# POISSON STRUCTURE OF THE THREE-DIMENSIONAL EULER EQUATIONS IN FOURIER SPACE

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Abstract. We derive a Poisson structure in the space of Fourier modes for the Euler equations in vorticity formulation on a three-dimensional periodic domain. This allows us to analyse the structure of the Euler equations using a Hamiltonian framework. The Poisson structure is then restricted to the divergence-free subspace on which the dynamics of the Euler equations takes place, reducing the size of the system of ODEs by a third. The divergence-free subspace is realised as a subspace define by sub-Casimirs, which are invariants which are Casimirs only after restriction to the subspace. The Poisson structure is shown to have the helicity as a Casimir invariant. We conclude by showing that periodic shear flows in three dimensions are equilibria that correspond to singular points of the Poisson structure, and hence the usual approach to study their nonlinear stability through the Energy-Casimir method fails.

## 1. Introduction

The field of Hamiltonian hydrodynamics has been a rich area of study [AK98, Mor98, Sal88]. Arnold described the dynamics of the Euler equations on a two-dimensional periodic domain as geodesics on the manifold of volume-preserving diffeomorphisms [Arn66b, AK98]. This was used to prove, using the energy-Casimir method, stability results for shear flows [Arn66a, AK98]. This structure also led to the truncated Poisson structure approximating the Euler equations which was developed by Zeitlin [Zei91, Zei05], which in turn led to the development of a Poisson integrator [McL93] and a number of stability and instability results for shear flows [DMW16, DW17]. For alternative approaches to geometric integration of incompressible fluid equations that are not in Fourier space see, e.g., [Sal04, PMT$^+$11, Mor17]. For more background on fluid brackets and in particular on the question on how to project onto the divergence free subspace see [CDGB$^+$13]. We present here a new Poisson structure in Fourier space for the Euler equations on a three-dimensional periodic domain. The Poisson structure encapsulates the geometry underlying the dynamics of the Fourier coefficients of the vorticity. It is hoped that this will lay the foundations for geometric integration of the three-dimensional incompressible fluid equations. As an application we study periodic shear flows in three dimensions, and show

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that for these equilibria the Poisson structure is singular. As a result, the Energy-Casimir method cannot be applied.

## 2. Vorticity Formulation for the Euler Equations

The vorticity form of the Euler equations for an incompressible, inviscid fluid on some three-dimensional domain $\mathcal{D} \subset \mathbb{R}^3$ is

$$\frac{\partial \Omega}{\partial t} = (\Omega \cdot \nabla)V - (V \cdot \nabla)\Omega. \tag{2.1}$$

Here $V : \mathcal{D} \times \mathbb{R} \to \mathbb{R}^3$ describes the instantaneous velocity of a fluid element at position $\mathbf{x} = (x, y, z) \in \mathcal{D}$ and time $t \in \mathbb{R}$, and the vorticity $\Omega(\mathbf{x}; t) = \nabla \times V$ measures the local rotational motion of a fluid element. One can think of each component of $\Omega$ representing the rotation of the fluid around the axis parallel to the corresponding coordinate.

We assume that the fluid density $\rho$ is constant — without loss of generality we can set $\rho = 1$ — and that the two vector fields satisfy the *divergence-free conditions*

$$\nabla \cdot V = 0, \quad \nabla \cdot \Omega = 0. \tag{2.2}$$

For the velocity field, this is a consequence of the assumed incompressibility; for the vorticity, this follows from the fact that the divergence of the curl of any vector must be zero. A recent review of the dynamics of the Euler equations is given in [Con07] for example.

We will consider these equations on the periodic domain

$$\mathcal{D} = \left[0, \frac{2\pi}{\kappa_x}\right) \times \left[0, \frac{2\pi}{\kappa_y}\right) \times \left[0, \frac{2\pi}{\kappa_z}\right)$$

for some $\kappa_x, \kappa_y, \kappa_z \in \mathbb{R}^+$. The *anisotropy matrix*

$$K := \begin{pmatrix} \kappa_x & 0 & 0 \\ 0 & \kappa_y & 0 \\ 0 & 0 & \kappa_z \end{pmatrix}$$

represents the unit wavenumbers in $\mathcal{D}$; it is diagonal, positive definite, and invertible. For an isotropic domain we can scale so that $K = \mathbb{I}_3$; in general we allow for anisotropy by letting the wavenumbers be arbitrary. The allowed wavenumbers on $\mathcal{D}$ are elements of the scaled integer lattice

$$D = \{K\mathbf{a} \mid \mathbf{a} \in \mathbb{Z}^3\}. \tag{2.3}$$

Compare this to the approach in [DW17], where the lattice $\mathbb{Z}^3$ is used for the mode numbers instead of $D$. By scaling the lattice instead, the influence of the domain size is mostly hidden from view. This should not be misinterpreted as meaning it is unimportant; for instance, the stability results of [DW17] depend on the domain size. However, "hiding" the aspect ratios in this way makes the exposition less cumbersome.

The velocity and vorticity Fourier coefficients for a wavenumber $\mathbf{j}$ in the lattice (2.3) are given by integrals over the domain $\mathcal{D}$:

$$\mathbf{v_j} = \frac{\kappa_x \kappa_y \kappa_z}{(2\pi)^3} \int_{\mathcal{D}} V(\mathbf{x}, t) e^{-i\mathbf{j}\cdot\mathbf{x}} \mathrm{d}\mathbf{x},$$
$$\boldsymbol{\omega}_\mathbf{j} = \frac{\kappa_x \kappa_y \kappa_z}{(2\pi)^3} \int_{\mathcal{D}} \Omega(\mathbf{x}, t) e^{-i\mathbf{j}\cdot\mathbf{x}} \mathrm{d}\mathbf{x}$$

where $\mathbf{v_j}(t), \boldsymbol{\omega}_\mathbf{j}(t) \in \mathbb{C}^3$. The inverse transform is

$$V = \sum_{\mathbf{j}\in D} \mathbf{v_j}(t) e^{i\mathbf{j}\cdot\mathbf{x}},$$
$$\Omega = \sum_{\mathbf{j}\in D} \boldsymbol{\omega}_\mathbf{j}(t) e^{i\mathbf{j}\cdot\mathbf{x}}. \tag{2.4}$$

Since $V$ and $\Omega$ are real, $\mathbf{v_{-j}} = \overline{\mathbf{v}}_\mathbf{j}$ and $\boldsymbol{\omega}_\mathbf{-j} = \overline{\boldsymbol{\omega}}_\mathbf{j}$.

The divergence free conditions (2.2) imply that for all $j \in D$,

$$\mathbf{j}\cdot\mathbf{v_j} = 0, \;\; \mathbf{j}\cdot\boldsymbol{\omega}_\mathbf{j} = 0. \tag{2.5}$$

Thus, all the dynamics of the Euler equations occur on the *divergence-free subspace*

$$\mathcal{M} = \{\boldsymbol{\omega}_\mathbf{j} \in \mathbb{R}^3 \;\mid\; \mathbf{j} \in D,\, \mathbf{j}\cdot\boldsymbol{\omega}_\mathbf{j} = 0\}. \tag{2.6}$$

When $\mathbf{v}$ is divergence free, this space is invariant under the Euler equations (2.1).

