



Invariant Manifolds of Traveling Waves of the 3D Gross–Pitaevskii Equation in the Energy Space

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Abstract: We study the local dynamics near general unstable traveling waves of the 3D Gross–Pitaevskii equation in the energy space by constructing smooth local invariant center-stable, center-unstable and center manifolds. We also prove that (i) the center-unstable manifold attracts nearby orbits exponentially before they go away from the traveling waves along the center or unstable directions and (ii) if an initial data is not on the center-stable manifolds, then the forward orbit leaves traveling waves exponentially fast. Furthermore, under an additional non-degeneracy assumption, we show the orbital stability of the traveling waves on the center manifolds, which also implies the uniqueness of the local invariant manifolds. Our method based on a geometric bundle coordinate system should work for a general class of Hamiltonian PDEs.

1. Introduction

Consider the Gross–Pitaevskii (GP) equation

$$iu_t + \Delta u + (1 - |u|^2)u = 0, \quad u = u_1 + iu_2 : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}, \quad (\text{GP})$$

where u satisfies the boundary condition $|u| \rightarrow 1$ as $|x| \rightarrow \infty$. The (GP) equation arises in various physical problems, such as superconductivity, superfluidity in Helium II, and Bose-Einstein condensate (for example [1, 8]). Formally, the (GP) equation is a Hamiltonian PDE associated to the energy

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} (1 - |u|^2)^2 dx, \quad (1.1)$$

and the energy space is

$$X_0 = \{u \in H_{loc}^1(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3), 1 - |u|^2 \in L^2(\mathbb{R}^3)\}. \quad (1.2)$$

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The global well-posedness of (GP) in X_0 was proved by Gérard [20]. From the definition of X_0 , it is clear that the real part and imaginary part of a function in X_0 may have different spatial decay rates, which makes the analysis of this equation quite different from the classical NLS.

Due to the translation invariance of (GP), the momentum $P(u) = \frac{1}{2} \int_{\mathbb{R}^3} \langle i \nabla u, u - 1 \rangle dx$ is also formally conserved. We denote each component of $P(u)$ as

$$P_j(u) = \frac{1}{2} \int_{\mathbb{R}^3} \langle i \partial_{x_j} u, u - 1 \rangle dx = - \int_{\mathbb{R}^3} \langle u_1 - 1, \partial_{x_j} u_2 \rangle dx \quad (1.3)$$

for $j = 1, 2, 3$. The corresponding relative equilibria are traveling wave solutions to (GP) in the form of $U_c(x - ct)$ where $c \in \mathbb{R}^3$ and U_c satisfies

$$-ic \cdot \nabla U_c + \Delta U_c + (1 - |U_c|^2)U_c = 0. \quad (1.4)$$

Due to the rotational invariance of (GP), we only need to consider those traveling waves traveling in x_1 direction, i.e., $u(t, x) = U_{ae_1}(x - ae_1 t)$, where $e_1 = (1, 0, 0)^T$.

Traveling waves with finite energy play a very important role in the dynamics of (GP). In a series of papers [2, 3, 23, 30, 31], the existence, some qualitative properties, and the stability of traveling waves have been studied formally. A rigorous mathematical study was initiated by Béthuel and Saut in [13], in which they proved the existence of traveling waves for 2D (GP) with small $|c| \ll 1$, followed by [10–12, 15, 24, 25, 39, 40]. In particular, Mariş [40] constructed a full branch of subsonic traveling waves for 3D (GP) by minimizing the energy-momentum functional subject to a Pohozaev type constraint.

Given any traveling wave solution $U_c = (u_c, v_c)$ of (GP) with traveling speed $c \in \mathbb{R}^3$, $|c| \in (0, \sqrt{2})$, its spatial translations form a 3D manifold

$$\mathcal{M} = \{U_c(\cdot + y) : y \in \mathbb{R}^3\}$$

of traveling waves. The *main goal* of this paper is to study the local dynamics of (GP) near such $\mathcal{M}$ of an unstable traveling wave U_c (see the remark on instability right after Theorem 1.4).

We rewrite (GP) in the traveling frame $u(t, x) = U(t, x - ct)$, where U satisfies

$$i \partial_t U - ic \cdot \nabla U + \Delta U + (1 - |U|^2)U = 0. \quad (1.5)$$

It is clear that U_c is a steady state of (1.5). Linearizing (1.5) at U_c , one has

$$\partial_t U = J L_c U, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad L_c = (E + c \cdot P)''(U_c), \quad U \in X_1 = H^1 \times \dot{H}^1. \quad (1.6)$$

A more explicit expression of L_c can be found in (2.11). Under a mild spatial decay assumption (2.6) of U_c , it is straightforward to verify that the tangent space of the energy space X_0 at U_c is X_1 , where naturally the linearized equation (1.6) should be considered. The linearized energy-momentum quadratic form $L_c : X_1 \rightarrow X_1^*$ is bounded, symmetric, and uniformly positive except in finitely many directions. Even though the symplectic operator $J^{-1} = J^* = -J : X_1^* \rightarrow X_1$ is not bounded and thus the classic framework of Grillakis–Shatah–Strauss [26, 27] does not apply to (1.6), the recent results in [37] are applicable to analyze $J L_c$. Consequently, (1.6) satisfies the following exponential trichotomy property (even without the non-degeneracy assumed in [36]).

Lemma 1.1. *There exist $C, \lambda, \tilde{d} > 0$ and closed subspaces X^c, X^u, X^s of X_1 invariant under e^{JL_c} such that $X_1 = X^u \oplus X^c \oplus X^s$, $d = \dim X^u = \dim X^s < \infty$, and*

$$\begin{aligned} \|e^{JL_c}|_{X^s}\| &\leq Ce^{\lambda t}, \quad \forall t \geq 0, \quad \|e^{JL_c}|_{X^u}\| \leq Ce^{\lambda t}; \quad \forall t \leq 0, \\ \|e^{JL_c}|_{X^c}\| &\leq C(1 + |t|^{\tilde{d}}), \quad \forall t \in \mathbb{R}. \end{aligned}$$

The exponential trichotomy both describes the linear dynamics near the traveling waves and provides a framework to analyze the local nonlinear dynamics.

If $X^{u,s} = \{0\}$, U_c is spectrally stable and it actually implies the nonlinear orbital stability of $\mathcal{M}$ under some additional assumptions (see, for example, [17, 36]). If U_c is spectrally unstable with $d > 0$, conceptually one expects the existence of locally invariant submanifolds which can be viewed as the deformation from the invariant subspaces under small nonlinear perturbations. Here the local invariance of a submanifold $\mathcal{N}$ means that, for any initial value $U(0)$ in the interior of $\mathcal{N}$, the solution $U(t) \in \mathcal{N}$, $t \in (-T, T)$, for some $T > 0$, and thus it can exit $\mathcal{N}$ only through its boundary. The locally invariant submanifolds related to the exponential trichotomy are the unstable and stable manifolds of U_c and the center-unstable, center-stable, and center manifolds of $\mathcal{M}$. The former two contain U_c and are tangent to X^u and X^s at U_c , while the latter three ones should contain $\mathcal{M}$, be translation invariant, and be tangent to $X^{cu} = X^u \oplus X^c$, $X^{cs} = X^s \oplus X^c$, and X^c . Some comments on their dynamic significance:

- 1.) The nonlinear dynamics in these invariant manifolds are reflected by the corresponding linear ones, or even exactly conjugate in the case of the unstable and stable manifolds. For example, the unstable manifold can be characterized as the set of initial data near U_c whose solutions converge to U_c as $t \rightarrow -\infty$ and go away from U_c at least at certain exponential growth rate as t increases. This immediately provides a stronger result than the mere nonlinear instability.
- 2.) These invariant manifolds provide a framework to organize the local dynamics. For a typical initial value near $\mathcal{M}$, its trajectory would first approach the center-unstable manifold along the direction of X^s and then exit the neighborhood of $\mathcal{M}$ along the X^u direction, constituting a saddle type dynamics.
- 3.) Numerics [9] indicate that after leaving a neighborhood of unstable traveling waves (upper branch), the orbits of (GP) scatter to either stable traveling waves (lower branch) or constant states. Under non-degeneracy conditions **(H1-2)** in Sect. 6, there is orbital stability on the center-stable manifolds which provides a third type of dynamics not easily observed in numerics, where orbits stay close to $\mathcal{M}$ for all $t > 0$. In the case where the center-stable manifold is co-dim 1, it very likely serves as the boundary between the first two types of asymptotic behaviors. Combined with other tools like the virial identity, such classification of dynamics near unstable solitons based on invariant manifolds has been obtained for models including the Klein-Gordon and NLS [41, 42].

Under some additional conditions, the 1-dim unstable and stable manifolds of an unstable traveling wave U_c were constructed in $X_0 \cap H^3$ in [36]. As they represent low dimensional special structures in the phase space, it is indeed more desirable for them to have extra properties such as higher H^k regularity, $k > 1$.

The main results of this paper are the existence, smoothness, and some dynamic properties of the center-stable, center-unstable, and center manifolds of unstable traveling waves of (GP). In contrast to the finite dimensional stable and unstable subspaces, as the center subspace has a finite codimension, it is more preferable for these invariant manifolds to be constructed and describe the dynamics in the energy space X_0 . Moreover, on a center manifold where its topology is the same as the one of the energy space, the energy conservation provides a crucial control on the nonlinear dynamics.

Theorem 1.2. *Let U_c be a traveling wave of (GP) satisfying the spatial decay condition (2.6) and that $\exists \lambda \in \sigma(JL_c)$ with $\operatorname{Re} \lambda > 0$, (i.e. $d = \dim X^{u,s} > 0$), then*

- (1) *There exist the locally invariant co-dim d center-unstable and center-stable manifolds $\mathcal{W}^{cu}$ and $\mathcal{W}^{cs}$ and the co-dim $2d$ center manifold $\mathcal{W}^c$, all containing $\mathcal{M}$.*
- (2) *$\mathcal{W}^{cu}$, $\mathcal{W}^{cs}$, $\mathcal{W}^c$ are translation invariant, i.e. if $U \in \mathcal{W}^{cu,cs,c}$, then $U(\cdot + y) \in \mathcal{W}^{cu,cs,c}$ for any $y \in \mathbb{R}^3$.*
- (3) *Let ψ be defined in (2.2), then $\psi^{-1}(\mathcal{W}^{cu,cs,c})$ are smooth submanifolds of X_1 . Moreover, the tangent spaces of $\psi^{-1}(\mathcal{W}^{cu,cs,c})$ at $\psi^{-1}(U_c)$ are equal to $(D\psi^{-1})(U_c)X^{cu,cs,c}$.*
- (4) *Orbits in a small neighborhood of $\mathcal{M}$ are exponentially attracted to $\mathcal{W}^{cu}$ as t increases (before they possibly exit the neighborhood in the center-unstable directions), while repelled exponentially by $\mathcal{W}^{cs}$.*
- (5) *$\mathcal{W}^c = \mathcal{W}^{cu} \cap \mathcal{W}^{cs} \supset \mathcal{M}$ is the transversal intersection of $\mathcal{W}^{cu}$ and $\mathcal{W}^{cs}$.*

Remark 1.3. In the above neither the non-degeneracy $\ker L_c = \operatorname{span}\{\partial_{x_j} U_c \mid j = 1, 2, 3\}$ of L_c nor that U_c is a ground state obtained in [40] is assumed. Without such non-degeneracy, traveling waves for the given wave speed c may not be locally unique. However, all nearby traveling waves with close wave speeds must belong to $\mathcal{W}^c$. See Proposition 4.19.

As X_0 is not a flat space, we identify X_0 with X_1 through a coordinate map $\psi : X_1 \rightarrow X_0$ given in (2.2), borrowed from [21]. So these invariant manifolds are smooth in the sense that their images under ψ^{-1} are smooth submanifolds in X_1 . The above statement (4) also implies that $\mathcal{W}^c$ exponentially attracts nearby orbits in $\mathcal{W}^{cs}$ and exponentially repels those in $\mathcal{W}^{cu}$ (before they possibly exit a neighborhood of $\mathcal{M}$ along X^c). Consequently $\mathcal{M}$ is orbitally unstable. More detailed statements of the results are given in Sects. 4 and 5.

It is well-known that center-unstable manifolds, et. al. are not unique even for ODEs and the dynamics, including the stability, on the center manifold is rather subtle. However, under non-degeneracy conditions (H1-2) which ensure the uniform positivity of L_c on $X^c / \ker L_c$, in Sect. 6 we prove

Theorem 1.4. *Assume (H1-2) in addition, then there exist $C, \delta > 0$ such that*

- (1) *Given any initial value $U(0)$ such that $\psi^{-1}(U(0))$ is in a $(C^{-2}\delta)$ -neighborhood of $\psi^{-1}(\mathcal{M})$, then $U(0) \in \mathcal{W}^{cu,cs}$ if and only if $\psi^{-1}(U(t))$ is in a δ -neighborhood of $\psi^{-1}(\mathcal{M})$ for all $\mp t \geq 0$, and $U(0) \in \mathcal{W}^c$ if and only if $\psi^{-1}(U(t))$ is in a δ -neighborhood of $\psi^{-1}(\mathcal{M})$ for all $t \in \mathbb{R}$.*
- (2) *$\mathcal{M}$ is orbitally stable in $\mathcal{W}^c$ and $\mathcal{W}^{cu,cs,c}$ are locally unique.*

Remarks on the Instability of Traveling Waves of (GP). Firstly, the existence of unstable traveling waves of the 3D (GP) was first suggested in [30], where a Derrick-type argument was used to show that the numerically derived traveling waves in the upper branch in the energy-momentum plane are not energy minimizers under fixed momentum. Such instability was further supported by numerical evidence in [30] that initial perturbations near the upper branch can evolve toward the stable lower branch. A Grillakis–Shatah–Strauss type stability criterion was formulated based on numerics and heuristic arguments [9, 30] and then later rigorously proved [36] for the traveling waves constructed by Mariş [40]. Moreover, a variation of this criterion was used to obtain rigorously the spectral instability of a slow traveling wave of the 2D NLS with

non-vanishing condition at infinity in [36]. Secondly, in [9], numerical computations with high resolutions were performed to find eigenvalues of the $10^5 \times 10^5$ matrices obtained from the discretization of the linearized (GP) along the whole upper branch. At each of such traveling waves, only one unstable eigenvalue was identified and it has cylindrical symmetric eigenfunctions (consistent with [36]). The maximal exponential growth rate is approximately 0.012. Thirdly, when $c \rightarrow \sqrt{2}-$, it was formally shown in [31] and later rigorously proved in [16] that properly rescaled traveling waves of 3D (GP) converge to solitary waves of 3D KP-I equation. A formal leading order relationship between the eigenvalues of (GP) and the 3D KP-I equation was also given in [9]. As the ground state solitary waves of the 3D KP-I display certain instability [38], it also indicates such for some traveling waves of (GP). On the one hand, we fully admit that we are not aware of any existing traveling waves of the 3D (GP) having been rigorously proved to be spectrally unstable. On the other hand, we feel that there have been strong enough indications, particularly the numerics in [9], of the existence of spectrally unstable traveling waves to warrant the study of the local dynamics of the 3D (GP) near such waves.

Among previous results on local invariant manifolds of relative equilibria of dispersive PDEs, Bates and Jones [4] proved a general theorem on the existence of Lipschitz locally invariant manifolds of equilibria for semilinear PDEs by the graph transform method and then applied it to the radial Klein-Gordon equation combined with an energy argument. In [44], Schlag constructed a co-dimension 1 center-stable manifold of the manifold of ground states for the 3D cubic NLS in $W^{1,1}(\mathbb{R}^3) \cap W^{1,2}(\mathbb{R}^3)$ under an assumption that the linearization of NLS at each ground state has no embedded eigenvalue in the essential spectrum and proved the scattering on the center-stable manifold. Later this result was improved by Beceanu [6,7] who constructed center-stable manifolds in $W^{1,2}(\mathbb{R}^3) \cap |x|^{-1}L^2(\mathbb{R}^3)$ and in critical space $\dot{H}^{1/2}(\mathbb{R}^3)$. Similar results were obtained in Krieger and Schlag [35] for the supercritical 1D NLS. Nakanishi and Schlag [41] constructed a center-stable manifold of ground states for 3D cubic NLS in the energy space with a radial assumption by using the framework in Bates and Jones [4]. Nakanishi and Schlag [43] constructed center-stable manifolds of ground states for non-linear Klein-Gordon equation without radial assumption, following a graph transform approach. Also, see [22,32–34,42] for related results.

At a rough conceptual level, our proof follows the framework as in [14,19]. However, instead of being near a steady state, our construction is around the 3-dim invariant manifold $\mathcal{M}$, which is qualitatively comparable to [18], a more general result in finite dimensions. A rather naive initial attempt may be to construct the local invariant manifolds near $U_c(\cdot + y)$ for each $y \in \mathbb{R}^3$ and then patch them together to obtain $\mathcal{W}^{cu,cs,c}$. While the local construction for each y may follow from the standard procedure combined with some space-time estimates, it is highly questionable whether such ‘patch-up’ is possible as these local invariant manifolds of each $U_c(\cdot + y)$ are not unique in the first place. Therefore as in [43] we construct $\mathcal{W}^{cu,cs,c}$ as the center-unstable, center-stable, and center manifolds of the whole $\mathcal{M}$ instead of the individual $U_c(\cdot + y)$. This requires a coordinate system in a neighborhood of $\mathcal{M}$. A rather natural option would be

$$U = \psi(\Phi(y, w^u, w^s, w^c)) = \psi(\psi^{-1}(U_c)(\cdot + y) + w^u(\cdot + y) + w^s(\cdot + y) + w^c(\cdot + y))$$

where $D\psi(U_c)w^{u,s} \in X^{u,s}$, $D\psi(U_c)w^c \in \tilde{X}^c$, and $\tilde{X}^c$ is a fixed subspace of X^c transversal to $\text{span}\{\partial_{x_j} U_c : j = 1, 2, 3\}$ with $\text{codim-}\tilde{X}^c = 2d + 3$. The possible polynomial growth of e^{JL_c} in some directions in $\tilde{X}^c$ are too weak compared to its exponential

decay along $X^{u,s}$ (as $t \rightarrow \mp\infty$) to cause any real trouble. However, a more serious issue is that the above local coordinate mapping Φ is a homeomorphism but *not a diffeomorphism* from its domain to its image in X_1 . In fact,

$$D_y\Phi(y, w^u, w^s, w^c) = \left(\partial_y(\psi^{-1}(U_c)) + \partial_y w^u + \partial_y w^s + \partial_y w^c \right) (\cdot + y).$$

While the first three terms on the right side belong to X_1 due to the high regularity of U_c and functions in $X^{u,s}$, in general the last term $\partial_y w^c \in L^2$ only as $w^c \in \tilde{X}^c$ and $\tilde{X}^c$ is of finite co-dim in $X_1 = H^1 \times \dot{H}^1$.

This issue of loss of regularity due to the translation parametrization also appeared in previous works on other PDE models such as [5] and [43], in the latter of which it was handled by a rather analytically oriented nonlinear ‘mobile distance’. Here instead we adopt a more geometric bundle coordinate system used in [5], based on the observation that, while the above parametrization by the spatial translation of y is not smooth with respect to y , the vector bundles $\{w \in X_1 : w(\cdot - y) \in X^{u,s,c}\}$ are smooth in y . In such a framework based on vector bundle coordinates, some second fundamental form type quantities are to be carefully treated in the rather technical but intuitive analysis. See also Remark 2.6. Based on this coordinate system, we decompose equation (GP) and, as in the standard procedure in constructing local invariant manifolds, we cut off the nonlinear terms (except the ones corresponding to the second fundamental form of the center bundle) outside a small neighborhood of $\mathcal{M}$. Our subsequent estimates are mainly based on the exponential trichotomy and the energy conservation and involve minimal amount of dispersive estimates in Sect. 3. In particular, no spectral assumptions such as the nonexistence of embedded eigenvalues or resonance are needed.

At this stage, with the estimates of non-homogeneous linear equations in Sect. 3, one may apply the usual Lyapunov-Perron integral equation method or the graph transform of Hadamard to obtain invariant manifolds for the modified system (with the cut-off) which coincides with the original one near $\mathcal{M}$. While we obtained the invariant manifolds by conceptually following the procedure in [14, 19] which is more in line with the Lyapunov-Perron method, one could also choose to estimate the time- T map of the modified equation and then apply the graph transform method as in [5, 18].

While we focus on local invariant manifolds of traveling waves of the 3D (GP) in this paper, we believe that this framework is rather general and it could be adapted with minimal modifications to yield local invariant manifolds of unstable relative equilibria, including ground states and excited states, of a class of Hamiltonian dispersive PDEs such as the NLS, nonlinear Klein-Gordon equations, etc., involving finite dimensional symmetry groups such as the phase rotations, spatial translation or rotations, etc. Generally, the main necessary assumptions would just be that the Hessian of the modified energy functional at the relative equilibria (like L_c) has only finitely many negative directions, so that the linear analysis in [37] is applicable. In fact, we also constructed local invariant manifolds of traveling waves of supercritical gKdV equation and analyze the nearby dynamics [29].

The paper is organized as follows. In Sect. 2 we set up the basic framework for the construction of local invariant manifolds of $\mathcal{M}$. Section 3 is on the estimates of non-homogeneous linear equations. The existence of Lipschitz local invariant manifolds and some of their properties related to the local dynamics are obtained in Sect. 4, while the smoothness is obtained in Sect. 5. The non-degenerate case under assumption (H1-2) is analyzed in Sect. 6. Finally, some tedious technical details are left in the Appendix.

Notations. Throughout the paper, we follow the following notations:

- $\dot{H}^s$: the homogeneous Sobolev space $\{u \mid |D|^s u \in L^2\}$.
- X_0 : the energy space defined in (1.2).
- $X_1 = H^1 \times \dot{H}^1$ defined in (1.6).
- $\langle \cdot, \cdot \rangle$: Euclidean or L^2 duality pair unless specified otherwise.
- The generic upper bound C is always independent of $y \in \mathbb{R}^3$.
- Differentiations are usually not with respect to c , unless specified.

2. A Coordinate System Near Traveling Waves

In this section, we rewrite equation (GP) in an appropriate local coordinate system near traveling waves.

2.1. Structure of X_0 and a generalization of the momentum. Any $u \in X_0$ can be written as $u = \alpha(1 + v)$ where $\alpha \in S^1$ and $v \in \dot{H}^1(\mathbb{R}^3)$ satisfying

$$|1 + v|^2 - 1 = 2\operatorname{Re}(v) + |v|^2 \in L^2(\mathbb{R}^3).$$

The distance on the energy space is introduced as following. Given $u = \alpha(1 + v)$ and $\tilde{u} = \tilde{\alpha}(1 + \tilde{v})$ in X_0 , we define the distance d by

$$d(u, \tilde{u}) = |\alpha - \tilde{\alpha}| + \|\nabla v - \nabla \tilde{v}\|_{L^2(\mathbb{R}^3)} + \||1 + v|^2 - |1 + \tilde{v}|^2\|_{L^2(\mathbb{R}^3)}. \quad (2.1)$$

Select $\chi(\xi) \in C_0^\infty(\mathbb{R})$ such that $\chi(\xi) = 1$ near $\xi = 0$ and define the Fourier multiplier $\chi(D)$ as

$$\widehat{\chi(D)u}(\xi) = \chi(|\xi|)\hat{u}(\xi).$$

Lemma 2.1. ([21]), *The mapping*

$$\begin{aligned} \psi_\chi : S^1 \times (H^1(\mathbb{R}^3) + i\dot{H}^1(\mathbb{R}^3)) &\rightarrow X_0 \\ (\alpha, w) &\mapsto \alpha \left(1 + w - \chi(D) \left(\frac{(\operatorname{Im}(w))^2}{2} \right) \right) \end{aligned} \quad (2.2)$$

is a homeomorphism.

Remark 2.2. Note that X_0 is not a linear space, but this homeomorphism between $H^1 \times \dot{H}^1$ and X_0 allows us to work in the linear space $H^1 \times \dot{H}^1$. Since $\dot{H}^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$, the above $\alpha \in S^1$ is invariant for any solution of (GP), which can be fixed to be 1 due to the phase invariance of (GP). Also, this structure of X_0 does not depend on the choice of the cut-off function χ . To simplify the notation, we will fix $\alpha = 1$ and wrote $\psi(w)$ for $\psi_\chi(1, w)$. Apparently, ψ^{-1} is given by

$$\psi^{-1}(u + iv) = \left(u - 1 + \frac{1}{2}\chi(D)(v^2), v \right). \quad (2.3)$$

The coordinate mapping ψ commutes with the translation and $SO(3)$ action. Namely, let $y \in \mathbb{R}^3$ and $Q_{3 \times 3}$ be an orthogonal matrix with $\det Q = 1$, then

$$\psi(\alpha, w(Q \cdot)) = \psi(\alpha, w)(Q \cdot), \quad \psi(\alpha, w(\cdot - y)) = \psi(\alpha, w)(\cdot - y). \quad (2.4)$$

This is useful as (GP) is invariant under the translation and $SO(3)$ action.

Let $X_1 = H^1 \times \dot{H}^1$. By Lemma 2.1, for any $u \in X_0$, there exists a unique $w = w_1 + i w_2 \in X_1$ such that $u = \psi(w)$. As in [36], extend the domain of the momentum to X_1 as

$$\tilde{P}(w) = - \int_{\mathbb{R}^3} \left[w_1 + (1 - \chi(D)) \frac{w_2^2}{2} \right] \nabla w_2 dx, \quad \tilde{P} \in C^\infty(X_1, \mathbb{R}). \quad (2.5)$$

One can see that $\tilde{P}(w) = P(u)$ when $u = \psi(w) \in 1 + H^1(\mathbb{R}^3)$.

2.2. A local form of the GP equation near a traveling wave manifold. Consider a smooth and bounded traveling wave solution $U_c = u_c + i v_c$ of (GP) with the traveling velocity $c \in \mathbb{R}^3$ satisfying $|c| \in (0, \sqrt{2})$, we first rewrite the equation in the traveling frame in a neighborhood of U_c . Assume

$$\lim_{|x| \rightarrow \infty} \left(|x|^2 (|\operatorname{Re} U_c(x) - 1| + |\nabla \operatorname{Re} U_c|) + |x| |\operatorname{Im} U_c(x)| \right) = 0. \quad (2.6)$$

Such traveling waves exist as proved [25, 40]. In terms of the coordinate mapping ψ given in (2.2), let

$$w_c := \psi^{-1}(U_c) = (w_{1c}, w_{2c}) \in X_1.$$

The traveling wave manifold $\{U_c(\cdot + y) \mid y \in \mathbb{R}^3\}$ with wave velocity c generated by U_c is invariant under (GP). To study the nearby dynamics, we rewrite solutions in the traveling frame $u(t, x) = U(t, x - ct)$ and then $U(t, x)$ satisfies

$$i \partial_t U - i c \cdot \nabla U + \Delta U + (1 - |U|^2)U = 0 \quad (2.7)$$

or in the abstract form

$$\partial_t U = J(E + c \cdot P)'(U), \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad (2.8)$$

where we recall E and P are the energy and momentum defined in (1.1) and (1.3), respectively, and J is the matrix representation of $-i$. The traveling wave U_c generates a manifold of equilibria of (2.8):

$$\mathcal{M} = \{U_c(\cdot + y) : y \in \mathbb{R}^3\}. \quad (2.9)$$

Our main goal is to construct local invariant manifolds of $\mathcal{M}$. For any $y \in \mathbb{R}^3$, let

$$K_{c,y} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = D\psi(w_c(\cdot + y)) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_1 - \chi(D) \frac{(v_c(\cdot + y)w_2)}{w_2} \end{pmatrix}. \quad (2.10)$$

and

$$\begin{aligned} L_{c,y} &= (E + c \cdot P)''(U_c(\cdot + y)) \\ &= \begin{bmatrix} -\Delta - 1 + (3u_c^2 + v_c^2)(\cdot + y) & -c \cdot \nabla + 2(u_c v_c)(\cdot + y) \\ c \cdot \nabla + (2u_c v_c)(\cdot + y) & -\Delta - 1 + (u_c^2 + 3v_c^2)(\cdot + y) \end{bmatrix}. \end{aligned} \quad (2.11)$$

Both $K_{c,y}$ and $L_{c,y}$ are conjugate to $K_{c,0}$ and $L_{c,0}$ through translation

$$K_{c,y} w = (K_{c,0} w(\cdot - y))(\cdot + y), \quad L_{c,y} U = (L_{c,0} U(\cdot - y))(\cdot + y). \quad (2.12)$$

To simply the notation, we denote

$$K_y = K_{0,y}, \quad L_c = L_{c,0}.$$

From (2.6) and Hardy's inequality, we have that $K_{c,y}$ is an isomorphism on X_1 with

$$K_{c,y}^{-1} w = \begin{pmatrix} w_1 + \chi(D)(v_c(\cdot + y)w_2) \\ w_2 \end{pmatrix}, \quad (2.13)$$

and $L_{c,y}$ induces a real valued symmetric bounded bilinear form on X_1 , namely,

$$K_{c,y}, K_{c,y}^{-1} \in L(X_1), \quad L_{c,y} \in L(X_1, X_1^*), \quad L_{c,y}^* = L_{c,y}.$$

Moreover, using the translation invariance (2.12) and Hardy's inequality, there exists $C > 0$ such that

$$\|K_{c,y}\|_{L(X_1)} + \|K_{c,y}^{-1}\|_{L(X_1)} \leq C, \quad \forall y \in \mathbb{R}^3. \quad (2.14)$$

Consequently, J is viewed as a closed operator

$$J : X_1^* \supset D(J) \rightarrow X_1 \quad \text{satisfying } J^* = -J$$

where

$$\begin{aligned} D(J) &= \{w = (w_1, w_2) \mid w_1 \in H^{-1} \cap \dot{H}^1 \text{ and } w_2 \in \dot{H}^{-1} \cap H^1\} \\ &= \{w = (w_1, w_2) \mid w_1 \in H^1 \text{ and } |\xi| \hat{w}_2, |\xi|^{-1} \hat{w}_2 \in L^2\}. \end{aligned}$$

Suppose for some $y(t) \in \mathbb{R}^3$ and $w(t) = (w_1(t), w_2(t)) \in X_1$ smooth in t ,

$$\begin{aligned} U(t) &= \psi(w_c(\cdot + y(t)) + w(t)) \\ &= U_c(\cdot + y) + K_{c,y} w - \left(\frac{1}{2} \chi(D)(w_2^2), 0 \right)^T \end{aligned} \quad (2.15)$$

is a solution to (2.7).

