Strictly commutative complex orientation theory

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Abstract

For a multiplicative cohomology theory E, complex orientations are in bijective correspondence with multiplicative natural transformations to E from complex bordism cohomology MU. If E is represented by a spectrum with a highly structured multiplication, we give an iterative process for lifting an orientation $MU \to E$ to a map respecting this extra structure, based on work of Arone–Lesh. The space of strictly commutative orientations is the limit of an inverse tower of spaces parametrizing partial lifts; stage 1 corresponds to ordinary complex orientations, and lifting from stage $(m-1)$ to stage m is governed by the existence of an orientation for a family of E-modules over a fixed base space F_m.

When E is p-local, we can say more. We find that this tower only changes when m is a power of p, and if E is $E(n)$-local the tower is constant after stage p^n. Moreover, if the coefficient ring E^* is p-torsion free, the ability to lift from stage 1 to stage p is equivalent to a condition on the associated formal group law that was shown necessary by Ando.

Characteristic classes play a fundamental role in algebraic topology, with the primary example being the family of Chern classes $c_i(\xi) \in H^{2i}(X)$ associated to a complex vector bundle $\xi \to X$. Not all generalized cohomology

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theories possess Chern classes, but they are present in important cases such as complex K-theory K and complex bordism theory MU. In fact, for a cohomology theory E taking values in graded-commutative rings, the following types of information are equivalent:

- a choice of characteristic class $c_1(\xi) \in \widetilde{E}^2(X)$ for complex line bundles $\xi \to X$ such that, for the canonical line bundle $\gamma_1 \to \mathbb{CP}^1$, the isomorphism $\widetilde{E}^2(\mathbb{CP}^1) \cong E^0$ carries $c_1(\gamma_1)$ to 1;
- a family of characteristic classes $c_i(\xi) \in \widetilde{E}^{2i}(X)$ for complex vector bundles $\xi \to X$ satisfying the above formula for $c_1(\gamma_1)$ and such that the Cartan formula for Whitney sums holds; or
- a natural transformation $MU \to E$ of multiplicative cohomology theories.

In the third case, the natural transformation $MU^{2i}(X) \to E^{2i}(X)$ allows us to push forward the characteristic classes $c_i^{MU}(\xi)$ to classes $c_i^E(\xi)$. Such a map $MU \to E$ is called a complex orientation of E, and as a result MU plays a fundamental role in the theory of Chern classes.

Moving from the homotopy category to the point-set level, the spectrum MU representing complex bordism is also one of the best known examples of a spectrum with a multiplication which is associative and commutative up to all higher coherences (an E_∞ ring structure). If we know that E is also equipped with an E_∞ ring structure, it is natural to ask whether a complex orientation $MU \to E$ can be lifted to a map respecting this E_∞ ring structure (an E_∞ orientation), and what data is necessary to describe this. This is a stubborn problem and in prominent cases the answer is unknown, such as when E is a Lubin–Tate spectrum.

For any complex orientation of E, there is a formal group law $\mathbb{G}$ expressing the first Chern class of a tensor product of two complex line bundles: we have

$$c_1(\xi' \otimes_{\mathbb{C}} \xi'') = c_1(\xi') +_{\mathbb{G}} c_1(\xi'')$$

for an associative, commutative, and unital power series $x +_{\mathbb{G}} y \in E^* [\![x, y]\!]$. Ando gave a necessary and sufficient condition for an orientation $MU \to E$ to be an H_∞ orientation [And92], a weaker structure than an E_∞ orientation which can be described as a natural transformation that respects geometric power operations. Ando's condition was that a certain natural power operation Ψ with source $E^*(X)$ should act as the canonical Lubin isogeny on the coordinate ring $E^*(\mathbb{CP}^\infty)$ of the formal group law $\mathbb{G}$ (see also [AHS04, §4.3]). In general, this is stronger than the data of a complex orientation alone [JN10], and very few E_∞ ring spectra are known to admit H_∞ orientations. (Ando also showed that Lubin–Tate spectra associated to the Honda formal group law have unique H_∞ orientations; in recent work, Zhu has generalized this to all of the Lubin–Tate spectra [Zhu].)

However, it is the case that the rationalization $MU_\mathbb{Q}$ is universal among rational, complex oriented E_∞ rings [BR14, 6.1]. Further, Walker studied orientations for the case of p-adic K-theory and the Todd genus [Wal08], and Möllers studied orientations in the case of $K(1)$-local spectra [Möl10]. Both gave proofs that Ando's condition for H_∞ orientations was also sufficient to produce E_∞ orientations.

The goal of this paper is to apply work by Arone–Lesh [AL07] to extend this procedure, giving an inductive approach to the construction of E_∞ orientations. Before getting into details, we will describe the motivation for this construction.

As an E_∞ ring spectrum, the fact that MU is a Thom spectrum gives it a universal property. There is a map of infinite loop spaces $U \to GL_1(\mathbb{S})$ from the infinite unitary group to the space of self-equivalences of the sphere spectrum $\mathbb{S}$, and MU is universal among E_∞ ring spectra E with a chosen nullhomotopy of the map of infinite loop spaces $U \to GL_1(\mathbb{S}) \to GL_1(E)$ [May77, §V].

We can recast this using the language of Picard groups. We consider two natural functors: one sends a complex vector space V to the spectrum $\Sigma^\infty S^V$, which has an inverse under the smash product; the second sends a smash-invertible spectrum I to a smash-invertible MU-module $MU \wedge I$. On restricting to the subcategory of weak equivalences and applying classifying spaces, we obtain maps of E_∞ spaces

$$\coprod_m BU(m) \to \mathbb{Z} \times BGL_1(\mathbb{S}) \to \mathbb{Z} \times BGL_1(MU).$$

Here the latter two are the Picard spaces $\mathrm{Pic}(\mathbb{S})$ and $\mathrm{Pic}(MU)$ [MS16, 2.2.1]. Passing through an infinite loop space machine, we obtain a sequence of maps of spectra

$$ku \to \mathrm{pic}(\mathbb{S}) \to \mathrm{pic}(MU),$$

where ku is the connective complex K-theory spectrum. The universal property of MU can then be rephrased: the spectrum MU is universal among E_∞ ring spectra E equipped with a coherently commutative diagram

$$\begin{tikzcd}
ku \arrow[r] \arrow[d] & \mathrm{pic}(\mathbb{S}) \arrow[d] \\
H\mathbb{Z} \arrow[r, dotted] & \mathrm{pic}(E).
\end{tikzcd}$$

(More concretely, this asks for a map $H\mathbb{Z} \to \mathrm{pic}(E)$ and a chosen homotopy between the two composites.)

