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Generalized surface tension bounds in vacuum decay

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Abstract

Coleman and De Luccia (CDL) showed that gravitational effects can prevent the decay by bubble nucleation of a Minkowski or AdS false vacuum. In their thin-wall approximation this happens whenever the surface tension in the bubble wall exceeds an upper bound proportional to the difference of the square roots of the true and false vacuum energy densities. Recently it was shown that there is another type of thin-wall regime that differs from that of CDL in that the radius of curvature grows substantially as one moves through the wall. Not only does the CDL derivation of the bound fail in this case, but also its very formulation becomes ambiguous because the surface tension is not well-defined. We propose a definition of the surface tension and show that it obeys a bound similar in form to that of the CDL case. We then show that both thin-wall bounds are special cases of a more general bound that is satisfied for all bounce solutions with Minkowski or AdS false vacua. We discuss the limit where the parameters of the theory attain critical values and the bound is saturated. The bounce solution then disappears and a static planar domain wall solution appears in its stead. The scalar field potential then is of the form expected in supergravity, but this is only guaranteed along the trajectory in field space traced out by the bounce.

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I. INTRODUCTION

In their classic study of vacuum decay via bubble nucleation, Coleman and De Luccia (CDL) [1] discovered a surprising feature of decays from a Minkowski vacuum to an anti-de-Sitter (AdS) vacuum or from one AdS vacuum to another. If a potential has two vacua of differing energy, decay from the higher energy false vacuum to the lower energy true vacuum is always possible, if gravitational effects are ignored. However, if the higher vacuum has either zero or negative energy, such decays are quenched if the two vacua are sufficiently close in energy. In the thin-wall approximation of CDL, bubble nucleation is only possible if

$$\sigma < \frac{2}{\sqrt{3\kappa}} \left(\sqrt{|U_{\rm tv}|} - \sqrt{|U_{\rm fv}|} \right) . \tag{1.1}$$

Here σ is the surface tension in the bubble wall, $U_{\rm fv}$ and $U_{\rm tv}$ are the energy densities of the false and true vacua, and $\kappa = 8\pi G_N$.

As in the non-gravitational case, the CDL thin-wall approximation requires that the two vacua be close in energy. In addition, one must require that the radius of curvature of the wall can be treated as being constant as one moves through the bubble wall. Recently it was shown [2] that there are regions of parameter space that allow a new type of thin-wall regime in which the latter requirement is violated. In this case not only does the CDL derivation of Eq. (1.1) fail, but also its very formulation becomes ambiguous, because the surface tension is not well-defined. In this paper we will show how this inequality can be generalized to this new thin-wall regime. Furthermore, we show how these bounds for the thin-wall cases can be seen as special cases of a more general bound, applicable even to bounce solutions that are in no sense thin-wall.

We also discuss the case where the parameters of the theory are taken to the boundary beyond which nucleation is quenched. As the boundary is approached, the bubble radius at nucleation increases without bound. When the critical values of the parameters are actually achieved, the bounce solution is absent. In its stead there is a static planar domain wall [3, 4]. Such walls have been constructed as BPS solutions in supergravity theories [5–9], but they can also arise as solutions that only possess what has been termed “fake supersymmetry” [10–12]. We will describe how this happens in our approach. We will also recall the related work of Abbott and Park [13, 14] connecting the existence of bounces to the vacuum stability results of Boucher [15]

The remainder of this manuscript is organized as follows. In Sec. II we review the CDL formalism, including their thin-wall approximation. In Sec. III the new thin-wall regime is described and the generalization of Eq. (1.1) to this new regime is derived. In Sec. IV we derive the more general bound that applies to all bounces. In Sec. V we discuss the approach to the critical quenching limit where the Euclidean bounce disappears and a static planar domain wall appears, and make connections to supersymmetry. Section VI summarizes our results and comments on the extension to theories with multiple scalar fields. There is an appendix that addresses some special issues that arise when the false vacuum is Minkowskian.

II. THE CDL FORMALISM

We consider a theory with a real scalar field ϕ governed by a potential $U(\phi)$ that has two metastable vacua at $\phi_{\rm tv}$ and $\phi_{\rm fv}$. The values of the potential at these vacua satisfy

$U_{\rm tv} < U_{\rm fv} \le 0$. Thus, the higher false vacuum can be either Minkowski or AdS, while the true vacuum is AdS.[1] The AdS vacua have characteristic lengths given by

$$\ell_{\rm fv} = \left(\frac{\kappa}{3}|U_{\rm fv}|\right)^{-1/2} = \left(-\frac{\kappa}{3}U_{\rm fv}\right)^{-1/2} \tag{2.1}$$

and similarly for $\ell_{\rm tv}$. Following CDL, we seek bounce solutions of the Euclidean field equations. Making the standard assumption of O(4) symmetry, we can write the Euclidean metric in the form

$$ds^2 = d\xi^2 + \rho(\xi)^2\, d\Omega_3^2\,. \tag{2.2}$$

For the cases we are considering, decays from a Minkowski or AdS vacuum, $\rho(\xi)$ has a single zero and ξ runs from 0 to ∞. The bounce thus has R^4 topology, in contrast with the de Sitter bounces that are topologically four-spheres.