Here (2.4) and the definition of $K_{c,y}$ are used. Substituting (2.15) into (2.8) and using the definition of $L_{c,y}$ and that U_c is an equilibrium of (2.8), we obtain

$$\begin{aligned} \partial_t y \cdot \nabla U_c(\cdot + y) + \partial_t (K_{c,y} w) - (\chi(D)(w_2 \partial_t w_2), 0)^T \\ = J L_{c,y} K_{c,y} w + J((E + c \cdot P)'(U) - L_{c,y} K_{c,y} w). \end{aligned}$$

The above equation of $w = (w_1, w_2)^T$ can be written as a system of 2 equations of $\partial_t w_1$ and $\partial_t w_2$, which can be solved easily due to the upper triangular structure of $K_{c,y}$. In particular we have the equation for w_2 ,

$$\partial_t w_2 = -(L_{c,y} K_{c,y} w)_1 - \partial_t y \cdot \nabla v_c(\cdot + y) + G_2(c, y, w) \quad (2.16)$$

where G_2 is given in the below. The above system of evolution equations for w can be written in a compact form

$$\partial_t y \cdot \nabla U_c(\cdot + y) + \partial_t (K_{c,y} w) = J L_{c,y} K_{c,y} w + G(c, y, \partial_t y, w), \quad (2.17)$$

where $G = (G_1(c, y, \partial_t y, w), G_2(c, y, w))^T$ are

$$\begin{aligned} G = & \left(|U|^2 - |U_c(\cdot + y)|^2 - 2U_c(\cdot + y) \cdot (K_{c,y} w) \right) J U_c(\cdot + y) \\ & + \left(|U|^2 - |U_c(\cdot + y)|^2 \right) J K_{c,y} w + \frac{1}{2} \begin{pmatrix} 2\chi(D)(w_2 \partial_t w_2 - w_2 \nabla w_2 \cdot c) \\ -(1 - |U|^2)\chi(D)(w_2^2) - \Delta \chi(D)(w_2^2) \end{pmatrix} \end{aligned}$$

and $\partial_t w_2$ in G_1 should be substituted by (2.16). This results in the dependence of G_1 on $\partial_t y$. The nonlinearity G is affine in $\partial_t y$ and contains terms of w of algebraic degrees between 2 and 6. Like $K_{c,y}$ and $L_{c,y}$, $G(c, y, \tilde{y}, w)$ is translation invariant in the sense of (2.12), namely,

$$G(c, y + x, \tilde{y}, w) = G(c, y, \tilde{y}, w(\cdot - x))(\cdot + x). \quad (2.18)$$

A more detailed form of and some basic estimates on G are straightforward but tedious and we leave them in the ‘‘Appendix’’.

2.3. Decomposition of X_1 and local coordinates near traveling waves. The free choices of $y \in \mathbb{R}^3$ and $w \in X_1$ are clearly redundant in the representation of the above U . We shall impose appropriate restrictions on w by analyzing the linearization of (GP) near U_c . We will focus on unstable traveling waves. Namely, we assume

(H) The spectrum $\sigma(JL_c) \not\subset i\mathbb{R}$.

Since $|c| \in (0, \sqrt{2})$, (2.6) and the explicit form (2.11) of L_c imply that $L_c : X_1 \rightarrow X_1^*$ is a compact perturbation to

$$L_{c,\infty} = \begin{bmatrix} -\Delta + 2 & -c \cdot \nabla \\ c \cdot \nabla & -\Delta \end{bmatrix} : X_1 \rightarrow X_1^*$$

which is an isomorphism as it induces a uniformly positive quadratic form on X_1 . This follows from a proof similar to the one in [36] and we skip the details. Therefore $\dim \ker L_c < \infty$ and it is uniformly positive on some finite co-dimensional subspace of X_1 . Let $n^-(L_c)$ be the Morse index of L_c , namely,

$$n^-(L_c) = \max\{\dim Y \mid L_c \text{ is negative on the subspace } Y \subset X_1\}. \quad (2.19)$$

According to the index formula of linear Hamiltonian systems [37], it holds that $n^-(L_c) > 0$ for any unstable traveling wave. We first cite Theorem 2.1 in [37] whose hypotheses are easily satisfied due to $\dim \ker L_c < \infty$ and Remark 2.2 in [37].

Theorem 2.1 in [37]. There exist closed subspaces Y_j , $j = 1, \dots, 6$, and $Y_0 = \ker L_c$ such that

- (1) $X_1 = \oplus_{j=0}^6 Y_j$, $Y_j \subset \cap_{k=1}^{\infty} D((JL_c)^k)$, $j \neq 3$, and

$$\dim Y_1 = \dim Y_4, \dim Y_5 = \dim Y_6, \dim Y_1 + \dim Y_2 + \dim Y_5 = n^-(L_c);$$

- (2) JL_c and L_c take the following forms in this decomposition

$$JL_c \longleftrightarrow \begin{pmatrix} 0 & A_{01} & A_{02} & A_{03} & A_{04} & 0 & 0 \\ 0 & A_1 & A_{12} & A_{13} & A_{14} & 0 & 0 \\ 0 & 0 & A_2 & 0 & A_{24} & 0 & 0 \\ 0 & 0 & 0 & A_3 & A_{34} & 0 & 0 \\ 0 & 0 & 0 & 0 & A_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_6 \end{pmatrix}, \quad (2.20)$$

$$L_c \longleftrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{14} & 0 & 0 \\ 0 & 0 & L_{Y_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_{Y_3} & 0 & 0 & 0 \\ 0 & B_{14}^* & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & B_{56} \\ 0 & 0 & 0 & 0 & 0 & B_{56}^* & 0 \end{pmatrix}. \quad (2.21)$$

- (3) $B_{14} : Y_4 \rightarrow Y_1^*$ and $B_{56} : Y_6 \rightarrow Y_5^*$ are isomorphisms and there exists $\varepsilon > 0$ satisfying $\mp \langle L_{Y_{2,3}} u, u \rangle \geq \varepsilon \|u\|^2$, for all $u \in Y_{2,3}$;
(4) all blocks of JL_c are bounded operators except A_3 , where A_{03} and A_{13} are understood as their natural extensions defined on Y_3 ;
(5) $A_{2,3}$ are anti-self-adjoint with respect to the equivalent inner product $\mp \langle L_{Y_{2,3}} \cdot, \cdot \rangle$ on $Y_{2,3}$;
(6) the spectra $\sigma(A_j) \subset i\mathbf{R}$, $j = 1, 2, 3, 4$, $\pm \operatorname{Re} \lambda > 0$ for all $\lambda \in \sigma(A_{5,6})$, and $\sigma(A_5) = -\sigma(A_6)$; and
(7) $n^-(L|_{Y_5 \oplus Y_6}) = \dim Y_5$ and $n^-(L|_{Y_1 \oplus Y_4}) = \dim Y_1$.
(8) $(u, v)_{X_1} = 0$ for all $u \in Y_1 \oplus Y_2 \oplus Y_3 \oplus Y_4$ and $v \in \ker L_c$.

We modify this decomposition of X_1 slightly for this paper. Let

$$X_c^T = \operatorname{span}\{\partial_{x_j} U_c \mid j = 1, 2, 3\}, \quad \tilde{Y}_0 = \{w \in \ker L_c \mid (w, \tilde{w}) = 0, \forall \tilde{w} \in X_c^T\}$$

and

$$X_c^{d1} = \tilde{Y}_0 \oplus Y_1 \oplus Y_2, \quad X_c^{e,d2,+,-} = Y_{3,4,5,6}.$$

For any $y \in \mathbb{R}^3$ and $\alpha \in \{T, d1, e, d2, +, -\}$, define

$$X_{c,y}^\alpha = \{w \in X_1 \mid w(\cdot - y) \in X_c^\alpha\}.$$

Recall the traveling wave manifold $\mathcal{M}$ defined in (2.9). Clearly

$$X_1 = X_{c,y}^T \oplus X_{c,y}^{d1} \oplus X_{c,y}^e \oplus X_{c,y}^{d2} \oplus X_{c,y}^+ \oplus X_{c,y}^-, \quad X_{c,y}^T = T_{U_c(\cdot+y)} \mathcal{M}, \quad (2.22)$$

with associated projection $\Pi_{c,y}^{T,d1,e,d2,+,-}$. Let

$$0 < \lambda < \min\{\operatorname{Re} \mu \mid \mu \in \sigma(A_5)\}.$$

Lemma 2.3. Assume (2.6) and (H), then there exists $C > 0$, such that, for any $y \in \mathbb{R}^3$,

- (1) $X_{c,y}^{+,-,T,d1,d2} \subset \dot{H}^k \cap X_1$ for any $k \geq 1$;
- (2) $0 < \dim X_{c,y}^\pm = d \leq n^-(L_c)$, $d_1 = \dim X_{c,y}^{d1} = n^-(L_c) + \dim \ker L_c - 3 - d$, and $d_2 = \dim X_{c,y}^{d2} \leq n^-(L_c) - d \leq d_1$;
- (3) there exist bases $V_{c,j}^\pm$, $j = 1, \dots, d$ of $X_{c,0}^\pm$, $V_{c,j}^{d1}$, $j = 1, \dots, d_1$, of $X_{c,0}^{d1}$, $V_{c,j}^{d2}$, $j = 1, \dots, d_2$ of $X_{c,0}^{d2}$, and $V_{c,j}^T = \partial_{x_j} U_c$, $j = 1, 2, 3$, of $X_{c,0}^T$, along with $\zeta_{c,j}^\pm$, $j = 1, \dots, d$, $\zeta_{c,j}^{d1}$, $j = 1, \dots, d_1$, $\zeta_{c,j}^{d2}$, $j = 1, \dots, d_2$, and $\zeta_{c,j}^T$, $j = 1, 2, 3$, belonging to $D((JL_{c,y})^*) \cap H^k = H^k \times (\dot{H}^{-1} \cap H^k)$ for any $k \geq 1$, such that

$$\Pi_{c,y}^\alpha w = \sum_{j=1}^{\dim X_{c,0}^\alpha} \langle \zeta_{c,j}^\alpha(\cdot + y), w \rangle V_{c,j}^\alpha(\cdot + y), \quad \alpha \in \{T, d1, d2, +, -\},$$

consequently, projections $\Pi_{c,y}^{T,d1,e,d2,+, -}$ are smooth in y with derivatives bounded uniformly in $y \in \mathbb{R}^3$;

- (4) In the decomposition $X_1 = \oplus_{\alpha \in \{T,d1,e,d2,+, -\}} X_{c,y}^\alpha$, $JL_{c,y}$ and the quadratic form $L_{c,y}$ take the form

$$L_{c,y} \longleftrightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_{11}(y) & 0 & L_{12}(y) & 0 & 0 \\ 0 & 0 & L^e(y) & 0 & 0 & 0 \\ 0 & L_{12}(y)^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & L_{+-}(y) \\ 0 & 0 & 0 & 0 & L_{+-}(y)^* & 0 \end{bmatrix},$$

$$JL_{c,y} \longleftrightarrow \begin{bmatrix} 0 & A_{T1}(y) & A_{Te}(y) & A_{T2}(y) & 0 & 0 \\ 0 & A_1(y) & A_{1e}(y) & A_{12}(y) & 0 & 0 \\ 0 & 0 & A_e(y) & A_{e2}(y) & 0 & 0 \\ 0 & 0 & 0 & A_2(y) & 0 & 0 \\ 0 & 0 & 0 & 0 & A_+(y) & 0 \\ 0 & 0 & 0 & 0 & 0 & A_-(y) \end{bmatrix}$$

where

- (a) all above blocks are translation invariant in the sense of (2.12);
- (b) all above blocks are bounded except $A_e(y)$;
- (c) $\|e^{tA_1(y)}\| + \|e^{tA_2(y)}\| \leq C(1 + |t|)^{d_1}$;
- (d) the quadratic form $\langle L^e(y)V^e, V^e \rangle \geq \frac{1}{C}\|V^e\|_{X_1}^2$ for any $V^e \in X_{c,y}^e$ and $A_e(y)$ is anti-self-adjoint with respect to $L^e(y) = L_{c,y}|_{X_{c,y}^e}$;
- (e) $\sigma(A_+(y)) = -\sigma(A_-(y))$, $\|e^{tA_\pm(y)}|_{X_{c,y}^\pm}\| \leq Ce^{\lambda t}$, for all $\mp t \geq 0$;

Proof. All the conclusions directly follow from Theorem 2.1 in [37] except those on the dual basis $\zeta_{c,j}^\alpha$ and the smoothness of $\Pi_{c,y}^\alpha$ in y . In particular $X_{c,y}^{+,-,T,d1,d2} \subset \dot{H}^k \cap X_1$ is due to $D((JL_c)^k) = \dot{H}^{1+2k} \cap X_1$. To complete the proof, we only need to show the smoothness of $\Pi_{c,y}^\alpha$ in y . Due to the translation invariance, we have

$$\Pi_{c,y+y'}^\alpha w = \left(\Pi_{c,y'}^\alpha (w(\cdot - y)) \right) (\cdot + y), \quad \alpha \in \{T, d1, e, d2, +, -\} \quad (2.23)$$

for any $y \in \mathbb{R}^3$, which also implies

$$D_y^n \Pi_{c,y} w = \left(D_y^n \Pi_{c,0} (w(\cdot - y)) \right) (\cdot + y).$$

Let

$$w^1, \dots, w^{d_0}, w^{d_0+1}, \dots, w^{d'} \text{ be a basis of } X_c^T \oplus X_c^{d1} \oplus X_c^{d2} \oplus X_c^+ \oplus X_c^-$$

formed by bases of $X_c^{T,d1,d2,+,-}$ such that $w^j = \partial_{x_j} U_c$, $j = 1, 2, 3$ and $w^1, \dots, w^{d_0}$ is a basis of $\ker L_c$ where $d_0 = \dim \ker L_c$.

Let

$$\tilde{w}^j = \begin{cases} \begin{bmatrix} 1 - \Delta & 0 \\ 0 & -\Delta \end{bmatrix} w^j, & j = 1, \dots, d_0; \\ L_c w^j, & j = d_0 + 1, \dots, d'. \end{cases}$$

Clearly $\tilde{w}^j \in X_1^* \cap H^k = H^k \times (\dot{H}^{-1} \cap H^k)$ for any k and $j = 1, \dots, d'$ and $\{\tilde{w}^1, \dots, \tilde{w}^{d_0}\}$ and $\{\tilde{w}^{d_0+1}, \dots, \tilde{w}^{d'}\}$ are both linearly independent. Moreover $\{\tilde{w}^1, \dots, \tilde{w}^{d'}\}$ are also linearly independent. In fact, assume

$$\tilde{w} = a_1 \tilde{w}^1 + \dots, a_{d_0} \tilde{w}^{d_0} = a_{d_0+1} \tilde{w}^{d_0+1} + \dots, a_{d'} \tilde{w}^{d'}$$

for some $a_1, \dots, a_{d'}$. Since $w^j \in \ker L_c$ for $j = 1, \dots, d_0$, we have

$$0 = \langle L_c \sum_{j=1}^{d_0} a_j w^j, \sum_{j=d_0+1}^{d'} a_j w^j \rangle = \langle \tilde{w}, \sum_{j=1}^{d_0} a_j w^j \rangle = (\tilde{w}, \tilde{w})_{X_1^*}.$$

Therefore $\tilde{w} = 0$ and we obtain the linear independence of $\{\tilde{w}^1, \dots, \tilde{w}^{d'}\}$.

For any $w \in X_c^e$, on the one hand, from the above L_c -orthogonality between X_c^e and $X_c^T \oplus X_c^{d1} \oplus X_c^{d2} \oplus X_c^+ \oplus X_c^-$, we have $\langle \tilde{w}^j, w \rangle = \langle L_c w^j, w \rangle = 0$, for $j = d_0+1, \dots, d'$. On the other hand, due to the orthogonality between X_c^e and $\ker L_c$ with respect to the $(\cdot, \cdot)_{X_1}$, for any $j = 1, \dots, d_0$, we have $\langle \tilde{w}^j, w \rangle = (w^j, w)_{X_1} = 0$. Counting the dimensions, we obtain that

$$\tilde{w}^1, \dots, \tilde{w}^{d'} \text{ form a basis of } \ker i_{X_c^e}^* = \{f \in X_1^* \mid \langle f, w \rangle = 0, \forall w \in X_c^e\}.$$

As $\ker i_{X_c^e}^*$ is isomorphic to $(X_c^T \oplus X_c^{d1} \oplus X_c^{d2} \oplus X_c^+ \oplus X_c^-)^*$, let $\gamma_1, \dots, \gamma_{d'} \in \ker i_{X_c^e}^*$ be the dual basis of $w^1, \dots, w^{d'}$. Since γ_j can be written as a linear combination of $\tilde{w}^1, \dots, \tilde{w}^{d'}$, we have $\gamma_j \in H^k \times (\dot{H}^{-1} \cap H^k)$ for any k and $j = 1, \dots, d'$. From (2.23) and the definition of γ_j , it is easy to verify that, for any $\alpha \in \{T, d1, d2, +, -\}$, $w \in X_1$, and $y \in \mathbb{R}^3$,

$$\Pi_{c,y}^\alpha w = \sum_{w^j \in X_c^\alpha} \langle \gamma_j(\cdot + y), w \rangle w^j(\cdot + y).$$

The smoothness of $\Pi_{c,y}^\alpha$ in y follows from the regularity $\gamma_j \in H^k \times (\dot{H}^{-1} \cap H^k)$ and $w^j \in \dot{H}^k \cap X_1$ for any k and $j = 1, \dots, d'$, which also implies the smoothness of $\Pi_{c,y}^e = I - \sum_{\alpha=T,d1,d2,+,-} \Pi_{c,y}^\alpha$. Divide $\{w^j, j = 1, \dots, d'\}$ and $\{\gamma_j, j = 1, \dots, d'\}$ according to $\alpha \in \{T, d1, d2, +, -\}$, we obtain $V_{c,j}^\alpha$ and $\zeta_{c,j}^\alpha$ and complete the proof of the lemma. $\square$

Remark 2.4. Under the following additional non-degeneracy assumptions

$$\ker(L_c) = \text{span}\{\nabla U_c\}, \quad n^-(L_c) = d = \dim X^+, \quad (2.24)$$

we have $X_{c,y}^{d1,d2} = \{0\}$ and the decomposition may be simplified. We shall discuss this case carefully in Sect. 6.

With respect to the bases $\{V_{c,j}^\alpha\}, \alpha \in \{d1, d2, +, -\}$, operators

$$A_{T1}(y), A_1(y), A_{T2}(y), A_{12}(y), A_2(y), A_+(y), A_-(y)$$

representing $\Pi_{c,y}^\beta J L_{c,y} | X_{c,y}^\alpha : X_{c,y}^\alpha \rightarrow X_{c,y}^\beta$ in the above block decomposition of $J L_{c,y}$ can be represented by matrices

$$M_{T1}, M_1, M_{T2}, M_{12}, M_2, M_+, M_-$$

which are independent of y due to the translation invariance (2.12) of $K_{c,y}$ and $L_{c,y}$. Namely,

$$A_{T1}(y)((V_{c,1}^\alpha, \dots, V_{c,d1}^\alpha)a) = (V_{c,1}^\beta, \dots, V_{c,3}^\beta)(M_{T1}a), \quad \forall a \in \mathbb{R}^{d1}, \dots \quad (2.25)$$

From Lemma 2.3,

$$\|e^{tM_1}\| + \|e^{tM_2}\| \leq C(1 + |t|)^{d1}, \quad \forall t \in \mathbb{R}; \quad \|e^{tM_\pm}\| \leq C e^{\pm \lambda t}, \quad \forall \mp t \geq 0. \quad (2.26)$$

A similar representation through translation in $x \rightarrow x + y$ of $A_e(y)$ would cause loss of regularity when ∂_y is carried out. Instead we will keep working with $A_e(y) = \Pi_{c,y}^e J L_{c,y} \Pi_{c,y}^e$. When viewed as an (unbounded) operator from X_1 to X_1 , it is a uniformly (in y) bounded perturbation to a constant coefficient operator and its derivatives of all orders are bounded operators. In fact, separating the terms in (2.11) with constant coefficients from those with spatially decaying variable coefficients implies

$$J L_{c,y} = J L_{c,\infty} + \tilde{Q}(y), \quad L_{c,\infty} = \begin{bmatrix} 2 - \Delta & -c \cdot \nabla \\ c \cdot \nabla & -\Delta \end{bmatrix}, \quad (2.27)$$

and

$$\tilde{Q}(y) = \begin{bmatrix} 2u_c v_c & u_c^2 - 1 + 3v_c^2 \\ 3(1 - u_c^2) - v_c^2 & -2u_c v_c \end{bmatrix} (\cdot + y). \quad (2.28)$$

Lemma 2.5. Fix $c \in (0, \sqrt{2})$. For any integer $k \geq 0$, there exists $C_k > 0$ such that for any $y \in \mathbb{R}^3$, it holds

$$\|D_y^k(A_e(y) - J L_{c,\infty})\|_{L((\otimes^k(\mathbb{R}^3)) \otimes X_1, X_1)} \leq C_k.$$

Proof. Clearly $\tilde{Q}(0) \in L^\infty$ and $\nabla^k \tilde{Q}(0) \in L^\infty \cap L^2$ for any $k \geq 1$, along with (2.6) and Hardy's inequality, it is straightforward to prove, for all $y \in \mathbb{R}^3$ and $V \in X_1$,

$$\|\tilde{Q}V\|_{X_1} + \|D_y^k \tilde{Q}V\|_{X_1} + \|J \tilde{Q}V\|_{X_1^*} + \|J D_y^k \tilde{Q}V\|_{X_1^*} \leq C_k \|V\|_{X_1}, \quad (2.29)$$

for some $C_k > 0$ independent of y . Write

$$A_e(y) - J L_{c,\infty} = A_e(y) - J L_{c,y} + \tilde{Q}(y).$$

Therefore, to complete the proof of the lemma, we only need to show the boundedness of $D_y^k(JL_{c,y} - A_e(y))$, which, according to Lemma 2.3, has the same blockwise decomposition except the $A_e(y)$ component replaced by 0. The uniform boundedness (in y) of $A_e(y) - JL_{c,y}$ follows from the boundedness of those blocks, where the uniformity in y is due to their translation invariance in the sense of (2.12). The uniform upper bounds of $D_y^k(JL_{c,y} - A_e(y))$ also follow from the translation invariance of these blocks and the extra regularity of $\zeta_{c,j}^\alpha$ and $V_{c,j}^\alpha$, $\alpha \in \{T, d1, d2, +, -\}$. $\square$

Following from that $X_{c,y}^T$ is the tangent space $T_{U_c(\cdot+y)}\mathcal{M}$ and $K_{c,y} = D\psi(w_c(\cdot+y))$ is an isomorphism, $K_{c,y}^{-1}X_{c,y}^T$ is the tangent space of

$$\psi^{-1}(\mathcal{M}) = \{w_c(\cdot+y) \mid y \in \mathbb{R}^3\}.$$

Based on the Implicit Function Theorem, it is straightforward to prove that, for small δ ,

$$\begin{aligned} & \{w_c(\cdot+y) + w^{d1} + w^e + w^{d2} + w^+ + w^- \mid \\ & w^\alpha \in K_{c,y}^{-1}X_{c,y}^\alpha, \|w^\alpha\| < \delta, \alpha \in \{d1, e, d2, +, -\}\} \end{aligned}$$

is a neighborhood of $\psi^{-1}(\mathcal{M}) \subset X_1$, where each point has a unique representation in the above form. Locally the total $\|w^{d1}\| + \|w^e\|_{X_1} + \|w^{d2}\| + \|w^+\| + \|w^-\|$ of the transversal components is equivalent to the X_1 distance to $\mathcal{M}$.

2.4. A local bundle coordinate system. Accordingly, we shall set up the bundle coordinates near $\psi^{-1}(\mathcal{M})$ precisely. Denote

$$\mathcal{X}^e = \{(y, V^e) \mid y \in \mathbb{R}^3, V^e \in X_{c,y}^e\}, \quad (2.30)$$

and balls on this bundle

$$\mathcal{X}^e(\delta) = \{(y, V) \in \mathcal{X}^e \mid \|V\|_{X_1} < \delta\}. \quad (2.31)$$

Let $y_\# \in \mathbb{R}^3$ and $B^3(\delta)$ be the open ball on $\mathbb{R}^3$ centered at $y_\#$ with radius δ . For $\delta \ll 1$, a smooth (due to the smoothness of $\Pi_{c,y}^e$ with respect to y) local trivialization from $B^3(\delta) \times X_{c,y_\#}^e$ to $\mathcal{X}^e$, thus a local coordinate system, of $\mathcal{X}^e$ is given by $(y, \Pi_{c,y}^e V)$, $V \in X_{c,y_\#}^e$. There is a natural translation on $\mathcal{X}^e$

$$(z, y, V^e) \longrightarrow (y+z, V^e(\cdot, +z)).$$

Along with other subspaces $X_{c,y}^{T,d1,d2,+, -}$, we will often consider bundles $\mathbb{R}^k \oplus \mathcal{X}^e$ over $\mathbb{R}^3$ with fibers $\mathbb{R}^k \oplus X_{c,y}^e$, as well as their balls

$$B^k(\delta_1) \oplus \mathcal{X}^e(\delta_2) = \{(y, a, V^e) \mid a \in \mathbb{R}^k, |a| < \delta_1, (y, V^e) \in \mathcal{X}^e(\delta_2)\}. \quad (2.32)$$

For any fixed $y_\#$, the smoothness of $\Pi_{c,y}^e$ with respect to y allows it to serve to as a local trivialization of the fibers $X_{c,y}^e$ for y near $y_\#$.

Define an embedding

$$Em : \mathbb{R}^{3+d1+d2+2d} \oplus \mathcal{X}^e \rightarrow X_1$$

as

$$\begin{aligned}
& Em(y, a^T, a^{d1}, a^{d2}, a^+, a^-, V^e) \\
&= \sum_{j=1}^3 a_j^T \partial_{x_j} w_c(\cdot + y) + K_{c,y}^{-1} \left(\sum_{j=1}^{d_1} a_j^{d1} V_{c,j}^{d1}(\cdot + y) \right. \\
&\quad \left. + \sum_{j=1}^{d_2} a_j^{d2} V_{c,j}^{d2}(\cdot + y) + \sum_{j=1}^d a_j^+ V_{c,j}^+(\cdot + y) + \sum_{j=1}^d a_j^- V_{c,j}^-(\cdot + y) + V^e \right) \\
&:= K_{c,y}^{-1} \left((a^T V_c^T + a^{d1} V_c^{d1} + a^{d2} V_c^{d2} + a^+ V_c^+ + a^- V_c^-)(\cdot + y) + V^e \right).
\end{aligned} \tag{2.33}$$

The embedding $Em^\perp : \mathbb{R}^{d_1+d_2+2d} \oplus \mathcal{X}^e$ defined on the transversal bundle will be used for the bundle coordinates near $\psi^{-1}(\mathcal{M})$

$$Em^\perp(y, a^{d1}, a^{d2}, a^+, a^-, V^e) = Em(y, 0, a^{d1}, a^{d2}, a^+, a^-, V^e). \tag{2.34}$$

Clearly $Em^\perp$ is translation invariant in the sense

$$\begin{aligned}
& Em^\perp(y + \tilde{y}, a^{d1}, a^{d2}, a^+, a^-, V^e(\cdot + \tilde{y})) \\
&= Em^\perp(y, a^{d1}, a^{d2}, a^+, a^-, V^e)(\cdot + \tilde{y}), \quad \forall \tilde{y} \in \mathbb{R}^3.
\end{aligned} \tag{2.35}$$

On the one hand, according to the above trivialization, given any Banach space Z , a mapping $f : Z \rightarrow \mathcal{X}^e$ is said to be smooth near some $z_0 \in Z$ if $y(z)$ and $V^e(z) \in X_{c,y(z)}^e$ are smooth in z near z_0 , where $f(z) = (y(z), \Pi_{c,y(z)}^e V^e(z))$. Due to the smoothness of $\Pi_{c,y}^e$, in fact this is equivalent to the smoothness of $y(z)$ and $V(z) \in X_1$ where $f(z) = (y(z), V(z))$.

On the other hand, for any Banach space Y , a mapping $g : \mathcal{X}^e \rightarrow Y$ is said to be smooth near some $(y_\#, V_\#)$ if

$$\tilde{g}(y, V) = g(y, \Pi_{c,y}^e V), \quad y \in \mathbb{R}^3, \quad V \in X_{c,y_\#}^e$$

is smooth in $(y, V) \in \mathbb{R}^3 \times X_{c,y_\#}$ near $(y_\#, V_\#)$. It is straightforward to verify

- g is smooth if and only if locally $g(y, \Pi_{c,y}^e V)$, $y \in \mathbb{R}^3$, $V \in X_1$, is smooth on $\mathbb{R}^3 \times X_1$.
- g is smooth if and only if locally it is the restriction to $\mathcal{X}^e$ of a smooth mapping defined on $\mathbb{R}^3 \times X_1$;
- g is smooth if and only if $g \circ f$ is smooth for any smooth $f : Z \rightarrow \mathcal{X}^e$ defined on any Banach space Z ;
- Em is smooth with respect to (y, V^e) , due to the smoothness of $K_{c,y}^{-1}$ and the basis $V_{c,j}^\alpha$, $\alpha \in \{T, d1, d2, +, -\}$.

We shall often work with $g(y, Em(y, a, V^e))$ with g smooth on $\mathbb{R}^3 \times X_1$.