To write (2.1) in Fourier space, it is convenient to define an antisymmetric matrix, the cross-product matrix, for a vector $\mathbf{a} = (a_x, a_y, a_z)^\mathsf{T}$ by

$$\widehat{\mathbf{a}} = \begin{pmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{pmatrix}. \tag{2.7}$$

The point is that for any vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, $\widehat{\mathbf{a}}\mathbf{b} = \mathbf{a}\times\mathbf{b}$. Thus for example, the condition $\Omega = \nabla \times V$ implies

$$\boldsymbol{\omega}_\mathbf{j} = i\mathbf{j}\times\mathbf{v_j} = i\widehat{\mathbf{j}}\mathbf{v_j}. \tag{2.8}$$

The properties of the cross product imply that for any invertible $3\times 3$ matrix $M$, $M\mathbf{a}\times M\mathbf{b} = \det(M)M^{-\mathsf{T}}(\mathbf{a}\times\mathbf{b})$. Thus we have

$$\widehat{M\mathbf{a}} = \det(M)M^{-\mathsf{T}}\widehat{\mathbf{a}}M^{-1}.$$

In the special case of a rotation matrix $R \in SO(3)$, this reduces to $\widehat{R\mathbf{a}} = R\widehat{\mathbf{a}}R^\mathsf{T}$.

We wish to write (2.1) as an infinite set of differential equations for the dynamics of the vorticity modes $\boldsymbol{\omega}_\mathbf{j}$ only. Combining (2.5) and (2.8) implies

$$\mathbf{j}\times\boldsymbol{\omega}_\mathbf{j} = -i|\mathbf{j}|^2\mathbf{v_j}.$$

Thus from (2.8), for all $\mathbf{j} \neq \mathbf{0}$, we can take the divergence-free inverse to the curl

$$\mathbf{v_j} = i\frac{\mathbf{j}\times\boldsymbol{\omega}_\mathbf{j}}{|\mathbf{j}|^2}. \tag{2.9}$$

In addition we assume that we are in a frame of reference where $\mathbf{v_0} = \mathbf{0}$. Note also that $\boldsymbol{\omega}_\mathbf{0} = \mathbf{0}$ by (2.8). This allows us to formally write $V = (\nabla\times)^{-1}\Omega$ on the divergence-free subspace.

Since $\rho = 1$, the energy per unit volume becomes

$$\begin{aligned} E &= \frac{\kappa_x\kappa_y\kappa_z}{2(2\pi)^3}\int_\mathcal{D} V\cdot V \mathrm{d}\mathbf{x} = \frac{1}{2}\sum_{\mathbf{j}\in D}\mathbf{v_j}\cdot\mathbf{v_{-j}} \\ &= \frac{1}{2}\sum_{\mathbf{j}\in D\setminus\{\mathbf{0}\}}\frac{\boldsymbol{\omega}_\mathbf{j}\cdot\boldsymbol{\omega}_\mathbf{-j}}{|\mathbf{j}|^2}, \end{aligned} \tag{2.10}$$

where we used (2.5) and (2.9) for the last step.

Now (2.9) implies that

$$\begin{aligned}(\Omega\cdot\nabla)V &= \sum_{\mathbf{j},\mathbf{k}\in D\setminus\{\mathbf{0}\}} i(\mathbf{j}\cdot\boldsymbol{\omega}_{\mathbf{k}})\mathbf{v}_{\mathbf{j}}e^{i(\mathbf{j}+\mathbf{k})\cdot\mathbf{x}}\\ &= -\sum_{\mathbf{j},\mathbf{k}\in D\setminus\{\mathbf{0}\}} (\mathbf{j}\cdot\boldsymbol{\omega}_{\mathbf{k}})\frac{\mathbf{j}\times\boldsymbol{\omega}_{\mathbf{j}}}{|\mathbf{j}|^2}e^{i(\mathbf{j}+\mathbf{k})\cdot\mathbf{x}}.\end{aligned}$$

Similarly

$$\begin{aligned}(V\cdot\nabla)\Omega &= \sum_{\mathbf{j},\mathbf{k}\in D\setminus\{\mathbf{0}\}} i(\mathbf{j}\cdot\mathbf{v}_{\mathbf{k}})\boldsymbol{\omega}_{\mathbf{j}}e^{i(\mathbf{j}+\mathbf{k})\cdot\mathbf{x}}\\ &= -\sum_{\mathbf{j},\mathbf{k}\in D\setminus\{\mathbf{0}\}} \frac{1}{|\mathbf{k}|^2}\mathbf{j}\cdot(\mathbf{k}\times\boldsymbol{\omega}_{\mathbf{k}})\boldsymbol{\omega}_{\mathbf{j}}e^{i(\mathbf{j}+\mathbf{k})\cdot\mathbf{x}}.\end{aligned}$$

Thus rewriting (2.1) in Fourier space leads to the system of ODEs

$$\begin{aligned}\dot{\boldsymbol{\omega}}_{\mathbf{j}} &= \sum_{\mathbf{k}+\mathbf{l}=\mathbf{j}} \frac{1}{|\mathbf{k}|^2}\left(\mathbf{l}\cdot(\mathbf{k}\times\boldsymbol{\omega}_{\mathbf{k}})\boldsymbol{\omega}_{\mathbf{l}} - (\mathbf{k}\cdot\boldsymbol{\omega}_{\mathbf{l}})\mathbf{k}\times\boldsymbol{\omega}_{\mathbf{k}}\right)\\ &= \sum_{\mathbf{k}\in D\setminus\{\mathbf{0}\}} A(\mathbf{j},\mathbf{k})\frac{\boldsymbol{\omega}_{-\mathbf{k}}}{|\mathbf{k}|^2},\end{aligned} \tag{2.11}$$

where

$$A(\mathbf{j},\mathbf{k}) = \left(\boldsymbol{\omega}_{\mathbf{j}+\mathbf{k}}(\mathbf{k}\times\mathbf{j})^{\mathsf{T}} - (\mathbf{k}\cdot\boldsymbol{\omega}_{\mathbf{j}+\mathbf{k}})\widehat{\mathbf{k}}\right). \tag{2.12}$$

Note that since $\mathbf{v}_{\mathbf{0}} = \boldsymbol{\omega}_{\mathbf{0}} = \mathbf{0}$, the term $\mathbf{k} = \mathbf{0}$ has been omitted in the summation so there is no singularity in (2.11). Thus we derived an infinite set of ordinary differential equation for the vorticity modes that does not depend on velocity.

## 3. Poisson Structure

To obtain a Poisson bracket for the Fourier space Euler equations (2.11) we define

$$\begin{aligned}J(\mathbf{j},\mathbf{k}) &:= A(\mathbf{j},\mathbf{k}) - \frac{1}{2}((\mathbf{j}+\mathbf{k})\cdot\boldsymbol{\omega}_{\mathbf{k}+\mathbf{j}})\widehat{\mathbf{j}-\mathbf{k}}\\ &= \boldsymbol{\omega}_{\mathbf{j}+\mathbf{k}}(\mathbf{k}\times\mathbf{j})^{\mathsf{T}} + \frac{1}{2}((\mathbf{j}-\mathbf{k})\cdot\boldsymbol{\omega}_{\mathbf{j}+\mathbf{k}})\widehat{\mathbf{k}} - \frac{1}{2}((\mathbf{j}+\mathbf{k})\cdot\boldsymbol{\omega}_{\mathbf{j}+\mathbf{k}})\widehat{\mathbf{j}}.\end{aligned} \tag{3.1}$$

On the subspace $\mathcal{M}$, (2.6), $A(\mathbf{j},\mathbf{k}) = J(\mathbf{j},\mathbf{k})$ and so for initial conditions in this space, the Euler equations (2.11) become

$$\dot{\boldsymbol{\omega}}_{\mathbf{j}} = \sum_{\mathbf{k}} J(\mathbf{j},\mathbf{k})\frac{\boldsymbol{\omega}_{-\mathbf{k}}}{|\mathbf{k}|^2}. \tag{3.2}$$

Notice that $A$ and $J$ are matrix valued functions depending on two lattice vectors $\mathbf{j},\mathbf{k}$ and a vorticity vector $\boldsymbol{\omega}_{\mathbf{j}+\mathbf{k}}$, but the dependence on the latter is suppressed in the notation. By writing the differential equations in this form, the Euler equations on a three-dimensional periodic domain are a Poisson system. To show this we first obtain a simple lemma.