The Euclidean action can then be written in the form[2]

$$S = 2\pi^2 \int_0^\infty d\xi \left\{ \rho^3 \left[\frac{1}{2}\phi'^2 + U(\phi)\right] - \frac{3}{\kappa}\left(\rho\rho'^2 + \rho\right)\right\} \tag{2.3}$$

and a bounce must satisfy

$$\phi'' + \frac{3\rho'}{\rho}\,\phi' = \frac{dU}{d\phi}\,, \tag{2.4}$$

$$\rho'^2 = 1 + \frac{\kappa}{3}\rho^2\left[\frac{1}{2}\phi'^2 - U(\phi)\right]\,, \tag{2.5}$$

subject to the boundary conditions

$$\phi'(0) = 0\,, \qquad \phi(\infty) = \phi_{\rm fv}\,, \qquad \rho(0) = 0\,, \tag{2.6}$$

where primes denote derivatives with respect to ξ. Equations (2.4) and (2.5) imply the useful equation

$$\rho'' = -\frac{\kappa}{3}\,\rho\left[\phi'^2 + U(\phi)\right]\,. \tag{2.7}$$

We now note that $\rho(\xi)$ is a monotonically increasing function. To establish this, note first that the boundary conditions imply that $\rho'(0) = 1$. Requiring that the bounce approach the pure false vacuum solution at large ξ implies that ρ must asymptotically increase with ξ either linearly (for a Minkowski false vacuum) or exponentially (for an AdS false vacuum). If ρ were not monotonic between these limits, it would have a local minimum at some finite $\bar\xi$. This would require that $\rho'(\bar\xi) = 0$ and $\rho''(\bar\xi) > 0$. However, this cannot be, since Eq. (2.5) shows that $U(\phi)$ must be positive at any zero of ρ', while Eq. (2.7) implies that ρ'' can only be positive if $U(\phi)$ is negative.

[1] Note that an AdS vacuum can correspond to a local maximum of $U(\phi)$, provided that the Breitenlohner-Freedman bound is respected.

[2] The Gibbons-Hawking boundary term [16] does not appear here because it is exactly canceled by the surface term from the integration by parts that removes the ρ'' that appears in the curvature scalar R. In fact, the tunneling rate is unaffected by the inclusion or omission of the boundary term, because its contributions to the bounce action and the false vacuum action are equal, and so cancel in the tunneling exponent B [2].

We will find it useful to rewrite some of these results in terms of a Euclidean pseudo-energy

$$E = \frac{1}{2}\phi'^2 - U(\phi)\,. \tag{2.8}$$

Because of the ϕ' "friction" term in Eq. (2.4), this is not conserved, but instead obeys

$$\begin{aligned} E' &= -\frac{3\rho'}{\rho}\,\phi'^2 \\ &= -\phi'^2\sqrt{\frac{9}{\rho^2} + 3\kappa E} \end{aligned} \tag{2.9}$$

with the second line following with aid of Eq. (2.5). We just showed that ρ is a monotonically increasing function of ξ. It then follows that E is monotonically decreasing from an initial maximum $E(0) < |U_{\rm tv}|$ to an asymptotic minimum $E(\infty) = |U_{\rm fv}|$.

The tunneling exponent B is obtained by subtracting the action of the homogeneous false vacuum solution from that of the bounce. For configurations that satisfy Eq. (2.5) the action can be rewritten as

$$\begin{aligned} S &= 4\pi^2 \int_0^\infty d\xi\, \left[\rho^3 U(\phi) - \frac{3}{\kappa}\rho\right] \\ &= 4\pi^2 \int_0^\infty d\rho\, \frac{1}{\rho'}\left[\rho^3 U(\phi) - \frac{3}{\kappa}\rho\right]\,. \end{aligned} \tag{2.10}$$

The actions of the bounce and the false vacuum are both divergent, so we must regulate the integrals. Thus, we define

$$S(L) = 4\pi^2 \int_0^L d\rho\, \frac{1}{\rho'}\left[\rho^3 U(\phi) - \frac{3}{\kappa}\,\rho\right] \tag{2.11}$$

and obtain a finite value for

$$B = \lim_{L\to\infty}\left[S_{\rm bounce}(L) - S_{\rm fv}(L)\right]\,. \tag{2.12}$$

In particular, for an AdS false vacuum, with $\phi = \phi_{\rm fv}$ everywhere,

$$\rho_{\rm fv} = \ell_{\rm fv}\sinh(\xi/\ell_{\rm fv})\,. \tag{2.13}$$

Integrating the action density gives

$$S_{\rm fv}(L) = A_{\rm fv}(0, L) \tag{2.14}$$

where

$$\begin{aligned} A_{\rm fv}(\rho_1,\rho_2) &= 4\pi^2 \int_{\rho_1}^{\rho_2} d\rho\, \frac{1}{\rho'}\left[\rho^3 U_{\rm fv} - \frac{3}{\kappa}\,\rho\right] \\ &= -\frac{4\pi^2}{\kappa}\ell_{\rm fv}^2\left[\left(1 + \frac{\rho_2^2}{\ell_{\rm fv}^2}\right)^{3/2} - \left(1 + \frac{\rho_1^2}{\ell_{\rm fv}^2}\right)^{3/2}\right]\,. \end{aligned} \tag{2.15}$$