This allows us to exploit Arone–Lesh's sequence of spectra interpolating between ku and $H\mathbb{Z}$, giving us an inductive sequence of obstructions to E_∞ orientations.

Theorem 1. *There exists a filtration of MU by E_∞ Thom spectra*

$$\mathbb{S} \to MX_1 \to MX_2 \to MX_3 \to \cdots \to MU$$

with the following properties.

1. *The map* $\mathrm{hocolim}\, MX_i \to MU$ *is an equivalence.*

2. *There is a canonical complex orientation of MX_1 such that, for all E_∞ ring spectra E, the space* $\mathrm{Map}_{E_\infty}(MX_1, E)$ *is homotopy equivalent to the space of ordinary complex orientations of E.*

3. *For all $m > 0$ and all maps of E_∞ ring spectra $MX_{m-1} \to E$, the space of extensions to a map of E_∞ ring spectra $MX_m \to E$ is a homotopy pullback diagram of the form*

$$\begin{tikzcd}
\mathrm{Map}_{E_\infty}(MX_m, E) \arrow[r] \arrow[d] & \mathrm{Map}_{E_\infty}(MX_{m-1}, E) \arrow[d] \\
\{*\} \arrow[r] & \mathrm{Map}_*(F_m, \mathrm{Pic}(E))
\end{tikzcd}$$

for a certain fixed space F_m, where $\mathrm{Pic}(E)$ *is the classifying space of the category of smash-invertible E-modules.*

More specifically, given an E_∞ map $MX_{m-1} \to E$, there is an E-module Thom spectrum $M_E\xi$ classified by a map $\xi\colon F_m \to \mathrm{Pic}(E)$. An extension to an E_∞ map $MX_m \to E$ exists if and only if there is an orientation $M_E\xi \to E \wedge S^{2m}$ in the sense of [ABG+14], and the space of extensions is naturally equivalent to the space $\mathrm{Or}(\xi)$ *of orientations.*

4. *The map $MX_{m-1} \to MX_m$ is a rational equivalence if $m > 1$, a p-local equivalence if m is not a power of p, and a $K(n)$-local equivalence if $m > p^n$.*

In particular, the spectrum MX_1 will be a universal complex oriented E_∞ ring spectrum described by Baker–Richter [BR14].

The suspended spaces F_m are explicitly described by [AL07] as being derived orbit spectra $(L_m)^\diamond \wedge^{\mathbb{L}}_{U(m)} S^{2m}$. Here L_m is the nerve of the (topologized) poset of proper direct-sum decompositions of $\mathbb{C}^m$, S^{2m} is the one-point compactification of $\mathbb{C}^m$, and $\diamond$ denotes unreduced suspension. Alternatively, the space F_m can be described as the homotopy cofiber of the map of Thom spaces

$$(L_m \times_{U(m)} EU(m))^{\gamma_m} \to BU(m)^{\gamma_m}$$

for the universal bundle γ_m.

In the particular case of an $E(1)$-local E_∞ ring spectrum E, such as a form of K-theory [LN14, Appendix A], this will allows us to verify that Ando's criterion is both necessary and sufficient if E^* is torsion-free. At higher chromatic levels there are expected to be secondary and higher obstructions involving relations between power operations.

Remark 2. Rognes had previously constructed a similar filtration on algebraic K-theory spectra [Rog92], further examined in the case of complex K-theory in [AL10]. This filtration gives rise to a sequence of spectra interpolating the map $* \to ku$ rather than $ku \to H\mathbb{Z}$. On taking Thom spectra of the resulting infinite loop maps to $\mathbb{Z} \times BU$, the result should be a construction of the periodic complex bordism spectrum MUP with very similar properties but slightly different subquotients, relevant to a more rigid orientation theory for 2-periodic spectra.

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1 The filtration of connective K-theory

In this section, we will give short background on the results that we require from [AL07, 3.9, 8.3, 9.4, 9.6, 11.3].

Proposition 3. *There exists a sequence of maps of E_∞ spaces*

$$B_0 \to B_1 \to B_2 \to B_3 \to \cdots$$

with the following properties.

1. *The space $B_\infty = \operatorname{hocolim} B_m$ is equivalent to the discrete E_∞ space $\mathbb{N}$, and the induced maps $\pi_0 B_m \to \mathbb{N}$ are isomorphisms.*
2. *The space B_0 is the nerve $\coprod BU(n)$ of a skeleton of the category of finite-dimensional vector spaces and isomorphisms, with the E_∞ structure induced by direct sum.*
3. *Let $\mathbb{P}$ denote the functor taking a space X to the free E_∞ space on X [May72, 3.5], with the homotopy type*
$$\mathbb{P}(X) = \coprod_{n \geq 0} (X^n)_{h\Sigma_n}.$$
For each $m > 0$, there is a homotopy pushout diagram of E_∞ spaces
$$\begin{tikzcd} \mathbb{P}(F_m) \arrow[r] \arrow[d] & B_{m-1} \arrow[d] \\ \mathbb{P}(*) \arrow[r] & B_m, \end{tikzcd}$$
where F_m is the path component of B_{m-1} mapping to $m \in \mathbb{N}$.
4. *The map $F_m \to *$ is an isomorphism in rational homology if $m > 1$, an isomorphism in p-local homology if m is not a power of p, and an isomorphism in $K(n)$-homology if $m > p^n$.*