**Lemma 3.1.** *The matrix* $J(\mathbf{j},\mathbf{k})$ *satisfies* $J(\mathbf{j},\mathbf{k}) + J(\mathbf{k},\mathbf{j})^{\mathsf{T}} = 0$ *and* $(\mathbf{j}^{\mathsf{T}}J(\mathbf{j},\mathbf{k}))^{\mathsf{T}} = J(\mathbf{j},\mathbf{k})\mathbf{k} = -\frac{1}{2}((\mathbf{j}+\mathbf{k})\cdot\omega_{\mathbf{j}+\mathbf{k}})\mathbf{j}\times\mathbf{k}$.

*Proof.* These can both be directly verified by computation. Note in particular that both results are valid for arbitrary values of $\boldsymbol{\omega}_{\mathbf{j}+\mathbf{k}}$. For values of $\boldsymbol{\omega}_{\mathbf{j}+\mathbf{k}}$ that are in the divergence free subspace and hence satisfy $(\mathbf{j}+\mathbf{k})\cdot\omega_{\mathbf{j}+\mathbf{k}} = 0$, both $J(\mathbf{j},\mathbf{k})\mathbf{k}$ and $\mathbf{j}^\mathsf{T} J(\mathbf{j},\mathbf{k})$ vanish. □

**Theorem 3.2** (Poisson Structure for Three-Dimensional Euler Equations)**.** *The dynamics of the three-dimensional Euler equations for an incompressible, inviscid flow on a periodic domain are a Poisson system with bracket*

$$\{f,g\} = \sum_{\mathbf{j},\mathbf{k}\in D} \left(\frac{\partial f}{\partial \boldsymbol{\omega}_{\mathbf{j}}}\right)^\mathsf{T} J(\mathbf{j},\mathbf{k}) \left(\frac{\partial g}{\partial \boldsymbol{\omega}_{\mathbf{k}}}\right) \tag{3.3}$$

*where $J(\mathbf{j},\mathbf{k})$ is given by* (3.1)*, and Hamiltonian*

$$H = \frac{1}{2}\sum_{\mathbf{j}\in D\setminus\{\mathbf{0}\}} \frac{\boldsymbol{\omega}_{-\mathbf{j}}\cdot\boldsymbol{\omega}_{\mathbf{j}}}{|\mathbf{j}|^2} \tag{3.4}$$

*on the space of Fourier modes $\boldsymbol{\omega}_{\mathbf{j}} \in \mathbb{R}^3$ for all $\mathbf{j}\in D$.*

*The physically relevant divergence free subspace is an invariant subspace of the dynamics, given by the Poisson submanifold* (2.6)*.*

*Proof.* For the Hamiltonian (3.4), the differential equations for $\boldsymbol{\omega}_{\mathbf{j}}$ are given by

$$\dot{\boldsymbol{\omega}}_{\mathbf{j}} = \{\boldsymbol{\omega}_{\mathbf{j}}, H\} = \sum_{\mathbf{k}\neq\mathbf{0}} J(\mathbf{j},\mathbf{k})\frac{\boldsymbol{\omega}_{-\mathbf{k}}}{|\mathbf{k}|^2}$$

which is the same as (3.2).

To be a Poisson bracket, (3.3) must be antisymmetric, bilinear, and satisfy the Jacobi identity [Olv00, Mei17]. One can check

$$\begin{aligned}
\{g,f\} &= \sum_{\mathbf{k},\mathbf{j}\in D} \left(\frac{\partial g}{\partial \boldsymbol{\omega}_{\mathbf{k}}}\right)^\mathsf{T} J(\mathbf{k},\mathbf{j}) \left(\frac{\partial f}{\partial \boldsymbol{\omega}_{\mathbf{j}}}\right)\\
&= -\sum_{\mathbf{k},\mathbf{j}\in D} \left(\frac{\partial g}{\partial \boldsymbol{\omega}_{\mathbf{k}}}\right)^\mathsf{T} J(\mathbf{j},\mathbf{k})^\mathsf{T} \left(\frac{\partial f}{\partial \boldsymbol{\omega}_{\mathbf{j}}}\right)\\
&= -\sum_{\mathbf{k},\mathbf{j}\in D} \left(\left(\frac{\partial f}{\partial \boldsymbol{\omega}_{\mathbf{j}}}\right)^\mathsf{T} J(\mathbf{j},\mathbf{k}) \left(\frac{\partial g}{\partial \boldsymbol{\omega}_{\mathbf{k}}}\right)\right)^\mathsf{T}\\
&= -\{f,g\}
\end{aligned}$$

as $J(\mathbf{k},\mathbf{j})^\mathsf{T} = -J(\mathbf{j},\mathbf{k})$ by Lemma 3.1. Thus the bracket is antisymmetric. It also is bilinear as the bracket is linear in the partial derivatives.

The final condition is the Jacobi identity, $\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\} = 0$. For a bracket of the form (3.3), this can be reduced to a condition on the structure matrix, [Olv00, Eq. (6.15)], which becomes

$$
\begin{aligned}
0 &= \sum_{\delta\in\{x,y,z\},\mathbf{l}\in D} \bigg[ J(\mathbf{i},\mathbf{l})_{\alpha,\delta} \frac{\partial J(\mathbf{j},\mathbf{k})_{\beta,\gamma}}{\partial(\omega_{\mathbf{l}})_\delta} \\
&\qquad + J(\mathbf{k},\mathbf{l})_{\gamma,\delta} \frac{\partial J(\mathbf{i},\mathbf{j})_{\alpha,\beta}}{\partial(\omega_{\mathbf{l}})_\delta} + J(\mathbf{j},\mathbf{l})_{\beta,\delta} \frac{\partial J(\mathbf{k},\mathbf{i})_{\gamma,\alpha}}{\partial(\omega_{\mathbf{l}})_\delta} \bigg] \\
&= \sum_{\delta\in\{x,y,z\}} \bigg[ J(\mathbf{i},\mathbf{j}+\mathbf{k})_{\alpha,\delta} \frac{\partial J(\mathbf{j},\mathbf{k})_{\beta,\gamma}}{\partial(\omega_{\mathbf{j}+\mathbf{k}})_\delta} \\
&\qquad + J(\mathbf{k},\mathbf{i}+\mathbf{j})_{\gamma,\delta} \frac{\partial J(\mathbf{i},\mathbf{j})_{\alpha,\beta}}{\partial(\omega_{\mathbf{i}+\mathbf{j}})_\delta} + J(\mathbf{j},\mathbf{k}+\mathbf{i})_{\beta,\delta} \frac{\partial J(\mathbf{k},\mathbf{i})_{\gamma,\alpha}}{\partial(\omega_{\mathbf{k}+\mathbf{i}})_\delta} \bigg]
\end{aligned} \tag{3.5}
$$

for all $\alpha,\beta,\gamma \in \{x,y,z\}$ and $\mathbf{i},\mathbf{j},\mathbf{k} \in D$. Here we have used $x,y,z$-subscripts to denote the components, e.g.,

$$\boldsymbol{\omega}_{\mathbf{j}} = ((\omega_{\mathbf{j}})_x, (\omega_{\mathbf{j}})_y, (\omega_{\mathbf{j}})_z),$$

and similarly for the matrix $J$. The relation (3.5) can be directly verified from the definition of $J(\mathbf{j},\mathbf{k})$ by a tedious calculation.