A. The CDL thin-wall approximation

In the CDL thin-wall approximation the bounce solution is divided into three parts: an exterior region of pure false vacuum, an interior region of pure true vacuum, and a thin wall that separates the two. For such a configuration we can write

$$B(\rho) = B_{\text{exterior}}(\rho) + B_{\text{interior}}(\rho) + B_{\text{wall}}(\rho) \tag{2.16}$$

with ρ being the curvature radius of the wall. In the false vacuum exterior region the actions of the bounce and the false vacuum cancel completely, and so $B_{\text{exterior}} = 0$. The contribution in the interior region is the difference of true- and false-vacuum terms,

$$B_{\text{interior}} = A_{\text{tv}}(0,\rho) - A_{\text{fv}}(0,\rho)\,. \tag{2.17}$$

Finally, we have the contribution from the wall, which can be written in the form

$$B_{\text{wall}} = 2\pi^2\rho^3\sigma \tag{2.18}$$

where the surface tension σ is given by the flat-spacetime expression[3]

$$\sigma = 2\int_{\text{wall}} d\xi\,[U(\phi_{\text{bounce}}) - U_{\text{fv}}] \tag{2.19}$$

and the integration over ξ is restricted to the wall region[4]

It is crucial here that the field profile in the wall, and hence σ, are to a good approximation independent of ρ, a consequence of the fact that the bounce radius is much greater than the thickness of the wall. This approximation becomes better as the difference between the true and false vacuum energies decreases, not because the wall gets thinner (it doesn't), but because the bounce gets bigger. Indeed, one might term this the "large-bounce approximation".

Note that Eq. (2.18) implicitly assumes that ρ is essentially constant as one moves through the wall. The CDL thin-wall analysis is only valid if this is the case. In Sec. III we will consider thin-wall configurations for which this assumption fails.

The bounce is obtained by requiring that the wall radius be a stationary point $\bar\rho$ of $B(\rho)$. Setting $dB/d\rho = 0$ leads to

$$\begin{aligned}\sigma &= \frac{2}{\kappa}\left(\sqrt{\frac{1}{\ell_{\text{tv}}^2} - \frac{1}{\bar\rho^2}} - \sqrt{\frac{1}{\ell_{\text{fv}}^2} - \frac{1}{\bar\rho^2}}\right)\\ &< \frac{2}{\kappa}\left(\frac{1}{\ell_{\text{tv}}} - \frac{1}{\ell_{\text{fv}}}\right)\,.\end{aligned} \tag{2.20}$$

with the bound being approached in the limit $\bar\rho \to \infty$. Using Eq. (2.1) to rewrite the inequality on the second line yields the bound in Eq. (1.1).

[3] The absence of gravitational corrections in the CDL expression for the surface tension will be justified in Sec. III.

[4] More precisely, CDL replace U in the integral by a function U_0 that has minima at ϕ_{tv} and ϕ_{fv} and is equal to U_{fv} at both minima. In the thin-wall limit the effect of this replacement is higher order.

III. THIN-WALL BOUNCES BEYOND CDL

Reference [2] examined tunneling in the more general case where the true and false vacua are not close in energy and the conditions for CDL's thin-wall approximation are not met. It was found that as the mass scales in the potential are increased, making gravitational effects stronger, a new type of thin-wall regime emerges. More specifically, for any given potential one can define a quantity β as the ratio of a mass scale in the potential to the Planck mass. Gravitational effects become stronger as β is increased. Eventually, as β approaches a critical value, the bounce radius tends to infinity. In the limit the bounce solution disappears, tunneling is completely quenched, and the false vacuum is stabilized.

In this new thin-wall regime the scalar field profiles are qualitatively similar to those in the CDL thin-wall bounces. There is an interior region, $0 < \xi < \xi_1$, that is approximately pure true vacuum, an exterior region, $\xi_2 < \xi < \infty$, that is almost pure false vacuum, and a narrow transition region, or wall, that separates the two, with the wall thickness[5] $\Delta\xi = \xi_2 - \xi_1$ being small compared to ξ_1. However, they differ from their CDL counterparts in that $\rho(\xi)$ grows considerably as one passes through the wall, and so cannot be approximated as being constant.

As with the CDL thin-wall approximation, it is convenient to write the tunneling exponent as the sum of three terms, each of which is the difference between a bounce action term and a corresponding false vacuum term. In order to be consistent with the form of the long-distance regulator of the action integrals, Eq. (2.12), the corresponding regions of the bounce and the false vacuum must be defined by values of ρ, rather than ξ. Thus, if the wall in the bounce solution runs between ξ_1 and ξ_2, then the corresponding false vacuum region is bounded by $\rho_1 \equiv \rho(\xi_1)$ and $\rho_2 \equiv \rho(\xi_2)$.

With this understanding, we obtain for the interior region, $\rho < \rho_1$,

$$\begin{aligned} B_{\text{interior}} &= A_{\text{tv}}(0,\rho_1) - A_{\text{fv}}(0,\rho_1) \\ &= -\frac{4\pi^2}{\kappa}\left\{ \ell_{\text{tv}}^2 \left[\left(1 + \frac{\rho_1^2}{\ell_{\text{tv}}^2}\right)^{3/2} - 1 \right] - \ell_{\text{fv}}^2 \left[\left(1 + \frac{\rho_1^2}{\ell_{\text{fv}}^2}\right)^{3/2} - 1 \right] \right\}; \end{aligned} \tag{3.1}$$

i.e., the CDL result with $\rho = \rho_1$. In the exterior region the actions of the bounce and the false vacuum exactly cancel, so $B_{\text{exterior}} = 0$.