5. *The spectrum $\Sigma^\infty F_m$ is $(2m-1)$-connnected.*

The explicit description allows analysis of the filtration quotients using [AL07, 2.5]. The space F_1 is the path component $BU(1) \subset \coprod BU(n)$. We then get a homotopy commutative diagram

$$\begin{tikzcd}
B(\Sigma_p \wr U(1)) \arrow[rr] \arrow[dd] \arrow[dr] & & BU(p) \arrow[d] \\
& \mathbb{P}(BU(1)) \arrow[r] \arrow[d] & \coprod BU(n) \arrow[d] \\
B\Sigma_p \arrow[r] & \mathbb{P}(*) \arrow[r] & B_1
\end{tikzcd} \tag{1}$$

with the top map induced by the inclusion of the monomial matrices in $U(p)$ and the left map induced by the projection $\Sigma_p \wr U(1) \to \Sigma_p$. The homotopy pushout of the subdiagram

$$B\Sigma_p \longleftarrow B(\Sigma_p \wr U(1)) \longrightarrow BU(p) \tag{2}$$

maps to F_p by a p-local homotopy equivalence.

Arone–Lesh then apply an infinite loop space machine to the sequence of Proposition 3, with the following result.

Corollary 4. *There exists a sequence of connective spectra*

$$b_0 \to b_1 \to b_2 \to b_3 \to \cdots \tag{3}$$

with the following properties.

1. *The spectrum $b_\infty = \operatorname{hocolim} b_m$ is equivalent to the spectrum $H\mathbb{Z}$, and the induced maps $\pi_0 b_n \to \mathbb{Z}$ are isomorphisms.*
2. *The spectrum b_0 is the complex K-theory spectrum ku.*
3. *For each $m > 0$, there is a homotopy pushout diagram*

$$\begin{tikzcd}
\Sigma^\infty (F_m)_+ \arrow[r] \arrow[d] & b_{m-1} \arrow[d] \\
\Sigma^\infty S^0 \arrow[r] & b_m.
\end{tikzcd} \tag{4}$$

4. *The map $b_{m-1} \to b_m$ is an isomorphism in rational homology if $m > 1$, an isomorphism in p-local homology if m is not a power of p, and an isomorphism in $K(n)$-homology if $m > p^n$.*

5. *The homotopy fiber $\Sigma^\infty F_m$ of the map $b_{m-1} \to b_m$ is $(2m-1)$-connected.*

2 The filtration of BU

Definition 5. For each $m \geq 0$, let x_m be the homotopy fiber $\mathrm{hofib}(ku \to b_m)$ of the maps from equation (3), and let $X_m = \Omega^\infty x_m$.

This gives rise to a sequence of maps

$$* \simeq x_0 \to x_1 \to x_2 \to x_3 \to \cdots, \tag{5}$$

with homotopy colimit $bu \simeq \Sigma^2 ku$ by Bott periodicity. For each $m > 0$, diagram (4) and the octahedral axiom imply that the homotopy fiber of $x_{m-1} \to x_m$ is equivalent to the desuspension $\Omega\Sigma^\infty F_m$ of the reduced suspension spectrum.

Applying Ω^∞ to the filtration of equation (5), we obtain a filtration of BU by infinite loop spaces:

$$* \to X_1 \to X_2 \to X_3 \to \cdots \tag{6}$$

The homotopy fiber of $X_{m-1} \to X_m$ is the space $\Omega Q F_m = \Omega^{\infty+1}\Sigma^\infty F_m$.

In order to analyze the effect of these maps in $K(n)$-local homology, we will require some preliminary results.

Lemma 6. *Suppose $x \to y \to z$ is a fiber sequence of spectra, $\Omega^\infty x$ is $K(n)$-locally trivial, and x is connective. Then the map $\Omega^\infty y \to \Omega^\infty z$ induces an isomorphism on $K(n)$-homology.*

Proof. By assumption $\pi_0 y \to \pi_0 z$ is surjective, so the map $\Omega^\infty y \to \Omega^\infty z$ is a principal fibration whose fiber over any point is $\Omega^\infty x$. Applying the (natural) generalized Atiyah–Hirzebruch spectral sequence, we obtain a spectral sequence

$$\mathcal{H}_*(\Omega^\infty z; K(n)_*(\Omega^\infty x)) \Rightarrow K(n)_*(\Omega^\infty y),$$

where the E_2-term may be homology with coefficients in a local coefficient system. By assumption, the edge morphism to the Atiyah–Hirzebruch spectral sequence

$$H_*(\Omega^\infty z; K(n)_*) \Rightarrow K(n)_*(\Omega^\infty z)$$

is an isomorphism on E_2-terms, so it converges to an isomorphism $K(n)_*(\Omega^\infty y) \to K(n)_*(\Omega^\infty z)$. $\square$

Proposition 7. *Let W be a based space whose suspension spectrum is at least k-connected, and define $\mathcal{N}$ be the family of spectra T such that $\Omega^\infty(T \wedge W)$ is $K(n)$-acyclic. The family $\mathcal{N}$ has the following properties:*

1. *$\mathcal{N}$ is closed under finite wedges.*
2. *$\mathcal{N}$ is closed under filtered homotopy colimits.*
3. *Suppose $T' \to T \to T''$ is a fiber sequence such that T' is a $(-k-1)$-connected spectrum in $\mathcal{N}$. Then T is in $\mathcal{N}$ if and only if T'' is in $\mathcal{N}$.*
4. *If $\widetilde{H}_{k+1}W$ is torsion, $\mathcal{N}$ contains the Eilenberg–Mac Lane spectrum $\Sigma^{n-k}HA$ for any abelian group A.*
5. *If W is $K(n)$-locally trivial, $\mathcal{N}$ contains S^0.*
6. *If W is $K(n)$-locally trivial and $\widetilde{H}_{k+1}W$ is torsion, $\mathcal{N}$ contains all $(n-k-1)$-connected spectra.*

Remark 8. In particular, since $\Sigma^\infty W$ is k-connected the assumption on $H_{k+1}W$ holds automatically with k replaced by $(k-1)$. Therefore, $\Sigma^{n-k+1}HA$ is in $\mathcal{N}$ for any A, and if W is $K(n)$-acyclic all $(n-k)$-connected spectra are in $\mathcal{N}$.