Near the 3-dim manifold $\mathcal{M}$ of traveling waves, we will work through the mapping Φ defined on $\mathbb{R}^{d_1+d_2+2d} \oplus \mathcal{X}^e$ which is diffeomorphic on $\mathbb{R}^{d_1+d_2+2d}(\delta) \oplus \mathcal{X}^e(\delta)$

$$U = \Phi(y, a, V^e) = \psi(w_c(\cdot + y) + Em^\perp(y, a, V^e)). \tag{2.36}$$

This is a smooth vector bundle coordinate system in a neighborhood of $\mathcal{M} \subset X_0$ for sufficiently small $\delta > 0$. From (2.33) and (2.34), Φ can be naturally extended into a smooth mapping on $\mathbb{R}^{3+d_1+d_2+2d} \oplus X_1$.

Remark 2.6. As the subspaces $X_{c,y}^{T,d1,e,d2,+,-}$ are obtained as the translations of $X_c^{T,d1,e,d2,+,-}$, it is tempting to use the coordinate system

$$U = \psi((w_c + w^{d1} + w^{d2} + w^+ + w^- + w^e)(\cdot + y))$$

where $w^\alpha \in X_c^\alpha$ and $y \in \mathbb{R}^3$. However, such translation parametrization is not smooth in X_1 because the differentiation in y causes a loss of one order regularity in $D_y w^e(\cdot + y)$. This is one of the main issues in Nakanishi and Schlag [43], where the authors constructed the center-stable manifolds of the manifold of ground states for the Klein-Gordon equation. They introduced a nonlinear “mobile distance” to overcome that difficulty. Instead, the above bundle coordinate system (2.36), where $V^e \in X_{c,y}^e$ is not directly parametrized by a translation in y , represents a different framework based on the observation that, while the parametrization by the spatial translation of y is not smooth in X_1 with respect to y , the vector bundles $X_{c,y}^{T,d1,e,d2,+,-}$ over $\mathcal{M}$ are smooth in y as given in Lemma 2.3. This approach was used also in [5]. While it avoids the loss of regularity when differentiating in y , it will involve more geometric calculation.

2.5. An equivalent form of the GP equation near traveling waves. Let $U(t, x)$ be any solution to (2.7). If $U(t, x)$ stays in a small neighborhood of $\mathcal{M}$, then we can express U in the coordinate system (2.36)

$$U(t) = \Phi(y(t), a(t), V^e(t)), \quad (y, a, V^e)(t) \in B^{d1+d2+2d}(\delta) \oplus \mathcal{X}^e(\delta) \quad (2.37)$$

for some $\delta > 0$. Substituting (2.37) into (2.17) and using (2.15), we obtain

$$\begin{aligned} & \partial_t y \cdot \nabla U_c(\cdot + y) + \partial_t V^e + ((\partial_t a^{d1})V_c^{d1} + (\partial_t a^{d2})V_c^{d2} + (\partial_t a^+)V_c^+ \\ & \quad + (\partial_t a^-)V_c^-)(\cdot + y) + (a^{d1}\partial_t y \cdot (\nabla V_c^{d1}) + a^{d2}\partial_t y \cdot (\nabla V_c^{d2}) \\ & \quad + a^+\partial_t y \cdot (\nabla V_c^+) + a^-\partial_t y \cdot (\nabla V_c^-))(\cdot + y) \\ & = J L_{c,y} K_{c,y} Em^\perp(y, a, V^e) + G(c, y, \partial_t y, Em^\perp(y, a, V^e)). \end{aligned} \quad (2.38)$$

Starting with $\partial_t y$, we identify the evolution equation of each coordinate component. Applying $\Pi_{c,y}^T$ and using Lemma 2.3 and (2.25), we have

$$\begin{aligned} & \partial_t y + \langle \zeta_c^T(\cdot + y), \partial_t V^e \rangle + \langle \zeta_c^T, \partial_t y \cdot \nabla(a^{d1}V_c^{d1} + a^{d2}V_c^{d2} + a^+V_c^+ + a^-V_c^-) \rangle \\ & = M_{T1}a^{d1} + M_{T2}a^{d2} - \langle L_{c,y} J \zeta_c^T(\cdot + y), V^e \rangle \\ & \quad + \langle \zeta_c^T(\cdot + y), G(c, y, \partial_t y, Em^\perp(y, a, V^e)) \rangle. \end{aligned}$$

Since $V^e \in X_{c,y}^e$ implies $\langle \zeta_c^\alpha(\cdot + y), V^e \rangle = 0$ for all t , we have

$$\langle \zeta_c^\alpha(\cdot + y), \partial_t V^e \rangle = -\langle (\partial_t y \cdot \nabla \zeta_c^\alpha)(\cdot + y), V^e \rangle, \quad \alpha \in \{T, d1, d2, +, -\}. \quad (2.39)$$

Therefore $\tilde{y} = \partial_t y$ satisfies the following equation

$$\begin{aligned} & \tilde{y} - \langle (\tilde{y} \cdot \nabla \zeta_c^T)(\cdot + y), \Pi_{c,y}^e K_{c,y} w \rangle + \langle \zeta_c^T, \tilde{y} \cdot \nabla((I - \Pi_{c,y}^T - \Pi_{c,y}^e) K_{c,y} w) \rangle \\ & = M_{T1}a^{d1} + M_{T2}a^{d2} - \langle L_{c,y} J \zeta_c^T(\cdot + y), V^e \rangle + \langle \zeta_c^T(\cdot + y), G(c, y, \tilde{y}, w) \rangle, \end{aligned}$$

where $w = Em^\perp(y, a, V^e)$. We actually note that the above equation is well-defined for any small $w \in X_1$. From Lemma A.1 and the regularity of V_c^α and ζ_c^α , when $\|w\|_{X_1}$ is sufficiently small, one may solve for $\tilde{y} = \partial_t y$ and obtain

$$\partial_t y = M_{T1}a^{d1} + M_{T2}a^{d2} - \langle L_{c,y} J \zeta_c^T(\cdot + y), V^e \rangle + G^T(c, y, w), \quad (2.40)$$

where

$$w = Em^\perp(y, a, V^e).$$

According to Lemma A.1 and the regularity of $\zeta_c^{T,d1,d2,+,-}$, $G^T(c, y, w)$ is smooth in y and $w \in X_1$ when $\|w\|_{X_1} \ll 1$. As we did not prove $G \in X_1$ in Lemma A.1, we used the extra regularity of $\zeta_c^\alpha \in H^k \times (\dot{H}^{-1} \cap H^k)$. Furthermore, there exists $C > 0$ such that, for any $y \in \mathbb{R}^3$ and small $w \in X_1$,

$$|D_w^k D_y^l G^T(c, y, w)| \leq C_{l,k} \|w\|_{X_1}^{\max\{2-k,0\}}. \quad (2.41)$$

Applying $\Pi_{c,y}^\alpha$, $\alpha \in \{d1, d2, +, -, e\}$, to (2.38) and using the basis V_c^α , Lemma 2.3, (2.25), (2.39) and (2.40), we obtain

$$\partial_t a^\pm = M_\pm a^\pm + G^\pm(c, y, w), \quad (2.42)$$

$$\partial_t a^{d1} = M_1 a^{d1} + M_{12} a^{d2} - \langle L_{c,y} J \zeta_c^{d1}(\cdot + y), V^e \rangle + G^{d1}(c, y, w), \quad (2.43)$$

$$\partial_t a^{d2} = M_2 a^{d2} + G^{d2}(c, y, w), \quad (2.44)$$

$$\Pi_{c,y}^e \partial_t V^e = A_e(y) V^e + a^{d2} A_{e2}(y) V_c^{d2}(\cdot + y) + G^e(c, y, w), \quad V^e \in X_{c,y}^e, \quad (2.45)$$

where $A_{e2}(y) = \Pi_{c,y}^e J L_{c,y} \Pi_{c,y}^{d2}$ is smooth in y and

$$w = Em^\perp(y, a, V^e). \quad (2.46)$$

Much as in the derivation of G^T , G^α is also well-defined for any small $w \in X_1$, $\alpha \in \{d1, d2, +, -, e\}$. Like $K_{c,y}$ and $L_{c,y}$, G^α is translation invariant,

$$\begin{aligned} G^\alpha(c, y + z, w(\cdot + z)) &= G^\alpha(c, y, w), \quad \alpha \in \{T, d1, d2, +, -\} \\ G^e(c, y + z, w(\cdot + z)) &= G^e(c, y, w)(\cdot + z) \end{aligned} \quad (2.47)$$

for all $z \in \mathbb{R}^3$. For $\|w\|_{X_1} \ll 1$, G^α , $\alpha \in \{T, d1, d2, e, +, -\}$, are quadratic in w . From Lemma A.1 and the regularity of $\zeta_c^{T,d1,d2,+,-}$, they are smooth in y and w and satisfy

$$|D_w^k D_y^l G^\alpha(c, y, w)| \leq C_{l,k} \|w\|_{X_1}^{\max\{2-k,0\}}, \quad \alpha \in \{T, d1, d2, +, -\}. \quad (2.48)$$

The multi-linear terms in G^e prevent it from belonging to X_1 (see Lemma A.1). However, due to the extra regularity of ζ_c^α , projections $\Pi_{c,y}^\alpha$, $\alpha \in \{d1, d2, +, -\}$, act on a larger class of functions than X_1 . From Lemma A.1, we have

$$(I - \Pi_{c,y}^e) G^e(c, y, w) = 0, \quad G^e(c, y, w) \in X_1 + W^{1, \frac{3}{2}} + (L^{\frac{3}{2}} \cap \dot{W}^{1, \frac{6}{5}}) \quad (2.49)$$

and

$$|D_w^k D_y^l G^e|_{X_1 + W^{1, \frac{3}{2}} + L^{\frac{3}{2}} \cap \dot{W}^{1, \frac{6}{5}}} \leq C_{l,k} \|w\|_{X_1}^{\max\{2-k,0\}}. \quad (2.50)$$

Transforming the V^e equation. Before we end this section, we transform (2.45) to an equivalent form. In fact, since $(I - \Pi_{c,y}^e)V^e \equiv 0$, we have

$$(I - \Pi_{c,y}^e)\partial_t V^e = D_y \Pi_{c,y}^e(\partial_t y)V^e. \quad (2.51)$$

Therefore, (2.45) implies

$$\partial_t V^e = A_e(y)V^e + \mathcal{F}(c, y)(\partial_t y, V^e) + a^{d2}A_{e2}(y)V_c^{d2}(\cdot + y) + G^e(c, y, w) \quad (2.52)$$

where

$$A_e(y) = \Pi_{c,y}^e J L_{c,y} \Pi_{c,y}^e$$

is given in Lemma 2.3 and the bounded bilinear operator $\mathcal{F}(c, y) : \mathbb{R}^3 \otimes X_1 \rightarrow X_1$ is given by

$$\mathcal{F}(c, y)(z, V) = D_y \Pi_{c,y}^e(z)(\Pi_{c,y}^e V - (I - \Pi_{c,y}^e)V). \quad (2.53)$$

Here we can take the last part of $\mathcal{F}$ in the form of $V - b(I - \Pi_{c,y}^e)V$ for any b , which would not change the validity of (2.53) for $V \in X_{c,y}^e$. The above choice of $\mathcal{F}$ would bring certain convenience in some calculation later. Using the smoothness of $\Pi_{c,y}^e$ in y given in Lemma 2.3, we obtain

$$\|\mathcal{F}(c, y)(z, V)\|_{X_1} \leq C|z|\|V\|_{X_1} \quad (2.54)$$

for some $C > 0$ independent of y . The bilinear operator $\mathcal{F}$ is a modification of the second fundamental form of the bundle $X_{c,y}^e$ over $\mathbb{R}^3$ as a sub-bundle of $X_1 = X_{c,y}^e \oplus ((I - \Pi_{c,y}^e)X_1)$ over $\mathbb{R}^3$.

While (2.52) is deduced from (2.45), actually the opposite also holds if $V(s) \in X_{c,y(s)}^e$ for some s . To see this, applying $I - \Pi_{c,y}^e$ to (2.52) we obtain

$$\partial_t((I - \Pi_{c,y}^e)V) = (I - \Pi_{c,y}^e)D_y \Pi_{c,y}^e(\partial_t y)(\Pi_{c,y}^e V - (I - \Pi_{c,y}^e)V) - D_y \Pi_{c,y}^e(\partial_t y)V.$$

Differentiating $\Pi_{c,y}^e \Pi_{c,y}^e = \Pi_{c,y}^e$ with respect to y we have

$$D_y \Pi_{c,y}^e(\cdot) \Pi_{c,y}^e + \Pi_{c,y}^e D_y \Pi_{c,y}^e(\cdot) = D_y \Pi_{c,y}^e(\cdot). \quad (2.55)$$

It follows that

$$\partial_t((I - \Pi_{c,y}^e)V) = -D_y \Pi_{c,y}^e(\partial_t y)(I - \Pi_{c,y}^e)V. \quad (2.56)$$

Since this is a well-posed homogeneous linear equation of $(I - \Pi_{c,y}^e)V$, which is finite dimensional, the solution has to vanish if we assume $V(s) \in X_{c,y(s)}^e$. Therefore

$$V(t) \in X_{c,y(t)}^e, \quad \forall t, \quad \text{if } V(s) \in X_{c,y(s)}^e \text{ and } V(t) \text{ solves (2.52)}. \quad (2.57)$$

Finally (2.45) follows from applying $\Pi_{c,y}^e$ to (2.52).

Compared to (2.45), equation (2.52) is more convenient as the latter may be posed on the whole space X_1 . Along with the boundedness of $\mathcal{F}$, it makes it easier to prove the local well-posedness and obtain estimates of (2.52) and thus we will mainly work with (2.52).

In summary, in a neighborhood of $\mathcal{M} \subset X_0$, equation (GP) written in the bundle coordinates $(y, a^{d1}, a^{d2}, a^+, a^-, V^e) \in B^{d1+d2+2d}(\delta) \oplus \mathcal{X}^e(\delta)$ is equivalent to the system consisting of (2.40), (2.42), (2.43), (2.44), and (2.52), along with (2.46).

3. Linear Analysis

We first analyze the linear part of (2.52) whose unknown is valued in a vector bundle $X_{c,y}^e$ over $\mathbb{R}^3$. However it is observed that (2.52) is well-posed with $V^e \in X_1$, we will consider this general situation as well as the case $V \in X_{c,y}^e$. Relaxing the restriction $V \in X_{c,y}^e$ would provide a little convenience in some estimates later. Moreover since G^e does not necessarily belong to X_1 , we give a space-time estimate which will be used to close the nonlinear estimates in later sections. Consider the following more general form of (2.52)

$$\partial_t V = \Pi_{c,y}^e J L_{c,y} \Pi_{c,y}^e V + \mathcal{F}(c, y)(\partial_t y, V) + f(t). \quad (3.1)$$

Here we assume $y(t)$, $-\infty \leq t_0 \leq t \leq t_1 \leq \infty$ satisfy

$$\sigma := |\partial_t y|_{L^\infty((t_0, t_1), \mathbb{R}^3)} \leq 1, \quad \forall t \in (t_0, t_1). \quad (3.2)$$

For the non-homogeneous term $f = (f_1(t, x), f_2(t, x))$, we need the norm

$$\|f\|_{\hat{X}_{(t_0, t_1)}^{p, q}} \triangleq \|f_1\|_{L_{(t_0, t_1)}^p B_{q, 2}^1} + \|f_2\|_{L_{(t_0, t_1)}^p \dot{B}_{q, 2}^1} \quad (3.3)$$

along with the associated spaces $\hat{X}_{(t_0, t_1)}^{p, q}$ and $\hat{X}_{(t_0, t_1), loc}^{p, q}$, where $B_{p, r}^s$ and $\dot{B}_{p, r}^s$ denote the standard Besov space as well as the homogeneous Besov space, respectively, and the “loc” denotes “local in t ”. In the standard terminology, an admissible Strichartz pair (p, q) and conjugate exponent p' of $p \in [1, \infty]$ are those satisfying

$$p, q \in [2, \infty], \quad 2/p + 3/q = 3/2; \quad 1/p' + 1/p = 1. \quad (3.4)$$

Our main goal in this section is to prove the following proposition.

Proposition 3.1. *Suppose (3.2) holds, (p, q) is a Strichartz pair, and $f \in \hat{X}_{(t_0, t_1), loc}^{\tilde{p}', q'}$ where $\tilde{p} \in [1, p]$. Then for any $s \in (t_0, t_1)$ and initial value $V(s) \in X_1$, (3.1) has a unique solution $V(t) \in X_1$. Moreover, there exists $C > 0$ independent of $t_0, t_1, s, \sigma, y(\cdot)$, and $f(\cdot)$, such that for any $t \in (t_0, t_1)$, and $\eta > C\sigma$, we have*

$$\begin{aligned} \langle L_{c, y(t)} V^e(t), V^e(t) \rangle^{\frac{1}{2}} &\leq e^{C\sigma|t-s|} \langle L_{c, y(s)} V^e(s), V^e(s) \rangle^{\frac{1}{2}} + C\eta^{-\frac{1}{\tilde{p}}} \|e^{\eta|t-\cdot|} f^e\|_{\hat{X}_{(s, t)}^{\tilde{p}', q'}} \\ |V^\perp(t)| &\leq e^{C\sigma|t-s|} |V^\perp(s)| + C\eta^{-\frac{1}{\tilde{p}}} \|e^{\eta|t-\cdot|} f^\perp\|_{L_{(s, t)}^{\tilde{p}'}} \end{aligned}$$

where

$$\begin{aligned} V^e(t) &= \Pi_{c, y(t)}^e V(t), \quad V^\perp(t) = (I - \Pi_{c, y(t)}^e) V(t), \\ f^e(t) &= \Pi_{c, y(t)}^e f(t), \quad f^\perp(t) = (I - \Pi_{c, y(t)}^e) f(t), \end{aligned} \quad (3.5)$$

satisfying

$$\begin{aligned} \partial_t V^e &= \Pi_{c, y}^e J L_{c, y} \Pi_{c, y}^e V^e + \mathcal{F}(c, y)(\partial_t y, V^e) + f^e(t), \\ \partial_t V^\perp &= -D_y \Pi_{c, y}^e (\partial_t y) V^\perp + f^\perp. \end{aligned} \quad (3.6)$$

Here one keeps in mind that $I - \Pi_{c,y}^e$ may be applied to a larger class of functions than X_1 and its range is finite dimensional. The above decoupling of $V^\perp$ and V^e is due to the choice (2.53) of $\mathcal{F}$. From the positivity of $L_{c,y}$ on $X_{c,y}^e$ (Lemma 2.3), we have

Corollary 3.2. *There exists $C > 0$ independent of $t_0, t_1, s, y(\cdot)$, and $f(\cdot)$, such that for any $t \in (t_0, t_1)$ and $\eta > C\sigma$, we have*

$$\|V(t)\|_{X_1} \leq C(e^{C\sigma|t-s|}\|V(s)\|_{X_1} + \eta^{-\frac{1}{p}}\|e^{\eta|t-s|}f(\cdot)\|_{\hat{X}_{(s,t)}^{\hat{p}',q'}}).$$

Moreover, $V(t) \in X_{c,y(t)}^e$, for almost all $t \in (t_0, t_1)$, if $V(s) \in X_{c,y(s)}^e$ and

$$(I - \Pi_{c,y(t)}^e)f(t, \cdot) = 0, \quad a.e. \ t \in (t_0, t_1). \quad (3.7)$$

The above estimates indicate that the linear equation (3.1) exhibits at most weak exponential growth due to $|\partial_t y|$.

Based on the regularity of the nonlinearity given in Lemma A.1, we also consider the space

$$\tilde{X}_{(t_0,t_1)} \triangleq L^2((t_0, t_1), X_1) + L^2((t_0, t_1), W^{1,\frac{3}{2}}) + L^2((t_0, t_1), L^{\frac{3}{2}} \cap \dot{W}^{1,\frac{6}{5}})$$

and $\tilde{X}_{(t_0,t_1),loc}$. The next proposition will be a simple consequence of Proposition 3.1.

Proposition 3.3. *Suppose (3.2) holds and $f \in \tilde{X}_{(t_0,t_1),loc}$. Then for any $s \in (t_0, t_1)$ and initial value $V(s) \in X_1$, (3.1) has a unique solution $V(t) \in X_1$. Moreover, there exists $C > 0$ independent of $t_0, t_1, s, \sigma, y(\cdot)$, and $f(\cdot)$, such that for any $t \in (t_0, t_1)$, and $\eta \in (C\sigma, 1)$, we have*

$$\begin{aligned} \|V^e(t)\|_{X_1} &\leq C(e^{C\sigma|t-s|}\|V^e(s)\|_{X_1} + \eta^{-\frac{1}{2}}\|e^{\eta|t-s|}f^e(\cdot)\|_{\tilde{X}_{(s,t)}}), \\ |V^\perp(t)| &\leq C(e^{C\sigma|t-s|}|V^\perp(s)| + \eta^{-\frac{1}{2}}\|e^{\eta|t-s|}f^\perp(\cdot)\|_{L_{(s,t)}^2}), \end{aligned}$$

where $V^{e,\perp}$ and $f^{e,\perp}$ are defined in (3.5) which satisfy (3.6).

These two propositions and Corollary 3.2 will be proved in the rest of the section.

Energy estimates of homogeneous linear equation. We start with the basic well-posedness and energy estimates of the homogeneous equation of (3.1) based on the uniform positivity of $L^e(y) = L_{c,y}|_{X_{c,y}^e}$.

Lemma 3.4. *Assume $f \equiv 0$, then (3.1) defines a bounded solution map*

$$S(t, s) \in L(X_1, X_1), \quad \forall t, s \in [t_0, t_1],$$

with initial value given at $t = s$, which satisfies

$$S(s, s) = I, \quad S(t, t')S(t', s) = S(t, s), \quad S(t, s) \in L(X_{c,y(s)}^e, X_{c,y(t)}^e). \quad (3.8)$$

Moreover there exists $C > 0$ independent of t_1, t_2, t, s , and $y(\cdot)$ such that

$$\langle L_{c,y(t)}S(t, s)V, S(t, s)V \rangle \leq e^{C\sigma|t-s|}\langle L_{c,y(s)}V, V \rangle, \quad \forall V \in X_{c,y(s)}^e. \quad (3.9)$$

As a consequence, the lemma implies that, under the assumption $f \equiv 0$, (3.1) preserves the constraint $V \in X_{c,y}^e$ if it holds initially. Later we will show that this holds for non-homogeneous equation as well. Furthermore the homogeneous equation induces possible exponential growth only due to $\|\partial_t y\|_{L^\infty}$. The coefficient 1 in front of the above exponential is important for future estimates.

Proof. From Lemma 2.5, $A_e(y) = \Pi_{c,y}^e J L_{c,y} \Pi_{c,y}^e$ is a bounded perturbation to $J L_{c,\infty}$ on X_1 . This, along with the boundedness of $\mathcal{F}$, implies that (3.1) is well-posedness on X_1 and thus the solution flow $S(t, s)$ of bounded linear operators is well-defined.

Since $(I - \Pi_{c,y}^e)X_1 \subset D(J L_{c,\infty})$, for any $V \in X_1$, by direct computation using (3.1), one finds that $(I - \Pi_{c,y}^e)V$ satisfies

$$\partial_t((I - \Pi_{c,y}^e)V) = (I - \Pi_{c,y}^e)D_y \Pi_{c,y}^e(\partial_t y)(\Pi_{c,y}^e V - (I - \Pi_{c,y}^e)V) - D_y \Pi_{c,y}^e(\partial_t y)V.$$

Following the same procedure as in Sect. 2.5, we obtain exactly the same equation as (2.56) which yields

$$(I - \Pi_{c,y(t)}^e)V(t) = 0, \quad \forall t, \quad \text{if } (I - \Pi_{c,y(s)}^e)V(s) = 0. \quad (3.10)$$

Finally we prove inequality (3.9). Let $V(t)$ be a solution of (3.1) with $V(s) \in X_{c,y(s)}^e \cap D(J L_{c,\infty})$, which yields $V(t) \in X_{c,y(t)}^e \cap D(J L_{c,\infty})$ for all t . By direct calculation using $J^* = -J = J^{-1}$, one has

$$\begin{aligned} \partial_t \langle L_{c,y} V, V \rangle &= \langle J^{-1} D_y \tilde{Q}(y)(\partial_t y) V, V \rangle + 2 \langle \partial_t V, L_{c,y} V \rangle \\ &= -\langle J D_y \tilde{Q}(y)(\partial_t y) V, V \rangle + 2 \langle \mathcal{F}(c, y)(\partial_t y, V), L_{c,y} V \rangle, \end{aligned}$$

where $\tilde{Q}$ is defined in (2.28). It follows from the bounds (2.54) and (2.29) that

$$|\partial_t \langle L_{c,y} V, V \rangle| \leq C \|\partial_t y\| \|V\|_{X_1}^2. \quad (3.11)$$

Recall from Lemma 2.3 that the bounded symmetric quadratic form $\langle L_{c,y} \cdot, \cdot \rangle$ satisfies $\langle L_{c,y} V, V \rangle \geq \varepsilon \|V\|_{X_1}^2$ for any $V \in X_{c,y}^e$. This uniform lower bound of $L_{c,y}$ on $X_{c,y}^e$, the above estimate, and the Gronwall inequality immediately imply (3.9) when $V(s) \in X_{c,y(s)}^e \cap D(J L_{c,\infty})$. Since $X_{c,y(s)}^e \cap D(J L_{c,\infty})$ is dense in $X_{c,y(s)}^e$, a standard density argument yields (3.9) for general solution $V(t) \in X_{c,y(t)}^e$. The proof of the lemma is complete. $\square$

Space-time estimates of (3.1). Given initial data at $t = s \in [t_0, t_1]$, the solution of (3.1) can be written as

$$V(t) = S(t, s)V(s) + \int_s^t S(t, \tau)f(\tau)d\tau. \quad (3.12)$$

Since $f(t)$ is not assumed to be in X_1 , we first prove the following lemma.

Lemma 3.5. *Suppose (3.2) and (3.7) hold, $(\tilde{p}, \tilde{q})$ is an admissible pair, and $f \in \hat{X}_{(t_0, t_1), loc}^{\tilde{p}', \tilde{q}'}$. Then for any given $s \in (t_0, t_1)$ and initial value $V(s) \in X_{c,y(s)}^e$, (3.12) has a unique solution $V(t)$ satisfying $V(t) \in X_{c,y(t)}^e$, for almost all $t \in (t_0, t_1)$. Moreover, for any admissible pair (p, q) , there exists $T, C > 0$ independent of f, t_0, t_1 , and $y(\cdot)$ such that, if $t_0 < t'_0 \leq s \leq t'_1 < t_1$ satisfy $t'_1 - t'_0 \leq T$, then*

$$\|V(t)\|_{\hat{X}_{(t'_0, t'_1)}^{p, q}} \leq C \left(\|V(s)\|_{X_1} + \|f\|_{\hat{X}_{(t'_0, t'_1)}^{\tilde{p}', \tilde{q}'}} \right). \quad (3.13)$$

In particular, if $(p, q) = (\infty, 2)$, then it holds, for $t \in (t'_0, t'_1)$

$$\langle L_{c,y(t)} V(t), V(t) \rangle^{\frac{1}{2}} \leq e^{C\sigma|t-s|} \langle L_{c,y(s)} V(s), V(s) \rangle^{\frac{1}{2}} + C \|f\|_{\hat{X}_{(t'_0, t'_1)}^{\bar{p}', \bar{q}'}}. \quad (3.14)$$

Proof. We prove the lemma in several steps.

• *Step 1. Change of variables and dispersive estimates of the constant coefficient homogeneous linear equation.* To make it more convenient to carry out the dispersive estimates, we first apply a similar transformation to diagonalize $JL_{c=0,\infty}$. Let

$$P = \sqrt{-\Delta(2-\Delta)^{-1}}, \quad R = \sqrt{-\Delta(2-\Delta)}, \quad \mathcal{P} = \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix}, \quad V = \mathcal{P}Z.$$

Apparently,

$$\mathcal{P} \text{ is an isomorphism from } \dot{H}^1 \text{ to } X_1. \quad (3.15)$$

From (3.1), (2.27), and (2.28), it is straight to compute that Z satisfies

$$Z_t = JHZ + Q(y(t), \partial_t y(t))Z + \tilde{f}(t), \quad (3.16)$$

where

$$H = \begin{bmatrix} R & -c \cdot \nabla \\ c \cdot \nabla & R \end{bmatrix}, \quad \tilde{f}(t) = \mathcal{P}^{-1} f(t),$$

$$Q(y, z) = \mathcal{P}^{-1} (\Pi_{c,y}^e J L_{c,y} \Pi_{c,y}^e - J L_{c,\infty} + \mathcal{F}(c, y)(z, \cdot)) \mathcal{P}.$$

Let $\mathcal{R} = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$. From (2.54), (2.29), (3.15), and our assumptions, we have

$$\|Q(y, z)Z\|_{\dot{H}^1} \leq C(1 + |z|) \|Z\|_{\dot{H}^1}, \quad \tilde{f} \in L_{(t_0, t_1)}^{\tilde{p}'} \dot{B}_{\tilde{q}', 2}^1 \quad (3.17)$$

for some $C > 0$ independent of y .