In the wall region we have

$$\begin{aligned} B_{\text{wall}} &= 4\pi^2 \int_{\rho_1}^{\rho_2} d\rho \left\{ \frac{1}{\rho_b'} \left[\rho^3 U(\phi_b) - \frac{3}{\kappa}\rho \right] - \frac{1}{\rho_{\text{fv}}'} \left[\rho^3 U_{\text{fv}} - \frac{3}{\kappa}\rho \right] \right\} \\ &= 4\pi^2 \int_{\xi_1}^{\xi_2} d\xi \left[\rho^3 U(\phi_b) - \frac{3}{\kappa}\rho \right] - A_{\text{fv}}(\rho_1,\rho_2)\,. \end{aligned} \tag{3.2}$$

In the first line the subscripts on ρ' and in the potential term indicate that in the first term these are to be evaluated from the bounce solution, and in the second term from the pure false vacuum solution.

[5] This definition of the wall thickness differs from that used in [2], which only included regions where $U(\phi) > U_{\text{fv}}$.

This expression for $B_{\rm wall}$ should reduce to the CDL result in the limit where $\Delta\rho = \rho_2 - \rho_1$ is small. To verify this, we write the false vacuum contribution to Eq. (3.2) as

$$- A_{\rm fv}(\rho_1, \rho_1 + \Delta\rho) = \frac{4\pi^2}{\kappa}(3\rho_1)\sqrt{1 + \frac{\rho_1^2}{\ell_{\rm fv}^2}}\,\Delta\rho + O[(\Delta\rho)^2]\,. \tag{3.3}$$

Now $\Delta\rho = \rho'(\xi_1)\Delta\xi$. In the false vacuum, $\rho' = \sqrt{1+\rho^2/\ell_{\rm fv}^2}$. Using these facts, we obtain

$$\begin{aligned} -A_{\rm fv}(\rho_1, \rho_1 + \Delta\rho) &= \frac{4\pi^2}{\kappa}(3\rho_1)\left(1 + \frac{\rho_1^2}{\ell_{\rm fv}^2}\right)\Delta\xi + O[(\Delta\xi)^2] \\ &= 4\pi^2\left(\frac{3\rho_1}{\kappa} - \rho_1^3 U_{\rm fv}\right)\Delta\xi + O[(\Delta\xi)^2]\,. \end{aligned} \tag{3.4}$$

Combining this result with the contribution from the bounce, and working to first order in $\Delta\xi$, we recover the CDL result, with the surface tension given by Eq. (2.19). Note that this justifies CDL's use of the flat-spacetime expression for the surface tension.

Let us now return to the more general case, with $\rho_2 - \rho_1$ not assumed to be small. We can no longer approximate ρ as being constant through the wall. One consequence is that the identification of a surface tension becomes problematic. One usually defines surface tension in terms of an energy per unit area (or action per unit hypersurface area). Because $\rho(\xi)$ grows in the wall, the area of the outer surface of the wall is larger than that of the inner surface of the wall. Which, if either, should be used? In fact, it is not even obvious that the wall action can be written as the product of an area and a radius-independent factor.

To answer these questions we need to examine the form of these new thin-wall solutions in more detail. The scalar field at the center of the bounce, $\phi(0)$, is very close to $\phi_{\rm tv}$. The field remains close to $\phi_{\rm tv}$ until $\xi \approx \xi_1$, so for the interior region, $\xi \lesssim \xi_1$, we have, analogously to Eq. (2.13),

$$\rho \approx \ell_{\rm tv} \sinh(\xi/\ell_{\rm tv})\,. \tag{3.5}$$

If gravitational effects are made stronger by increasing β, ξ_1 increases and ρ_1 grows exponentially.

In the near critical regime the growth of ρ in the interior region is such that at ξ_1 the first term on the right-hand side of Eq. (2.5) can be neglected. If $\rho_1 \gg \ell_{\rm fv}$ this remains true throughout the wall, and beyond. We can then write

$$\frac{\rho'}{\rho} = \sqrt{\frac{\kappa}{3}}\sqrt{\frac{1}{2}\phi'^2 - U(\phi)} \tag{3.6}$$

so that Eq. (2.4) becomes

$$\phi'' + \sqrt{3\kappa}\sqrt{\frac{1}{2}\phi'^2 - U(\phi)}\;\phi' = \frac{dU}{d\phi}\,. \tag{3.7}$$

Note that ρ does not appear in this equation. Hence the profile of $\phi(\xi)$ in the wall is independent of ρ.

Furthermore, integration of Eq. (3.6) gives

$$\rho(\xi) = \rho_1\, e^{G(\xi)} \tag{3.8}$$

where

$$G(\xi) = \sqrt{\frac{\kappa}{3}} \int_{\xi_1}^{\xi} d\xi \sqrt{\frac{1}{2}\phi'^2 - U(\phi)} \tag{3.9}$$

is also independent of ρ. This allows us to rewrite Eq. (3.2) as

$$B_{\rm wall} = 4\pi^2 \int_0^{\ln(\rho_2/\rho_1)} dG \left\{ \frac{1}{G'_b}\left[\rho_1^3 U(\phi_b)\, e^{3G} - \frac{3}{\kappa}\rho_1\, e^G\right] - \sqrt{\frac{3}{\kappa}}\frac{1}{\sqrt{-U_{\rm fv}}}\left[\rho_1^3 U_{\rm fv} e^{3G} - \frac{3}{\kappa}\rho_1 e^G\right]\right\} . \tag{3.10}$$