The Jacobi property and antisymmetry are both satisfied for all $\boldsymbol{\omega}_{\mathbf{j}} \in \mathbb{R}^3$; however the only physically relevant dynamics occur on the submanifold $\mathcal{M}$, the divergence free subspace. This is an invariant subspace because Lemma 3.1 gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{j}\cdot\boldsymbol{\omega}_{\mathbf{j}}) = \{\mathbf{j}\cdot\boldsymbol{\omega}_{\mathbf{j}}, H\} = \sum_{\mathbf{j},\mathbf{k}} \mathbf{j}^{\mathsf{T}} J(\mathbf{j},\mathbf{k}) \frac{\partial H}{\partial \boldsymbol{\omega}_k} = 0 \tag{3.6}$$

assuming $\mathbf{j}\cdot\boldsymbol{\omega}_{\mathbf{j}} = 0$ for all $\mathbf{j}$ . We can thus consider the Poisson system on the submanifold $\mathcal{M}$ an invariant subsystem of a larger system without the divergence-free restriction.

Thus the system given by (3.3), (3.4) and (2.6) is a Poisson system, and generates the dynamics of the Euler equations on a three-dimensional periodic domain. □

It appears surprising that the Poisson structure in Fourier space is not "tainted" in the terminology of [CDGB+13]. The particular extension from $A$ to $J$ that we have chosen does turn $J$ into a true Poisson structure, without the necessity to employ projectors as in [CDGB+13]. In the next section we will explicitly reduce the Poisson structure to the divergence free subspace, by employing a local transformation that allows for the explicit projection and restriction to the subspace.

It may seem as if the combinations $\mathbf{j}\cdot\boldsymbol{\omega}_{\mathbf{j}}$ should be Casimirs of the Poisson structure. Recall that a function $f$ is a Casimir of a bracket if $\{f,g\} = 0$ for any function $g$. If $f$ is a Casimir then it is an invariant of any dynamics generated by the bracket, independent of the choice of Hamiltonian. In our case, (3.6) is valid for any $H$ on the invariant subspace (2.6), but it does not hold for an arbitrary Hamiltonian outside the divergence free subspace. Such functions are often called *sub-Casimirs*, see, e.g. [HMRW85, OPBR05].

Having a Poisson formulation of our system is very useful, as it allows us to utilise the ideas and language of Hamiltonian mechanics. However, as it stands analysing the system is hindered by the fact that the physical dynamics takes place on the invariant manifold $\mathcal{M}$ rather than $\mathbb{R}^3$, where the bracket is defined. Fortunately, $\mathcal{M}$ is simple enough so that we can restrict the Poisson structure to the subspace.

## 4. Poisson Structure on the Divergence-Free Subspace

We wish to explicitly restrict our system to the manifold $\mathcal{M}$ (2.6), the divergence free subspace where $\mathbf{j}\cdot\boldsymbol{\omega}_{\mathbf{j}} = 0$ for all $\mathbf{j}$. We will write the Poisson system as a lower-dimensional, unrestricted system, thereby reducing the dimensionality of the system by one-third.

To do so, we introduce a rotation matrix $R_{\mathbf{j}} \in SO(3)$ for each $\mathbf{j} \in \mathbb{R}^3$ with the goal of making $\mathbf{j} \cdot \boldsymbol{\omega}_{\mathbf{j}}$ the new $x$-component of a rotated vorticity; i.e., we rotate each $\mathbf{j} \in \mathbb{R}^3$ to a vector parallel to the $x$-axis. In principle we can assign a different rotation matrix to each lattice point $\mathbf{j}$. Such a rotation matrix can be constructed using a non-zero vector $\mathbf{n} \in \mathbb{R}^3$ and set

$$R_{\mathbf{j}} = \begin{pmatrix} \mathbf{j}^{\mathsf{T}}/|\mathbf{j}| \\ (\mathbf{j}\times\mathbf{n})^{\mathsf{T}}/|\mathbf{j}\times\mathbf{n}| \\ (\mathbf{j}\times(\mathbf{j}\times\mathbf{n}))^{\mathsf{T}}/(|\mathbf{j}\times\mathbf{n}||\mathbf{j}|) \end{pmatrix} = \begin{pmatrix} \mathbf{j}^{\mathsf{T}}/|\mathbf{j}| \\ (\mathbf{j}\times\mathbf{n})^{\mathsf{T}}/|\mathbf{j}|_2 \\ (\mathbf{j}\times(\mathbf{j}\times\mathbf{n}))^{\mathsf{T}}/(|\mathbf{j}|_2|\mathbf{j}|) \end{pmatrix}, \quad \mathbf{j}\times\mathbf{n} \neq 0, \tag{4.1}$$

so that the rows of $R_{\mathbf{j}}$ are orthonormal. Here we defined the "2-norm" $|\mathbf{j}|_2 = |\mathbf{j}\times\mathbf{n}|$. Notice that we have

$$R_{-\mathbf{j}}R_{\mathbf{j}}^{\mathsf{T}} = S = \operatorname{diag}(-1,-1,1)\,. \tag{4.2}$$

The "signature" matrix $S$ will remain in the transformed Hamiltonian.

The most general case would be to consider $\mathbf{n}$ as a function of $\mathbf{j}$. It could be convenient to choose $\mathbf{n} = \hat{e}_x$, the unit vector in the $x$-direction. When $\mathbf{j}$ and $\mathbf{n}$ are parallel, namely for $\mathbf{j} = a\hat{e}_x$, this definition fails. To allow for this, we define $R_{a\mathbf{n}} = \mathbb{I}$ and and $R_{-a\mathbf{n}} = S$ for all $a > 0$. We then define a rotated vorticity by

$$\breve{\boldsymbol{\omega}}_{\mathbf{j}} = R_{\mathbf{j}}\boldsymbol{\omega}_{\mathbf{j}} \tag{4.3}$$

so that $(\breve{\boldsymbol{\omega}}_{\mathbf{j}})_x \propto \mathbf{j}^{\mathsf{T}}\boldsymbol{\omega}_{\mathbf{j}} = 0$ on the divergence free subspace. One may try to define the special cases for $R_{\mathbf{j}}$ by a limit where $\mathbf{n}$ tends to a vector parallel to $\mathbf{j}$. However, the result depends on how the limit is taken, and this is why we resorted to the definition of $R_{\pm a\mathbf{n}}$ above.