Proof. We will prove these items individually.

1. There is a weak equivalence

$$\Omega^\infty(\vee_{i=1}^N T_i \wedge W) \to \prod_{i=1}^N \Omega^\infty(T_i \wedge W),$$

and so this follows from Morava K-theory's Künneth formula

$$K(n)_*(X \times Y) \cong K(n)_*X \underset{K(n)_*}{\otimes} K(n)_*Y.$$

2. The functor $\Omega^\infty(T \wedge W)$ preserves filtered homotopy colimits in T, and so there is an isomorphism

$$\operatorname{colim} K(n)_*(\Omega^\infty(T_\alpha \wedge W)) \cong K(n)_*(\Omega^\infty((\operatorname{hocolim} T_\alpha) \wedge W)).$$

   Therefore, $\operatorname{hocolim} T_\alpha$ is in $\mathcal{N}$ if the T_α are.

3. The spectrum $T' \wedge W$ is connective, and so the result follows from a direct application of Lemma 6.

4. The spectrum $\Sigma^{n-k} H\mathbb{Z} \wedge W$ is an n-connected generalized Eilenberg–Mac Lane spectrum, and so there is a weak equivalance

$$\Omega^\infty(\Sigma^{n-k} H\mathbb{Z} \wedge W) \xrightarrow{\sim} \prod_{i=n+1}^{\infty} K(\widetilde{H}_{k-n+i} W, i).$$

   By the work of Ravenel–Wilson [RW80], $K(n)_* K(A, i)$ is trivial if $i > n+1$ or if $i = n+1$ and A is a torsion abelian group, and so by the Künneth formula $\Sigma^{n-k} H\mathbb{Z}$ is in $\mathcal{N}$.

   Applying item 1, we find $\Sigma^{n-k} H(\mathbb{Z}^N)$ is in $\mathcal{N}$; applying item 2, we find $\Sigma^{n-k} HF$ is in $\mathcal{N}$ whenever F is free abelian; applying item 3 to the fiber sequence

$$\Sigma^{n-k} HR \to \Sigma^{n-k} HF \to \Sigma^{n-k} HA$$

   associated to a free resolution $0 \to R \to F \to A \to 0$, we find that $\Sigma^{n-k} HA$ is in $\mathcal{N}$.

5. The Snaith splitting [Sna74] shows that we have a decomposition

$$\Sigma^\infty (QW)_+ \simeq \bigvee_{k \geq 0} (\Sigma^\infty W^{\wedge k})_{h\Sigma_k}.$$

   The spectra $\Sigma^\infty W^{\wedge k} \to *$ are $K(n)$-locally trivial for $k > 0$, and so the same is true of the homotopy orbit spectra.

Note that Snaith splitting is required to deduce this equivalence on the level of Morava K-theory because the equivalence does not hold on the level of mod-p homology. In particular, for symmetric smash powers the $K(n)$-homology $K(n)_*((Z^{\wedge k})_{h\Sigma_k})$ is not a functor of $K(n)_*(Z)$.

6. First suppose that T is connective. By items 5 and 3, $\mathcal{N}$ contains any sphere S^i for $i \geq 0$. By items 1 and 2, $\mathcal{N}$ contains any wedge $\vee S^i$ for $i \geq 0$. By item 3, induction on the dimension shows that $\mathcal{N}$ contains any connective finite-dimensional CW-spectrum. By item 2, $\mathcal{N}$ then contains any connective spectrum.

   We can now prove the general case. Suppose T is $(n-k-1)$-connected with $n-k<0$, and consider the following portion of the Whitehead tower of T:

$$\begin{tikzcd} T[0,\infty) \arrow[r] & T[-1,\infty) \arrow[r] \arrow[d] & \dots \arrow[r] & T[n-k+1,\infty) \arrow[r] \arrow[d] & T \arrow[d] \\ & \Sigma^{-1} H\pi_{-1}T & & \Sigma^{n-k+1} H\pi_{n-k+1}T & \Sigma^{n-k} H\pi_{n-k}T \end{tikzcd}$$

   We have just shown that $T[0,\infty)$ is in $\mathcal{N}$ because it is connective, and the spectra $\Sigma^i H\pi_i T$ are in $\mathcal{N}$ for $n-k \leq i \leq -1$ by item 4. By inductively applying item 3 we find that T is in $\mathcal{N}$. □

Proposition 9. *The natural map $\Omega Q F_m \to *$ is a rational homology equivalence for $m > 1$, a p-local equivalence for m not a power of p, and a $K(n)$-local equivalence if $m > p^n$.*

Proof. When $m = 1$ there is nothing to show. When $m > 1$ the suspension spectrum of F_m is k-connected for some $k \geq 2m-1$. The space F_m is also rationally trivial and p-locally trivial unless m is a power of p, so the rational homology and homotopy groups are always torsion and have p-torsion only when m is a power of p; hence the same is true for both $\Sigma^\infty F_m$ and $\Omega Q F_m$. If $m > p^n$ then

$$n-k-1 \leq n-(2m-1)-1 < n-2p^n \leq 1-2p < -2,$$

so S^{-1} is $(n-k-1)$-connected. By Proposition 7 item 6, we then find that $\Omega Q F_m = \Omega^\infty(S^{-1} \wedge F_m)$ is $K(n)$-locally trivial. □

3 Decomposition of MU

Definition 10. For each $m \geq 0$, let MX_m be the Thom spectrum of the infinite loop map $X_m \to BU$.

From the sequence (6) of infinite loop spaces over BU, we obtain a filtration of MU by E_∞ ring spectra:

$$\mathbb{S} \to MX_1 \to MX_2 \to MX_3 \to \cdots \tag{7}$$

Proposition 11. *For any associative MU-algebra E such that $\Omega QF_m \to *$ is an E_*-isomorphism, the map $MX_{m-1} \to MX_m$ induces an isomorphism in E-homology.*

Proof. After smashing with MU, the Thom diagonal makes the sequence of equation (7) equivalent to the sequence of MU-algebras

$$MU \to MU[X_1] \to MU[X_2] \to \cdots \to MU[BU],$$

where for an E_∞ space M we define $MU[M]$ to be E_∞ ring spectrum $MU \wedge M_+$.