In the limit of large bounce radius ($\rho_1 \gg l_{\rm fv}$), the terms cubic in ρ_1 dominate. Keeping only these, we have

$$B_{\rm wall} = 4\pi^2\rho_1^3\sqrt{\frac{3}{\kappa}} \int_0^{\ln(\rho_2/\rho_1)} dG\, e^{3G}\left[\frac{U(\phi_b)}{\sqrt{\frac{1}{2}\phi_b'^2 - U(\phi_b)}} + \sqrt{-U_{\rm fv}}\right] . \tag{3.11}$$

This suggests that we write

$$B_{\rm wall} = 2\pi^2\rho_1^3\tilde{\sigma} \tag{3.12}$$

where $2\pi^2\rho_1^3$ is the area of the inner surface of the wall and

$$\tilde{\sigma} = \sqrt{\frac{12}{\kappa}} \int_0^{\ln(\rho_2/\rho_1)} dG\, e^{3G}\left[\frac{U(\phi_b)}{\sqrt{\frac{1}{2}\phi_b'^2 - U(\phi_b)}} + \sqrt{-U_{\rm fv}}\right] \tag{3.13}$$

can be viewed as a generalization of the CDL surface tension σ.[6] (Note that, like σ, it is independent of ρ.) With this definition, the total expression for B takes the same form as in the CDL thin-wall limit, but with the replacements $\bar{\rho} \to \rho_1$ and $\sigma \to \tilde{\sigma}$. The line of reasoning that led to Eq. (2.20) and then to Eq. (1.1) now leads to

$$\tilde{\sigma} < \frac{2}{\sqrt{3\kappa}}\left(\sqrt{|U_{\rm tv}|} - \sqrt{|U_{\rm fv}|}\right) . \tag{3.14}$$

IV. A BOUND FOR ALL BOUNCES

We have obtained upper bounds on the surface tension for both the thin-wall approximation of CDL and the generalized thin-wall regime of [2]. However, thin-wall bounces of either type are special cases. There are bounce solutions that are not in any sense thin-wall, including some for which it is difficult to even define a surface tension. This raises the question of whether there is a more general bound that applies to all bounces and that reduces to Eqs. (1.1) and (3.14) in the appropriate limits.

[6] In the CDL expression for the surface tension, Eq. (2.19), the integrand is everywhere positive so σ is manifestly positive. This is not the case for $\tilde{\sigma}$. Indeed, the integrand in Eq. (3.13) is negative in the lower part of the integration range and positive in the upper part. In the next section we will show that this expression for $\tilde{\sigma}$ is a special case of a more general expression that is manifestly positive.

We now show that there is. To begin, we recall the definition of the pseudo-energy, Eq. (2.8), and the expression for its derivative, Eq. (2.9). If follows from the latter that

$$\frac{d\sqrt{E}}{d\xi} = -\frac{\sqrt{3\kappa}}{2}\sqrt{1+\frac{3}{\kappa E\rho^2}}\;\phi'^2 \,. \tag{4.1}$$

Integrating this, we find that

$$\begin{aligned}\sqrt{E(0)}-\sqrt{E(\infty)} &= \frac{\sqrt{3\kappa}}{2}\int_0^\infty d\xi\,\sqrt{1+\frac{3}{\kappa E\rho^2}}\;\phi'^2 \\ &> \frac{\sqrt{3\kappa}}{2}\int_0^\infty d\xi\,\phi'^2 \,.\end{aligned} \tag{4.2}$$

Noting that $E(0) \le |U_{\rm tv}|$ and recalling that $E(\infty) = |U_{\rm fv}|$, we have

$$\int_0^\infty d\xi\,\phi_b'^2 < \frac{2}{\sqrt{3\kappa}}\left(\sqrt{|U_{\rm tv}|}-\sqrt{|U_{\rm fv}|}\right) \,. \tag{4.3}$$

This inequality is exact, and does not depend on any approximations. It therefore applies to any bounce solution. In particular, it should reduce to our previous results for thin-wall bounces. In these bounces ϕ' is taken to vanish outside the wall region, so the integration can be restricted to the range $\xi_1 < \xi < \xi_2$. In the CDL thin-wall approximation the bounce profile in the wall region is approximately that of a (1+1)-dimensional kink. Equation (2.19) gives the surface tension in terms of an integral of the potential $U(\phi_b)$. A virial theorem [17] relates this to the integral of ϕ'^2 and shows that the bounds of Eqs. (1.1) and (4.3) are equivalent within the accuracy of the approximation.

For the new thin-wall case, demonstrating the equivalence of Eqs. (1.1) and (3.14) requires a bit more work. We begin by noting the identity

$$\frac{1}{3\kappa}\int_{\xi_1}^{\xi_2} d\xi\,\frac{d}{d\xi}\left(\rho^3\sqrt{E}\right) = \int_{\xi_1}^{\xi_2} d\xi\,\rho^3 U(\phi_b)\sqrt{1+\frac{3}{\kappa E\rho^2}}\,, \tag{4.4}$$

which is obtained by evaluating the derivative inside the integral on the left-hand side and using Eq. (4.1).