It was proved in [28] that for any $q \in [2, \infty]$, one has

$$\|e^{tJ\mathcal{R}}\phi\|_{\dot{B}_{q,2}^r} \lesssim t^{-3(\frac{1}{2}-\frac{1}{q})} \|\phi\|_{\dot{B}_{q',2}^r}.$$

Furthermore, for any admissible pairs (p_j, q_j) , $j = 1, 2$, it holds

$$\|e^{tJ\mathcal{R}}\phi\|_{L^{p_1} \dot{B}_{q_1,2}^r} \lesssim \|\phi\|_{\dot{H}^r}, \quad \left\| \int_0^t e^{(t-\tau)J\mathcal{R}} f(\tau) d\tau \right\|_{L^{p_1} \dot{B}_{q_1,2}^r} \lesssim \|f\|_{L^{p'_2} \dot{B}_{q'_2,2}^r}.$$

These estimates lead to

$$\|e^{tJH}\phi\|_{\dot{B}_{q,2}^r} \lesssim t^{-3(\frac{1}{2}-\frac{1}{q})} \|\phi\|_{\dot{B}_{q',2}^r}, \quad \|e^{tJH}\phi\|_{L^{p_1} \dot{B}_{q_1,2}^r} \lesssim \|\phi\|_{\dot{H}^r},$$

$$\left\| \int_0^t e^{(t-\tau)JH} g(\tau) d\tau \right\|_{L^{p_1} \dot{B}_{q_1,2}^r} \lesssim \|g\|_{L^{p'_2} \dot{B}_{q'_2,2}^r}. \quad (3.18)$$

In fact, since $JH - J\mathcal{R} = c \cdot \nabla$ which commutes with $J\mathcal{R}$, we have

$$e^{tJH}Z = (e^{tJ\mathcal{R}}Z)(\cdot + ct) = e^{tJ\mathcal{R}}(Z(\cdot + ct)).$$

The first two of the inequalities in (3.18) follow immediately due to the translation invariance of the Besov norms. To see the last one in (3.18),

$$\begin{aligned} & \left\| \int_0^t e^{(t-\tau)JH} g(\tau) d\tau \right\|_{L^{p_1} \dot{B}_{q_1,2}^r} = \left\| \int_0^t e^{(t-\tau)J\mathcal{R}} (g(\tau)(\cdot + c(t-\tau))) d\tau \right\|_{L^{p_1} \dot{B}_{q_1,2}^r} \\ & = \left\| \int_0^t e^{(t-\tau)J\mathcal{R}} (g(\tau)(\cdot - c\tau)) d\tau \right\|_{L^{p_1} \dot{B}_{q_1,2}^r} \lesssim \|g(t)(\cdot - ct)\|_{L^{p_2'} \dot{B}_{q_2',2}^r} = \|g\|_{L^{p_2'} \dot{B}_{q_2',2}^r}. \end{aligned}$$

• *Step 2. Space-time estimate.* In the next step, instead of (3.12), we will obtain the space-time estimate of solutions of (3.16) based on (3.18) with

$$\begin{aligned} Z(t) - e^{(t-s)JH} Z(s) &= \int_s^t e^{(t-\tau)JH} [Q(y(\tau), \partial_t y(\tau)) Z(\tau) + \tilde{f}(\tau)] d\tau \\ &= \int_0^{t-s} e^{(t-s-\tau)JH} [Q(y(s+\tau), \partial_t y(s+\tau)) Z(s+\tau) + \tilde{f}(s+\tau)] d\tau. \end{aligned} \quad (3.19)$$

By (3.18), (2.29), (3.17), (3.15), and $\sigma = |\partial_t y|_{L^\infty} \leq 1$, for admissible pairs (p, q) , $(\tilde{p}, \tilde{q})$ and $(\infty, 2)$, and $t_0 < t'_0 \leq s \leq t'_1 < t_1$, we have

$$\begin{aligned} & \|Z(t) - e^{(t-s)JH} Z(s)\|_{L_{(t'_0, t'_1)}^p \dot{B}_{q,2}^1} \leq C(\|Z\|_{L_{(t'_0, t'_1)}^1 \dot{H}^1} + \|\tilde{f}\|_{L_{(t'_0, t'_1)}^{\tilde{p}'} \dot{B}_{\tilde{q}',2}^1}) \\ & \leq C((t'_1 - t'_0)\|Z\|_{L_{(t'_0, t'_1)}^\infty \dot{H}^1} + \|\tilde{f}\|_{L_{(t'_0, t'_1)}^{\tilde{p}'} \dot{B}_{\tilde{q}',2}^1}). \end{aligned} \quad (3.20)$$

Consider the standard splitting of $Z(t)$ into

$$Z(t) = Z_h(t) + Z_{in}(t), \quad V(t) = \mathcal{P}Z_h(t) + \mathcal{P}Z_{in}(t),$$

where $Z_h(t)$ satisfies the corresponding homogeneous equation of (3.16) (i.e. without $\tilde{f}$) and $Z_h(s) = Z(s)$, and $Z_{in}(t)$ solves (3.16) and $Z_{in}(s) = 0$.

• *Step 3. Non-homogeneous part Z_{in} .* On the one hand, applying (3.20) to $Z_{in}(t)$ with the admissible pair $(p = \infty, q = 2)$, we obtain that there exists $T > 0$ independent of t_0, t_1, t'_0, t'_1 , and $y(t)$ such that, if $t'_1 - t'_0 \leq T$, it holds

$$\|Z_{in}\|_{L_{(t'_0, t'_1)}^\infty \dot{H}^1} \leq C\|\tilde{f}\|_{L_{(t'_0, t'_1)}^{\tilde{p}'} \dot{B}_{\tilde{q}',2}^1}.$$

Substituting this back into (3.20), we have that for any $t'_1 - t'_0 \leq T$ and admissible pairs (p, q) ,

$$\|Z_{in}\|_{L_{(t'_0, t'_1)}^p \dot{B}_{q,2}^1} \leq C\|\tilde{f}\|_{L_{(t'_0, t'_1)}^{\tilde{p}'} \dot{B}_{\tilde{q}',2}^1}. \quad (3.21)$$

We claim that $(I - \Pi_{c,y(t)}^e) f(t) = 0$ implies

$$\mathcal{P}Z_{in}(t) = \int_s^t S(t, \tau) f(\tau) d\tau \in X_{c,y(t)}^e, \quad a.e. \ t \in (t'_0, t'_1). \quad (3.22)$$

In fact, let

$$Y = \{\tilde{g} \in L_{(t'_0, t'_1)}^{\tilde{p}'} \dot{B}_{\tilde{q}',2}^1 \mid (I - \Pi_{c,y(t)}^e) \mathcal{P}g(t) = 0, \ \forall t \in (t'_0, t'_1)\} \subset L_{(t'_0, t'_1)}^{\tilde{p}'} \dot{B}_{\tilde{q}',2}^1.$$

Since $\zeta_c^\alpha \in H^k \times (H^k \cap H^{-1})$, for any $k \geq 1$ and $\alpha \in \{T, d1, d2, +, -\}$, $I - \Pi_{c,y}^e$ actually applies to $\mathcal{P}g(t) \in B_{\tilde{q}',2}^1 \times \dot{B}_{\tilde{q}',2}^1$. Consider the mapping Γ

$$W(t) = (\Gamma g)(t) \triangleq \int_s^t S(t, \tau) \mathcal{P} \tilde{g}(\tau) d\tau.$$

Inequality (3.21) and the definition of $\mathcal{P}$ imply that $\Gamma : L_{(t'_0, t'_1)}^{\tilde{p}'} \dot{B}_{\tilde{q}',2}^1 \rightarrow L_{(t'_0, t'_1)}^\infty X_1$ is a bounded operator, and thus also bounded when restricted to Y . Since Lemma 3.4 implies

$$(I - \Pi_{c,y(t)}^e)(\Gamma g)(t) = 0, \quad \forall t \in (t'_0, t'_1) \quad \text{if } g \in Y \cap L_{(t'_0, t'_1)}^\infty X_1$$

and $Y \cap L_{(t'_0, t'_1)}^\infty X_1$ is dense in Y , we obtain that $(I - \Pi_{c,y(t)}^e)\Gamma$ vanishes on Y . Therefore $\mathcal{P}Z_{in}(t) \in X_{c,y(t)}^e$ for almost all $t \in (t'_0, t'_1)$.

• *Step 4. Homogeneous part and the completion of the proof of the lemma.* On the other hand, it is clear

$$Z_h(t) = \mathcal{P}^{-1}S(t, s)V(s), \quad \mathcal{P}Z_h(t) \in X_{c,y(t)}^e,$$

which also implies $V(t) = \mathcal{P}Z_h(t) + \mathcal{P}Z_{in}(t) \in X_{c,y(t)}^e$ for almost all $t \in (t'_0, t'_1)$. Applying Lemma 3.4 to the $\|\cdot\|_{L^\infty \dot{H}^1}$ on the right side of (3.20) for Z_h , we have

$$\|Z_h - e^{(t-s)JH}Z(s)\|_{L_{(t'_0, t'_1)}^p \dot{B}_{q,2}^1} \leq C(t'_1 - t'_0)\|V(s)\|_{X_1}$$

for $t'_0 < s < t'_1 < t'_0 + T$. From (3.18), we obtain

$$\|Z_h\|_{L_{(t'_0, t'_1)}^p \dot{B}_{q,2}^1} \leq C\|V(s)\|_{X_1}$$

and inequality (3.13) follows immediately from the above estimates.

To derive (3.14) in the case of $(p = \infty, q = 2)$, we apply (3.9) instead, along with (3.21), (3.22), and (3.15) and the uniform positivity of $L_{c,y}$ on $X_{c,y}^e$, to compute, for $t \in (t'_0, t'_1)$,

$$\begin{aligned} \langle L_{c,y(t)}V(t), V(t) \rangle &= \langle L_{c,y(t)}S(t, s)V(s), S(t, s)V(s) \rangle \\ &\quad + 2\langle L_{c,y(t)}S(t, s)V(s), \mathcal{P}Z_{in}(t) \rangle + \langle L_{c,y(t)}\mathcal{P}Z_{in}(t), \mathcal{P}Z_{in}(t) \rangle \\ &\leq (\langle L_{c,y(t)}S(t, s)V(s), S(t, s)V(s) \rangle^{\frac{1}{2}} + \langle L_{c,y(t)}\mathcal{P}Z_{in}(t), \mathcal{P}Z_{in}(t) \rangle^{\frac{1}{2}})^2 \\ &\leq (e^{C\sigma|t-s|} \langle L_{c,y(s)}V(s), V(s) \rangle^{\frac{1}{2}} + C\|\tilde{f}\|_{L_{(t'_0, t'_1)}^{\tilde{p}'} \dot{B}_{\tilde{q}',2}^1})^2. \end{aligned}$$

This implies (3.14). Finally as T is independent of f and $y(t)$, a standard continuation argument extends the domain of solutions to (t_0, t_1) and thus completes the proof of the lemma. $\square$

In the next step, we iterate the above small time estimates.

Lemma 3.6. *Suppose (3.2) and (3.7) hold, (p, q) is an admissible pair, and $f \in \hat{X}_{(t_0, t_1)}^{\tilde{p}', q'}$, where $\tilde{p} \in [1, p]$. Then there exists $C > 0$ independent of f, t_0, t_1, s, t , such that for any $\eta > C\sigma$, every solution $V(t)$ to (3.1) satisfies*

$$\langle L_{c,y(t)}V(t), V(t) \rangle^{\frac{1}{2}} \leq e^{C\sigma|t-s|} \langle L_{c,y(s)}V(s), V(s) \rangle^{\frac{1}{2}} + C\eta^{-\frac{1}{\tilde{p}}} \|e^{\eta|t-s|} f\|_{\hat{X}_{(s,t)}^{\tilde{p}', q'}}$$

for any $t_0 < t, s < t_1$.

Proof. We only prove the estimates for $t > s$, the estimates for negative time can be obtained similarly. Suppose $t = s + kT + t'$ where $t' \in [0, T)$, we use (3.14) repeatedly to compute

$$\begin{aligned}
\langle L_{c,y(t)} V(t), V(t) \rangle^{\frac{1}{2}} &\leq e^{C\sigma t'} (\langle L_{c,y(\tau)} V(\tau), V(\tau) \rangle|_{\tau=s+kT})^{\frac{1}{2}} + C \|f\|_{\hat{X}_{(s+kT,t)}^{p',q'}} \\
&\leq e^{C\sigma(t-s)} \langle L_{c,y(s)} V(s), V(s) \rangle^{\frac{1}{2}} + C (\|f\|_{\hat{X}_{(s+kT,t)}^{p',q'}} + \sum_{j=1}^k e^{C\sigma(t-s-jT)} \|f\|_{\hat{X}_{(s+(j-1)T,s+jT)}^{p',q'}}) \\
&\leq e^{C\sigma(t-s)} \langle L_{c,y(s)} V(s), V(s) \rangle^{\frac{1}{2}} + C (\sum_{j=0}^k e^{\tilde{p}(C\sigma-\eta)(t-s-jT)})^{\frac{1}{\tilde{p}}} \|e^{\eta(t-\cdot)} f\|_{\hat{X}_{(s,t)}^{p',q'}}.
\end{aligned}$$

Summing up the exponentials completes the proof of the lemma. $\square$

Finally we drop the assumption (3.7).

Proof of Proposition 3.1. We split (3.1) into the $X_{c,y}^e$ component and its complementary component as in (3.5). Much as the calculation in the derivation of (2.56), we have that $V^\perp$ satisfies (3.6).

The remaining estimate of $V^\perp(t)$ is similar to the above. In fact, for $t > s$,

$$\begin{aligned}
|V^\perp(t)| &\leq e^{C\sigma(t-s)} |V^\perp(s)| + \int_s^t e^{C\sigma(t-\tau)} |f^\perp(\tau)| d\tau \\
&\leq e^{C\sigma(t-s)} |V^\perp(s)| + \int_s^t e^{(C\sigma-\eta)(t-\tau)} e^{\eta(t-\tau)} |f^\perp(\tau)| d\tau
\end{aligned}$$

which implies the desired estimate on $V^\perp(t)$.

Due to the choice of $\mathcal{F}$, it is straightforward to compute that $V^e(t) = V(t) - V^\perp(t)$ satisfies (3.1) with the non-homogeneous term $f(t)$ replaced by $f^e(t)$. Lemma 3.6 implies the estimate on $V^e(t)$ which completes the proof of the proposition. $\square$

Finally, we apply Lemma 3.6 to prove Proposition 3.3.

Proof of Proposition 3.3. We first decompose $f \in \tilde{X}_{(t_0,t_1),loc}$ into the sum of several terms satisfying the assumptions in Lemma 3.6. In fact, by the definition of $\tilde{X}$, we can write

$$f = \phi + \psi + \gamma$$

where

$$\phi \in L_{loc}^2 W^{1,\frac{3}{2}} \subset L_{loc}^2 B_{\frac{3}{2},2}^1 \subset \hat{X}_{(t_0,t_1),loc}^{2,\frac{3}{2}}, \quad \gamma \in L_{loc}^2 (H^1 \times \dot{H}^1) \subset \hat{X}_{(t_0,t_1),loc}^{2,2},$$

and

$$\psi \in L_{loc}^2 (L^2 \cap L^{\frac{3}{2}}), \quad \nabla \psi \in L_{loc}^2 L^{\frac{6}{5}}.$$

Let χ be the same smooth cut-off function used in Sect. 2.1. Clearly

$$\chi(D)\psi \in L_{loc}^2 W^{1,\frac{3}{2}} \subset L_{loc}^2 B_{\frac{3}{2},2}^1 \subset \hat{X}_{(t_0,t_1),loc}^{2,\frac{3}{2}}, \quad \text{and} \quad \nabla(1 - \chi(D))\psi \in L_{loc}^2 L^{\frac{6}{5}}.$$

Moreover, since the inverse Fourier transform of $\frac{1-\chi(|\xi|)}{|\xi|}$ is in L^1 , we have

$$(1 - \chi(D))\psi = \frac{1 - \chi(D)}{|D|} |D|\psi \in L_{loc}^2 L^{\frac{6}{5}}$$

and thus $(1 - \chi(D))\psi \in \hat{X}_{(t_0, t_1), loc}^{2, \frac{6}{5}}$.

The desired estimate follows immediately from applying Proposition 3.6 to each of these terms. $\square$

4. Construction of Lipschitz Local Invariant Manifolds of $\mathcal{M}$

Based on the space-time estimates developed in Sect. 3, we construct the center-unstable manifold $\mathcal{W}^{cu}(\mathcal{M})$ of $\mathcal{M}$, while the center-stable manifold $\mathcal{W}^{cs}(\mathcal{M})$ can be constructed similarly. The intersection of the center-unstable and the center-stable manifolds yields the center manifold of $\mathcal{M}$.

4.1. Outline of the construction of the center-unstable manifold of $\mathcal{M}$. Our construction roughly follows the procedure in [14]. The codim- d local center-unstable manifold are over the directions of $X_{c,y}^{d1} \oplus X_{c,y}^e \oplus X_{c,y}^{d2} \oplus X_{c,y}^+$ along $\mathcal{M}$. In coordinate system (2.36) $\mathcal{W}^{cu}(\mathcal{M})$ is represented as the graph of some mapping h^{cu}

$$\begin{aligned} \mathcal{W}^{cu}(\mathcal{M}) = \Phi(\{ & a^- = h^{cu}(y, a^{d1}, a^{d2}, a^+, V^e) \mid \\ & (y, a^{d1}, a^{d2}, a^+, V^e) \in B^{d1+d2+d}(\delta) \oplus \mathcal{X}^e(\delta) \}) \end{aligned} \quad (4.1)$$

where the above sets are defined in (2.32). Even though $\mathcal{W}^{cu}(\mathcal{M})$ is local, by using a standard cut-off technique, we will carry out the construction on $\mathbb{R}^{d1+d2+d} \oplus \mathcal{X}^e(\delta)$. Moreover, for technical convenience, we shall work with $h(y, a^{d1}, a^{d2}, a^+, V)$ defined on $\mathbb{R}^{3+d1+d2+d} \times X_1(\delta)$ to avoid the non-flat bundle. However, only the value of h on $\mathbb{R}^{d1+d2+d} \oplus \mathcal{X}^e(\delta)$ matters.

Let

$$X^{cu} = \mathbb{R}^{3+d1+d2+d} \times X_1, \quad X^{cu}(\delta) = \{(y, a^{d1}, a^{d2}, a^+, V) \in X^{cu} : \|V\|_{X_1} < \delta\}.$$

The following projection $\tilde{\Pi}^e$, linear except in y , will be used often

$$\tilde{\Pi}^e(y, a^{d1}, a^{d2}, a^+, V) = (y, a^{d1}, a^{d2}, a^+, \Pi_{c,y}^e V) \in \mathbb{R}^{d1+d2+d} \oplus \mathcal{X}^e. \quad (4.2)$$

We shall modify equations (2.40), (2.42), (2.43), (2.44), and (2.52), along with (2.46), into a system defined on $X^{cu} \times \mathbb{R}^d$. As a standard technique in local analysis, we first cut-off the nonlinearities as well as the off-diagonal linear terms in the direction transversal to $\mathcal{M}$. Take a cut-off function

$$\gamma \in C_0^\infty(\mathbb{R}), \quad \text{s. t. } \gamma(x) = 1, \quad \forall |x| \leq 1, \quad \gamma(x) = 0, \quad \forall |x| \geq 3, \quad |\gamma'|_{C^0(\mathbb{R})} \leq 1 \quad (4.3)$$

and for $\delta > 0$, $a^- \in \mathbb{R}^d$, and $W = (y, a^{d1}, a^{d2}, a^+, V) \in X^{cu}$, let

$$\gamma_\delta(W, a^-) = \gamma(3\delta^{-1}(|a^{d1}| + |a^{d2}| + |a^+| + |a^-| + \|V\|_{X_1})).$$

For $a^- \in \mathbb{R}^d$ and $W \in X^{cu}$, let

$$\begin{aligned}\hat{G}^\alpha(W, a^-) &= \gamma_\delta(W, a^-)G^\alpha(c, y, w), \quad \alpha \in \{+, -, d2\} \\ \hat{G}^{d1}(W, a^-) &= \gamma_\delta(W, a^-)(\langle \zeta_c^{d1}(\cdot + y), A_{1e}(y)K_{c,y}w \rangle + M_{12}a^{d2} + G^{d1}(c, y, w)) \\ \hat{G}^T(W, a^-) &= \gamma_\delta(W, a^-)(\langle \zeta_c^T(\cdot + y), A_{Te}(y)K_{c,y}w \rangle + M_{T1}a^{d1} + M_{T2}a^{d2} \\ &\quad + G^T(c, y, w)) \\ \hat{G}^e(W, a^-) &= \gamma_\delta(W, a^-)(A_{e2}(y)K_{c,y}w + G^e(c, y, w))\end{aligned}$$

where functions $\zeta_c^T = (\zeta_{c,1}^T, \zeta_{c,2}^T, \zeta_{c,3}^T)$, $\zeta_c^{d1} = (\zeta_{c,1}^{d1}, \dots, \zeta_{c,d_1}^{d1})$, and operators A_{1e} , A_{e2} , and A_{Te} are given in Lemma 2.3, matrices M_{12} , M_{T1} , M_{T2} in (2.25), and

$$\begin{aligned}w &= \Lambda(W, a^-) \triangleq Em^\perp(y, a^{d1}, a^{d2}, a^+, a^-, \Pi_{c,y}^e V) \\ &= K_{c,y}^{-1}((a^{d1}V_c^{d1} + a^{d2}V_c^{d2} + a^+V_c^+ + a^-V_c^-)(\cdot + y) + \Pi_{c,y}^e V).\end{aligned}\quad (4.4)$$

From the definitions of $\hat{G}^\alpha$, it clearly holds that they are independent of the extra component $(I - \tilde{\Pi}^e)W$ added to avoid the non-flat bundle $\mathbb{R}^{d1+d2+d} \oplus \mathcal{X}^e$. In particular, $\hat{G}^e$ satisfies

$$(I - \Pi_{c,y}^e)\hat{G}^e = 0, \quad \hat{G}^e \in \tilde{X} = X_1 + W^{1, \frac{3}{2}} + (L^{\frac{3}{2}} \cap \dot{W}^{1, \frac{6}{5}}). \quad (4.5)$$

Denote

$$\begin{aligned}\tilde{X}^{cu} &= \mathbb{R}^{3+d1+d2+d} \times \tilde{X}, \quad \tilde{X}^{cu}(\delta) = \{(y, a^{d1}, a^{d2}, a^+, V) \in \tilde{X}^{cu} : \|V\|_{\tilde{X}} < \delta\}, \\ \hat{G}^{cu}(W, a^-) &= (\hat{G}^T, \hat{G}^{d1}, \hat{G}^{d2}, \hat{G}^+, \hat{G}^e)(W, a^-), \\ A^{cu}(y, \tilde{y}) &= \text{diag}(0, M_1, M_2, M_+, \Pi_{c,y}^e J L_{c,y} \Pi_{c,y}^e + \mathcal{F}(c, y)(\tilde{y}, \cdot)).\end{aligned}$$

We shall consider, for $W = (y, a^{d1}, a^{d2}, a^+, V) \in X^{cu}$ and $a^- \in \mathbb{R}^d$,

$$\partial_t W = A^{cu}(y, \hat{G}^T(W, a^-))W + \hat{G}^{cu}(W, a^-) \quad (4.6a)$$

$$\partial_t a^- = M_- a^- + \hat{G}^-(W, a^-) \quad (4.6b)$$

which, for $\|w\|_{X_1} \leq \delta$, coincides with the system consisting of equations (2.40), (2.42), (2.43), (2.44), and (2.52), along with (2.46), the representation of (GP) in the local coordinate system near $\mathcal{M}$.

As the off-diagonal linear blocks in $J L_{c,y}$ are incorporated into $\hat{G}^{cu}$, the latter does not have small Lipschitz constants, which is often a necessity in constructing local invariant manifolds. Accordingly, we introduce metrics involving a scale constant $Q > 1$

$$\begin{aligned}\|(y, a^{d1}, a^{d2}, a^+, V)\|_{X_1, Q} &\triangleq |y| + Q|a^{d1}| + Q^3|a^{d2}| + |a^+| + Q^2\|V\|_{X_1}, \\ \|(y, a^{d1}, a^{d2}, a^+, V)\|_{\tilde{X}, Q} &\triangleq |y| + Q|a^{d1}| + Q^3|a^{d2}| + |a^+| + Q^2\|V\|_{\tilde{X}}\end{aligned}\quad (4.7)$$

to make Lipschitz constants of $\hat{G}^{cu,-}$ small (Lemma 4.3).

We shall construct the local center-unstable manifold $\mathcal{W}^{cu}(\mathcal{M})$ as the graph $\{a^- = h^{cu}(W)\}$ of some $h : X^{cu}(\delta) \rightarrow \mathbb{R}^d$. Since $\mathcal{W}^{cu}(\mathcal{M})$ is expect to be translation invariant, we will only consider translation invariant mappings $h : X^{cu}(\delta) \rightarrow \mathbb{R}^d$, which satisfy, for any $z \in \mathbb{R}^3$,

$$h(y + z, a^{d1}, a^{d2}, a^+, V(\cdot + z)) = h(y, a^{d1}, a^{d2}, a^+, V). \quad (4.8)$$

Fix constants Q, δ, μ such that

$$\delta < 1, \quad Q > 1, \quad \mu < \frac{1}{5}, \quad (4.9)$$

whose additional assumptions will be given later. Let

$$\begin{aligned} \Gamma_{\mu,\delta} = \{h : X^{cu}(\delta) \rightarrow \mathbb{R}^d \mid h(y, 0, 0, 0, 0) = 0, \|h\|_{C^0} \leq \delta/15, \\ h \text{ satisfies (4.8), and } Lip_{\|\cdot\|_{X_{1,Q}}} \leq \mu\}. \end{aligned} \quad (4.10)$$

Here $h(y, 0, 0, 0, 0) = 0$ is required as $\mathcal{W}^{cu}(\mathcal{M})$ should contain $\mathcal{M}$. Clearly $\Gamma_{\mu,\delta}$ equipped with $\|\cdot\|_{C^0}$ is a complete metric space.

We will define a transform on $\Gamma_{\mu,\delta}$ based on (GP). For any $h \in \Gamma_{\mu,\delta}$ and $\bar{W} \in X^{cu}(\delta)$, consider the solution $W(t) = (y, a^{d1}, a^{d2}, a^+, V)(t) \in X^{cu}$ of

$$\partial_t W = A^{cu}(y, \hat{G}^T(W, h(W)))W + \hat{G}^{cu}(W, h(W)), \quad W(0) = \bar{W}. \quad (4.11)$$

Remark 4.1. Even though h is defined only on $X^{cu}(\delta)$, due to the cut-off function γ_δ , for any $h \in \Gamma_{\mu,\delta}$, $\alpha \in \{T, d1, d2, \pm, e\}$, it holds $\hat{G}^\alpha(W, h(W)) = 0$ whenever $W \in X^{cu} \setminus X^{cu}(\delta)$. Consequently, the right side of (4.11) is well-defined for all $W \in X^{cu}$.

We define $\tilde{h}(\bar{W})$ as

$$\tilde{h}(\bar{W}) = \bar{a}^- = \int_{-\infty}^0 e^{-tM^-} \hat{G}^-(W(t), h(W(t))) dt. \quad (4.12)$$

We denote this transformation $h \rightarrow \tilde{h}$ as

$$\mathcal{T}(h) = \tilde{h}.$$

In order to construct the center-unstable manifold, in the following subsections, under suitable assumptions on Q, δ , and μ we will show $\tilde{h} \in \Gamma_{\mu,\delta}$ is well-defined and that $\mathcal{T}$ is a contraction on $\Gamma_{\mu,\delta}$. The graph of the unique fixed point, restricted to the set

$$B^{d1+d2+d}(\delta) \oplus \mathcal{X}^e(\delta) = \{(y, a^{d1}, a^{d2}, a^+, V) \in X^{cu}(\delta) \mid |(a^{d1}, a^{d2}, a^+)| < \delta, V \in X_{c,y}^e\}$$

would be the desired center-unstable manifold $\mathcal{W}^{cu}(\mathcal{M})$. To end this subsection, we give the following lemma to show that working on systems (4.6) or (4.11) on the expanded domain X^{cu} , only to avoid the non-flat bundle $\mathbb{R}^{d1+d2+d} \oplus \mathcal{X}^e$, does not change the local invariant manifolds.

Lemma 4.2. *The following statements hold.*

- (1) Suppose $W(t)$ satisfies (4.6a) on $[t_1, t_2]$ for some $a^- \in C^0([t_1, t_2], \mathbb{R}^d)$ and $\tilde{\Pi}^e W(t_0) = W(t_0)$ for some $t_0 \in [t_1, t_2]$, then $\tilde{\Pi}^e W(t) = W(t)$ for all $t \in [t_1, t_2]$.
- (2) Assume $h_j \in \Gamma_{\mu,\delta}$, $j = 1, 2$, satisfy $h_1(W) = h_2(W)$ for all $W \in X^{cu}(\delta)$ with $\tilde{\Pi}^e W = W$. Then $\tilde{h}_j$, $j = 1, 2$, defined in (4.12) satisfy the same property.

Proof. For the first statement, we observe that a direct consequence of (4.5), (3.6), and our assumption is $(I - \tilde{\Pi}^e)W(t) = 0$, for all $t \in [t_1, t_2]$, which implies $W(t) = \tilde{\Pi}^e W(t)$. The second statement of the lemma is just a simple corollary of part (1) and the definition of $\tilde{h}$. $\square$

4.2. A priori estimates. Following the construction outlined in Sect. 4.1, in order to prove that $\tilde{h} \in \Gamma_\mu$ is well-defined for any given $h \in \Gamma_{\mu,\delta}$, we start with the following preliminary estimates.

Lemma 4.3. $\hat{G}^{cu,-} : X^{cu} \times \mathbb{R}^d \rightarrow \tilde{X}^{cu} \times \mathbb{R}^d$ are smooth, $\hat{G}^{cu,-}(y, 0, 0, 0, 0, 0) = 0$. Moreover, there exists $C > 0$ independent of Q and δ such that

$$\|D\hat{G}^{cu}\|_{L_Q(X^{cu} \times \mathbb{R}^d, \tilde{X}^{cu})} \leq C(Q^{-1} + \delta Q^3), \quad \|D\hat{G}^-\|_{L(X^{cu} \times \mathbb{R}^d, \mathbb{R}^d)} \leq C\delta$$

where $\|\cdot\|_{L_Q(X^{cu} \times \mathbb{R}^d, \tilde{X}^{cu})}$ denote the operator norm when evaluated in $\|\cdot\|_{X_1, Q}$ and $\|\cdot\|_{\tilde{X}, Q}$.