Instead of a rotation, it might have seemed convenient to choose a matrix so that $R_{-\mathbf{j}}$ and $R_{\mathbf{j}}^{\mathsf{T}}$ are inverses of each other, unlike the relation (4.2), for in this case $H$, (3.4), would be invariant under the transformation (4.3). It is possible to change the definition of $R_{\mathbf{j}}$ such that $S$ becomes the identity. For this one has to divide the set $D$ into positive and negative lattice points, and use this sign in the definition of $R_{\mathbf{j}}$; see [Wor17] for the details. Since the lattice $D$, (2.3), is discrete, one could in principle use a different $R_{\mathbf{j}}$ for each lattice point, as long as the first row is proportional to $\mathbf{j}$. Specifically, each $R_{\mathbf{j}}$ could include an additional rotation about the $\mathbf{j}$-axis by an angle that depends on $\mathbf{j}$. For simplicity, we assume here that all these angles are equal to 0. Note that the definition of $\breve{\boldsymbol{\omega}}_{\mathbf{j}}$ implies the reality condition $\breve{\boldsymbol{\omega}}_{-\mathbf{j}} = S\overline{\breve{\boldsymbol{\omega}}_{\mathbf{j}}}$.

Since the first element of $\breve{\boldsymbol{\omega}}_{\mathbf{j}}$ must be zero on the divergence-free subspace, we can project down to this subspace and still capture the full dynamics.

**Proposition 4.1** (Reduced Poisson structure of three-dimensional Euler equations)**.** *Defining* $\tilde{\boldsymbol{\omega}}_{\mathbf{j}} = (\breve{\omega}_{\mathbf{j},y}, \breve{\omega}_{\mathbf{j},z})$ *and the reduced manifold*

$$\tilde{\mathcal{M}} = \{\tilde{\boldsymbol{\omega}}_{\mathbf{j}} \in \mathbb{R}^2 \mid \mathbf{j} \in D\},$$

*the Poisson Structure of Theorem 3.2 reduces to*

$$\{f,g\} = \sum_{\mathbf{j},\mathbf{k}\in D} \left(\frac{\partial f}{\partial \tilde{\boldsymbol{\omega}}_{\mathbf{j}}}\right)^{\mathsf{T}} \tilde{J}(\mathbf{j},\mathbf{k}) \left(\frac{\partial g}{\partial \tilde{\boldsymbol{\omega}}_{\mathbf{k}}}\right) \tag{4.4}$$

*for a matrix valued function* $\tilde{J} : D \times D \times \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$. *The new Hamiltonian becomes*

$$\tilde{H} = \frac{1}{2} \sum_{\mathbf{j}\in D\setminus\{\mathbf{0}\}} \frac{1}{|\mathbf{j}|^2} \tilde{\boldsymbol{\omega}}_{-\mathbf{j}}^{\mathsf{T}} \tilde{S} \tilde{\boldsymbol{\omega}}_{\mathbf{j}} \tag{4.5}$$

*with* $\tilde{S} = \operatorname{diag}(-1,1)$. *Then the dynamics of*

$$\dot{\tilde{\boldsymbol{\omega}}}_{\mathbf{j}} = \sum_{\mathbf{k}\neq\mathbf{0}} \tilde{J}(\mathbf{j},\mathbf{k}) \tilde{S} \frac{\tilde{\boldsymbol{\omega}}_{-\mathbf{k}}}{|\mathbf{k}|^2} \tag{4.6}$$

*are equivalent to the dynamics of* (3.2).

*Proof.* Using the matrix (4.1) $R_{\mathbf{j}}$ and the definition (4.3) for $\breve{\boldsymbol{\omega}}_{\mathbf{j}} = (\breve{\omega}_{\mathbf{j},x}, \breve{\omega}_{\mathbf{j},z}, \breve{\omega}_{\mathbf{j},y})$, the divergence-free condition implies that $\breve{\omega}_{\mathbf{j},x}$ is constant and zero for all $\mathbf{j}$. We first look, however, at the equations for all three components of $\breve{\boldsymbol{\omega}}_{\mathbf{j}}$.

Define

$$\begin{aligned} \breve{J}(\mathbf{j},\mathbf{k}) &= R_{\mathbf{j}} J(\mathbf{j},\mathbf{k}) R_{\mathbf{k}}^{\mathsf{T}} \\ &= R_{\mathbf{j}} \Big[ (R_{\mathbf{j}+\mathbf{k}}^{\mathsf{T}} \breve{\boldsymbol{\omega}}_{\mathbf{j}+\mathbf{k}})(\mathbf{k}\times\mathbf{j})^{\mathsf{T}} - (\mathbf{k}\cdot(R_{\mathbf{j}+\mathbf{k}}^{\mathsf{T}} \breve{\boldsymbol{\omega}}_{\mathbf{j}+\mathbf{k}}))\widehat{\mathbf{k}} \\ &\quad - \frac{1}{2}(\mathbf{j}+\mathbf{k})\cdot(R_{\mathbf{j}+\mathbf{k}}^{\mathsf{T}} \breve{\boldsymbol{\omega}}_{\mathbf{j}+\mathbf{k}})\widehat{(\mathbf{j}-\mathbf{k})} \Big] R_{\mathbf{k}}^{\mathsf{T}}. \end{aligned} \tag{4.7}$$

Then the dynamics of $\breve{\boldsymbol{\omega}}_{\mathbf{j}}$ are

$$\dot{\breve{\boldsymbol{\omega}}}_{\mathbf{j}} = \sum_{\mathbf{j},\mathbf{k}} \breve{J}(\mathbf{j},\mathbf{k}) S \frac{\breve{\boldsymbol{\omega}}_{-\mathbf{k}}}{|\mathbf{k}|^2}. \tag{4.8}$$

Since the first column of $R_{\mathbf{k}}^{\mathsf{T}}$ is $\mathbf{k}$ and the first row of $R_{\mathbf{j}}$ is $\mathbf{j}^{\mathsf{T}}$, we can conclude by Lemma 3.1 that when $\boldsymbol{\omega}_{\mathbf{j}+\mathbf{k}} \in \mathcal{M}$,

$$\breve{J}(\mathbf{j},\mathbf{k})_{x,\beta} = \breve{J}(\mathbf{j},\mathbf{k})_{\beta,x} = 0, \quad \beta\in\{x,y,z\}, \tag{4.9}$$

which is expected since the divergence-free subspace is invariant. This remains true in the most general case where a different vector $\mathbf{n}$ is used for the matrices $R_{\mathbf{j}}$, $R_{\mathbf{k}}$, and $R_{\mathbf{j}+\mathbf{k}}$. We already noted after the main theorem that the conditions $\mathbf{j}\cdot\boldsymbol{\omega}_{\mathbf{j}} = 0$ are sub-Casimirs of the system, and now we have explicitly reduced the dynamics to the invariant submanifold that they define. Note that this does not imply that the divergence is constant for initial conditions that are not on the divergence-free subspace.

Thus we can ignore the dynamics of the coordinate $\breve{\omega}_{\mathbf{j},x}$, and define

$$\tilde{J}(\mathbf{j},\mathbf{k}) = \begin{pmatrix} \breve{J}(\mathbf{j},\mathbf{k})_{y,y} & \breve{J}(\mathbf{j},\mathbf{k})_{y,z} \\ \breve{J}(\mathbf{j},\mathbf{k})_{z,y} & \breve{J}(\mathbf{j},\mathbf{k})_{z,z} \end{pmatrix}. \tag{4.10}$$

Note that the reality condition for $\tilde{\boldsymbol{\omega}}_{\mathbf{j}}$ becomes $\tilde{\boldsymbol{\omega}}_{-\mathbf{j}} = \tilde{S}\overline{\tilde{\boldsymbol{\omega}}}_{\mathbf{j}}$.