Alternatively, using the fact that the integrand is a total derivative gives

$$\begin{aligned}\frac{1}{3\kappa}\int_{\xi_1}^{\xi_2} d\xi\,\frac{d}{d\xi}\left(\rho^3\sqrt{E}\right) &= \frac{1}{3\kappa}\left[-(\rho_2^3-\rho_1^3)\sqrt{E_2}-\rho_1^3(\sqrt{E_2}-\sqrt{E_1})\right] \\ &= \frac{1}{3\kappa}\left[-(\rho_2^3-\rho_1^3)\sqrt{E_2}-\rho_1^3\int_{\xi_1}^{\xi_2} d\xi\,\frac{d\sqrt{E}}{d\xi}\right] \\ &= -\frac{1}{\kappa\ell_{\rm fv}}(\rho_2^3-\rho_1^3)-\frac{1}{2}\rho_1^3\int_{\xi_1}^{\xi_2} d\xi\,\phi'^2\sqrt{1+\frac{3}{\kappa E\rho^2}}\,.\end{aligned} \tag{4.5}$$

In the last equality we have used the definition of the AdS length, Eq. (2.1), and the fact that $E_2 = -U_{\rm fv}$. Comparing Eqs. (4.4) and (4.5), we have

$$\int_{\xi_1}^{\xi_2} d\xi\,\rho^3 U(\phi_b)\sqrt{1+\frac{3}{\kappa E\rho^2}}+\frac{1}{\kappa\ell_{\rm fv}}(\rho_2^3-\rho_1^3) = -\frac{1}{2}\rho_1^3\int_{\xi_1}^{\xi_2} d\xi\,\phi'^2\sqrt{1+\frac{3}{\kappa E\rho^2}}\,. \tag{4.6}$$

In the range of integration, $\xi_1 \le \xi \le \xi_2$, the pseudo-energy satisfies $E \ge E_2 = |U_{\rm fv}|$, while $\rho \ge \rho_1$. It follows that

$$\frac{3}{\kappa E \rho^2} < \frac{\ell_{\rm fv}^2}{\rho_1^2}\,. \tag{4.7}$$

Hence in the limit of large bounce radius, $\rho_1 \gg \ell_{\rm fv}$, the square roots in Eq. (4.6) can be set equal to unity, giving

$$\int_{\xi_1}^{\xi_2} d\xi\, \rho^3 U(\phi_b) + \frac{1}{\kappa \ell_{\rm fv}}(\rho_2^3 - \rho_1^3) = -\frac{1}{2}\rho_1^3 \int_{\xi_1}^{\xi_2} d\xi\, \phi'^2\,. \tag{4.8}$$

In this same limit Eq. (3.2) reduces to

$$\begin{aligned} B_{\rm wall} &= 4\pi^2 \int_{\xi_1}^{\xi_2} d\xi\, \rho^3 U(\phi_b) + \frac{4\pi^2}{\kappa \ell_{\rm fv}}(\rho_2^3 - \rho_1^3) \\ &= 2\pi^2 \rho_1^3 \int_{\xi_1}^{\xi_2} d\xi\, \phi'^2 \end{aligned} \tag{4.9}$$

where the second line follows from Eq. (4.8). Dividing by the surface area, $2\pi^2\rho_1^3$, gives

$$\tilde\sigma = \int_{\xi_1}^{\xi_2} d\xi\, \phi'^2 \tag{4.10}$$

and demonstrates the equivalence of Eqs. (3.14) and (4.3) in this regime.

V. BOUNCES AND WALLS IN THE CRITICAL LIMIT

It is instructive to examine the behavior of the bounce solution as the parameters of the theory approach the critical limit where the nucleation of AdS bubbles is totally quenched. As this limit is approached the wall radius of the bounce solution grows without bound, with both ξ_1 and ρ_1 diverging. When the parameters are actually at their limiting values, there is no Euclidean bounce at all.

Now let us focus on the fixed time slice through the center of the bounce, $t = 0$, when the bubble is nucleated. The bubble is instantaneously at rest, while the spatial part of the metric is

$$d\ell^2 = d\xi^2 + \rho(\xi)^2 d\Omega_2^2 \tag{5.1}$$

with $\rho(\xi)$ taken over from the bounce and ξ again being a radial coordinate. The limit of infinite bubble radius can be viewed, in a certain sense, as a planar wall separating two metastable vacua.

The metric for a static planar wall can be written as

$$ds^2 = A(z)(-dt^2 + dx^2 + dy^2) + dz^2 \tag{5.2}$$

with z being the spatial coordinate orthogonal to the wall. For our theory with a single scalar field, the field equations are

$$A'^2 = \frac{\kappa}{3} A^2 \left(\frac{1}{2}\phi'^2 - U\right) \tag{5.3}$$

$$\phi'' + \frac{3A'}{A} = \frac{dU}{d\phi} \tag{5.4}$$

with primes indicating differentiation with respect to z. These are to be solved subject to the boundary conditions that ϕ take its false (true) vacuum value at $z = \infty$ ($z = -\infty$).

If we make the correspondence

$$\xi \longleftrightarrow z\,, \qquad \rho \longleftrightarrow A\,, \tag{5.5}$$

these equations differ from Eqs. (2.4) and (2.5) only by the omission of the factor of unity on the right-hand side of Eq. (2.5). The boundary conditions at $\xi = \infty$ and $z = \infty$ agree. Although those at $\xi = 0$ and $z = -\infty$ differ, they coincide in the limit of infinite bubble radius.