The fiber sequence $\Omega QF_m \to X_{m-1} \to X_m$ of E_∞ spaces implies that there are equivalences

$$MU \underset{MU[\Omega QF_m]}{\wedge} (MU \wedge MX_{m-1}) \simeq MU \wedge MX_m.$$

Smashing this identification over MU with E translates this into an identity

$$E \underset{E[\Omega QF_m]}{\wedge} (E \wedge MX_{m-1}) \simeq E \wedge MX_m.$$

Since the natural map $E[\Omega QF_m] \to E$ is an equivalence by assumption, the result follows. $\square$

By work of Lazarev [Laz01], $K(n)$ admits the structure of an associative MU-algebra and so we can specialize this result to the case where E is a Morava K-theory. Combined with Proposition 9, this gives the following result.

Corollary 12. *The map $MX_{m-1} \to MX_m$ is a rational equivalence for $m > 1$, a p-local equivalence for m not a power of p, and a $K(n)$-local equivalence if $m > p^n$.*

In particular, we have the following equivalences:

$$\begin{aligned}(MX_1)_{\mathbb{Q}} &\simeq MU_{\mathbb{Q}}\\ L_{K(n)}MX_{p^n} &\simeq L_{K(n)}MU\\ L_{E(n)}MX_{p^n} &\simeq L_{E(n)}MU\end{aligned}$$

Remark 13. This filtration on MU relies only on the existence of the map $\coprod BU(n) \to \mathbb{N}$ of E_∞ spaces. In particular, this construction is naturally equivariant for the action of the cyclic group C_2, determines an equivariant filtration of the Real K-theory spectrum, and a sequence of C_2-equivariant E_∞ Thom spectra filtering the Real bordism spectrum $MU_{\mathbb{R}}$. However, the $K(n)$-local properties of this filtration appear to be less straightforward.

4 Picard groups

As described in the introduction, for an E_∞ ring spectrum E we let $\mathrm{Pic}(E)$ be the nerve of the symmetric monoidal category of smash-invertible E-modules and weak equivalences [HMS94, MS16]. The symmetric monoidal structure makes $\mathrm{Pic}(E)$ into a grouplike E_∞ space, and we write $\mathrm{pic}(E)$ for the associated spectrum. The 0-connected cover of $\mathrm{pic}(E)$ is $bgl_1(E)$.

As in [ABG+14], a map $\xi\colon X \to \mathrm{Pic}(E)$ over the path component of an invertible E-module E^ζ parametrizes families of E-modules over X with fibers equivalent to E^ζ, and there is an associated E-module Thom spectrum $M_E\xi$. (Technically, to apply the results of [ABG+14] we first need to smash with the element $E^{-\zeta} \in \pi_0\mathrm{Pic}(E)$ to move the target to $BGL_1(E)$.)

The functor sending a complex vector space to the suspension spectrum of its one-point compactification gives a map of E_∞ spaces $\coprod BU(m) \to \mathrm{Pic}(\mathbb{S})$, and the associated map of spectra is a map $ku \to \mathrm{pic}(\mathbb{S})$.

The space $\mathrm{Map}_{E_\infty}(MX_m, E)$ is naturally equivalent to the space of nullhomotopies of the composite map $x_m \to bu \to bgl_1(E)$ [AHR, ABG+]. However, the spectra x_m are 0-connected, so this is equivalent to the space of extensions in the diagram

$$\begin{tikzcd} ku \arrow[r] \arrow[d] & \mathrm{pic}(\mathbb{S}) \arrow[d] \\ b_m \arrow[r, dotted] & \mathrm{pic}(E). \end{tikzcd}$$

If we have already fixed an extension $b_{m-1} \to \mathrm{pic}(E)$, the pushout diagram (4) expresses the space of compatible extensions to b_m as the space of commutative diagrams

$$\begin{tikzcd} F_m \arrow[r] \arrow[d] & B_{m-1} \arrow[d] \\ CF_m \arrow[r, dotted] & \mathrm{Pic}(E). \end{tikzcd} \tag{8}$$

We write ξ for the diagonal composite $F_m \to \mathrm{Pic}(E)$ in this diagram.

Proposition 14. *Given an extension of $ku \to \mathrm{pic}(E)$ to a map $b_{m-1} \to \mathrm{pic}(E)$, the space of extensions to a map $b_m \to \mathrm{pic}(E)$ is equivalent to the space $\mathrm{Or}(\xi)$ of orientations of the E-module Thom spectrum $M_E\xi$ over F_m.*

Proof. We must show that the space of homotopies from ξ to a constant map is equivalent to the space $\mathrm{Or}(\xi)$ of orientations: maps of E-modules $M_E\xi \to E \wedge S^{2m}$ which restrict to an equivalence on Thom spectra at each point.

In our case, we may choose a basepoint $* \in BU(m)$ classifying the vector bundle $\mathbb{C}^m \to *$, whose image $* \to F_m \to \mathrm{Pic}(E)$ corresponds to the E-module $E \wedge S^{2m}$. We construct the following diagram of pullback squares.

$$\begin{tikzcd} \mathrm{Or}(\xi) \arrow[r] \arrow[d] & \{\xi\} \arrow[d] & \\ \mathrm{Map}_*(CF_m, \mathrm{Pic}(E)) \arrow[r] \arrow[d] & \mathrm{Map}_*(F_m, \mathrm{Pic}(E)) \arrow[r] \arrow[d] & \{E \wedge S^{2m}\} \arrow[d] \\ \mathrm{Map}(CF_m, \mathrm{Pic}(E)) \arrow[r] & \mathrm{Map}(F_m, \mathrm{Pic}(E)) \arrow[r] & \mathrm{Map}(*, \mathrm{Pic}(E)) \end{tikzcd}$$

Here the bottom map, of necessity, lands in the path component of the E-module $E \wedge S^{2m}$, and the upper-left pullback is the space $\mathrm{Or}(\xi)$ because