The form of $\tilde{J}$ is quite complicated. We can directly check that it satisfies the conditions for a Poisson structure matrix (antisymmetry and Jacobi identity). However, this also follows from the fact that the matrix $\breve{J}$ has vanishing first row and column, so that the Jacobi-identity is inherited from the matrix $J$, in particular because the transformation to the new coordinates is linear. Similarly, the antisymmetry is not changed by a linear transformation. The Hamiltonian in the new coordinates is obtained by substituting $\tilde{\boldsymbol{\omega}}_{\mathbf{j}}$ into (3.4) to obtain

$$\tilde{H} = \frac{1}{2} \sum_{\mathbf{j} \in D \setminus \{\mathbf{0}\}} \frac{\tilde{\boldsymbol{\omega}}_{-\mathbf{j}}^{\mathsf{T}} \tilde{S} \tilde{\boldsymbol{\omega}}_{\mathbf{j}}}{|\mathbf{j}|^2}. \tag{4.11}$$

As the coordinates $\tilde{\boldsymbol{\omega}}_{\mathbf{j}}$ are no longer restricted, the Poisson manifold is the full space of $\tilde{\boldsymbol{\omega}}_{\mathbf{j}} \in \mathbb{R}^2$ for all $\mathbf{j} \in D$. Then the Poisson bracket with structure matrix $\tilde{J}(\mathbf{j}, \mathbf{k})$ and the Hamiltonian $\tilde{H}$ generate the dynamics of the Euler equations in the new coordinates $\tilde{\boldsymbol{\omega}}_{\mathbf{j}}$. ◻

Before we list the general formulas for $\tilde{J}$ we give the three special cases that occur when either $\mathbf{j}$, $\mathbf{k}$, or $\mathbf{j}+\mathbf{k}$ are proportional to $\mathbf{n} = \hat{e}_x$. For example, when $\mathbf{j} = sa\mathbf{n}$ the corresponding rotation matrix is replaced by $\mathbb{I}$ or $S$, depending on whether $s = 1$ or $s = -1$, respectively. Analogous rules hold for $\mathbf{k}$ and $\mathbf{j}+\mathbf{k}$. The results are

$$\mathbf{j}||\mathbf{n}: \quad \tilde{J} = \frac{j_x}{|\mathbf{j}+\mathbf{k}|} \begin{pmatrix} (j_x k_y \tilde{\boldsymbol{\omega}}_{\mathbf{j}+\mathbf{k},z} + k_z \tilde{\boldsymbol{\omega}}_{\mathbf{j}+\mathbf{k},y} |\mathbf{j}+\mathbf{k}|) s & k_z \tilde{\boldsymbol{\omega}}_{\mathbf{j}+\mathbf{k},z} |\mathbf{k}| s \\ j_x k_z \tilde{\boldsymbol{\omega}}_{\mathbf{j}+\mathbf{k},z} - k_y \tilde{\boldsymbol{\omega}}_{\mathbf{j}+\mathbf{k},y} |\mathbf{j}+\mathbf{k}| & -k_y \tilde{\boldsymbol{\omega}}_{\mathbf{j}+\mathbf{k},z} |\mathbf{k}| \end{pmatrix},$$

$$\mathbf{k}||\mathbf{n}: \quad \tilde{J} = \begin{pmatrix} -(j_x + k_x)(j_y \tilde{\boldsymbol{\omega}}_{\mathbf{j}+\mathbf{k},z} - j_z \tilde{\boldsymbol{\omega}}_{\mathbf{j}+\mathbf{k},y} s) & |\mathbf{k}| (j_z \tilde{\boldsymbol{\omega}}_{\mathbf{j}+\mathbf{k},z} + j_y \tilde{\boldsymbol{\omega}}_{\mathbf{j}+\mathbf{k},y} s) \\ |\mathbf{j}| (j_z \tilde{\boldsymbol{\omega}}_{\mathbf{j}+\mathbf{k},z} + j_y \tilde{\boldsymbol{\omega}}_{\mathbf{j}+\mathbf{k},y} s) & 0 \end{pmatrix},$$

$$\mathbf{j}+\mathbf{k}||\mathbf{n}: \quad \tilde{J} = \frac{k_x}{|\mathbf{j}+\mathbf{k}|} \begin{pmatrix} -(j_y k_x \tilde{\boldsymbol{\omega}}_{\mathbf{j}+\mathbf{k},z} + j_z \tilde{\boldsymbol{\omega}}_{\mathbf{j}+\mathbf{k},y} |\mathbf{j}+\mathbf{k}|) s & j_y \tilde{\boldsymbol{\omega}}_{\mathbf{j}+\mathbf{k},y} |\mathbf{j}+\mathbf{k}| - j_z k_x \tilde{\boldsymbol{\omega}}_{\mathbf{j}+\mathbf{k},z} \\ -j_z \tilde{\boldsymbol{\omega}}_{\mathbf{j}+\mathbf{k},z} |\mathbf{j}| s & j_y \tilde{\boldsymbol{\omega}}_{\mathbf{j}+\mathbf{k},z} |\mathbf{j}| \end{pmatrix}.$$

Now we proceed to the general case where none of $\mathbf{j}$, $\mathbf{k}$, $\mathbf{j}+\mathbf{k}$ are parallel to $\mathbf{n}$. Since $\tilde{J}(\mathbf{j}, \mathbf{k})$ is linear in $\tilde{\boldsymbol{\omega}}_{\mathbf{j}+\mathbf{k}}$, and the $x$ component of $\tilde{\boldsymbol{\omega}}_{\mathbf{j}+\mathbf{k}}$ is zero, we can write

$$\begin{aligned} \tilde{J}(\mathbf{j}, \mathbf{k}) = & \begin{pmatrix} \tilde{J}_y(\mathbf{j}, \mathbf{k})_{y,y} & \tilde{J}_y(\mathbf{j}, \mathbf{k})_{y,z} \\ \tilde{J}_y(\mathbf{j}, \mathbf{k})_{z,y} & \tilde{J}_y(\mathbf{j}, \mathbf{k})_{z,z} \end{pmatrix} \tilde{\omega}_{\mathbf{j}+\mathbf{k},y} \\ & + \begin{pmatrix} \tilde{J}_z(\mathbf{j}, \mathbf{k})_{y,y} & \tilde{J}_z(\mathbf{j}, \mathbf{k})_{y,z} \\ \tilde{J}_z(\mathbf{j}, \mathbf{k})_{z,y} & \tilde{J}_z(\mathbf{j}, \mathbf{k})_{z,z} \end{pmatrix} \tilde{\omega}_{\mathbf{j}+\mathbf{k},z} \end{aligned} \tag{4.12}$$

where $\tilde{J}_y(\mathbf{j}, \mathbf{k})_{i,j}$, $\tilde{J}_z(\mathbf{j}, \mathbf{k})_{i,j}$ are functions of $\mathbf{j}, \mathbf{k}$ only.