Proof. From the definitions of $\hat{G}^{cu,-}$ and (2.48) and (2.50) which in turn are derived from Lemma A.1, the smoothness of $\hat{G}^{cu,-}$ and $\hat{G}^{cu,-}(y, 0, 0, 0, 0, 0) = 0$ follows immediately. Moreover, it is straightforward to obtain the following estimates. Firstly, for $l, k \geq 0$,

$$\|D_{(a^{d1}, a^{d2}, a^+, a^-, V)}^k D_y^l (\hat{G}^{cu}, \hat{G}^-)(W, a^-)\| \leq C_{k,l} \delta^{1-k} \quad (4.13)$$

for some $C_{k,l} > 0$. When we exclude the off-diagonal terms in $\hat{G}^{cu}$, the estimates may be improved to

$$\begin{aligned} & |D_{(a^{d1}, a^{d2}, a^+, a^-, V)}^k D_y^l \hat{G}^{d2,+, -}| + |D_{(a^+, a^-)}^k D_y^l \hat{G}^T| \\ & + \|D_{(a^{d1}, a^+, a^-, V)}^k D_y^l \hat{G}^e\| + |D_{(a^{d1}, a^+, a^-)}^k D_y^l \hat{G}^{d1}| \leq C_{k,l} \delta^{2-k}, \end{aligned} \quad (4.14)$$

for some $C_{k,l} > 0$. In the above $\hat{G}^e$ is always evaluated in the $\|\cdot\|_{\tilde{X}}$ norm. The desired estimates on $D\hat{G}^{cu,-}$ follow from straightforward calculations based on the above inequalities. $\square$

The following lemma is a simple corollary of Proposition 3.1.

Lemma 4.4. *There exists $C > 0$ such that, for any $y \in C^1((-\infty, 0], \mathbb{R})$ satisfying $|\partial_t y|_{L^\infty} \leq \sigma$, $f \in L^2((-\infty, 0], \tilde{X}^{cu})$, $Q > 1$, $\eta \in (C\sigma, 1)$, and $W(t)$, $t \in (-\infty, 0]$, solving*

$$\partial_t W = A^{cu}(y, \partial_t y)W + f,$$

we have

$$\|W(t)\|_{X_1, Q}^2 \leq C\eta^{-2d_1} e^{-\eta t} \|W(0)\|_{X_1, Q}^2 + C\eta^{-2d_1-1} \int_t^0 e^{\eta(\tau-t)} \|f(\tau)\|_{\tilde{X}, Q}^2 d\tau.$$

Proof. Since A^{cu} takes a diagonal form, we may consider each component individually. For the $a^{d1, d2}$ and y components, in addition to applying (2.26) we also use

$$|e^{tM_1}| + |e^{tM_2}| \leq C(1 + |t|)^{d_1} \leq C\eta^{-d_1} e^{\frac{\eta}{3}|t|}$$

and the following estimate based on the Cauchy-Schwartz inequality

$$\left| \int_t^0 e^{\frac{\eta}{3}(\tau-t)} g(\tau) d\tau \right|^2 \leq C\eta^{-1} \int_t^0 e^{\eta(\tau-t)} |g(\tau)|^2 d\tau$$

to obtain the desired inequality. The estimate on the V component is a direct consequence of Proposition 3.1 and the estimate of the a^+ and y component trivially follows from (2.26). $\square$

Proposition 4.5. Let $\eta \in (0, 1)$, $T \in [-\infty, 0)$, $h \in \Gamma_{\mu, \delta}$, and $\tilde{a}_2^-(\cdot) \in C^0([T, 0], \mathbb{R}^d)$. Suppose $W_1(t) = (y_1, a_1^{d1}, a_1^{d2}, a_1^+, V_1)(t) \in X^{cu}$ is a solution to (4.11) and $W_2 = (y_2, a_2^{d1}, a_2^{d2}, a_2^+, V_2)(t) \in X^{cu}$ solves

$$\partial_t W = A^{cu}(y, \hat{G}^T(W, \tilde{a}_2^-))W + \hat{G}^{cu}(W, \tilde{a}_2^-), \quad (4.15)$$

with initial values $\tilde{W}_j = (\bar{y}_j, \bar{a}_j^{d1}, \bar{a}_j^{d2}, \bar{a}_j^+, \bar{V}_j) \in X^{cu}(\delta)$, $j = 1, 2$. Then these solutions exist for all $t \in [T, 0]$ and there exists $C > 0$ independent of μ, T, η, Q , and δ , such that if (4.9) is satisfied and

$$C\eta^{-(1+d_1)}(Q^{-1} + Q^3\delta) < 1 \quad (4.16)$$

then $W_j(t) \in X^{cu}(C\delta)$ for all $t \in [T, 0]$ and

$$\begin{aligned} \|(W_2 - W_1)(t)\|_{X_1, Q}^2 &\leq C\eta^{-2d_1} \left(e^{-2\eta t} \|(W_2 - W_1)(0)\|_{X_1, Q}^2 \right. \\ &\quad \left. + \eta^{-1} Q^6 \delta^2 \int_t^0 e^{2\eta(\tau-t)} |(\tilde{a}_2^- - h(W_2))(\tau)|^2 d\tau \right). \end{aligned}$$

Proof. To analyze solutions to (4.11) and (4.15), we first note

$$\hat{G}^T(W, a^-) = 0, \quad \hat{G}^e(W, a^-) = 0, \quad \text{if } \|V\|_{X_1} \geq \delta.$$

Therefore if $\|V_j\|_{X_1} \geq \delta$, (3.11), (4.11), and decomposition (3.6) yield

$$\partial_t y_j = 0, \quad \partial_t \langle L_{c, y_j} \Pi_{c, y_j}^e V_j, \Pi_{c, y_j}^e V_j \rangle = 0, \quad \partial_t (I - \Pi_{c, y_j}^e) V_j = 0,$$

which along with the initial condition and Lemma 2.3 yield

$$\|V_j(t)\|_{X_1} \leq C\delta, \quad \forall t \in [T, 0]. \quad (4.17)$$

To estimate the difference, let

$$\begin{aligned} B(t) &= \mathcal{F}(c, y_2)(\hat{G}^T(W_2, \tilde{a}_2^-), \cdot) - \mathcal{F}(c, y_1)(\hat{G}^T(W_1, h(W_1)), \cdot) + A_e(y_2) - A_e(y_1) \\ B^{cu}(t) &\triangleq A^{cu}(y_2, \hat{G}^T(W_1, \tilde{a}_2^-)) - A^{cu}(y_1, \hat{G}^T(W_1, h(W_1))) \\ &= \text{diag}(0, 0, 0, 0, B(t)). \end{aligned}$$

The cut-off in the definition of $\hat{G}^T$, (4.11), and (4.13) imply

$$|\partial_t y_j| = |\hat{G}^T| \leq C\delta.$$

From Lemma 2.3, Lemma 2.5, (4.13), (4.17), and (4.17), we can estimate

$$\begin{aligned} \|B^{cu}(t)W_2(t)\|_{X_1, Q} &= Q^2 \|B(t)V_2(t)\|_{X_1} \\ &\leq C\delta Q^2 (|y_2 - y_1|(1 + |\hat{G}^T(W_1, h(W_1))|) + |\hat{G}^T(W_2, \tilde{a}_2^-) - \hat{G}^T(W_1, h(W_1))|) \\ &\leq C\delta Q^2 (\|W_2 - W_1\|_{X^{cu}} + |\tilde{a}_2^- - h(W_1)|). \end{aligned}$$

From the definitions of W_j , $j = 1, 2$, and decomposition (3.6), we have

$$\begin{aligned} \partial_t(W_2 - W_1) &= A^{cu}(y_1, \hat{G}^T(W_1, h(W_1)))(W_2 - W_1) + B^{cu}W_2 \\ &\quad + \hat{G}^{cu}(W_2, \tilde{a}_2^-) - \hat{G}^{cu}(W_1, h(W_1)). \end{aligned}$$

Applying Lemma 4.4 we obtain

$$\begin{aligned} \|(W_2 - W_1)(t)\|_{X_{1,Q}}^2 &\leq C\eta^{-2d_1}e^{-\eta t}\|(W_2 - W_1)(0)\|_{X_{1,Q}}^2 \\ &\quad + \eta^{-2d_1-1} \int_t^0 e^{\eta(\tau-t)} (\|(B^{cu}W_2)(\tau)\|_{\tilde{X},Q}^2 \\ &\quad + \|(\hat{G}^{cu}(W_2, \tilde{a}_2^-) - \hat{G}^{cu}(W_1, h(W_1)))(\tau)\|_{\tilde{X},Q}^2) d\tau. \end{aligned}$$

The above estimate on B^{cu} and Lemma 4.3 including (4.14) imply

$$\begin{aligned} \|(W_2 - W_1)(t)\|_{X_{1,Q}}^2 &\leq C\eta^{-2d_1}e^{-\eta t}\|(W_2 - W_1)(0)\|_{X_{1,Q}}^2 + \eta^{-2d_1-1} \int_t^0 e^{\eta(\tau-t)} \\ &\quad \times ((Q^{-1} + \delta Q^3)^2 \|(W_2^e - W_1^e)(\tau)\|_{X_{1,Q}}^2 + \delta^2 Q^6 |(\tilde{a}_2^- - h(W_1))(\tau)|^2) d\tau. \end{aligned}$$

Since h is Lipschitz with Lipschitz constant $\mu < 1$, we obtain

$$\begin{aligned} e^{\eta t} \|(W_2 - W_1)(t)\|_{X_{1,Q}}^2 &\leq C\eta^{-2d_1} \|(W_2 - W_1)(0)\|_{X_{1,Q}}^2 + \eta^{-2d_1-1} \int_t^0 e^{\eta\tau} \\ &\quad \times ((Q^{-1} + \delta Q^3)^2 \|(W_2 - W_1)(\tau)\|_{X_{1,Q}}^2 + \delta^2 Q^6 |(\tilde{a}_2^- - h(W_2))(\tau)|^2) d\tau. \end{aligned}$$

The estimate on $W_2 - W_1$ follows from the Gronwall inequality. $\square$

Remark 4.6. It is worth pointing out that the $\mathcal{F}$ term in the equation of $\partial_t V$ can not be cut off as it ensures $V \in X_{c,y}^e$ if this holds initially. In the proof of the above proposition, this term was under control since it vanishes when $\|V\|_{X_1} = C\delta$ which implies $\partial_t y = 0$. Seemingly this argument heavily depends on the lack of growth of $e^{tA_e(y)}$ for any fixed y . If $e^{tA_e(y)}$ indeed induces some weak exponential growth backward in t , instead of the cut-off applied to the V equation, a standard trick is to add a bump function to modify the V equation so that it is actually slightly inflowing/decaying backward in t for $\|V\|_{X_1} \geq C\delta$. The same estimates could be obtained subsequently.

4.3. Lipschitz center-unstable manifold. In this subsection, we will show that the transformation outlined in Sect. 4.1 is well-defined and is a contraction on $\Gamma_{\mu,\delta}$ for appropriate μ , Q , and δ , which would imply the existence of a fixed point and thus a center-unstable manifold. For any $h \in \Gamma_{\mu,\delta}$, recall we attempted to define a new mapping $\tilde{h} = \mathcal{T}(h)$ whose value $\tilde{h}(W) = \tilde{a}^-$ at $W = (\bar{y}, \tilde{a}^{d_1}, \tilde{a}^{d_2}, \tilde{a}^+, \bar{V})$ is given by (4.12).

Lemma 4.7. Fix $\eta \in (0, 1) \cap (0, \lambda)$. There exists $C > 0$ independent of Q , μ , δ and η , such that if (4.9), (4.16), and

$$C(\lambda - \eta)^{-1}\eta^{-(d_1+1)}Q^3\delta^2 < 1, \quad C(\lambda - \eta)^{-1}\eta^{-d_1}\delta < \mu, \quad (4.18)$$

are satisfied, then $\mathcal{T}$ is a contraction on $\Gamma_{\mu,\delta}$.

Proof. We first prove that $\tilde{h} \in \Gamma_{\mu,\delta}$. Since $h(y, 0) = 0$ and $\hat{G}^{cu,-}(y, 0) = 0$, if its initial data $\bar{W} = (\bar{y}, 0)$, then the solution to (4.11) apparently is $W(t) = (\bar{y}, 0)$. Therefore $\tilde{h}(\bar{y}, 0) = 0$. From (4.14) and (2.26), it is easy to estimate

$$\|\tilde{h}\|_{C^0} \leq C\lambda^{-1}\delta^2 \leq \delta/15,$$

where (4.18) is used.

From (2.47), (4.8), and that $\|\cdot\|_{X_1}$ is translation invariant, the cut-off is not affected by any spatial translation. Therefore

$$(y(t) + z, a^{d1}(t), a^{d2}(t), a^+(t), V(t, \cdot + z))$$

is a solution to (4.11) for any $z \in \mathbb{R}^3$ and any solution $(y, a^{d1}, a^{d2}, a^+, V)(t)$ to (4.11). Therefore the definition of $\tilde{h}$ implies that it also satisfies (4.8).

Finally, we show the Lipschitz property for $\tilde{h}$. For any $\tilde{W}_j \in X^{cu}(\delta)$, let $W_j(t), t \leq 0$, be the corresponding solutions to (4.11) and $\tilde{a}_j^-$ be given as in (4.12), for $j = 1, 2$. Applying Proposition 4.5 to these two solutions, we have, for any $t \leq 0$,

$$\|(W_2 - W_1)(t)\|_{X_1, Q} \leq C\eta^{-d_1} e^{-\eta t} \|\tilde{W}_2 - \tilde{W}_1\|_{X_1, Q}.$$

Therefore, we obtain from (4.12), (2.26), and (4.14) that

$$|\tilde{a}_2^- - \tilde{a}_1^-| \leq C\delta \int_{-\infty}^0 e^{\lambda\tau} \|(W_2 - W_1)(\tau)\|_{X_1, Q} d\tau \leq \frac{C\eta^{-d_1}\delta}{\lambda - \eta} \|\tilde{W}_2 - \tilde{W}_1\|_{X_1, Q}.$$

The desired Lipschitz property of $\tilde{h}$ follows immediately from (4.18) and thus $\tilde{h} \in \Gamma_{\mu, \delta}$.

To see the transformation $h \rightarrow \tilde{h}$ is a contraction, given any $h_1, h_2 \in \Gamma_{\mu, \delta}$ and initial value $W \in X^{cu}(\delta)$, let $W_j(t), t \leq 0, j = 1, 2$, be the solutions to (4.11) associated to h_j , with the initial value W . In applying Proposition 4.5, we notice the corresponding

$$|(a_2 - h_1(W))(t)| \leq \|h_2 - h_1\|_{C^0}, \quad (W_1 - W_2)(0) = 0,$$

and thus, for any $t \leq 0$,

$$\|(W_2 - W_1)(t)\|_{X_1, Q} \leq C\eta^{-d_1-1} Q^3 \delta e^{-\eta t} \|h_2 - h_1\|_{C^0}.$$

Therefore (4.12), (2.26), and (4.14) again imply

$$|\tilde{a}_2^- - \tilde{a}_1^-| \leq C\eta^{-(1+d_1)} Q^3 \delta^2 \int_{-\infty}^0 e^{(\lambda-\eta)\tau} d\tau \|h_2 - h_1\|_{C^0}.$$

Therefore (4.18) implies the contraction property. $\square$

Under conditions (4.9), (4.16), and (4.18), which can apparently be satisfied by choosing μ, δ, Q , and η carefully, Lemma 4.7 implies

$$\exists! h^{cu} \in \Gamma_{\mu, \delta}, \text{ s. t. } \mathcal{T}(h^{cu}) = h^{cu}.$$

We are only concerned with h^{cu} restricted to $\mathbb{R}^{d_1+d_2+d} \oplus \mathcal{X}^e(\delta)$. Let

$$\begin{aligned} W^{cu} &= \text{graph}(h^{cu}) = \{(y, a^{d1}, a^{d2}, a^+, a^-, V^e) \mid \\ a^- &= h^{cu}(y, a^{d1}, a^{d2}, a^+, V^e), (y, a^{d1}, a^{d2}, a^+, V^e) \in \mathbb{R}^{d_1+d_2+d} \times \mathcal{X}^e(\delta)\} \end{aligned}$$

and an even smaller submanifold for (2.7) and (GP)

$$\begin{aligned} \mathcal{W}^{cu}(\mathcal{M}) &= \Phi(\{(y, a^{d1}, a^{d2}, a^+, a^-, V^e) \in W^{cu} \mid \\ &|a^{d1}|, |a^{d2}|, |a^+|, \|V^e\|_{X_1} < \delta/15\}). \end{aligned} \tag{4.19}$$

Clearly W^{cu} is a Lipschitz manifold due to the Lipschitz property of $h^{cu} \in \Gamma_{\mu, \delta}$.

For any $\tilde{a}^- = h^{cu}(\tilde{W})$, $\tilde{W} = (\tilde{y}, \tilde{a}^{d1}, \tilde{a}^{d2}, \tilde{a}^+, \tilde{V}^e) \in \mathbb{R}^{d_1+d_2+d} \oplus \mathcal{X}^e(\delta)$ corresponding to a point on W^{cu} , let $W(t) = (y, a^{d1}, a^{d2}, a^+, V^e)(t)$ be the corresponding solution of (4.11) where $V^e(t) \in X_{c, y(t)}^e(C\delta)$ due to Lemma 4.2 and Proposition 4.5. Let $a^-(t) = h^{cu}(W(t))$ whenever t satisfies $V^e(t) \in X_{c, y(t)}^e(\delta)$. We can extend $a^-(t)$ to be a bounded function for $t \in \mathbb{R}$. For any $|t_0| \ll 1$ such that $\tilde{W} = W(t_0) \in X^{cu}(\delta)$, since (4.11) is autonomous, $W(t+t_0)$ is its solution with initial value $\tilde{W}$. Since $\mathcal{T}(h^{cu}) = h^{cu}$, we can compute $\tilde{a}^- = a^-(t_0) = h^{cu}(\tilde{W})$ satisfies

$$\begin{aligned} \tilde{a}^- &= \int_{-\infty}^0 e^{-\tau M_-} \hat{G}^-((W, a^-)(t_0 + \tau)) d\tau = \int_{-\infty}^{t_0} e^{-(\tau-t_0)M_-} \hat{G}^-((W, a^-)(\tau)) d\tau \\ &= e^{t_0 M_-} \tilde{a}^- + \int_0^{t_0} e^{(t_0-\tau)M_-} \hat{G}^-((W, a^-)(\tau)) d\tau. \end{aligned}$$

As in Remark 4.1, the above equality does not depend on the extension of $a^-(t)$ as $\hat{G}^-(W, a^-) = 0$ whenever $W \in X^{cu} \setminus X^{cu}(\delta)$. Therefore $(W, a^-)(t) = (W(t), h^{cu}(W(t)))$ is a solution to (4.6). Along with the translation invariance (4.8) of h^{cu} , we have proved the local invariance of W^{cu} under (4.6). Since h^{cu} is translation invariant, we obtain

Lemma 4.8. *W^{cu} is locally invariant under (4.6), i.e. if $w(t)$ is a solution to (4.6) and $w(0) \in W^{cu}$, then exists $\epsilon > 0$ such that $w(t) \in W^{cu}$ for all $t \in (-\epsilon, \epsilon)$. Moreover W^{cu} satisfies, for any $z \in \mathbb{R}^3$,*

$$w(\cdot + z) \in W^{cu} \text{ if } w \in W^{cu}.$$

Solutions starting on W^{cu} might leave W^{cu} through its boundary $\overline{h^{cu}(X^{cu}(\delta))} \setminus h^{cu}(X^{cu}(\delta))$.

Since (4.6) coincides with the original system (2.40), (2.42), (2.43), (2.44), and (2.52) when

$$|a^{d1}| + |a^{d2}| + |a^+| + |a^-| + \|V^e\|_{X_1} \leq \delta/3,$$

$\mathcal{W}^{cu}$ is a locally invariant manifold of (2.7) and (GP). Namely

Proposition 4.9. *If $U(t) = \Phi(w(t))$ solves (2.7), satisfies $U(0) \in \mathcal{W}^{cu}$, and $w(t) \in B^{d_1+d_2+2d}(\frac{\delta}{15}) \oplus \mathcal{X}^e(\frac{\delta}{15})$ for all $t \in [-T, T]$, $T > 0$, then $U(t) \in \mathcal{W}^{cu}$, $t \in [-T, T]$.*

4.4. Local dynamics related to the center-unstable manifold. We start with the local stability of the center-unstable manifold, which means that if a solution to (2.7) stays in a δ_0 -neighborhood of $\mathcal{M}$ over a time interval, then its distance to $\mathcal{W}^{cu}$ shrinks exponentially. Since (2.7) is equivalent to (4.6) for U near $\mathcal{M}$, we only need to work with (4.6). More precisely,

Lemma 4.10. *There exists $C > 0$ independent of Q, μ, δ and η , such that if (4.9), (4.16), (4.18), and*

$$C(\eta^{-(2d_1+1)} Q^6 \delta^2 + \delta^2 \eta^{-1} + \eta^{-2(d_1+1)} Q^6 \delta^4 (\lambda - 2\eta)^{-1}) < \eta \quad (4.20)$$

are satisfied, then for any $T > 0$ and solution

$$(W, a^-)(t) = (y, a^{d1}, a^{d2}, a^+, a^-, V^e)(t) \in \mathbb{R}^{d1+d2+2d} \oplus \mathcal{X}^e(\delta),$$

$t \in [0, T]$, to (4.6), we have

$$|a^-(t) - h^{cu}(W(t))| \leq C e^{-(\lambda-2\eta)t} |a^-(0) - h^{cu}(W(0))|$$

for any $t \in [0, T]$.

Proof. Let

$$W_1(t) = W(t), \quad \tilde{a}_1^-(t) = a^-(t), \quad \Delta^-(t) = a^-(t) - h^{cu}(W(t)).$$

Fix $t \in (0, T]$ and let $((W_2, \tilde{a}_2^-)(\tau))$ be the solution to (4.6) with initial value $((W(t), h^{cu}(W(t)))$ at $\tau = t$. The invariance of W^{cu} under (4.6) implies that, for all $\tau \leq t$,

$$\tilde{a}_2^-(\tau) = h^{cu}(W_2(\tau)), \quad \Delta^-(t) = \tilde{a}_1^-(t) - \tilde{a}_2^-(t),$$

where we note the latter holds at $\tau = t$ only. Denote

$$l(\tau) = \|(W_2 - W_1)(\tau)\|_{X_1, Q}.$$

Lemma 4.2 implies $\tilde{\Pi}^e W_j(\tau) = W_j(\tau)$ for $\tau \leq t$, $j = 1, 2$, since it holds at $\tau = t$. From Proposition 4.5, we have, for any $\tau \leq t$,

$$l(\tau)^2 \leq C \eta^{-(2d1+1)} Q^6 \delta^2 \int_{\tau}^t e^{2\eta(\tau'-\tau)} |\Delta^-(\tau')|^2 d\tau'. \quad (4.21)$$

Using the variation of parameter formula, we have

$$\begin{aligned} \Delta^-(t) &= (\tilde{a}_1^- - \tilde{a}_2^-)(t) \\ &= e^{tM^-} (\tilde{a}_1^- - \tilde{a}_2^-)(0) + \int_0^t e^{(t-\tau)M^-} \left(\hat{G}^-((W_1, \tilde{a}_1^-)(\tau),) - \hat{G}^-((W_2, \tilde{a}_2^-)(\tau)) \right) d\tau. \end{aligned}$$

It follows from (2.26) and (4.14) that

$$|\Delta^-(t)| \leq C e^{-\lambda t} |(\tilde{a}_1^- - \tilde{a}_2^-)(0)| + C \delta \int_0^t e^{-\lambda(t-\tau)} (l(\tau) + |(\tilde{a}_1^- - \tilde{a}_2^-)(\tau)|) d\tau.$$

Since

$$|(\tilde{a}_1^- - \tilde{a}_2^-)(\tau)| \leq |\Delta^-(\tau)| + \mu l(\tau),$$

we obtain

$$|\Delta^-(t)| \leq C e^{-\lambda t} (l(0) + |\Delta^-(0)|) + C \delta \int_0^t e^{-\lambda(t-\tau)} (l(\tau) + |\Delta^-(\tau)|) d\tau.$$

We use (4.21) to proceed

$$\begin{aligned}
|\Delta^-(t)|^2 &\leq C e^{-2(\lambda-\eta)t} |\Delta^-(0)|^2 + C \eta^{-(2d_1+1)} Q^6 \delta^2 e^{-2\lambda t} \int_0^t e^{2\eta\tau} |\Delta^-(\tau)|^2 d\tau \\
&\quad + C \delta^2 \left(\int_0^t e^{-\lambda(t-\tau)} |\Delta^-(\tau)| d\tau \right)^2 + C \eta^{-(2d_1+1)} Q^6 \delta^4 \\
&\quad \times \left(\int_0^t e^{-\lambda(t-\tau)} \left(\int_\tau^t e^{2\eta(\tau'-\tau)} |\Delta^-(\tau')|^2 d\tau' \right)^{\frac{1}{2}} d\tau \right)^2 \\
&\triangleq C e^{-2(\lambda-\eta)t} |\Delta^-(0)|^2 + I_1 + I_2 + I_3.
\end{aligned}$$

The above integrals are estimated by Cauchy-Schwartz inequality. Firstly,

$$\begin{aligned}
I_2 &\leq C \delta^2 \int_0^t e^{-2\eta(t-\tau)} d\tau \int_0^t e^{-2(\lambda-\eta)(t-\tau)} |\Delta^-(\tau)|^2 d\tau \\
&\leq C \delta^2 \eta^{-1} \int_0^t e^{-2(\lambda-\eta)(t-\tau)} |\Delta^-(\tau)|^2 d\tau.
\end{aligned}$$

Secondly,

$$\begin{aligned}
I_3 &\leq C \eta^{-(2d_1+1)} Q^6 \delta^4 e^{-2\lambda t} \int_0^t e^{2\eta\tau} d\tau \int_0^t \int_\tau^t e^{2(\lambda-\eta)\tau+2\eta(\tau'-\tau)} |\Delta^-(\tau')|^2 d\tau' d\tau \\
&\leq C \eta^{-(2d_1+1)} Q^6 \delta^4 e^{-2(\lambda-\eta)t} \int_0^t |\Delta^-(\tau')|^2 \int_0^{\tau'} e^{2\eta\tau'+2(\lambda-2\eta)\tau} d\tau d\tau' \\
&\leq C \eta^{-(2d_1+1)} Q^6 \delta^4 (\lambda-2\eta)^{-1} \int_0^t e^{-2(\lambda-\eta)(t-\tau)} |\Delta^-(\tau)|^2 d\tau.
\end{aligned}$$

Finally, it is also easy to see

$$I_1 \leq C \eta^{-(2d_1+1)} Q^6 \delta^2 \int_0^t e^{-2(\lambda-\eta)(t-\tau)} |\Delta^-(\tau)|^2 d\tau.$$

Therefore we obtain

$$|\Delta^-(t)|^2 \leq C e^{-2(\lambda-\eta)t} |\Delta^-(0)|^2 + \eta \int_0^t e^{-2(\lambda-\eta)(t-\tau)} |\Delta^-(\tau)|^2 d\tau$$

where assumption (4.20) is used. The desired estimates follow immediately from the Gronwall inequality and the proof is complete. $\square$

Remark 4.11. The proof of the local asymptotic stability of the center-unstable manifold could have been much simpler if h^{cu} had been smooth, which will be proved in the next section. In that case, one could obtain the decay estimate using certain property derived by differentiating the invariance equation of h^{cu} . In this subsection, even though we went a greater length to obtain the result, it has the benefit to show that the local asymptotic stability still holds even if h^{cu} is only Lipschitz, which is the case when G is only Lipschitz.

A direct corollary of Lemma 4.10 is the following condition for a point to belong to W^{cu} .

Lemma 4.12. *There exists $C > 0$ such that if $\eta \in (C\delta, 1)$ and Q, μ, δ satisfy (4.9), (4.16), (4.18), and (4.20), then a solution of (4.6)*

$$(W, a^-)(t) = (y, a^{d1}, a^{d2}, a^+, a^-, V^e)(t) \in W^{cu}$$

if $(W, a^-)(t) \in \mathbb{R}^{d1+d2+2d} \oplus \mathcal{X}^e(\delta)$ for all $t \leq 0$ and satisfies $\sup_{t \leq 0} |a^-(t)| < \infty$.

Since (GP), or equivalently (2.7), is equivalent to (4.6) in a neighborhood of $\mathcal{M}$, we have

Corollary 4.13. *If $U(t) = \Phi(w(t))$ is a solution of (2.7) satisfying*

$$w(t) \in B^{d1+d2+2d}(\frac{\delta}{15}) \oplus \mathcal{X}^e(\frac{\delta}{15})$$

for all $t \leq 0$, then $U(t) \in \mathcal{W}^{cu}$, $t \leq 0$.

Remark 4.14. Note that the assumption in the above lemma is satisfied if a solution stays in a neighborhood of $\mathcal{M}$ for all $t \leq 0$, which is the case of neighboring traveling waves.

More precisely, consider another travel wave $U_{\tilde{c}} = (u_{\tilde{c}}, v_{\tilde{c}}) = \psi(w_{\tilde{c}})$ with traveling velocity $\tilde{c} \in \mathbb{R}^3$. Here we recall ψ and ψ^{-1} are given in (2.2) and (2.3), respectively.

Lemma 4.15. *There exists $\delta_0 > 0$ such that*

- (1) *if $\tilde{c} \wedge c = 0$, $\|w_{\tilde{c}} - w_c\|_{X_1} < \delta_0$, then $U_{\tilde{c}} \in \mathcal{W}^{cu}$;*
- (2) *or without $\tilde{c} \wedge c = 0$, but instead with the additional $\|x \nabla(w_{\tilde{c}} - w_c)\|_{X_1} < \delta_0$ and that the angel between c and $\tilde{c}$ is bounded by δ_0 , then $u_{\tilde{c}} \in \mathcal{W}^{cu}$.*

Proof. The solution to (2.17) corresponding to $U_{\tilde{c}}$ is given by $w_{\tilde{c}}(x - \tilde{c}t)$. If $\tilde{c}$ and c are parallel, then $w_c(\cdot - \tilde{c}t) \in \psi^{-1}(\mathcal{M})$ and $\|w_{\tilde{c}}(\cdot - \tilde{c}t) - w_c(\cdot - \tilde{c}t)\|_{X_1} < \delta_0$ for all $t \leq 0$, then $U_{\tilde{c}} \in \mathcal{W}^{cu}$ by Corollary 4.13.