F_m is connected. Therefore, given this map $\xi\colon F_m \to \mathrm{Pic}(E)$, the space of extensions is equivalent to the space of orientations of the E-module Thom spectrum $M_E\xi$. $\square$

5 Orientation towers

The space of E_∞ orientations $MU \to E$ can now be expressed as the homotopy limit of the tower

$$\cdots \to \mathrm{Map}_{E_\infty}(MX_3, E) \to \mathrm{Map}_{E_\infty}(MX_2, E) \to \mathrm{Map}_{E_\infty}(MX_1, E) \to *.$$

The description of the space of extension diagrams from equation (8) is equivalent to a homotopy pullback square

$$\begin{tikzcd}
\mathrm{Map}_{E_\infty}(MX_m, E) \arrow[r] \arrow[d] & \mathrm{Map}_{E_\infty}(MX_{m-1}, E) \arrow[d] \\
\{*\} \arrow[r, "E\wedge S^{2m}"] & \mathrm{Map}_*(F_m, \mathrm{Pic}(E)),
\end{tikzcd}$$

where the bottom arrow classifies a constant map to the component of $E \wedge S^{2m}$ in $\mathrm{Pic}(E)$. The space of lifts is the space of orientations of the Thom spectrum on F_m, and so the unique obstruction to a lifting is the existence of a Thom class.

When $m = 1$, the space $\mathrm{Map}_{E_\infty}(MX_1, E)$ is the space of orientations of the Thom spectrum classified by the composite

$$BU(1) \to \coprod BU(n) \to \mathrm{Pic}(E).$$

More specifically, the Thom spectrum of this composite is $E \wedge MU(1) \simeq E \wedge BU(1)$. Orientations of this are classical complex orientations: the space of orientations of this Thom spectrum is the space of maps of E-modules $c_1\colon E \wedge BU(1) \to E \wedge S^2$ which restrict to the identity map of $E \wedge S^2$. Therefore, $\mathrm{Map}_{E_\infty}(MX_1, E)$ is naturally the space of ordinary complex orientations of E.

6 Symmetric power operations

In order to study p-local orientations by MX_p, we will need to recall the construction of power operations.

Associated to a complex vector bundle $\xi \to X$, we have the Thom space $\mathrm{Th}(\xi)$. This is functorial in maps of vector bundles which are fiberwise injections, and for the exterior Whitney sum $\boxplus$ there is a natural isomorphism

$$\mathrm{Th}(\xi \boxplus \xi') \cong \mathrm{Th}(\xi) \wedge \mathrm{Th}(\xi')$$

that is part of a strong monoidal structure on Th.

Definition 15. We define the following symmetric power functors:

$$\begin{aligned}
\mathbb{P}^{\times}_m(X) &= (X^{\times m})_{h\Sigma_m} && \text{for } X \text{ a space.}\\
\mathbb{P}^{\wedge}_m(X) &= (X^{\wedge m})_{h\Sigma_m} && \text{for } X \text{ a based space.}\\
\mathbb{P}^{\wedge_E}_m(X) &= (X^{\wedge_E m})_{h\Sigma_m} && \text{for } X \text{ an } E\text{-module.}
\end{aligned}$$

For any of the symmetric monoidal structures $\owedge$ above, we will write

$$D^{\owedge}_m \colon X^{\owedge m} \to \mathbb{P}^{\owedge}_m(X)$$

and

$$\Delta^{\owedge}_m \colon \mathbb{P}^{\owedge}_m(X \owedge Y) \to \mathbb{P}^{\owedge}_m(X) \owedge \mathbb{P}^{\owedge}_m(Y)$$

for the associated natural transformations.

For a vector bundle $\xi \to X$, there is a natural vector bundle structure on the map $\mathbb{P}^{\times}_m(\xi) \to \mathbb{P}^{\times}_m(X)$, and we have a pullback diagram of vector bundles

$$\begin{tikzcd}
\xi^{\boxplus m} \arrow[r, "D^{\times}_m"] \arrow[d] & \mathbb{P}^{\times}_m(\xi) \arrow[d] \\
X^{\times m} \arrow[r, "D^{\times}_m"'] & \mathbb{P}^{\times}_m(X).
\end{tikzcd}$$

Proposition 16. *There are natural isomorphisms:*

$$\begin{aligned}
\mathrm{Th}(\mathbb{P}^{\times}_m \xi) &\cong \mathbb{P}^{\wedge}_m \mathrm{Th}(\xi)\\
E \wedge \mathrm{Th}(\xi) &\cong M_E(\xi)\\
E \wedge \mathbb{P}^{\wedge}_m(Y) &\cong \mathbb{P}^{\wedge_E}_m(E \wedge Y)
\end{aligned}$$

Definition 17. Suppose E has a chosen complex orientation u, and let ε be the trivial complex vector bundle over a point. We write

$$t_u(\xi)\colon\ M_E\xi \to M_E(\varepsilon^{\dim(\xi)})$$

for the natural E-module complex orientation.

The map $t_u(\varepsilon)$ is the identity map of $M_E(\varepsilon) \cong E \wedge S^2$, and $t_u(\gamma_1) = c_1$ for the tautological bundle $\gamma_1 \to BU(1)$. Orientations commute with exterior sum: the strong monoidal structure of Th gives us an identification

$$t_u(\xi \boxplus \xi') \cong t_u(\xi) \wedge_E t_u(\xi').$$

Naturality of these orientations in pullback diagrams holds: for any map $f\colon\ X \to Y$ and vector bundle $\xi \to Y$ we have

$$t_u(f^*\xi) = t_u(\xi) \circ M_E(f).$$

In particular, this implies that $t_u(\xi^{\boxplus m}) = t_u(\mathbb{P}^{\times}_m\xi) \circ M_E(D^{\times}_m)$.

Definition 18. Write ρ_m for the vector bundle $\mathbb{P}^{\times}_m(\varepsilon)$ on $B\Sigma_m$, associated to the permutation representation of Σ_m on $\mathbb{C}^m$.