**Lemma 4.1.** *The Poisson structure matrix $\tilde{J}$ after restriction to the divergence free subspace satisfies*

$$\begin{aligned} \frac{1}{|\mathbf{k}|} \tilde{J}_y(\mathbf{j}, \mathbf{k})_{b,y} + \frac{1}{|\mathbf{j}+\mathbf{k}|} \tilde{J}_z(\mathbf{j}, -\mathbf{j}-\mathbf{k})_{b,z} &= 0, \\ \frac{1}{|\mathbf{k}|} \tilde{J}_y(\mathbf{j}, \mathbf{k})_{b,z} + \frac{1}{|\mathbf{j}+\mathbf{k}|} \tilde{J}_y(\mathbf{j}, -\mathbf{j}-\mathbf{k})_{b,z} &= 0, \\ \frac{1}{|\mathbf{k}|} \tilde{J}_z(\mathbf{j}, \mathbf{k})_{b,y} + \frac{1}{|\mathbf{j}+\mathbf{k}|} \tilde{J}_z(\mathbf{j}, -\mathbf{j}-\mathbf{k})_{b,y} &= 0, \end{aligned} \tag{4.13}$$

*for $b = y$ and $b = z$.*

*Proof.* This follows by direction computation from the explicit formulas given below.

$$\tilde{J}_y(\mathbf{j}, \mathbf{k})_{y,y} = \mathbf{n} \cdot \Big( (\mathbf{j} \times \mathbf{k}) \times (\mathbf{k}\, |\mathbf{j}|_2^2 + \mathbf{j}\, |\mathbf{k}|_2^2) \Big) \frac{1}{|\mathbf{j}|_2 |\mathbf{k}|_2 |\mathbf{j}+\mathbf{k}|_2}, \tag{4.14}$$

$$\tilde{J}_y(\mathbf{j},\mathbf{k})_{y,z} = -\mathbf{n}\cdot(\mathbf{j}\times\mathbf{k})\frac{|\mathbf{j}|_2|\mathbf{k}|}{|\mathbf{j}+\mathbf{k}|_2|\mathbf{k}|_2} \tag{4.15}$$

$$\tilde{J}_y(\mathbf{j},\mathbf{k})_{z,y} = -\mathbf{n}\cdot(\mathbf{j}\times\mathbf{k})\frac{|\mathbf{k}|_2|\mathbf{j}|}{|\mathbf{j}+\mathbf{k}|_2|\mathbf{j}|_2} \tag{4.16}$$

$$\tilde{J}_y(\mathbf{j},\mathbf{k})_{z,z} = 0, \tag{4.17}$$

$$\tilde{J}_z(\mathbf{j},\mathbf{k})_{y,y} = \mathbf{n}\cdot(\mathbf{j}\times\mathbf{k})\,|\mathbf{n}\times(\mathbf{j}\times\mathbf{k})|^2\frac{1}{|\mathbf{j}|_2|\mathbf{k}|_2|\mathbf{j}+\mathbf{k}|_2|\mathbf{j}+\mathbf{k}|}, \tag{4.18}$$

$$\tilde{J}_z(\mathbf{j},\mathbf{k})_{y,z} = -\frac{|\mathbf{k}|}{|\mathbf{j}+\mathbf{k}|}\tilde{J}_y(\mathbf{j},-\mathbf{j}-\mathbf{k})_{yy}, \tag{4.19}$$

$$\tilde{J}_z(\mathbf{j},\mathbf{k})_{z,y} = -\tilde{J}_z(\mathbf{k},\mathbf{j})_{y,z}, \tag{4.20}$$

$$\tilde{J}_z(\mathbf{j},\mathbf{k})_{z,z} = -\mathbf{n}\cdot(\mathbf{j}\times\mathbf{k})\frac{|\mathbf{j}||\mathbf{k}||\mathbf{j}+\mathbf{k}|_2}{|\mathbf{j}|_2|\mathbf{k}|_2|\mathbf{j}+\mathbf{k}|}. \tag{4.21}$$

□

The condition for antisymmetry now reads

$$\tilde{J}_y(\mathbf{j},\mathbf{k}) + \tilde{J}_y(\mathbf{k},\mathbf{j})^T = \tilde{J}_z(\mathbf{j},\mathbf{k}) + \tilde{J}_z(\mathbf{k},\mathbf{j})^T = 0\,. \tag{4.22}$$

## 5. Helicity

The three-dimensional Euler equations have a constant of motion, called the *helicity*, given by

$$h = \frac{\kappa_x\kappa_y\kappa_z}{(2\pi)^3}\int_{\mathcal{D}} V\cdot(\nabla\times V)\,\mathrm{d}x = \frac{\kappa_x\kappa_y\kappa_z}{(2\pi)^3}\int_{\mathcal{D}} V\cdot\Omega\,\mathrm{d}x.$$

see, e.g., [Gib08]. In terms of the Fourier modes on the periodic domain, this quantity becomes

$$h = \sum \mathbf{v}_\mathbf{k}\cdot\boldsymbol{\omega}_{-\mathbf{k}} = \sum \frac{i}{|\mathbf{k}|^2}\mathbf{k}\cdot(\boldsymbol{\omega}_\mathbf{k}\times\boldsymbol{\omega}_{-\mathbf{k}}).$$

Note that $h\in\mathbb{R}$, as $\boldsymbol{\omega}_{-\mathbf{k}} = \overline{\boldsymbol{\omega}}_\mathbf{k}$ so $\boldsymbol{\omega}_\mathbf{k}\times\boldsymbol{\omega}_{-\mathbf{k}}\in i\mathbb{R}^3$.

By transforming to the coordinates introduced in Section 4 and noting that

$$\begin{aligned}
\boldsymbol{\omega}_\mathbf{k}\times\boldsymbol{\omega}_{-\mathbf{k}} &= R_\mathbf{k}^\mathsf{T}\breve{\boldsymbol{\omega}}_\mathbf{k}\times R_{-\mathbf{k}}^\mathsf{T}\breve{\boldsymbol{\omega}}_{-\mathbf{k}}\\
&= R_\mathbf{k}^\mathsf{T}(\breve{\boldsymbol{\omega}}_\mathbf{k}\times S\breve{\boldsymbol{\omega}}_{-\mathbf{k}})\\
&= R_\mathbf{k}^\mathsf{T}(\breve{\boldsymbol{\omega}}_\mathbf{k}\times\overline{\breve{\boldsymbol{\omega}}}_\mathbf{k})\\
&= i\frac{\mathbf{k}}{|\mathbf{k}|}2\Im(\tilde{\omega}_{\mathbf{k},y}\overline{\tilde{\omega}}_{\mathbf{k},z}) = \frac{\mathbf{k}}{|\mathbf{k}|}(\tilde{\omega}_{\mathbf{k},y}\tilde{\omega}_{-\mathbf{k},z} + \tilde{\omega}_{\mathbf{k},z}\tilde{\omega}_{-\mathbf{k},y}),
\end{aligned} \tag{5.1}$$

we see that the helicity can be rewritten as

$$h = \sum_\mathbf{k}\frac{1}{|\mathbf{k}|}2\Im(\overline{\tilde{\omega}}_{\mathbf{k},y}\tilde{\omega}_{\mathbf{k},z}). \tag{5.2}$$

We now confirm that this is a Casimir of the Poisson system described in Proposition 4.1.