In the general case, there exists a near identity orthogonal matrix $O_{3 \times 3}$ such that $O\tilde{c} = |\tilde{c}| |c|^{-1} c$. It is easy to verify that $w_c(O(x - \tilde{c}t))$ is a traveling wave of (2.17) with traveling velocity $|\tilde{c}| |c|^{-1} c$. As $|O - I| \ll 1$ yields that $w_c(Ox)$ is close to w_c satisfying the assumption of case (1), therefore $U_c(Ox) \in \mathcal{W}^{cu}$. Our assumptions imply $\|w_{\tilde{c}} - w(O \cdot)\|_{X_1} \ll 1$ and thus Corollary 4.13 implies $U_{\tilde{c}} \in \mathcal{W}^{cu}$. $\square$

4.5. Construction of local center-stable manifolds. Basically by reversing the time in the previous procedure, we can construct a local Lipschitz center-stable manifold W^{cs} of $\mathcal{M}$. It is given by the graph of a function $h^{cs} : B^{d1+d2+d}(\delta) \oplus \mathcal{X}^e(\delta) \rightarrow \mathbb{R}^d$,

$$\mathcal{W}^{cs}(\mathcal{M}) = \Phi(\{a^+ = h^{cs}(y, a^{d1}, a^{d2}, a^-, V^e) \mid (y, a^{d1}, a^{d2}, a^-, V^e) \in B^{d1+d2+d}(\delta) \oplus \mathcal{X}^e(\delta)\}).$$

We briefly outline the steps here. Let $X^{cs} = \mathbb{R}^{3+d1+d2+d} \times X_1$ be same as X^{cu} and equipped with the same $\|\cdot\|_{X_1, Q}$ metric as in (4.7). The set $\Gamma_{\mu, \delta}^{cs}$ of mappings $h : X^{cs}(\delta) \rightarrow \mathbb{R}^d$ also takes the same form as $\Gamma_{\mu, \delta}^{cu}$.

On $X^{cs} \times \mathbb{R}^d$, we rewrite (4.6) as

$$\partial_t W = A^{cs}(y, \hat{G}^T(W, a^+))W + \hat{G}^{cs}(W, a^+) \quad (4.22a)$$

$$\partial_t a^+ = M_+ a^+ + \hat{G}^+(W, a^+) \quad (4.22b)$$

where $\hat{G}^{cs} = (\hat{G}^T, \hat{G}^{d1}, \hat{G}^{d2}, \hat{G}^-, \hat{G}^e)$ are as defined in Sect. 4.1 and

$$A^{cs}(y, \tilde{y}) = \text{diag}(0, M_1, M_2, M_-, \Pi_{c,y}^e J L_{c,y} \Pi_{c,y}^e + \mathcal{F}(c, y)(\tilde{y}, \cdot)).$$

For any h and $\bar{W} \in X^{cs}(\delta)$, let $W(t)$, $t \geq 0$, be the solution to

$$\partial_t W = A^{cs}(y, \hat{G}^T(W, h(W)))W + \hat{G}^{cs}(W, h(W)), \quad W(0) = \bar{W}$$

and define $\tilde{h}(\bar{W})$ as

$$\tilde{h}(\bar{W}) = \bar{a}^+ = - \int_0^\infty e^{-\tau M_+} \hat{G}^+(W(t), h(W(t))) dt$$

and $\mathcal{T}^{cs}(h) = \tilde{h}$. Following exactly the same procedure, one proves that this defines a contraction mapping on $\Gamma_{\mu,\delta}^{cs}$, the graph of whose fixed point, restricted to $\mathbb{R}^{d_1+d_2+d}(\frac{\delta}{15}) \oplus \mathcal{X}^e(\frac{\delta}{15})$, leads to the locally invariant Lipschitz center stable manifold of $\mathcal{M}$.

Proposition 4.16. *There exist $\delta > 0$ such that there exists $h^{cs} \in \Gamma_{\mu,\delta}^{cs}$ and*

(1) *the center-stable manifold*

$$\mathcal{W}^{cs} = \Phi(\{(W, a^+) \in W^{cs} \mid W \in B^{d_1+d_2+d}(\delta/15) \oplus \mathcal{X}^e(\delta/15)\})$$

is locally invariant under (2.7), where

$$W^{cs} = \text{graph}(h^{cs}) = \{a^+ = h^{cs}(W) \mid W \in \mathbb{R}^{d_1+d_2+d} \oplus \mathcal{X}^e(\delta)\},$$

is locally invariant under (4.22).

(2) *There exists $C > 0$ independent of Q, μ, δ and $\eta \in (0, 1)$, such that if (4.9), (4.16), (4.18), and (4.20) are satisfied, then for any $T > 0$ and any solution $U(t) = \Phi((W, a^+)(t))$ to (4.22) with $(W, a^+)(t) \in \mathbb{R}^{d_1+d_2+2d} \oplus \mathcal{X}^e(\delta/15)$, $t \in [0, T]$, we have*

$$|a^+(t) - h^{cs}(W(t))| \geq C e^{(\lambda-2\eta)t} |a^+(0) - h^{cs}(W(0))|$$

for any $t \in [0, T]$.

(3) *A solution of (2.7) $\Phi((W, a^+)(t)) \in \mathcal{W}^{cs}$ for all $t \geq 0$ if*

$$(W, a^+)(t) \in B^{d_1+d_2+2d}(\delta/15) \oplus \mathcal{X}^e(\delta/15)$$

for all $t \geq 0$.

The estimate in part (2) on the growth of $a^+(t)$, $t > 0$, for any solution follows directly from the decay estimate of $a^+(t)$, $t < 0$, which is parallel to Lemma 4.10 for W^{cu} in the opposite time evolution direction.

Remark 4.17. The local invariance of $\mathcal{W}^{cs}$ is in the same sense as in Proposition 4.9.

Like $\mathcal{W}^{cu}$, $\mathcal{W}^{cs}$ is translation invariant in the sense as in Lemma 4.8, and $\mathcal{W}^{cs}$ is Lipschitz. As in Lemma 4.15, all neighboring traveling waves belong to W^{cs} under the same assumptions.

Remark 4.18. The above statement (2) implies that, if the initial value is not on the center-stable manifold, then the solution would eventually leave the $\frac{\delta}{15}$ -neighborhood of $\mathcal{M}$, and thus $\mathcal{M}$ is orbitally unstable.

4.6. Local center manifolds. A center manifold $\mathcal{W}^c = \Phi(W^c)$ is given by the intersection of a center-unstable and a center-stable manifold, and thus it is also locally invariant and extends in the directions of the center subspace $X_{c,y}^T \oplus X_{c,y}^{d1} \oplus X_{c,y}^{d2} \oplus X_{c,y}^e$ at any y . For $0 < \delta \ll 1$, $(y, a^{d1}, a^{d2}, a^+, a^-, V^e) \in \mathbb{R}^{d1+d2+2d} \oplus \mathcal{X}^e(\delta)$ belongs to W^c if and only if

$$a^- = h^{cu}(y, a^{d1}, a^{d2}, a^+, V^e), \quad a^+ = h^{cs}(y, a^{d1}, a^{d2}, a^+, V^e). \quad (4.23)$$

The Lipschitz property implies that (4.23) is equivalent to that (a^+, a^-) is the fixed point of a contraction (h^{cu}, h^{cs}) with Lipschitz constant μ . Therefore we obtain

Proposition 4.19. *There exists $C > 0$ independent of Q, μ, δ and $\eta \in (0, 1)$, such that if (4.9), (4.16), (4.18), and (4.20) are satisfied, then there exists $h^c : \mathbb{R}^{d1+d2} \oplus \mathcal{X}^e(\delta) \rightarrow \mathbb{R}^{2d}$ such that*

(1) *the center manifold*

$$\begin{aligned} \mathcal{W}^c = \Phi(\{(y, a^{d1}, a^{d2}, a^+, a^-, V^e) \in W^c \mid \\ (y, a^{d1}, a^{d2}, V^e) \in B^{d1+d2}(\delta/15) \oplus \mathcal{X}^e(\delta/15)\}) \end{aligned}$$

is locally invariant under (2.7), where

$$\begin{aligned} W^c = \text{graph}(h^c) = \text{graph}(h^{cu}) \cap \text{graph}(h^{cs}) = \{(y, a^{d1}, a^{d2}, a^+, a^-, V^e) \\ \in \mathbb{R}^{d1+d2+2d} \oplus \mathcal{X}^e(\delta) \mid (a^+, a^-) = h^c(y, a^{d1}, a^{d2}, V^e)\}, \end{aligned}$$

is locally invariant under (4.22).

(2) *h^c satisfies (4.8), $h^c(y, 0, 0, 0) = 0$, and*

$$\begin{aligned} |h^c(y_2, a_2^{d1}, a_2^{d2}, V_2^e) - h^c(y_1, a_1^{d1}, a_1^{d2}, V_1^e)| \\ \leq \frac{\mu}{1-\mu} (|y_2 - y_1| + Q|a_2^{d1} - a_1^{d1}| + Q^3|a_2^{d2} - a_1^{d2}| + Q^2\|V_2^e - V_1^e\|_{X_1}). \end{aligned}$$

(3) *a solution $\Phi((y, a^{d1}, a^{d2}, V, a^+, a^-)(t)) \in \mathcal{W}^c$ if $(y, a^{d1}, a^{d2}, a^+, a^-, V)(t) \in B^{d1+d2+2d}(\delta/15) \oplus \mathcal{X}^e(\delta/15)$ for all $t \in \mathbb{R}$.*

(4) *There exists $\delta > 0$ such that any traveling wave solution satisfying assumptions in Lemma 4.15 belongs to $\mathcal{W}^c$.*

The following lemma states that, as a submanifold, the center manifold attracts orbits on the center-unstable and center-stable manifolds.

Lemma 4.20. *There exists $C > 0$ independent of Q, μ, δ and $\eta \in (0, 1)$, such that if (4.9), (4.16), (4.18), and (4.20) are satisfied, then for any $T > 0$ the following hold.*

(1) *Let $U(t) = \Phi((W, a^+, a^-)(t)) \in \mathcal{W}^{cs}$ be a solution to (2.7), where $W = (y, a^{d1}, a^{d2}, V^e)$, satisfying $(W, a^+, a^-)(t) \in B^{d1+d2+2d}(\delta/15) \oplus \mathcal{X}^e(\delta/15)$, $t \in [0, T]$, then we have*

$$|(a^+, a^-)(t) - h^c(W(t))| \leq C e^{-(\lambda-2\eta)t} |(a^+, a^-)(0) - h^c(W(0))|$$

for any $t \in [0, T]$.

(2) Let $U(t) = \Phi((W, a^+, a^-)(t)) \in \mathcal{W}^{cu}$ be a solution to (2.7), where $W = (y, a^{d1}, a^{d2}, V^e)$, satisfying $(W, a^+, a^-)(t) \in B^{d1+d2+2d}(\delta/15) \oplus \mathcal{X}^e(\delta/15)$, $t \in [-T, 0]$, then we have

$$|(a^+, a^-)(t) - h^c(W(t))| \leq C e^{(\lambda-2\eta)t} |(a^+, a^-)(0) - h^c(W(0))|$$

for any $t \in [-T, 0]$.

Proof. Let us denote the components of h^c by $h^c = (h_+^c, h_-^c)$. From (4.23), we have

$$h_-^c(W) = h^{cu}(W, h_+^c(W)), \quad h_+^c(W) = h^{cu}(W, h_-^c(W)). \quad (4.24)$$

We shall only prove part (1) of the lemma, where $a^+ = h^{cs}(W, a^-)$, as part (2) is verbatim. One may compute

$$|a^- - h_-^c(W)| = |a^- - h^{cu}(W, h_+^c(W))| \leq |a^- - h^{cu}(W, a^+)| + \mu |a^+ - h_+^c(W)|$$

and

$$|a^+ - h_+^c(W)| = |h^{cs}(W, a^-) - h^{cs}(W, h_-^c(W))| \leq \mu |a^- - h_-^c(W)|.$$

Therefore

$$|(a^+, a^-)(t) - h^c(W(t))| \leq (1 - \mu)^{-1} |a^- - h^{cu}(W, a^+)|$$

which along with Lemma 4.10 implies the desired estimates. $\square$

5. Smoothness of the Center-Unstable Manifold

We will prove the smoothness of the local center-unstable/center-stable/center manifolds roughly following the approach in [19].

Proposition 5.1. *For any $k > 0$, there exists $C > 0$ such that if $\eta \in (C\delta, 1)$ and Q, μ, δ satisfy (4.9), (4.16), (4.18), (4.20), (5.18), and (5.25), then $h^{cu}, h^{cs}, h^c \in C^k$ and $Dh^{cu}(y, 0, 0, 0, 0)$, $Dh^{cs}(y, 0, 0, 0, 0)$, and $Dh^c(y, 0, 0, 0, 0)$ are equal to 0.*

Unlike in [19], however, A^{cu} in (4.6) depends on the unknowns and extra care has to be taken. Without loss of generality, we will work on $h^{cu} \in \Gamma_{\mu, \delta}$, which is defined on $X^{cu}(\delta)$, and the proof of h^{cs} is verbatim. The smoothness of the center manifold, as the intersection of the center-unstable and center-stable manifolds, follows subsequently.

5.1. Outline of the framework of the smoothness proof. We first introduce some notations to simplify the presentations. Consider (4.11) with $h = h^{cu}$. For $t \leq 0$, let

$$\Psi(t, W) = (y, a^{d1}, a^{d2}, a^+, V)(t), \quad W = (y, a^{d1}, a^{d2}, a^+, V) \in X^{cu}(\delta),$$

be the solution with initial value W . We have from Lemma 4.2 that

$$\tilde{\Gamma}^e \Psi(t, W) = \Psi(t, W), \quad \forall t \leq 0 \quad \text{if} \quad \tilde{\Gamma}^e W = W. \quad (5.1)$$

Moreover, assuming (4.9) and (4.16), Proposition 4.5 implies, for all $t \leq 0$,

$$Lip_{\|\cdot\|_{X_1, Q}} \Psi(t, \cdot) \leq C \eta^{-d1} e^{-\eta t}, \quad \Psi(t, W) \in X^{cu}(C\delta), \quad \forall W \in X^{cu}(\delta). \quad (5.2)$$

As the fixed point of the transformation $\mathcal{T}$, h^{cu} satisfies

$$h^{cu}(W) = \int_{-\infty}^0 e^{-tM_-} \hat{G}^- \left(\Psi(t, W), h^{cu}(\Psi(t, W)) \right) dt. \quad (5.3)$$

Since (4.11) is autonomous, a time translation of (5.3) implies, for $t \leq 0$,

$$h^{cu}(\Psi(t, W)) = \int_{-\infty}^t e^{(t-\tau)M_-} \hat{G}^- \left(\Psi(\tau, W), h^{cu}(\Psi(\tau, W)) \right) d\tau. \quad (5.4)$$

By differentiating (5.3) formally, we obtain, for any $\tilde{W} \in X^{cu}$,

$$\begin{aligned} Dh^{cu}(W)\tilde{W} &= \int_{-\infty}^0 e^{-tM_-} \left(D_{a^-} \hat{G}^- \left(\Psi(t, W), h^{cu}(\Psi(t, W)) \right) Dh^{cu}(\Psi(t, W)) \right. \\ &\quad \left. + D_W \hat{G}^- \left(\Psi(t, W), h^{cu}(\Psi(t, W)) \right) \right) D\Psi(t, W) \tilde{W} dt. \end{aligned}$$

Here $D\Psi$ also depends on Dh^{cu} as it solves the following system of equation derived by differentiating (4.11)

$$\partial_t D\Psi = A^{cu}(y(t), \hat{G}^T) D\Psi + \mathcal{G}_1(\Psi) D\Psi + \tilde{\mathcal{G}}_1(\Psi) Dh^{cu} D\Psi, \quad (5.5)$$

where Ψ and $D\Psi$ are evaluated at (t, W) , $\hat{G}^{cu}$ at (Ψ, h^{cu}) , h^{cu} and $D^{cu}h$ at Ψ . In the above $\mathcal{G}_1 \in C^k(X^{cu}, L(X^{cu}))$ and $\tilde{\mathcal{G}}_1 \in C^k(X^{cu}, L(\mathbb{R}^d, X^{cu}))$ are given by

$$\begin{aligned} \tilde{\mathcal{G}}_1(W)\tilde{a}^- &= D_{a^-} (A^{cu}(y, G^T(W, a^-)))|_{a^- = h^{cu}(W)} (\tilde{a}^-) W + D_{a^-} \hat{G}^{cu}(\tilde{a}^-) \\ &= (0, 0, 0, 0, \mathcal{F}(c, y)(D_{a^-} \hat{G}^T(\tilde{a}^-), V)) + D_{a^-} \hat{G}^{cu}(\tilde{a}^-), \end{aligned} \quad (5.6)$$

$$\begin{aligned} \mathcal{G}_1(W)\tilde{W} &= D_W (A^{cu}(y, G^T(W, a^-)))|_{a^- = h^{cu}(W)} (\tilde{W}) W + D_W \hat{G}^{cu}(\tilde{W}) \\ &= (0, 0, 0, 0, (D_y A^e(y)\tilde{y})V + \mathcal{F}(c, y)(D_W \hat{G}^T(\tilde{W}), V) \\ &\quad + (D_y \mathcal{F}(c, y)(\tilde{y}))(\hat{G}^T, V)) + D_W \hat{G}^{cu}(\tilde{W}) \end{aligned} \quad (5.7)$$

where $W = (y, a^{d1}, a^{d2}, a^-, V)$, $\tilde{W} = (\tilde{y}, \tilde{a}^{d1}, \tilde{a}^{d2}, \tilde{a}^-, \tilde{V}) \in X^{cu}$ and $\hat{G}^{cu}$ are evaluated at $(W, h^{cu}(W))$.

Motivated by the above formally derived equations, we define a linear transformation $\mathcal{T}_1$ on

$$Y_1 = C^0(X^{cu}(\delta), L(X^{cu}, \mathbb{R}^d))$$

as, for any $\mathcal{H} \in Y_1$, $W \in X^{cu}(\delta)$, and $\tilde{W} \in X^{cu}$,

$$\begin{aligned} (\mathcal{T}_1 \mathcal{H})(W)\tilde{W} &= \int_{-\infty}^0 e^{-tM_-} \left(D_W \hat{G}^- (\Psi, h^{cu}(\Psi)) \right. \\ &\quad \left. + D_{a^-} \hat{G}^- (\Psi, h^{cu}(\Psi)) \mathcal{H}(\Psi) \right) \Psi_1(t) \tilde{W} dt \end{aligned} \quad (5.8)$$

where Ψ is evaluated at (t, W) . Operator $\Psi_1(t) \in L(X^{cu})$ satisfies $\Psi_1(0) = I$ and

$$\partial_t \Psi_1 = A^{cu}(y(t), \hat{G}^T) \Psi_1 + \mathcal{G}_1(\Psi) \Psi_1 + \tilde{\mathcal{G}}_1(\Psi) \mathcal{H}(\Psi) \Psi_1, \quad (5.9)$$

where $\tilde{\mathcal{G}}_1$ and $\mathcal{G}_1$ are given in (5.6) and (5.7), $\hat{G}^{cu}$ is evaluated at $(\Psi, h^{cu}(\Psi))$, and $\mathcal{H}$ at $\Psi(t, W)$. Just as in Remark 4.1, the right side of (5.9) and the integrand in (5.8) are well-defined. Since (4.11) is autonomous, when W is shifted to $\Psi(t_0, W)$, the principle fundamental solution to the associated (5.9) becomes $\Psi_1(t + t_0)\Psi_1(t_0)^{-1}$. Therefore we obtain

$$\begin{aligned} (\mathcal{T}_1 \mathcal{H})(\Psi(t_0, W))\Psi_1(t_0)\tilde{W} &= \int_{-\infty}^{t_0} e^{(t_0-t)M_-} \left(D_W \hat{G}^-(\Psi, h^{cu}(\Psi)) \right. \\ &\quad \left. + D_{a^-} \hat{G}^-(\Psi, h^{cu}(\Psi)) \mathcal{H}(\Psi) \right) \Psi_1(t) \tilde{W} dt, \end{aligned} \quad (5.10)$$

where Ψ is still evaluated at (t, W) and Ψ_1 defined for W .

If $h^{cu} \in C^1$, then Dh^{cu} must be the fixed point of $\mathcal{T}_1$. Therefore, our strategy to prove $h^{cu} \in C^1$ is to show 1.) $\mathcal{T}_1$ is a well-defined contraction and 2.) the fixed point of $\mathcal{T}_1$ is indeed Dh^{cu} (Sect. 5.2). In the proof of the C^k and higher order C^k smoothness of h^{cu} we shall need the following spaces $Y_k, k \geq 1$, of symmetric k -linear transformations depending smoothly on the base points,

$$Y_k = \left(C^0(X^{cu}(\delta), L(\otimes_{sym}^k(X^{cu}), \mathbb{R}^d)) \right), \quad k \geq 0.$$

We equip Y_k with the norm

$$\begin{aligned} \|\mathcal{H}\|_{Y_k} &= \sup\{\|\mathcal{H}(W)\|_{L_Q^k} : W \in X^{cu}(\delta)\}, \\ \|\mathcal{H}(W)\|_{L_Q^k} &= \sup\left\{ \frac{|\mathcal{H}(W)(\tilde{W}_1, \dots, \tilde{W}_k)|}{\|\tilde{W}_1\|_Q \dots \|\tilde{W}_k\|_Q} : \tilde{W}_1, \dots, \tilde{W}_k \in X^{cu} \setminus \{0\} \right\}. \end{aligned} \quad (5.11)$$

We also use the $\|\cdot\|_{L_Q^k}$ norm of multilinear transformations in $L(\otimes_{sym}^k(X^{cu}), X^{cu})$ where $\|\cdot\|_{X_1, Q}$ is used in both the domain and the range.

Formally differentiate (5.3) twice, we see $D^2 h^{cu}$ is a fixed point of the following affine transformation $\mathcal{T}_2$ on the space Y_2 of symmetric k -linear (with $k = 2$) transformations depending continuously on the base points, where

$$Y_k = \left(C^0(X^{cu}(\delta), L(\otimes_{sym}^k(X^{cu}), \mathbb{R}^d)) \right).$$

Here for any $\mathcal{H} \in Y_2, W \in X^{cu}(\delta)$, and $\tilde{W}_1, \tilde{W}_2 \in X^{cu}$,

$$\begin{aligned} (\mathcal{T}_2 \mathcal{H})(W)(\tilde{W}_1, \tilde{W}_2) &= \int_{-\infty}^0 e^{-tM_-} \left(\left(D_{a^-} \hat{G}^- \mathcal{H}(\Psi) + D_{WW} \hat{G}^- \right) \right. \\ &\quad \left(D\Psi \tilde{W}_1, D\Psi \tilde{W}_2 \right) + D_{a^-} \hat{G}^- (Dh^{cu} D\Psi \tilde{W}_1, Dh^{cu} D\Psi \tilde{W}_2) \\ &\quad + 2D_{Wa^-} \hat{G}^- (D\Psi \tilde{W}_1, Dh^{cu} D\Psi \tilde{W}_2) \\ &\quad \left. + (D_{a^-} \hat{G}^- Dh^{cu} + D_W \hat{G}^-) \Psi_2(t)(\tilde{W}_1, \tilde{W}_2) \right) dt, \end{aligned} \quad (5.12)$$

where Ψ and $D\Psi$ are evaluated at (t, W) , h^{cu} and Dh^{cu} at Ψ , $\hat{G}^-$ and $D\hat{G}^-$ at (Ψ, h^{cu}) , and the symmetric bilinear transformation $\Psi_2(t) \in L(\otimes_{sym}^2 X^{cu}, X^{cu})$ satisfies $\Psi_2(0) = 0$ and

$$\begin{aligned} \partial_t \Psi_2 &= \left(A^{cu}(y(t), \hat{G}^T) + \mathcal{G}_1(\Psi) + \tilde{\mathcal{G}}_1(\Psi) Dh^{cu} \right) \Psi_2 \\ &\quad + \tilde{\mathcal{G}}_1(\Psi) \mathcal{H}(\Psi) (D\Psi, D\Psi) + \mathcal{G}_2(\Psi, D\Psi, Dh^{cu}). \end{aligned} \quad (5.13)$$

Here $\mathcal{G}_2(\Psi, D\Psi, Dh^{cu}) \in L(\otimes_{sym}^2 X^{cu}, X^{cu})$ is given by

$$\begin{aligned} \mathcal{G}_2(\Psi, D\Psi, Dh^{cu})(\tilde{W}_1, \tilde{W}_2) &= D_W(A^{cu}(y, G^T) + \mathcal{G}_1(\Psi))(\tilde{W}_1)(D\Psi(\tilde{W}_2)) \\ &\quad + D_W(\tilde{\mathcal{G}}_1(\Psi))(\tilde{W}_1)Dh^{cu}D\Psi(\tilde{W}_2). \end{aligned}$$

In order to prove $h^{cu} \in C^2$, we shall 1.) show that $\mathcal{T}_2$ is a well-defined affine contraction and 2.) its fixed point is D^2h^{cu} .

The general C^k smoothness of h^{cu} (Sect. 5.3) follows much as $h^{cu} \in C^2$ by 1.) differentiating (5.3) repeatedly to obtain an affine operator on the space Y_k of multilinear transformations, and 2.) proving its fixed point is indeed $D^k h^{cu}$.

Remark 5.2. A possible alternative adopted approach to prove the $C^k, k \geq 1$, smoothness of h^{cu} is to prove that iteration sequences of the transformation $\mathcal{T}$ defined in (4.11) and (4.12) actually converge in C^k topology. That proof usually required the $C^{k,1}$ bound on nonlinearity. Even though $\hat{G}$ is indeed smooth in our problem, this proof in Sect. 5 shows that the C^k smoothness holds as long as $\hat{G} \in C^k$.

5.2. C^1 smoothness of h^{cu} . We first prove the following estimate on equation (5.9) where $\Psi = \Psi(t, W)$, $W \in X^{cu}(\delta)$.

Lemma 5.3. *There exists $C > 0$ such that, if $\eta \in (C\delta, 1)$ and μ, δ, Q satisfy (4.9) and (4.16), then for any $B \in C^0([T, 0], L(X^{cu}, \mathbb{R}^d))$ with $\|B\|_{C_t^0 L_Q^1} \leq 1$, any solution $\tilde{W}(t) \in X^{cu}$ of*

$$\partial_t \tilde{W} = (A^{cu}(y(t), \hat{G}^T) + \mathcal{G}_1(\Psi) + \tilde{\mathcal{G}}_1(\Psi)B)\tilde{W} + f(t),$$

satisfies

$$\|\tilde{W}(t)\|_{X_{1,Q}}^2 \leq C\eta^{-2d_1}e^{-2\eta t}\|W(0)\|_{X_{1,Q}}^2 + C\eta^{-2d_1-1}\int_t^0 e^{2\eta(\tau-t)}\|f(\tau)\|_{\tilde{X},Q}^2 d\tau.$$

Proof. From Lemma 2.5, (2.53), (4.13), and (4.14), we have, for any

$$W = (y, a^{d_1}, a^{d_2}, a^+, V), \quad \tilde{W} = (y, a^{d_1}, a^{d_2}, a^+, V) \in X^{cu},$$

it holds that

$$\|\tilde{\mathcal{G}}_1(W)\tilde{a}^-\|_{\tilde{X},Q} \leq C\delta(1 + \|V\|_{X_1})|\tilde{a}^-|, \quad (5.14)$$

$$\|\mathcal{G}_1(W)\tilde{W}\|_{\tilde{X},Q} \leq C(\|V\|_{X_1} + Q^{-1} + Q^3\delta)\|\tilde{W}\|_{X_{1,Q}}. \quad (5.15)$$

Lemma 4.4, (5.2), and the above inequalities imply that, for any $t \in [T, 0]$,

$$\begin{aligned} \|\tilde{W}(t)\|_{X_{1,Q}}^2 &\leq C\eta^{-2d_1}e^{-\eta t}\left(\|\tilde{W}(0)\|_{X_{1,Q}}^2 \right. \\ &\quad \left. + \eta^{-1}\int_t^0 e^{\eta\tau}((Q^{-1} + Q^3\delta)^2\|\tilde{W}(\tau)\|_{X_{1,Q}}^2 + \|f(\tau)\|_{\tilde{X},Q}^2)d\tau\right). \end{aligned}$$

The lemma follows from the Gronwall inequality. $\square$

Recall the Lipschitz constant μ in the definition of $\Gamma_{\mu,\delta}$, which naturally should be an upper bound of $\|Dh^{cu}\|_{L_Q^1}$.

Lemma 5.4. *There exists $C > 0$ such that, if $\eta \in (C\delta, 1)$ and μ, δ, Q satisfy (4.9), (4.16), (4.18), and (4.20), then $\mathcal{T}_1$ defines a mapping on the closed μ -ball $Y_1(\mu) = \{\mathcal{H} \in Y_1 : \|\mathcal{H}\|_{Y_1} \leq \mu\}$ with Lipschitz constant $C\delta\eta^{-d_1}(\lambda - 2\eta)^{-1}$.*

Proof. Let $\mathcal{H} \in Y_1(\mu)$. Lemma 5.3 implies that, for any $W \in X^{cu}(\delta)$, $t \leq 0$, the operator $\Psi_1(t)$ defined in (5.9) satisfies

$$\|\Psi_1(t)\|_{L_Q(X^{cu})} \leq C\eta^{-d_1}e^{-\eta t}. \quad (5.16)$$

Therefore (2.26), definition (5.8) of $\mathcal{H}_1$, and (4.13) imply

$$\|\mathcal{T}_1(\mathcal{H})\|_{Y_1} \leq C\delta\eta^{-d_1} \int_{-\infty}^0 e^{(\lambda-\eta)t} dt = C\delta\eta^{-d_1}(\lambda - \eta)^{-1} \leq \mu \quad (5.17)$$

due to (4.18). To prove $\mathcal{T}_1(\mathcal{H}) \in Y_1$, it remains to show $\mathcal{H}(W)$ is continuous in W . In fact, the above estimate implies that $\mathcal{T}_1^{(n)}(\mathcal{H}) \rightarrow \mathcal{T}_1(\mathcal{H})$ uniformly, where

$$\begin{aligned} (\mathcal{T}_1^{(n)}(\mathcal{H}))(W)\tilde{W} &= \int_{-n}^0 e^{-tM_-} \left(D_W \hat{G}^-(\Psi, h^{cu}(\Psi)) \right. \\ &\quad \left. + D_{a-} \hat{G}^-(\Psi, h^{cu}(\Psi)) \mathcal{H}(\Psi) \right) \Psi_1(t) \tilde{W} dt. \end{aligned}$$

From the continuity of $D\hat{G}^{cu,-}$, it is easy to verify that $(\mathcal{T}_1^{(n)}(\mathcal{H}))(W)$ is C^0 in W . Therefore $\mathcal{T}_1(\mathcal{H})$ is also continuous and thus $\mathcal{T}_1(\mathcal{H}) \in Y_1(\mu)$.