For this vector bundle, the naturality of $D^{\wedge_E}_m$ implies that the Thom class $t_u(\rho_m^{\boxplus k}) \circ D^{\wedge_E}_m$ is the identity map.

The identities above allow us to verify several relations between Thom classes.

Proposition 19. *For a complex vector bundle $\xi \to X$ and a point $i\colon\ * \to X$ with a chosen lift to $i\colon\ \varepsilon \to \xi$, we have the following.*

$$\begin{aligned}
t_u(\xi^{\boxplus m}) &= t_u(\mathbb{P}^{\times}_m(\xi)) \circ D^{\wedge_E}_m \\
t_u(\xi^{\boxplus m}) &= t_u(\rho_m^{\oplus \dim(\xi)}) \circ \mathbb{P}^{\wedge_E}_m(t_u(\xi)) \circ D^{\wedge_E}_m \\
t_u(\rho_m^{\oplus \dim(\xi)}) &= t_u(\mathbb{P}^{\times}_m(\xi)) \circ \mathbb{P}^{\wedge_E}_m(M_E i) \\
t_u(\rho_m^{\oplus \dim(\xi)}) &= t_u(\rho_m^{\oplus \dim(\xi)}) \circ \mathbb{P}^{\wedge_E}_m(t_u(\xi)) \circ \mathbb{P}^{\wedge_E}_m(M_E i)
\end{aligned}$$

Corollary 20. *For a complex vector bundle $\xi \to X$, the map $t_u(\rho_m) \circ \mathbb{P}^{\wedge_E}_m(t_u(\xi))$ is an orientation of the Thom spectrum $M_E(\mathbb{P}^{\times}_m(\xi))$ over $\mathbb{P}^{\times}_m X$ which coincides with $t_u(\mathbb{P}^{\times}_m(\xi))$ after restriction to $X^{\times m}$ or $B\Sigma_m$.*

7 Power operations

Assume that p is a fixed prime and E is an E_∞ ring spectrum with a chosen complex orientation u. In this section we will recall power operations on even-degree cohomology classes [Rez]; in the case $E = MU$ these were constructed by tom Dieck and Quillen, and used by Ando in his characterization of H_∞ structures [And00].

From here on, we will write $\rho = \rho_p$ for the permutation representation of Σ_p.

Definition 21. For an E-module spectrum M, we define

$$\mathcal{P}_u \colon [M, E \wedge S^{2k}]_E \to [\mathbb{P}_p^{\wedge_E} M, E \wedge S^{2pk}]_E$$

by the formula $\mathcal{P}_u(\alpha) = t_u(\rho^{\oplus k}) \circ \mathbb{P}_p^{\wedge_E}(\alpha)$.

These power operations satisfy a multiplication formula. For E-module spectra M and N with maps $\alpha \in [M, E \wedge S^{2k}]$ and $\beta \in [N, E \wedge S^{2l}]$, we can form

$$\alpha \wedge_E \beta \in [M \wedge_E N, E \wedge S^{2(k+l)}].$$

Then there is a natural identity

$$\mathcal{P}_u(\alpha \wedge_E \beta) \circ \Delta_p^{\wedge_E} = \mathcal{P}_u(\alpha) \wedge_E \mathcal{P}_u(\beta).$$

These power operations also depend on u, except when $k = 0$ where they agree with the ordinary extended power construction.

Remark 22. While the formula for the power operations is given using even spheres, it implicitly relies on a fixed identification of S^{2k} with the one-point compactification of a complex vector space.

Definition 23. For a complex vector bundle $\xi \to X$, let $j \colon B\Sigma_p \times X \to \mathbb{P}_p^{\times}(X)$ be the diagonal, and let $\rho \boxtimes \xi \to B\Sigma_p \times X$ be the external tensor vector bundle $j^*\mathbb{P}_p^{\times}(\xi)$. We define

$$P_u \colon E^{2k}(\mathrm{Th}(\xi)) \to E^{2pk}(\mathrm{Th}(\rho \boxtimes \xi))$$

by the formula $P_u(\alpha) = \mathcal{P}_u(\alpha) \circ M_E(j)$.

Definition 24. For a p-local, complex orientable multiplicative cohomology theory F, the transfer ideal $I_{tr} \subset F^*(B\Sigma_p)$ is the image of the transfer map $F^* \to F^*(B\Sigma_p)$, generated by the image of 1 under the transfer. For any Y equipped with a chosen map to $B\Sigma_p$, we also write I_{tr} for the ideal of $F^*(Y)$ generated by the image of I_{tr}.

The natural transformations P_u are multiplicative but not additive, instead satisfying a Cartan formula. The terms in the Cartan formula which obstruct additivity are transfers from the cohomology of proper subgroups of the form $\Sigma_k \times \Sigma_{p-k} \subset \Sigma_p$. If E is p-local, in the evenly-graded ring $E^{2*}(\mathrm{Th}(\rho \boxtimes \xi))$ the mixed terms in the Cartan formula are contained inside the transfer ideal $I_{tr} \cdot E^{2*}(\mathrm{Th}(\rho \boxtimes \xi))$.

Proposition 25. *The maps P_u reduce to natural maps*

$$\Psi_u \colon E^{2*}(\mathrm{Th}(\xi)) \to E^{2p*}(\mathrm{Th}(\rho \boxtimes \xi))/I_{tr}$$

that are additive and take any Thom class for ξ to a Thom class for $\rho \boxtimes \xi$. The maps Ψ_u are multiplicative, in the sense that for elements $\alpha \in E^{2*}(\mathrm{Th}(\xi))$ and $\beta \in E^{2*}(\mathrm{Th}(\xi'))$ we have $\Psi_u(\alpha\beta) = \Psi_u(\alpha)\Psi_u(\beta)$.

8 Cohomology calculations

In this section, we fix a p-local, complex orientable multiplicative cohomology theory F. Choosing a complex orientation of F, we use $\mathbb{G}$ to denote the associated formal group law over F^*, and $[n]_{\mathbb{G}}(x)$ the power series representing the associated n-fold sum $(x +_{\mathbb{G}} x +_{\mathbb{G}} \cdots +_{\mathbb{G}} x)$.