**Lemma 5.1.** *The helicity $h$ given by* (5.2) *is a Casimir of the reduced Poisson structure described in Proposition 4.1.*

*Proof.* To show that $h$ is a Casimir, we must show that $\{g, h\} = 0$ for any function $g$. This requires that $\sum_{\mathbf{k}\in D} \tilde{J}(\mathbf{j},\mathbf{k})\frac{\partial \tilde{h}}{\partial \tilde{\omega}_{\mathbf{k}}} = 0$ for all $\mathbf{j},\mathbf{k} \in D$. Now, $\frac{\partial \tilde{h}}{\partial \tilde{\omega}_{\mathbf{k}}} = \frac{2i}{|\mathbf{k}|}\begin{pmatrix}\tilde{\omega}_{-\mathbf{k},z}\\ \tilde{\omega}_{-\mathbf{k},y}\end{pmatrix}$. By left-multiplying by $\tilde{J}(\mathbf{j},\mathbf{k})$, and separating out the terms, the required conditions are exactly those stated in (4.13). We can thus conclude that the helicity is a Casimir. □

Note that $h$ is not a Casimir of the full system before restriction to the divergence free subspace.

## 6. An Example: Shear Flows

A shear flow on the torus is an equilibrium solution $V^e$ of Euler's equations with the direction of the velocity constant and $(V^e \cdot \nabla)V^e = 0$. The vorticity $\Omega^e = \nabla \times V^e$ then must have the form

$$\Omega^e(\mathbf{x}) = \mathbf{\Gamma} C(\mathbf{p}\cdot\mathbf{x})$$

where $C$ is an arbitrary periodic function with period $2\pi$, $\mathbf{p} = (p_1, p_2, p_3)^\mathsf{T}$ is a fixed vector with components that are co-prime integers, and $\mathbf{\Gamma}$ is a real fixed vector satisfying $\mathbf{\Gamma}\cdot\mathbf{p} = 0$. Let the Fourier coefficients of $C$ be $c_n$, then the corresponding Fourier modes are $\boldsymbol{\omega}^e_{\mathbf{k}} = \mathbf{\Gamma} c_n \delta_{\mathbf{k}-n\mathbf{p}}$. Thus

$$\dot{\boldsymbol{\omega}}_{\mathbf{j}}|_{\tilde{\boldsymbol{\omega}}^e} = \sum_{\mathbf{k}} J(\mathbf{j},\mathbf{k},\mathbf{\Gamma} c_n \delta_{\mathbf{k}+\mathbf{j}-n\mathbf{p}})\frac{\mathbf{\Gamma} c_{-m}\delta_{-\mathbf{k}+m\mathbf{p}}}{|\mathbf{k}|^2}$$

where we have written $J$ with three arguments to make the linear dependence on the last argument explicit. The Kronnecker $\delta$'s pick out the terms $\mathbf{k} = m\mathbf{p}$ and $\mathbf{j} = (n-m)\mathbf{p}$. Relabelling $n \to n+m$, the non-zero equations become

$$\dot{\boldsymbol{\omega}}_{n\mathbf{p}}|_{\tilde{\boldsymbol{\omega}}^e} = \sum_{m} J(n\mathbf{p}, m\mathbf{p}, \mathbf{\Gamma} c_{m+n})\frac{\mathbf{\Gamma} c_{-m}}{|m\mathbf{p}|^2}\,.$$

Now $J$ vanishes because $\mathbf{p}\cdot\mathbf{\Gamma} = 0$ (the divergence free condition) and because $\mathbf{j} = n\mathbf{p}$ and $\mathbf{k} = m\mathbf{p}$ are parallel. Hence $\Omega^e$ is indeed an equilibrium, since either $J$ vanishes or is multiplied by zero coming from the gradient of the Hamiltonian. A similar argument applies to the reduced system with $\tilde{J}$.

A point of a Poisson structure at which the co-rank of the Poisson tensor is less than the maximal co-rank is called a *singular point* of the Poisson structure. If the equilibrium is a regular point of the Poisson structure then the gradient of the Hamiltonian is a linear combination of gradients of the Casimirs, proving that the gradient of the Hamiltonian is in the kernel of the Poisson structure, and hence the vector field vanishes at the equilibrium point. For example, this happens for the two-dimensional case and has been used in [DW17] to study the nonlinear stability of two-dimensional shear flows through the Energy-Casimir method.

In the three-dimensional case at hand here, the gradient of the Hamiltonian $\tilde{H}$ is not proportional to the gradient of the Casimir $\tilde{h}$, as we now show. At the equilibrium point the gradient $\nabla\tilde{H}$ has components

$$\left.\frac{\partial \tilde{H}}{\partial \tilde{\boldsymbol{\omega}}_{\mathbf{k}}}\right|_{\tilde{\boldsymbol{\omega}}^e} = \frac{1}{|n\mathbf{p}|^2}\tilde{S}\tilde{\mathbf{\Gamma}} c_{-n}\delta_{\mathbf{k}+n\mathbf{p}}$$

where $\tilde{\mathbf{\Gamma}}$ are the last two components of $R_{n\mathbf{p}}\mathbf{\Gamma}$. The gradient of $\tilde{h}$ at the equilibrium is similar,

$$\left.\frac{\partial \tilde{h}}{\partial \tilde{\boldsymbol{\omega}}_{\mathbf{k}}}\right|_{\tilde{\boldsymbol{\omega}}^e} = \frac{2i}{|n\mathbf{p}|}\tilde{T}\tilde{\mathbf{\Gamma}}c_{-n}\delta_{\mathbf{k}+n\mathbf{p}}$$

where $\tilde{T}$ is the anti-diagonal matrix. But clearly $\tilde{S}$ and $\tilde{T}$ are not proportional, so the two gradients are not proportional. Assuming that there is no additional (unknown) Casimir function this implies that the equilibrium point is a singular point of the Poisson structure, i.e. a point where the co-rank drops.

When the gradient of the Hamiltonian is a linear combination of gradients of Casimirs the linearisation of the vector field can be found by multiplying the Poisson structure at the equilibrium by a certain Hessian. This is, however, not true at a singular point of the Poisson structure. We refrain from giving explicit formulas for the linearisation here, but we conclude this example by observing that because periodic shear flows in three dimensions are singular points of the Poisson structure the Energy-Casimir method cannot be applied in this case. This is a notable difference from shear flows in two dimensions.

## 7. Conclusion

By describing the dynamics of the vorticity Fourier modes as a Poisson system, we have opened the possibility of future study analysing and exploiting this structure.

One possible use for this structure could be the development of a structure preserving integrator akin to the integrator for the two-dimensional Euler equations in [McL93]. Even though that integrator in practice requires the finite-dimensional truncation developed in [Zei91], one may hope to also find such a truncation and hence a similar integrator for the three-dimensional problem. In the two-dimensional case the analogue of our $\tilde{J}$ is simply the $z$-component of the cross product of $\mathbf{j} \times \mathbf{k}$, and the finite-dimensional truncation leads to the sine-bracket [Zei91]. Our Poisson structure $\tilde{J}$ for the three-dimensional case is already so much more complicated algebraically that we were not able to find an analogue of the sine-bracket. Nevertheless, we suspect that such a structure preserving truncation exists.

The reduced Poisson structure $\tilde{J}$ can also be the basis for nonlinear stability analysis. In the two-dimensional case [DW17] nonlinear stability results were obtained using the Energy-Casimir method. But we have shown above that for periodic shear flows in three dimensions the Energy-Casimir fails, because the Poisson structure is singular at the equilibrium. It would be interesting to find other equilibria for which the gradient of the Hamiltonian $\tilde{H}$ and the gradient of the helicity $\tilde{h}$ are proportional (and hence these are regular points of the Poisson structure), and for such equilibria the Energy-Casimir method may work.

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