In the following we estimate the Lipschitz constant of $\mathcal{T}_1$. Let $\mathcal{H}_j \in Y_1(\mu)$ and $\Psi_{1,j}(t)$ be defined in (5.9) for $\mathcal{H}_j$, $j = 1, 2$, which satisfy

$$\begin{aligned} \partial_t(\Psi_{1,2} - \Psi_{1,1}) &= (A^{cu}(y, G^T) - \mathcal{G}_1(\Psi) - \tilde{\mathcal{G}}_1(\Psi)\mathcal{H}_1)(\Psi_{1,2} - \Psi_{1,1}) \\ &\quad + \tilde{\mathcal{G}}_1(\Psi)(\mathcal{H}_2 - \mathcal{H}_1)(\Psi)\Psi_{1,2} \end{aligned}$$

and $(\Psi_{1,2} - \Psi_{1,1})(0) = 0$. Using Lemma 5.3 and (5.6), we obtain

$$\|(\Psi_{1,2} - \Psi_{1,1})(t)\|_{L_Q(X^{cu})} \leq C\eta^{-2d_1 - \frac{1}{2}}\delta|t|^{\frac{1}{2}}e^{-\eta t}\|\mathcal{H}_2 - \mathcal{H}_1\|_{Y_1}.$$

From the definition of $\mathcal{T}_1$, we have, for any $W \in X^{cu}(\delta)$,

$$\begin{aligned} (\mathcal{T}_1(\mathcal{H}_1) - \mathcal{T}_1(\mathcal{H}_2))(W) &= \int_{-\infty}^0 e^{-tM_-} (D_{a-} \hat{G}^-(\mathcal{H}_2 - \mathcal{H}_1)\Psi_{1,2}(t) \\ &\quad + (D_W \hat{G}^- + D_{a-} \hat{G}^- \mathcal{H}_1)(\Psi_{1,2} - \Psi_{1,1})(t)) dt, \end{aligned}$$

where $D\hat{G}$ is evaluated at $(\Psi, h^{cu}(\Psi))$, $\mathcal{H}_j$ at Ψ , and Ψ at (t, W) . Using (2.26), (4.14), and the above estimates on $\Psi_{1,j}$ and $\Psi_{1,2} - \Psi_{1,1}$, it follows that

$$\begin{aligned} \|\mathcal{T}_1(\mathcal{H}_1) - \mathcal{T}_1(\mathcal{H}_2)\|_{Y_1} &\leq C\delta \int_{-\infty}^0 e^{(\lambda-\eta)t} \eta^{-d_1} (1 + \eta^{-d_1 - \frac{1}{2}}\delta|t|^{\frac{1}{2}}) dt \|\mathcal{H}_2 - \mathcal{H}_1\|_{Y_1} \\ &\leq C\delta\eta^{-d_1}(\lambda - 2\eta)^{-1} \|\mathcal{H}_2 - \mathcal{H}_1\|_{Y_1}. \end{aligned}$$

The proof of the lemma is complete. $\square$

Assume

$$C\delta\eta^{-d_1}(\lambda - 2\eta)^{-1} < 1, \quad (5.18)$$

then $\mathcal{T}_1$ is a contraction mapping on $Y_1(\mu)$. Let $\mathcal{H}^{cu} \in Y_1(\mu)$ be the unique fixed point of $\mathcal{T}_1$. In the rest of this subsection, we will prove

Lemma 5.5. *There exists $C > 0$ such that if $\eta \in (C\delta, 1)$ and Q, μ, δ satisfy (4.9), (4.16), (4.18), (4.20), and (5.18), then $h^{cu} \in C^1(X^{cu}, \mathbb{R}^d)$ and $Dh^{cu}(W) = \mathcal{H}^{cu}(W)$ for any $W \in X^{cu}$.*

Proof. Since $\mathcal{H}(W)$ is continuous in W , it suffices to show $Dh^{cu}(W_0)\tilde{W} = D\mathcal{H}(W_0)\tilde{W}$ at any fixed $W_0 \in X^{cu}(\delta)$ and $\tilde{W} \in X^{cu} \setminus \{0\}$. Let $\Psi_1(t)$ be defined as in (5.9) associated to $\mathcal{H}^{cu}$ and W_0 and

$$\begin{aligned} R_\Psi(t) &= \Psi(t, W_0 + \tilde{W}) - \Psi(t, W_0) - \Psi_1(t)\tilde{W}, \\ R_h(t) &= h^{cu}(\Psi(t, W_0 + \tilde{W})) - h^{cu}(\Psi(t, W_0)) - \mathcal{H}^{cu}(\Psi(t, W_0))\Psi_1(t)\tilde{W}. \end{aligned}$$

According to (5.2), (5.16), $\|\mathcal{H}^{cu}\|_{Y_1} \leq \mu$, and the Lipschitz property of h^{cu} , R_Ψ and R_h satisfy the rough estimates

$$\|R_\Psi(t)\|_{X_1, Q} + |R_h(t)| \leq C\eta^{-d_1}e^{-\eta t}\|\tilde{W}\|_{X_1, Q}, \quad (5.19)$$

for $t \leq 0$. Our goal is to show $\|R_\Psi(0)\|_{X_1, Q}/\|\tilde{W}\|_{X_1, Q} \rightarrow 0$ as $\|\tilde{W}\|_{X_1, Q} \rightarrow 0$.

To analyze R_h and R_Ψ , denote

$$\begin{aligned} W(s, t) &= (1-s)\Psi(t, W_0) + s\Psi(t, W_0 + \tilde{W}), \\ a^-(s, t) &= (1-s)h^{cu}(\Psi(t, W_0)) + sh^{cu}(\Psi(t, W_0 + \tilde{W})) \end{aligned}$$

and for $\alpha \in \{T, d1, d2, \pm, V, cu\}$, let

$$\begin{aligned} R^\alpha(t) &= \hat{G}^\alpha(W(1, t), a^-(1, t)) - [\hat{G}^\alpha + D_W \hat{G}^\alpha(W(1, t) - W(0, t)) \\ &\quad + D_{a^-} \hat{G}^\alpha(a^-(1, t) - a^-(0, t))] \end{aligned}$$

where $\hat{G}^\alpha$ and $D\hat{G}^\alpha$ in the brackets [...] are evaluated at

$$(W(0, t), a^-(0, t)) = (\Psi(t, W_0), h^{cu}(\Psi(t, W_0))).$$

From (5.16), we have

$$\|R^{cu}(t)\|_{\tilde{X}, Q} + |R^-(t)| \leq r(t)\|\tilde{W}\|_{X_1, Q} \quad (5.20)$$

where $r(t) > 0$ satisfies

$$r(t) \leq C\eta^{-d_1}e^{-\eta t}, \quad \|r\|_{C^0([t_1, t_2], \mathbb{R})} \rightarrow 0 \text{ as } \|\tilde{W}\|_{X_1, Q} \rightarrow 0 \quad (5.21)$$

for any $t_1 \leq t_2 \leq 0$.¹

From (5.3) and $\mathcal{T}_1(\mathcal{H}^{cu}) = \mathcal{H}^{cu}$, we have

$$R_h(0) = \int_{-\infty}^0 e^{-tM^-} (R^-(t) + D_W \hat{G}^- R_\Psi(t) + D_{a^-} \hat{G}^- R_h(t)) dt.$$

¹ Here we only need some uniform continuity of $D\hat{G}$, instead of $\hat{G} \in C^2$ or $C^{1,1}$.

Moreover, using (5.3) and (5.10) instead, we obtain

$$R_h(t) = \int_{-\infty}^0 e^{-\tau M_-} (R^- + D_W \hat{G}^- R_\Psi + D_{a^-} \hat{G}^- R_h)|_{\tau+t} d\tau, \quad t \leq 0, \quad (5.22)$$

where again the above $D\hat{G}^-$ are evaluated at $(\Psi(t + \tau, W_0), h^{cu}(\Psi(t + \tau, W_0)))$.

From (4.11), $R_\Psi(t)$ satisfies $R_\Psi(0) = 0$ and

$$\begin{aligned} \partial_t R_\Psi &= \mathcal{A}_0^{cu}(t) R_\Psi + \mathcal{A}_0^-(t) R_h + R^{cu} + D_W \hat{G}^{cu} R_\Psi + D_{a^-} \hat{G}^{cu} R_h + \int_0^1 (\mathcal{A}_s^{cu} \\ &\quad - \mathcal{A}_0^{cu})(t) (W(1, t) - W(0, t)) + (\mathcal{A}_s^- - \mathcal{A}_0^-)(t) (a^-(1, t) - a^-(0, t)) ds \end{aligned}$$

where $D\hat{G}^{cu}$ is evaluated at $(W(0, t), a^-(0, t))$, operators $\mathcal{A}_s^{cu}(t) \in L(X^{cu})$ and $\mathcal{A}_s^-(t) \in L(\mathbb{R}^d, X^{cu})$ are given by

$$D_{(W, a^-)}(A^{cu}(y, G^T(W, a^-))W)|_{(W(s, t), a^-(s, t))}(\tilde{W}, \tilde{a}^-) = \mathcal{A}_s^{cu}(t)\tilde{W} + \mathcal{A}_s^-(t)\tilde{a}^-,$$

or more explicitly,

$$\begin{aligned} \mathcal{A}_s^{cu}(t)\tilde{W} &= A^{cu}(y(s, t), G^T(W(s, t), a^-(s, t)))\tilde{W} \\ &\quad + D_W(A^{cu}(y, G^T(W, a^-))|_{(W(s, t), a^-(s, t))}(\tilde{W})W(s, t) \\ \mathcal{A}_s^-(t)\tilde{a}^- &= D_{a^-}(A^{cu}(y, G^T(W, a^-))|_{(W(s, t), a^-(s, t))}(\tilde{a}^-)W(s, t) \end{aligned}$$

with $W(s, t)$ and $a^-(s, t)$ defined in the above and $y(s, t)$ being the y component of $W(s, t)$ (so the D_W also acts on the y component in A^{cu}). Note DA^{cu} acts only on the V component of $\tilde{W}$. From Lemma 2.5, (5.2), (5.6), (5.7), (5.14), (5.15), and (5.20), it is straightforward to obtain

$$\begin{aligned} \|\partial_t R_\Psi - (A^{cu} + \mathcal{G}_1)R_\Psi\|_{\tilde{X}, Q} &= \|\partial_t R_\Psi - \mathcal{A}_0^{cu}R_\Psi - D_W \hat{G}^{cu}R_\Psi\|_{\tilde{X}, Q} \\ &\leq C\delta|R_h| + \|R^{cu}\|_{\tilde{X}, Q} + r(t)\|\tilde{W}\|_{X_1, Q} \leq C\delta|R_h| + r(t)\|\tilde{W}\|_{X_1, Q} \end{aligned}$$

where A^{cu} and $\mathcal{G}_1$ are evaluated based on $\Psi(t, W_0)$ and $r(t)$ satisfies (5.21). Lemma 5.3 implies

$$\|R_\Psi(t)\|_{X_1, Q}^2 \leq C\delta^2\eta^{-2d_1-1} \int_t^0 e^{2\eta(\tau-t)} |R_h(\tau)|^2 d\tau + r_1(t)\|\tilde{W}\|_{X_1, Q}^2 \quad (5.23)$$

where A^{cu} and $\mathcal{G}_1$ are evaluated based on $\Psi(t, W_0)$ and $r_1(t)$ satisfies

$$r_1(t) \leq C\eta^{-4d_1-2}(1+|t|)e^{-2\eta t}, \quad \|r_1\|_{C^0([t_1, t_2], \mathbb{R})} \rightarrow 0 \text{ as } \|\tilde{W}\|_{X_1, Q} \rightarrow 0 \quad (5.24)$$

for any $t_1 \leq t_2 \leq 0$.

Finally, let

$$\tilde{R}_h = \sup_{t \leq 0} e^{2\eta t} \frac{|R_h(t)|}{\|\tilde{W}\|_{X_1, Q}}, \quad \tilde{R}_\Psi = \sup_{t \leq 0} e^{2\eta t} \frac{\|R_\Psi(t)\|_{X_1, Q}}{\|\tilde{W}\|_{X_1, Q}}.$$

Inequality (5.19) implies $\tilde{R}_h, \tilde{R}_\Psi < \infty$. We will prove $\tilde{R}_h, \tilde{R}_\Psi \rightarrow 0$ as $\|\tilde{W}\|_{X_1, Q} \rightarrow 0$. In fact, (5.23) and (5.22) along with (5.20) and Lemma 4.3 imply

$$\tilde{R}_\Psi \leq C\delta\eta^{-d_1-1}\tilde{R}_h + \sup_{t \leq 0} r_1(t)^{\frac{1}{2}}e^{2\eta t}$$

and

$$\begin{aligned} \tilde{R}_h &\leq C \int_{-\infty}^0 \delta e^{(\lambda-2\eta)\tau} (\tilde{R}_\Psi + \tilde{R}_h) + e^{\lambda\tau+2\eta t} r(t+\tau) d\tau \\ &\leq C(\lambda-2\eta)^{-1} (\delta(\tilde{R}_\Psi + \tilde{R}_h) + \sup_{\tau \leq 0} r(\tau)e^{2\eta\tau}). \end{aligned}$$

Therefore

$$\tilde{R}_\Psi + \tilde{R}_h \leq C \left(\sup_{t \leq 0} r_1(t)^{\frac{1}{2}}e^{2\eta t} + \sup_{\tau \leq 0} r(\tau)e^{2\eta\tau} \right).$$

From (5.21) and (5.24), we obtain that $\tilde{R}_h, \tilde{R}_\Psi \rightarrow 0$ as $\|\tilde{W}\|_{X_1, Q} \rightarrow 0$. Consequently $Dh^{cu}(W_0) = \mathcal{H}(W_0)$ and $D\Psi(t, W_0) = \Psi_1(t, W_0)$. $\square$

Finally we prove that, at any traveling wave, the center-unstable manifold is tangent to the center-unstable subspace.

Lemma 5.6. *There exists $C > 0$ such that if $\eta \in (C\delta, 1)$ and Q, μ, δ satisfy (4.9), (4.16), (4.18), (4.20), and (5.18), then $Dh^{cu}(y, 0, 0, 0, 0) = 0$ at any $y \in \mathbb{R}^3$.*

Proof. In the proof of this lemma, we adopt the notation $(y, 0) = (y, 0, 0, 0, 0) \in X^{cu}$. Observe that (4.11) and the definition of $\hat{G}^{cu}$ implies $\Psi(t, (y, 0)) = (y, 0)$ for all $t \leq 0$. For any $\mathcal{H} \in Y_1$, (4.12), the fact $D\hat{G}^-(y, 0) = 0$ and the above observation imply $\mathcal{T}_1(\mathcal{H})(y, 0) = 0$. The conclusion of the lemma follows immediately. $\square$

5.3. Higher order smoothness of h^{cu} . In this subsection, we shall prove

Proposition 5.7. *For any $k \geq 1$, there exists $C > 0$ such that if $\eta \in (C\delta, 1)$ and Q, μ, δ satisfy (4.9), (4.16), (4.18), (4.20), (5.18), and*

$$C\delta\eta^{-kd_1}(\lambda - k\eta)^{-1} \leq 1 \tag{5.25}$$

then $h^{cu} \in C^k$ and

$$\|D^k h^{cu}\|_{Y_k} + \sup_{t \leq 0} e^{k\eta t} \|D^k \Psi(t, \cdot)\|_{C^0 L_Q^k} < \infty.$$

Here $D^k \Psi$ denotes the differentiation with respect to W only. In particular $\|Dh^{cu}\|_{Y_1} \leq \mu$ and $\|D^k h^{cu}\|_{Y_k}$ may depend on δ for $k > 1$. In the rest of this subsection, C as usual denotes a generic upper bound independent of t , $W \in X^{cu}(\delta)$, and δ, Q, μ , while $\tilde{C}$ is independent of t and $W \in X^{cu}(\delta)$, but may depend on δ, Q, μ .

Formally differentiating (4.11) and (4.12) k times implies that $D^k h^{cu}$ should be a fixed point of the following affine transformation $\mathcal{T}_k$ on the space Y_k . Here for $k \geq 2$, any $\mathcal{H} \in Y_k$, $W \in X^{cu}(\delta)$, and $\tilde{W}_1, \dots, \tilde{W}_k \in X^{cu}$,

$$\begin{aligned} & (\mathcal{T}_k \mathcal{H})(W)(\tilde{W}_1, \dots, \tilde{W}_k) \\ &= \int_{-\infty}^0 e^{-tM-} \left(((D_{a-} \hat{G}^- D h^{cu} + D_W \hat{G}^-) \Psi_k(t) + \mathcal{L}_k(t))(\tilde{W}_1, \dots, \tilde{W}_k) \right. \\ & \quad \left. + D_{a-} \hat{G}^- \mathcal{H}(\Psi)(D\Psi \tilde{W}_1, \dots, D\Psi \tilde{W}_k) \right) dt, \end{aligned} \quad (5.26)$$

where $D^l \Psi$ is evaluated at (t, W) , $D^l h^{cu}$ at Ψ , $D^l \hat{G}^l$ at (Ψ, h^{cu}) , the symmetric multilinear mapping $\mathcal{L}_k(t) \in Y_k$ is an algebraic combination involving $D^l \Psi$ and $D^l h^{cu}$, $D^k \hat{G}^-$, and $D^l \hat{G}^-$, $0 \leq l \leq k-1$, and the symmetric multilinear $\Psi_k(t) \in L(\otimes_{sym}^k X^{cu}, X^{cu})$ satisfies $\Psi_k(0) = 0$ and

$$\begin{aligned} \partial_t \Psi_k &= \left(A^{cu}(y(t), \hat{G}^T) + \mathcal{G}_1(\Psi) + \tilde{\mathcal{G}}_1(\Psi) D h^{cu}(\Psi) \right) \Psi_k \\ & \quad + \tilde{\mathcal{G}}_1(\Psi) \mathcal{H}(\Psi)(D\Psi, \dots, D\Psi) + \mathcal{G}_k(t). \end{aligned} \quad (5.27)$$

Here $\mathcal{G}_k(t) \in C^0(X^{cu}(\delta), L(\otimes_{sym}^k X^{cu}, X^{cu}))$ is again an algebraic combination involving $D^l \Psi$ and $D^l h^{cu}$, $D^k \hat{G}^{cu}$, and $D^l \hat{G}^{cu}$, $0 \leq l \leq k-1$. These terms $\mathcal{G}_k$ and $\mathcal{L}_k$ are the lower order term in the higher order differentiation of compositions of mappings. The explicit forms of $\mathcal{T}_2$, $\mathcal{G}_2$, and $\mathcal{L}_2$ can be found in (5.8) and (5.27).

The proof of Proposition 5.7 is inductive in k . The case of $k = 1$ has been proved in Sect. 5.2. Assume it holds for $1 \leq l < k$, we will prove it for k . As outlined in Sect. 5.1, we shall prove by showing that $D^k h^{cu}$ is given by the fixed point of the contraction $\mathcal{T}_k$. Based on the usual formula of higher order derivatives of compositions of mappings, the induction assumptions imply

$$\sup_{t \leq 0} e^{k\eta t} \|\mathcal{L}_k(t)\|_{Y_k} + \sup_{t \leq 0} e^{k\eta t} \|\mathcal{G}_k(t)\|_{C^0 L_Q^k} < \infty. \quad (5.28)$$

In the following proof we will skip some details which are similar to those in Sect. 5.2.

For $k \geq 2$, as $\mathcal{T}_k$ is an affine transformation on Y_k , we first consider its homogeneous part $\tilde{\mathcal{T}}_k \in L(Y_k)$

$$\begin{aligned} & (\tilde{\mathcal{T}}_k \mathcal{H})(W)(\tilde{W}_1, \dots, \tilde{W}_k) \\ &= \int_{-\infty}^0 e^{-tM-} \left((D_{a-} \hat{G}^- D h^{cu} + D_W \hat{G}^-) \tilde{\Psi}_k(t)(\tilde{W}_1, \dots, \tilde{W}_k) \right. \\ & \quad \left. + D_{a-} \hat{G}^- \mathcal{H}(\Psi)(D\Psi \tilde{W}_1, \dots, D\Psi \tilde{W}_k) \right) dt, \end{aligned} \quad (5.29)$$

with the same convention of the notations and

$$\begin{aligned} \partial_t \tilde{\Psi}_k &= \left(A^{cu}(y(t), \hat{G}^T) + \mathcal{G}_1(\Psi) + \tilde{\mathcal{G}}_1(\Psi) D h^{cu}(\Psi) \right) \tilde{\Psi}_k \\ & \quad + \tilde{\mathcal{G}}_1(\Psi) \mathcal{H}(\Psi)(D\Psi, \dots, D\Psi). \end{aligned} \quad (5.30)$$

Lemma 5.8. *Let $k \geq 2$. There exists $C > 0$ such that if $\eta \in (C\delta, 1)$ and Q, μ, δ satisfy (4.9), (4.16), (4.18), (4.20), and (5.18), then*

$$\|\tilde{\mathcal{T}}_k\|_{L(Y_k)} \leq C\delta\eta^{-kd_1}(\lambda - k\eta)^{-1}.$$

Proof. Lemma 5.3 and (5.6) imply, for $t \leq 0$,

$$\|\tilde{\Psi}_k(t)\|_{L_Q^k} \leq C\delta\eta^{-(k+1)d_1-1}e^{-k\eta t}\|\mathcal{H}\|_{Y_k}. \quad (5.31)$$

Substituting it into (5.29) yields the lemma. $\square$

Lemma 5.9. *Let $k \geq 2$ and assume Proposition 5.7 holds for each l , $0 \leq l \leq k$. There exists $C > 0$ such that if $\eta \in (C\delta, 1)$ and Q, μ, δ satisfy (4.9), (4.16), (4.18), (4.20), (5.18), and (5.25), then $\mathcal{T}_k$ is a contraction on Y_k . Moreover, for any $\mathcal{H} \in Y_k$ and $W \in X^{cu}(\delta)$, the $\Psi_k(t)$ defined in (5.27) satisfies*

$$\sup_{t \leq 0, W \in X^{cu}(\delta)} e^{k\eta t} \|\Psi_k(t)\|_{L_Q^k} < \infty.$$

Proof. Firstly, the $\|\cdot\|_{Y_k}$ bound of $\mathcal{T}_k(0)$ can be easily obtained using (5.28), which along with Lemma 5.8 implies the $\|\cdot\|_{Y_k}$ bound of $\mathcal{T}_k(\mathcal{H})$ for any $\mathcal{H} \in Y_k$. The continuity of $\mathcal{T}_k(\mathcal{H})(W)$ with respect to W follows from the same argument as in the proof of Lemma 5.4. Therefore $\mathcal{T}_k(\mathcal{H}) \in Y_k$ and thus (5.25) and Lemma 5.8 imply that $\mathcal{T}_k$ is a contraction. $\square$

Recall that the Lipschitz property of h^{cu} was used in the proof of $h^{cu} \in C^1$ in Sect. 5.2. Similarly, before we proceed to prove $D^k h^{cu}$ is equal to the fixed point of $\mathcal{T}_k$ and thus $h^{cu} \in C^k$, we first take a step back to prove $\text{Lip } D^{k-1} h^{cu} < \infty$ using the above lemma.

Lemma 5.10. *Let $k \geq 2$ and assume Proposition 5.7 holds for each l , $0 \leq l \leq k-1$. There exists $C > 0$ such that if $\eta \in (C\delta, 1)$ and Q, μ, δ satisfy (4.9), (4.16), (4.18), (4.20), (5.18), and (5.25), then*

$$\text{Lip } D^{k-1} h^{cu} + \sup_{t \leq 0} e^{k\eta t} \text{Lip } D^{k-1} \Psi(t, \cdot) < \infty.$$

Proof. From the induction assumption, $D^{k-1} h^{cu} \in Y_{k-1}$ and thus $\mathcal{T}_{k-1}(D^{k-1} h^{cu}) = D^{k-1} h^{cu}$. To prove the lemma, we shall show that, for some C_{k-1} which might depend on δ, Q, μ , the closed subset

$$\tilde{Y}_k = \{\mathcal{H} \in Y_{k-1} : \text{Lip } \mathcal{H} \leq C_{k-1}\}$$

of Y_{k-1} is invariant under $\mathcal{T}_{k-1}$, which implies $D^{k-1} h^{cu} \in \tilde{Y}_{k-1}$ and thus Lipschitz.

Since $\mathcal{L}_{k-1}$ and $\mathcal{G}_{k-1}$, appearing in (5.26) and (5.27), involve only $D^l \Psi$ and $D^l h^{cu}$, $D^{k-1} \hat{G}^-$, and $D^l \hat{G}^-$, $0 \leq l \leq k-2$, the induction assumptions imply, for $t \leq 0$,

$$\|D_W \mathcal{L}_{k-1}(t)\|_{Y_k} + \|D_W \mathcal{G}_{k-1}(t)\|_{C^0 L_Q^k} \leq \tilde{C} e^{-k\eta t}. \quad (5.32)$$

Due to the slightly different forms, one has to proceed separately in the cases of $k = 2$ and $k > 2$, even though the estimates in proving these cases are essentially the same.

Case 1: $k-1 \geq 2$. For $\mathcal{H} \in \tilde{Y}_{k-1}$ and $W_j \in X^{cu}(\delta)$, $j = 0, 1$, let $\Psi_{k-1}^j(t) = D^{k-1} \Psi(t, W_j)$, which are also the solutions to (5.27) where Ψ is evaluated at (t, W_j) . From Lemma 5.3, (5.31), (5.32), the induction assumptions, and the Lipschitz bound on $\mathcal{H}$, it is straightforward to obtain the desired Lipschitz estimate on $D^{k-1} \Psi(t, \cdot)$

$$\|\Psi_{k-1}^1 - \Psi_{k-1}^0\|_{L_Q^{k-1}} \leq (\tilde{C} + C\delta\eta^{-1-(k+1)d_1} C_{k-1}) e^{-k\eta t} \|W_1 - W_0\|_{X_1, Q}.$$

Therefore, using (4.14), (5.32), (4.16), and the induction assumptions, we can estimate (5.26) as

$$\begin{aligned} & \|\mathcal{T}_{k-1}(\mathcal{H})(W_0) - \mathcal{T}_{k-1}(\mathcal{H})(W_1)\|_{L_Q^{k-1}} \\ & \leq \int_{-\infty}^0 e^{(\lambda-k\eta)t} (C\delta\eta^{-kd_1}(1+\eta^{-d_1-1}\delta)C_{k-1} + \tilde{C})dt \|W_1 - W_0\|_{X_1, Q} \\ & \leq (\lambda - k\eta)^{-1} (C\delta\eta^{-kd_1}C_{k-1} + \tilde{C}) \|W_1 - W_0\|_{X_1, Q}. \end{aligned}$$

From (5.25), there exists $C_{k-1} > 0$ such that $\mathcal{T}_{k-1}(\mathcal{H}) \in \tilde{Y}_k$ for any $\mathcal{H} \in \tilde{Y}_k$.

Case 2: $k-1 = 1$. In this case, one considers (5.8) and (5.9) instead. The estimates are similar and we omit the details. $\square$

Assume (5.25) and let $\mathcal{H}_k \in Y_k$ be the fixed point of $\mathcal{T}_k$. We will prove

Lemma 5.11. *There exists $C > 0$ such that if $\eta \in (C\delta, 1)$ and Q, μ, δ satisfy (4.9), (4.16), (4.18), (4.20), (5.18), and (5.25), then $D^k h^{cu} = \mathcal{H}_k$.*

Proof. As in the proof of Lemma 5.5, for any fixed $W_0 \in X^{cu}(\delta)$ and $\tilde{W} \in X^{cu} \setminus \{0\}$, let $\Psi_k(t) \in L(\otimes_{sym}^k X^{cu}, X^{cu})$ be defined as in (5.27) associated to $\mathcal{H}_k$ and W_0 and

$$\begin{aligned} R_\Psi(t) &= D^{k-1}\Psi(t, W_0 + \tilde{W}) - D^{k-1}\Psi(t, W_0) - \Psi_k(t)(\tilde{W}, \dots) \\ R_h(t) &= \left(D^{k-1}h^{cu}(\Psi(t, W_0 + \tilde{W})) - D^{k-1}h^{cu}(\Psi(t, W_0)) \right) (D\Psi(\cdot), \dots, D\Psi(\cdot)) \\ &\quad - \mathcal{H}_k(\Psi(t, W_0))(D\Psi(\tilde{W}), D\Psi(\cdot), \dots, D\Psi(\cdot)), \end{aligned}$$

where all above $D\Psi$ are evaluated at (t, W_0) . Note in the above $\Psi_k(t)(\tilde{W}, \dots) \in L(\otimes_{sym}^{k-1} X^{cu}, X^{cu})$ and

$$\mathcal{H}_k(\Psi(t, W_0))(D\Psi(\tilde{W}), D\Psi(\cdot), \dots, D\Psi(\cdot)) \in L(\otimes_{sym}^{k-1} X^{cu}, \mathbb{R}^d),$$

consistent with the other terms. According to (5.2) and Lemma 5.10, R_Ψ and R_h satisfy the rough estimates

$$\|R_\Psi(t)\|_{L_Q^{k-1}} + |R_h(t)|_{L_Q^{k-1}} \leq \tilde{C}e^{-k\eta t} \|\tilde{W}\|_{X_1, Q}, \quad (5.33)$$

for $t \leq 0$. Our goal is to show $\|R_{\Psi, h}(0)\|_{X_1, Q} / \|\tilde{W}\|_{X_1, Q} \rightarrow 0$ as $\|\tilde{W}\|_{X_1, Q} \rightarrow 0$.