There is an interesting connection with supersymmetry when the parameters are near the critical limit. Let us suppose that we are given $\phi(\xi)$ and $\rho(\xi)$ satisfying the bounce equations Eqs. (2.4) and (2.5). Because ϕ is a monotonic function of ξ, we can view ξ, and therefore ϕ', E, and ρ, as functions of ϕ in the neighborhood of the bounce solution. We can then define a function $f(\phi)$ by

$$f(\phi)^2 = \frac{1}{3\kappa}\left(\frac{1}{2}\phi'^2 - U\right) = \frac{1}{3\kappa}E\,. \tag{5.6}$$

The derivative of f with respect to ϕ is

$$\begin{aligned}\frac{df}{d\phi} &= \frac{df/d\xi}{d\phi/d\xi}\\ &= \frac{1}{\sqrt{3\kappa}}\frac{d\sqrt{E}}{d\xi}\frac{1}{\phi'}\\ &= -\frac{1}{2}\phi'\alpha^{-1}\end{aligned} \tag{5.7}$$

where the last line follows from Eq. (4.1) and

$$\alpha(\phi) = \left(1 + \frac{1}{\kappa^2\rho^2 f^2}\right)^{-1/2}\,. \tag{5.8}$$

Solving Eq. (5.7) for ϕ' and substituting the result into Eq. (5.6) leads to

$$U = 2\alpha^2\left(\frac{df}{d\phi}\right)^2 - 3\kappa f^2\,. \tag{5.9}$$

If α were equal to unity, this would be the form for the potential in a supergravity theory, with f being the superpotential. In fact, $\alpha \approx 1$ wherever $\rho \gg \ell_{\rm fv}$. Thus, for a near-critical bounce the deviation from the supersymmetric form is confined to a region of approximate true vacuum in the center of the bounce.[7]

Repeating the calculation for the static planar wall, one finds that $\alpha = 1$ everywhere in space. The domain wall would then have the form of a supersymmetric wall interpolating

[7] This assumes that the false vacuum is AdS. The case of a Minkowski false vacuum is addressed in the appendix.

between isolated vacua of an $\mathcal{N}=1$ supergravity potential, with the disappearance of the bounce solution guaranteeing the stability of each of these vacua against decay by nucleation of bubbles of the other. Note, however, that our construction is restricted to the interval of field space between the two vacua; the form of $U(\phi)$ outside this interval is unconstrained and need not be derivable from a superpotential.

An alternative path to demonstrating vacuum stability is to prove a positive energy theorem or a BPS bound. In the presence of gravity, Boucher [15], generalizing Witten's work [18, 19], gave the following criteria: an extremum $\bar{\phi}$ of a potential $U(\phi)$ with $U(\bar{\phi})<0$ is stable if there exists a real function $W(\phi)$ such that

$$2\left(\frac{dW}{d\phi}\right)^2 - 3\kappa W^2 = U(\phi) \qquad \forall\phi \tag{5.10}$$

and

$$W(\bar{\phi}) = \left[-U(\bar{\phi})/3\kappa\right]^{1/2} . \tag{5.11}$$

These criteria make no direct reference to bubble nucleation. That connection was drawn by Abbott and Park [14]. Given a potential $U(\phi)$, one can obtain $W(\phi)$ by integrating Eq. (5.10), provided that $U + 3\kappa W^2$ remains positive. Abbott and Park showed that if the latter becomes negative at some $\phi = \phi_s$ before one reaches an extremum of U, then ϕ_s is the starting point $\phi(0)$ for a bounce solution that governs the decay via bubble nucleation of the $\bar{\phi}$ vacuum.

VI. CONCLUDING REMARKS

Gravity can quench the nucleation of bubbles in a Minkowski or AdS false vacuum. This result was proven analytically by Coleman and De Luccia in a thin-wall limit where the energy difference between the true and false vacua is small. Their analysis implies an inequality, Eq. (1.1), that relates the surface tension σ and the true- and false-vacuum energies. It is essential for this inequality that σ is independent of the radius of curvature of the bounce wall, and that the contribution of the wall to the Euclidean action is the product of σ and the surface area of the wall.

Subsequently, numerical analyses have generalized this claim to a wider variety of potentials [2, 20–22]. As the boundary beyond which nucleation is quenched is approached, the bubble radius at nucleation increases without bound and a new thin-wall regime emerges [2]. This new thin-wall regime differs from the CDL thin-wall regime in that the wall radius of curvature ρ grows exponentially as one moves through the bubble wall. It is then far from obvious that the wall action can be decomposed as the product of a surface tension and an area. Indeed, it is not even clear how the area should be defined. In Sec. III we used the fact that the matter field profile in the wall is independent of the curvature of the wall to define a modified surface tension $\tilde{\sigma}$ that is ρ-independent. We were then able to show analytically that the wall action is $\tilde{\sigma}$ times the area of the inner surface of the bounce wall. Closely following the reasoning of CDL then led to the bound of Eq. (3.14) on $\tilde{\sigma}$.

In Sec. IV we proved that the upper bounds on the surface tensions in the two thin-wall regimes are limiting cases of a more general bound, Eq. (4.3), that is satisfied by all bounces. Gravity sources a frictional force that depletes the pseudo-energy as it evolves through the radial direction of the bounce. This bound expresses the constraint on this frictional force that is required if the bounce is to interpolate between the two vacua.

