j - n_{j-1} \geq n_j - n_{j-1} - j + \xi n \geq$$
$$\frac{n_\ell - n_j}{n_j}\left(\frac{3}{2}n_j - 2n_\ell + n_{\ell-1} - \xi n\right) \geq \frac{n_\ell - n_j}{n_j}\left(\left(1 + \frac{1}{6}\right)n_j - (2n_\ell - n_{\ell-1})\right).$$

We can then apply Lemma 13 to $n_1, \ldots, n_k$ to deduce that, with $r = 19/18$, we have $k \leq (\log_r n_k)^2 + 2\log_r n_k + m + 1$. We can assume that $\log_r n_k \geq 1$, so if we let $C = 5(\log_r 2)^2$, then

$$C\,(\log n_k)^2 = 5\,(\log_r n_k)^2 \geq (\log_r n_k)^2 + 2\log_r n_k + 2.$$

Therefore, since $m < 3\xi n + 1$, we have that F_k is a spanning rainbow forest with

$$k \leq (\log_r n_k)^2 + 2\log_r n_k + m + 1 \leq C\,(\log n_k)^2 + 3\xi n,$$

paths. This completes the proof of Theorem 12. □

Proof of Lemma 13.

Claim 13.1. *For all $m \leq j < \ell \leq k$, if $n_\ell \leq r n_j$, then $n_\ell - n_j \leq r^{-1}(n_\ell - n_{j-1})$.*

Proof. We have that

$$2n_\ell - n_{\ell-1} \leq 2n_\ell - n_j \leq (2r - 1)n_j.$$

With the fact that $c = 3r - 3$, this implies that

$$(1 + c)n_j - (2n_\ell - n_{\ell-1}) \geq (1 + c - (2r - 1))n_j = (r - 1)n_j.$$

Combining this with (11) gives us that $(r - 1)(n_\ell - n_j) \leq n_j - n_{j-1}$, so

$$r(n_\ell - n_j) = n_\ell - n_j + (r - 1)(n_\ell - n_j) \leq n_\ell - n_j + n_j - n_{j-1} = n_\ell - n_{j-1},$$

which proves the claim. □

Claim 13.2. *For all $m \leq j < \ell \leq k$, if $n_\ell \leq r n_j$, then $\ell - j < \log_r n_j$.*

Proof. Note that if $n_\ell \le rn_j$ and $j \le \ell - 1$, then for every i such that $j \le i \le \ell - 1$, we have that $n_\ell \le rn_j \le rn_i$. Therefore, Claim 13.1 implies $n_\ell - n_i \le r^{-1}(n_\ell - n_{i-1})$, which further implies that

$$1 \le n_\ell - n_{\ell-1} \le r^{-(\ell-(j+1))}(n_\ell - n_j),$$

and $\ell - (j+1) \le \log_r(n_\ell - n_j)$. Using this and the fact that our assumption $n_\ell \le rn_j$ implies that $n_\ell - n_j \le (r-1)n_j$ we have that

$$\ell - (j+1) \le \log_r(n_\ell - n_j) \le \log_r((r-1)n_j).$$

Therefore, because $1 < r \le 4/3$ implies that $r - 1 < r^{-1}$, i.e., $\log_r(r-1) < -1$, we have

$$\log_r n_j \ge \ell - j - 1 - \log_r(r-1) > \ell - j,$$

and the claim holds. □

Let $t = \lceil \log_r n_k \rceil$ and $s = \left\lfloor \frac{k-m}{t} \right\rfloor$. Note that, for every $1 \le i \le s$,

$$(it + m) - ((i-1)t + m) = t \ge \log_r n_k,$$

and, therefore, Claim 13.2 implies that

$$n_{it+m} \ge rn_{(i-1)t+m},$$

so $n_{it+m} \ge r^i n_m$, and

$$n_k \ge n_{st+m} \ge r^s n_m \ge r^s.$$

Therefore,

$$\log_r n_k \ge s \ge \frac{k-m}{t} - 1 \ge \frac{k-m}{(\log_r n_k) + 1} - 1,$$

which implies

$$k \le (\log_r n_k)^2 + 2\log_r n_k + m + 1. \quad \square$$

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