Using $\mathcal{T}_{k-1}(D^{k-1}h^{cu}) = D^{k-1}h^{cu}$, Lemma 5.10, and the induction assumptions, much as the derivation of (5.22) and (5.23), we obtain

$$\begin{aligned} R_h(t) &= \int_{-\infty}^0 e^{-\tau M^-} ((D_W \hat{G}^- + D_{a^-} \hat{G}^- D h^{cu}) R_\Psi \\ &\quad + D_{a^-} \hat{G}^- R_h + R_1)|_{t+\tau} d\tau \\ \partial_t R_\Psi &= \left(A^{cu}(y(t), \hat{G}^T) + \mathcal{G}_1(\Psi) + \tilde{\mathcal{G}}_1(\Psi) D h^{cu}(\Psi) \right) R_\Psi + \tilde{\mathcal{G}}_1(\Psi) R_h + R_2(t), \end{aligned} \quad (5.34)$$

where $D\hat{G}^-$ and $D\hat{G}^{cu}$ are evaluated at (Ψ, h^{cu}) , h^{cu} at Ψ , and Ψ at (t, W_0) , followed by the shift in the integral of R_h . Here the norms $r_j(t) = \|R_j(t)\|_{L_Q^{k-1}}$, $j = 1, 2$, of the remainder terms $R_1(t)$ and $R_2(t)$ satisfy

$$r_1(t) + r_2(t) \leq \tilde{C}e^{-k\eta t} \|\tilde{W}\|_{X_1, Q}, \quad \lim_{\|\tilde{W}\|_{X_1, Q} \rightarrow 0} \frac{\|r_1 + r_2\|_{C^0([t_1, t_2])}}{\|\tilde{W}\|_{X_1, Q}} = 0 \quad (5.35)$$

for any $t_1 \leq t_2 \leq 0$. Lemma 5.3 and (5.2) imply

$$\|R_\Psi(t)\|_{L_Q^{k-1}}^2 \leq C\delta^2\eta^{-2d_1-1} \int_t^0 e^{2\eta(\tau-t)} \|R_h(\tau)\|_{L_Q^{k-1}}^2 d\tau + r_3(t) \quad (5.36)$$

where

$$r_3(t) \leq \tilde{C}e^{-2k\eta t} \|\tilde{W}\|_{X_1,Q}^2, \quad \lim_{\|\tilde{W}\|_{X_1,Q} \rightarrow 0} \frac{\|r_3\|_{C^0([t_1,t_2])}}{\|\tilde{W}\|_{X_1,Q}^2} = 0. \quad (5.37)$$

Finally, let

$$\tilde{R}_h = \sup_{t \leq 0} e^{(k+1)\eta t} \frac{|R_h(t)|}{\|\tilde{W}\|_{X_1,Q}}, \quad \tilde{R}_\Psi = \sup_{t \leq 0} e^{(k+1)\eta t} \frac{\|R_\Psi(t)\|_{L_Q^{k-1}}}{\|\tilde{W}\|_{X_1,Q}}.$$

Inequality (5.33) implies $\tilde{R}_h, \tilde{R}_\Psi < \infty$. Inequalities (5.36) and (5.34) along with (5.35), (5.37), and Lemma 4.3 imply

$$\tilde{R}_\Psi \leq C\delta\eta^{-d_1-1} \tilde{R}_h + \sup_{t \leq 0} r_3(t)^{\frac{1}{2}} e^{(k+1)\eta t}$$

and

$$\tilde{R}_h \leq C(\lambda - (k+1)\eta)^{-1} (\delta(\tilde{R}_\Psi + \tilde{R}_h) + \sup_{\tau \leq 0} r_1(\tau) e^{(k+1)\eta \tau}).$$

Therefore

$$\tilde{R}_\Psi + \tilde{R}_h \leq C \left(\sup_{t \leq 0} r_3(t)^{\frac{1}{2}} e^{(k+1)\eta t} + \sup_{\tau \leq 0} r_1(\tau) e^{(k+1)\eta \tau} \right).$$

From (5.35) and (5.37), we obtain that $\tilde{R}_h, \tilde{R}_\Psi \rightarrow 0$ as $\|\tilde{W}\|_{X_1,Q} \rightarrow 0$. $\square$

In the last step of the above proof, we may define $\tilde{R}_h$ and $\tilde{R}_\Psi$ by using a weight $e^{a\eta t}$ with any $a > k$ and thus we do not have to assume $\lambda > (k+1)\eta$ additionally.

6. A Non-degeneracy Case

In this section, we consider a traveling wave $U_c = u_c + i v_c$ satisfying the following non-degeneracy conditions. Recall $L_{c,y}$ and L_y defined in (2.11), its Morse index $n^-(L_c)$ in (2.19), and the dimensions d_1, d_2, d in Lemma 2.3. Assume

(H1) $\ker L_c = \text{span}\{\partial_{x_j} U_c \mid j = 1, 2, 3\}$;

(H2) $d = \dim X^{u,s} = n^-(L_c)$.

Remark 6.1. Assumption (H1) is a linearized elliptic problem. Usually (H2) is not easy to verify directly. A special situation is when $n^-(L_c) = 1$, which is often the case when U_c is derived from the Mountain Pass or a constrained minimization process with one constraint. In this case, according to Theorem 2.3 and Proposition 2.2 in [37], (H2) is satisfied if $\langle L_c V, V \rangle > 0$ for all $V \in \ker(JL_c)^2 \setminus \ker(JL_c)$. More specifically, it was proved in [36] that, if $c_0 \in \mathbb{R}^3$ and $U_{ac_0}(x)$ is a family of traveling waves depending on a smoothly, then $\frac{d}{da} P(U_{ac_0}) < 0$ along with $n^-(L_c) = 1$ implies (H2).

Under these hypotheses, among the subspaces in the decomposition given in Lemma 2.3, statement (2) there implies $X_{c,y}^{d1} = X_{c,y}^{d2} = \{0\}$ and thus, in the same notations, we have the following decomposition.

Lemma 6.2. *Assume (H), (H1-2), and (2.6), then for any $y \in \mathbb{R}^3$, it holds that*

- (1) $X = X_{c,y}^T \oplus X_{c,y}^e \oplus X_{c,y}^+ \oplus X_{c,y}^-$,
 (2) $JL_{c,y}$ and $L_{c,y}$ take the forms

$$L_{c,y} \longleftrightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & L^e(y) & 0 & 0 \\ 0 & 0 & 0 & L_{+-}(y) \\ 0 & 0 & L_{+-}(y)^* & 0 \end{bmatrix},$$

$$JL_{c,y} \longleftrightarrow \begin{bmatrix} 0 & A_{Te}(y) & 0 & 0 \\ 0 & A_e(y) & 0 & 0 \\ 0 & 0 & A_+(y) & 0 \\ 0 & 0 & 0 & A_-(y) \end{bmatrix}.$$

Here the above blocks satisfy the same properties as in Lemma 2.3.

In this non-degenerate case, we shall carefully consider the energy-momentum functional $E + c \cdot \tilde{P}$ invariant under (GP) and (2.7), where E and $\tilde{P}$ are defined in (1.1) and (2.5). Let $\tilde{E}_c(y, a^+, a^-, V^e)$ be defined as

$$\tilde{E}_c = (E + c \cdot \tilde{P}) \circ \Phi \in C^\infty(\mathbb{R}^{2d} \oplus \mathcal{X}^e, \mathbb{R}),$$

where the coordinate mapping Φ is defined in (2.36), whose domain can also be extended to $\mathbb{R}^{3+2d} \times X_1$. The smoothness of $\tilde{E}_c$ follows from Lemmas 2.2 and 2.3 of [36] along with Lemma 2.3. Using (2.10) and (2.11) it is straightforward to obtain the leading order expansion of $\tilde{E}$ at $(y, 0, 0, 0) = \Phi^{-1}(U_c(\cdot + y))$

$$\begin{aligned} \tilde{E}(y, a^+, a^-, V^e) &= \tilde{E}_c(y, a^+, a^-, V^e) - \tilde{E}_c(y, 0, 0, 0) \\ &= \langle L_e(y)V^e, V^e \rangle + 2\langle \tilde{L}_{+-}a^-, a^+ \rangle + O(|a^+| + |a^-| + \|V^e\|_{X_1})^3 \end{aligned} \quad (6.1)$$

when $|a^+|$, $|a^-|$, and $\|V^e\|_{X_1}$ are small. Here $L_e(y)$ is given in Lemma 2.3, uniformly positive, and translation invariant, i.e.

$$\langle L_e(0)V^e, V^e \rangle = \langle L_e(y)V^e(\cdot + y), V^e(\cdot + y) \rangle.$$

The $d \times d$ matrix $\tilde{L}_{+-}$ is defined by

$$\langle \tilde{L}_{+-}a^-, a^+ \rangle = \langle L_{+-}(y)a^- \xi_c^-(\cdot + y), a^+ \xi_c^-(\cdot + y) \rangle,$$

where $\xi_c^- = (\xi_{c,1}^-, \dots, \xi_{c,d}^-)$ and $L_{+-}(y)$ are given in Lemma 2.3. Operators $\tilde{L}_{+-}$ are independent of y since $L_{c,y}$ and thus $L_{+-}(y)$ are translation invariant.

Let $\mathcal{W}^{cu,cs,c}$, $h^{cu,cs}$, $h^c = (h_+^c, h_-^c)$ be given in Sect. 4, whose smoothness are established in Sect. 5, and the parameters Q, μ, δ, η satisfy (4.9), (4.16), (4.18), (4.20), and (5.25). For any $(y, a^+ = h^{cs}(y, a^-, V^e), a^-, V^e) \in \mathcal{W}^{cs}$, since $Dh^{cs}(y, 0, 0, 0) = 0$, we have

$$|\tilde{E}(y, h^{cs}(y, a^-, V^e), a^-, V^e) - \langle L_e(y)V^e, V^e \rangle| \leq C_0(|a^-| + \|V^e\|_{X_1})^3 \quad (6.2)$$

for some $C_0 > 0$. Based on the expansion (6.1), we can prove the exponential stability of $\mathcal{W}^c$ inside $\mathcal{W}^{cs}$.

Lemma 6.3. *There exists $C > 1$ such that if $\eta \in (C\delta, 1)$ and Q, μ, δ satisfy (4.9), (4.16), (4.18), (4.20), and (5.25), then for any initial value*

$$\bar{W} = (\bar{y}, \bar{a}^+ = h^{cs}(\bar{y}, \bar{a}^-, \bar{V}^e), \bar{a}^-, \bar{V}^e) \in \mathcal{W}^{cs}$$

with $|\bar{a}^-| + \|\bar{V}^e\|_{X_1} < C^{-2}\delta$, its corresponding solution $W(t) = (y, a^+, a^-, V^e)(t)$ satisfies, for all $t \geq 0$,

$$\begin{aligned} |a^-(t)| + \|V^e(t)\|_{X_1} &< \delta/15, \quad a^+(t) = h^{cs}((y, a^-, V^e)(t)), \\ |a^-(t) - h_-^c((y, V^e)(t))| &\leq C e^{-(\lambda-2\eta)t} |a^-(0) - h_-^c((y, V^e)(0))|. \end{aligned} \quad (6.3)$$

Proof. The assumptions on $\bar{W}$, the conservation of $\tilde{E}$, and (6.2) imply

$$|\tilde{E}(W(t))| = |\tilde{E}(\bar{W})| \leq C^{-2}\delta^2.$$

Let

$$T = \sup\{t > 0 : |a^-(t')| + \|V^e(t')\|_{X_1} < \delta/15, \forall t' \in [0, t]\} > 0.$$

On $[0, T]$, Proposition 4.16 implies (6.3) holds, which along with $Dh^c(y, 0) = 0$ implies

$$|a^-(t)| \leq C \|V^e(t)\|_{X_1}^2 + C^{-1} e^{-(\lambda-2\eta)t} \delta, \quad t \in [0, T].$$

Applying (6.2) again, we obtain

$$\|V^e(t)\|_{X_1}^2 \leq C(\tilde{E}(W(t)) + |a^-(t)|^3).$$

The above inequalities imply

$$|a^-(t)| + \|V^e(t)\|_{X_1}^2 \leq C^{-1}\delta^2 < \delta/15, \quad t \in [0, T].$$

Therefore $T = \infty$. From Propositions 4.16 and 4.19 and Lemma 4.20, the rest of the lemma follows. $\square$

Following exactly the same arguments, we also obtain the exponential stability of $\mathcal{W}^c$ backward in time inside $\mathcal{W}^{cu}$.

Lemma 6.4. *There exists $C > 1$ such that if $\eta \in (C\delta, 1)$ and Q, μ, δ satisfy (4.9), (4.16), (4.18), (4.20), and (5.25), then for any initial value*

$$\bar{W} = (\bar{y}, \bar{a}^+, \bar{a}^- = h^{cu}(\bar{y}, \bar{a}^+, \bar{V}^e), \bar{V}^e) \in \mathcal{W}^{cu}$$

with $|\bar{a}^+| + \|\bar{V}^e\|_{X_1} < C^{-2}\delta$, its corresponding solution $W(t) = (y, a^+, a^-, V^e)(t)$ satisfies, for all $t \leq 0$,

$$\begin{aligned} |a^+(t)| + \|V^e(t)\|_{X_1} &< \delta/15, \quad a^-(t) = h^{cu}((y, a^+, V^e)(t)), \\ |a^+(t) - h_+^c((y, V^e)(t))| &\leq C e^{(\lambda-2\eta)t} |a^+(0) - h_+^c((y, V^e)(0))|. \end{aligned} \quad (6.4)$$

Consequently, we also obtain the stability of $\mathcal{M}$ in $\mathcal{W}^c$.

Proposition 6.5. *There exist $C > 1$ and $\delta > 0$ such that, for any initial value $\bar{W} = (\bar{y}, (\bar{a}^+, \bar{a}^-) = h^c(\bar{y}, \bar{V}^e), \bar{V}^e) \in \mathcal{W}^c$ with $\|\bar{V}^e\|_{X_1} < C^{-2}\delta$, its corresponding solution $W(t) = (y, a^+, a^-, V^e)(t)$ satisfies, for all $t \in \mathbb{R}$,*

$$\|V^e(t)\|_{X_1} < \delta/15, \quad (a^+, a^-)(t) = h^c((y, V^e)(t)).$$

Combine the above results and Corollary 4.13, Propositions 4.16 and 4.19, we obtain the following characterization of $\mathcal{W}^{cu}$, $\mathcal{W}^{cs}$, and $\mathcal{W}^c$.

Proposition 6.6. *There exist $C > 1$ and $\delta > 0$ such that the following hold. Let $U(t) = \Phi(W(t))$, where $W(t) = (y, a^+, a^-, V^e)(t)$, is a solutions to (2.7) with initial value*

$$\bar{W} = (\bar{y}, \bar{a}^+, \bar{a}^-, \bar{V}^e) \in B^{2d}(C^{-2}\delta) \oplus \mathcal{X}^e(C^{-2}\delta),$$

then

- (1) $\bar{W} \in \mathcal{W}^{cu}$ and thus $W(t) \in \mathcal{W}^{cu}$ for all $t \leq 0$, if and only if $W(t) \in B^{2d}(\delta/15) \oplus \mathcal{X}^e(\delta/15)$ for all $t \leq 0$.
- (2) $\bar{W} \in \mathcal{W}^{cs}$ and thus $W(t) \in \mathcal{W}^{cu}$ for all $t \geq 0$, if and only if $W(t) \in B^{2d}(\delta/15) \oplus \mathcal{X}^e(\delta/15)$ for all $t \geq 0$.
- (3) $\bar{W} \in \mathcal{W}^c$ and thus $W(t) \in \mathcal{W}^c$ for all $t \in \mathbb{R}$, if and only if $W(t) \in B^{2d}(\delta/15) \oplus \mathcal{X}^e(\delta/15)$ for all $t \in \mathbb{R}$.

Remark 6.7. Note that when we construct the local invariant manifolds, we cut off the nonlinearities to focus on the local dynamics. Different choices of the cut-off could yield different local invariant manifolds. Therefore local center-stable, center-unstable, and center manifolds are usually not unique. However, under the non-degeneracy conditions (H1-2), we obtain the above characterization of the local invariant manifolds which are independent of the cut-off. Therefore the local manifolds are unique in this case.

Appendix A

In the Appendix, we give some estimates of the nonlinear term G in (2.17). One may compute in (2.16)

$$\begin{aligned} G_2(c, y, w) = & - \left(|U|^2 - |U_c(\cdot + y)|^2 - 2U_c(\cdot + y) \cdot (K_{c,y}w) \right) u_c(\cdot + y) \\ & - \left(|U|^2 - |U_c(\cdot + y)|^2 \right) (w_1 - \chi(D)(v_c(\cdot + y)w_2)) \\ & - \frac{1}{2} \left((1 - |U|^2)\chi(D)(w_2^2) + \Delta\chi(D)(w_2^2) \right), \end{aligned} \quad (\text{A.1})$$

where U is given in (2.15) as

$$U = \psi(w_c(\cdot + y) + w) = U_c(\cdot + y) + K_{y,c}w - \left(\frac{1}{2}\chi(D)(w_2^2), 0 \right)^T. \quad (\text{A.2})$$

Substituting this into G_1 we obtain

$$G_1 = G_1(c, y, \partial_t y, w) = G_{11}(c, y, w) + G_{12}(c, y, \partial_t y, w)$$

where

$$\begin{aligned} G_{11} = & \left(|U|^2 - |U_c(\cdot + y)|^2 - 2U_c(\cdot + y) \cdot (K_{c,y}w) \right) v_c(\cdot + y) \\ & + \left(|U|^2 - |U_c(\cdot + y)|^2 \right) w_2 - \frac{1}{2}\chi(D)(w_2 \nabla w_2 \cdot c) \end{aligned} \quad (\text{A.3})$$

and by substituting (2.16) into G_1

$$G_{12} = G_{12}(c, y, \tilde{y}, w) = \chi(D) \left(w_2 \left(- (L_{c,y} K_{c,y} w)_1 - \tilde{y} \cdot \nabla v_c(\cdot + y) \right) + G_2(c, y, w) \right). \quad (\text{A.4})$$

Here $(L_{c,y} K_{c,y} w)_1$ denotes the first component of $L_{c,y} K_{c,y} w$. For fixed c and y , G is a polynomial of w and $\tilde{y}$. More precisely, it is the sum of some multi-linear transformations on w and $\tilde{y}$ of degrees between 2 and 6.

Lemma A.1. *Fix c . It holds that*

$$G(c, \cdot, \cdot, \cdot) \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3 \times X_1, W^{1, \frac{3}{2}}) + C^\infty(\mathbb{R}^3 \times X_1, L^{\frac{3}{2}} \cap \dot{W}^{1, \frac{6}{5}}),$$

and

$$G(c, y, 0) = 0, \quad D_w G(c, y, 0) = 0.$$

In particular, the only term G_{12} containing $\tilde{y}$ belongs to $C^\infty(\mathbb{R}^3 \times \mathbb{R}^3 \times X_1, W^{1, \frac{3}{2}})$. More refined estimates on G can be found in (A.10), (A.11), (A.12), (A.13), (A.14), and (A.15), where the generic constants C in those inequalities are independent of y .

Proof. Due to the polynomial form of G in $w \in X_1$ and $\tilde{y} \in \mathbb{R}^3$, we only need to estimate the boundedness of each monomial, i.e. multi-linear transformation. To handle the terms with $\chi(D)$, we will repeatedly use

$$\|\nabla^s \chi(D) f\|_{L^p \cap L^\infty} \leq C_{s,p} \|f\|_{L^p}, \quad \forall k \geq 0, \quad 1 \leq p \leq \infty. \quad (\text{A.5})$$

We start with the consideration on $|U|^2 - |U_c(\cdot + y)|^2$. Let

$$\rho = \frac{1}{2} \chi(D) \left(w_2^2 + 2w_2 w_{2c}(\cdot + y) \right) \implies D_y^k \rho = \chi(D) \left(w_2 D^k w_{2c}(\cdot + y) \right)$$

for any $k \geq 1$ and

$$\nabla \rho = \chi(D) (w_2 \nabla w_2 + \nabla w_2 w_{2c}(\cdot + y) + w_2 \nabla w_{2c}(\cdot + y)).$$

Using $w_{2c} = v_c \in \dot{H}^1$, (A.5) implies, for any $s + k \geq 1$

$$\|\nabla^s D_y^k \rho\|_{L^{\frac{3}{2}} \cap L^\infty} \leq C_{s,k} \|w_2\|_{\dot{H}^1} (\|w_2\|_{\dot{H}^1} + 1) \quad (\text{A.6})$$

where $C_{s,k}$ is independent of y . Here we used the property $D^k w_{2c} = D^k v_c \in L^2 \cap L^\infty$ for all $k \geq 1$ due to equation (1.4) and also the embedding $\dot{W}^{1, \frac{3}{2}}(\mathbb{R}^3) \rightarrow L^3(\mathbb{R}^3)$ on ρ . The second term in ρ actually has a better spatial decay estimate by using Hardy's inequality and (2.6). Namely,

$$\|w_2 w_{2c}(\cdot + y)\|_{L^2} = \|w_2(\cdot - y) v_c\|_{L^2} \leq C \left\| \frac{w_2(\cdot - y)}{|x|} \right\|_{L^2} \leq C \|w_2\|_{\dot{H}^1}.$$

On some occasions, we also need to consider $(I - \chi(D))(f_1 f_2)$, for $f_{1,2} \in \dot{H}^1$. Since $\nabla(f_1 f_2) \in L^{\frac{3}{2}}$, we have

$$\begin{aligned} \|(I - \chi(D))(f_1 f_2)\|_{L^{\frac{3}{2}}} &= \left\| \frac{I - \chi(D)}{|D|} |D|(f_1 f_2) \right\|_{L^{\frac{3}{2}}} \\ &\leq C \|\nabla(f_1 f_2)\|_{L^{\frac{3}{2}}} \leq C \|f_1\|_{\dot{H}^1} \|f_2\|_{\dot{H}^1} \end{aligned}$$

where we used that the inverse Fourier transform of $\frac{1 - \chi(\xi)}{|\xi|}$ is in L^1 . It implies

$$\|(I - \chi(D))(f_1 f_2)\|_{W^{1, \frac{3}{2}}} \leq C \|f_1\|_{\dot{H}^1} \|f_2\|_{\dot{H}^1}. \quad (\text{A.7})$$

One may compute that

$$\begin{aligned} |U|^2 - |U_c|^2 &= w_1^2 + 2u_c(\cdot + y)(w_1 - \rho) + \rho^2 - 2w_1\rho + 2v_c(\cdot + y)w_2 + w_2^2 \\ &= w_1^2 + 2w_1 + 2(u_c(\cdot + y) - 1)(w_1 - \rho) + \rho^2 - 2w_1\rho \\ &\quad + 2(I - \chi(D))(v_c(\cdot + y)w_2 + w_2^2). \end{aligned}$$

By using the above inequalities and (2.6), we obtain, through straightforward calculations, for $k \geq 1$,

$$\begin{aligned} &\| |U|^2 - |U_c|^2 - 2w_1 \|_{W^{1, \frac{3}{2}}} + \| D_y^k (|U|^2 - |U_c|^2) \|_{W^{1, \frac{3}{2}} \cap \dot{H}^1} \\ &\leq C \|w\|_{X_1} (\|w\|_{X_1}^3 + 1). \end{aligned} \quad (\text{A.8})$$

Similarly,

$$\begin{aligned} &|U|^2 - |U_c|^2 - 2U_c(\cdot + y) \cdot (K_{c,y}w) \\ &= w_1^2 + (1 - \chi(D))(w_2^2) + (1 - u_c(\cdot + y))\chi(D)(w_2^2) + \rho^2 - 2w_1\rho \end{aligned}$$

and along with the above inequalities, it implies for $k \geq 1$

$$\begin{aligned} &\| D_y^k (|U|^2 - |U_c|^2 - 2U_c(\cdot + y) \cdot (K_{c,y}w)) \|_{W^{1, \frac{6}{5}} \cap \dot{H}^1} \\ &+ \| |U|^2 - |U_c|^2 - 2U_c(\cdot + y) \cdot (K_{c,y}w) \|_{W^{1, \frac{3}{2}}} \leq C \|w\|_{X_1}^2 (\|w\|_{X_1}^2 + 1). \end{aligned} \quad (\text{A.9})$$

Substituting (A.8) and (A.9) into (A.1) and using (A.5), we obtain through straightforward calculations

$$\|G_2\|_{L^{\frac{3}{2}} \cap L^2} + \|\nabla G_2\|_{L^{\frac{3}{2}} + L^{\frac{6}{5}}} \leq C \|w\|_{X_1}^2 (\|w\|_{X_1}^3 + 1). \quad (\text{A.10})$$

Here we have to estimate ∇G_2 in $L^{\frac{3}{2}} + L^{\frac{6}{5}}$ since

$$\|\nabla \Delta \chi(D)(w_2^2)\|_{L^{\frac{3}{2}}} \leq C \|w_2\|_{X_1}^2$$

does not seem to have better decay and

$$\| (|U|^2 - |U_c|^2) \nabla w_1 \|_{L^{\frac{6}{5}}} \leq C \|w\|_{X_1}^2 (\|w\|_{X_1}^3 + 1),$$

does not seem to have better regularity. Similarly, for any $k \geq 1$,

$$\| D_y^k G_2 \|_{W^{1, \frac{6}{5}} \cap \dot{W}^{1, \frac{3}{2}}} \leq C \|w\|_{X_1}^2 (\|w\|_{X_1}^3 + 1). \quad (\text{A.11})$$

The estimates for G_{11} are

$$\|G_{11}\|_{L^{\frac{3}{2}} \cap L^2} + \|\nabla G_{11}\|_{L^{\frac{3}{2}} \cap L^\infty + L^1 \cap L^{\frac{6}{5}}} \leq C \|w\|_{X_1}^2 (\|w\|_{X_1}^3 + 1). \quad (\text{A.12})$$

Again, we have to estimate ∇G_2 in this norm as

$$\|\nabla \chi(D)(w_2 \nabla w_2 \cdot c)\|_{L^{\frac{3}{2}} \cap L^\infty} \leq C \|w\|_{X_1}^2$$

does not seem to have better decay and

$$\|\nabla \left((|U|^2 - |U_c(\cdot + y)|^2) w_2 \right)\|_{L^1 \cap L^{\frac{6}{5}}} \leq C \|w\|_{X_1}^2 (\|w\|_{X_1}^3 + 1)$$

does not seem to have better regularity. Differentiating in y implies that

$$\|D_y^k G_{11}\|_{W^{1, \frac{6}{5}} \cap \dot{W}^{1, \frac{3}{2}}} \leq C \|w\|_{X_1}^2 (\|w\|_{X_1}^3 + 1), \quad k \geq 1. \quad (\text{A.13})$$

Next we consider G_{12} . Recall from (2.11), for any $f = (f_1, f_2) \in X_1$,

$$(L_{c,y} f)_1 = (2 - \Delta) f_1 + (3(u_c^2 - 1) + v_c^2)(\cdot + y) f_1 - c \cdot \nabla f_2 + 2(u_c v_c)(\cdot + y) f_2.$$

Using Hardy's inequality, and the fact $K_{c,y}$ being an isomorphism, we obtain

$$\|(L_{c,y} K_{c,y} w)_1 + \Delta w_1\|_{L^2} \leq C \|w\|_{X_1}.$$

From $w_2 \Delta w_1 = \nabla \cdot (w_2 \nabla w_1) - \nabla w_2 \cdot \nabla w_1$ and (A.5), we have, for any $s \geq 0$,

$$\|\nabla^s \chi(D)(w_2 \Delta w_1)\|_{L^{\frac{3}{2}} \cap L^\infty} \leq C \|w_2\|_{\dot{H}^1} \|w_1\|_{\dot{H}^1}.$$

Therefore, (A.4), (A.10), and the above inequalities yield

$$\|\nabla^s G_{12}\|_{L^{\frac{3}{2}} \cap L^\infty} \leq C \|w_2\|_{\dot{H}^1} (|\tilde{y}| + \|w_1\|_{\dot{H}^1} + \|w\|_{X_1}^2 (\|w\|_{X_1}^3 + 1)). \quad (\text{A.14})$$

Differentiating in y , we have, for $k \geq 1$,

$$\begin{aligned} D_y^k (L_{c,y} K_{c,y} w)_1 &= 2D^k (u_c v_c)(\cdot + y) w_2 \\ &+ D^k (3(u_c^2 - 1) + v_c^2)(\cdot + y) w_1 - (2 - \Delta) \chi(D) ((D^k v_c)(\cdot + y) w_2) \\ &- \sum_{k_1+k_2=k} D^{k_1} (3(u_c^2 - 1) + v_c^2)(\cdot + y) \chi(D) ((D^{k_2} v_c)(\cdot + y) w_2). \end{aligned}$$

By using (A.5) and (A.11), we obtain, for any $k \geq 1$,

$$\|\nabla^s D_y^k G_{12}\|_{L^{\frac{3}{2}} \cap L^\infty} \leq C \|w_2\|_{\dot{H}^1} (|\tilde{y}| + \|w_1\|_{\dot{H}^1} + \|w\|_{X_1}^2 (\|w\|_{X_1}^3 + 1)). \quad (\text{A.15})$$

Finally, we note that G_{11} , G_{12} , and G_2 are polynomials of w and $\tilde{y}$ consisting of monomials of degrees between 2 and 6 with coefficients depending on $U_c(\cdot + y)$. Therefore, one may regroup those monomials so that some of them belong to $W^{1, \frac{3}{2}}$ while others to $\dot{W}^{1, \frac{6}{5}}$. Moreover it is easy to obtain the estimates on $D_w^l D_y^k G$ and the proof is complete. $\square$

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