The work presented here was limited to single-field potentials, but the proof of the bound of Eq. (4.3), the existence of the generalized thin-wall regime [2] and the definition of surface tension in this regime, Eq. (3.13), should extend to multifield potentials. With N scalar fields ϕ_i the bounce should satisfy

$$\phi_i'' + \frac{3\rho'}{\rho}\phi_i' = \frac{dU}{d\phi_i}\,, \tag{6.1}$$

$$\rho'^2 = 1 + \frac{\kappa}{3}\rho^2\left[\sum_i \frac{1}{2}\phi_i'^2 - U(\phi_i)\right]\,. \tag{6.2}$$

Solving these will lead to a trajectory through field space of the form $\phi_i(\xi) = g_i(\xi)$. In the neighborhood of this bounce trajectory we can introduce a coordinate Φ along the trajectory, defined by

$$d\Phi^2 = \sum_{i=1}^{N} dg_i^2 \tag{6.3}$$

together with $N-1$ fields normal to the trajectory.

In these new coordinates the action along the bounce

$$S = 2\pi^2\int_0^\infty d\xi\,\left\{\rho^3\left[\frac{1}{2}\Phi'^2 - U(\Phi)\right] - \frac{3}{\kappa}(\rho\rho'^2 + \rho)\right\} \tag{6.4}$$

only depends on Φ. The form of $U(\Phi)$ depends on the explicit bounce solution, but the only information that was assumed about the potential was that it was a continuous function interpolating between $U_{\rm tv}$ and $U_{\rm fv}$. The formal arguments made in Secs. III and IV should extend to the field Φ, and the bound of Eq. (4.3) will be satisfied.

In the single-field case $U(\phi)$ approached a supersymmetric form as the parameters of the theory approached the boundary where nucleation was quenched. Precisely on this boundary, where the false vacuum becomes stable against decay by tunneling, a static planar domain wall appears. $U(\phi)$ takes on the supergravity form and can be written in terms of a fake superpotential $W(\phi)$. However, this form is only guaranteed on the interval in field space lying between the true and false vacua. For the multifield case similar arguments allow one to define a fake superpotential, but only along the trajectory of the bounce in field space and only encoding a dependence on Φ, but not on the fields normal to the trajectory.

Even though these restrictions weaken the connection between stability and supersymmetry, recent claims [23–25] based on the weak gravity conjecture [26] assert the instability of all non-supersymmetric vacua in a UV complete theory. Said differently, theories for which stable domain walls do not obey Eq. (1.1) live in the swampland, with gravity never strong enough to quench a decay. See [27–30] for more examples of this instability.

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Appendix: Minkowski false vacuum

At some points in our analysis we have relied on the condition that $\rho_1 \gg \ell_{\rm fv}$. With an AdS false vacuum this can always be achieved by going sufficiently close to the critical limit. This is not the case if the false vacuum is Minkowskian, with $U_{\rm fv} = 0$ implying $\ell_{\rm fv} = \infty$. Here we examine the consequences of this fact.

In our analysis we used this inequality to argue that in the wall and in the entire exterior region the right-hand side of Eq. (2.5) was dominated by the second term. This led to the conclusion that the field profile in the wall was independent of ρ. It also allowed us to approximate the square roots in Eq. (4.6) as unity, which led to Eq. (4.10) and the equivalence of Eqs. (3.14) and (4.3).

Now suppose that the false vacuum is Minkowskian. By going sufficiently close to the critical limit we can always ensure that the second term dominates the right-hand side of Eq. (2.5) near the inner edge of the wall, but as E decreases with increasing ξ there comes a $\bar\xi$ where

$$\frac{\kappa}{3}E = \frac{\kappa}{3}\left[\frac{1}{2}\phi'^2 - U\right] = \frac{1}{\rho^2}\,. \tag{A.1}$$

For a thin-wall bubble (of either kind) ρ is already exponentially large at ξ_1, which means that E, $U(\phi_b)$, and ϕ'^2 will all have become exponentially small by the time that $\bar\xi$ is reached. Thus, the nontrivial part of the field profile will lie in the region $\xi < \bar\xi$ and will be independent of ρ, just as with an AdS false vacuum. Near the critical limit $\bar\xi$ will lie outside the wall (i.e., $\bar\xi > \xi_2$), so the square roots in Eq. (4.6) will remain close to unity for the entire range of integration.

We saw in Sec. V that the potential closely approximates the supergravity form wherever $\alpha \approx 1$ or, equivalently, $\kappa E\rho^2 \gg 1$. With an AdS false vacuum E has a nonzero lower bound, and so for near-critical bounces these conditions hold everywhere except for a region of approximate true vacuum in the center of the bounce. With a Minkowski false vacuum E tends to zero as $\rho \to \infty$, so the large-distance behavior requires closer examination. To begin, note that

$$\begin{aligned}(E\rho^2)' &= 2\rho\rho' E + \rho^2 E' \\ &= \rho\rho'\left(2E - 3\phi'^2\right) \\ &= -2\rho\rho'\left(U + \phi'^2\right)\,.\end{aligned} \tag{A.2}$$

When ϕ is close to its value at the Minkowski false vacuum minimum, $U(\phi)$ is positive, so the quantity in parentheses on the last line is positive definite, but exponentially decreasing in magnitude. Near the false vacuum ρ grows linearly with ξ, so the integrated decrease in $E\rho^2$ as $\xi \to \infty$ is finite. Hence, for near-critical bounces where $E\rho^2$ is large even near the end of the wall region, it will remain large as ξ increases. Just as with an AdS false vacuum, in a near-critical bounce α will only deviate from unity near the center of the bounce, far from the wall.

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