Proposition 26. *The restriction map*

$$F^*(B(\Sigma_p \wr U(1))) \to F^*(BU(1)^p) \times F^*(B\Sigma_p \times BU(1))$$

is injective.

We will discuss a proof of this result that requires more multiplicative structure from E but applies to a wider variety of objects than $BU(1)$ in Section 10.

Proof. Writing $B(\Sigma_p \times U(1)) \to B(\Sigma_p \wr U(1))$ as a map of homotopy orbit spaces $BU(1)_{h\Sigma_p} \to (BU(1)^p)_{h\Sigma_p}$, we obtain a diagram of function spectra

$$\begin{array}{ccc} F(B(\Sigma_p \wr U(1)), F) & \xrightarrow{\sim} & F(BU(1)^p, F)^{h\Sigma_p} \\ \downarrow & & \downarrow \\ F(B(\Sigma_p \times U(1)), F) & \xrightarrow[\sim]{} & F(BU(1), F)^{h\Sigma_p}. \end{array}$$

Therefore, the map on cohomology is the abutment of a map of homotopy fixed-point spectral sequences:

$$\begin{array}{ccc} H^s(\Sigma_p, F^t(BU(1)^p)) & \Longrightarrow & F^{t+s}(B(\Sigma_p \wr U(1))) \\ \downarrow & & \downarrow \\ H^s(\Sigma_p, F^t(BU(1))) & \Longrightarrow & F^{t+s}(B(\Sigma_p \times U(1))) \end{array}$$

The composite $F^*(BU(p)) \to F^*(B(\Sigma_p \wr U(1))) \to F^*(BU(1)^p)^{\Sigma_p}$ is an isomorphism. The latter map is the edge morphism in the above spectral sequence, and so the line $s = 0$ consists of permanent cycles.

As a module acted on by the group $C_p \subset \Sigma_p$, the ring $F^*(BU(1)^p) \cong F^* [\![\alpha_1, \ldots, \alpha_p]\!]$ is a direct sum of two submodules: the subring $F^* [\![c_p]\!]$ generated by the monomials $(\prod \alpha_i)^k$, and a free C_p-module with no higher cohomology. Therefore, for $s > 0$ the map

$$H^s(\Sigma_p, F^* [\![c_p]\!]) \to H^s(\Sigma_p, F^*(BU(1)^p))$$

is an isomorphism. The composite $F^* [\![c_p]\!] \to F^*(BU(1)^p) \to F^*(BU(1))$ induces an injection on cohomology. The above spectral sequences are, in positive cohomological degree, the tensor products of this injective map of groups (which consist of permanent cycles) with the cohomology spectral sequence for $F^*(B\Sigma_p)$, and so converge to an injective map. □

Corollary 27. *For any complex vector bundle $\xi \to B(\Sigma_p \wr U(1))$, two Thom classes for ξ are equivalent if and only if their restrictions to $BU(1)^p$ and $B\Sigma_p \times BU(1)$ are equivalent.*

Proof. The product of the restriction maps on the F-cohomology of Thom spaces is injective by naturality of the Thom isomorphism. □

Proposition 28. *If F^* is torsion-free, the map*

$$F^*(B\Sigma_p \times BU(1)) \to F^*(BU(1)) \times F^*(B\Sigma_p \times BU(1))/I_{tr}$$

is injective.

(This is similar to results from [HKR00], though here we do not assume that the coefficient ring F^* is local.)

Proof. The natural Künneth isomorphisms on the skeleta $\mathbb{CP}^k \subset BU(1)$ take the form

$$F^*(B\Sigma_p \times \mathbb{CP}^k) \cong F^*(B\Sigma_p) \otimes_{F^*} F^*(\mathbb{CP}^k).$$

This inverse system in k also satisfies the Mittag–Leffler condition, and so it suffices to show that the map $F^*(B\Sigma_p) \to F^* \times F^*(B\Sigma_p)/I_{tr}$ is injective. As the cyclic group $C_p \subset B\Sigma_p$ has index relatively prime to p, the left-hand map is injective in the commutative diagram

$$\begin{tikzcd}
F^*(B\Sigma_p) \arrow[r] \arrow[d, hook] & F^* \times F^*(B\Sigma_p)/I_{tr} \arrow[d] \\
F^*(BC_p) \arrow[r] & F^* \times F^*(BC_p)/I_{tr}.
\end{tikzcd}$$

It therefore suffices to prove that the bottom map is injective.

As p is not a zero divisor in F^*, the p-series $[p]_{\mathbb{G}}(x)$ is not a zero divisor in $F^*(BU(1))$, and so the cohomology ring $F^*(BC_p)$ is the quotient $F^* [\![c_1]\!] /[p]_{\mathbb{G}}(c_1)$ [HMS94]. The kernel of the map $F^*(BC_p) \to F^*$ is generated by c_1, while the transfer ideal is generated by the divided p-series $\langle p\rangle_{\mathbb{G}}(c_1) = [p]_{\mathbb{G}}(c_1)/c_1$ by naturality of the map $MU^* \to F^*$ [Qui71, 4.2]. The intersection of the ideals (c_1) and $(\langle p\rangle_{\mathbb{G}}(c_1))$ in the power series ring consists of elements $g(c_1) \cdot \langle p\rangle_{\mathbb{G}}(c_1)$ such that the constant coefficient of g is annihilated by the constant coefficient p of $\langle p\rangle_{\mathbb{G}}(c_1)$. As F^* is torsion-free, this ideal is generated by $[p]_{\mathbb{G}}(c_1)$. $\square$

Corollary 29. *For any complex vector bundle $\xi \to B\Sigma_p \times BU(1)$, two orientations for ξ are equivalent if and only if their restrictions to $BU(1)$ are equal and their images in $F^*(B\Sigma_p)/I_{tr} \otimes_{F^*(B\Sigma_p)} F^*(\mathrm{Th}(\xi))$ are equal.*

9 Orientations by MX_p

In this section, we fix a p-local E_∞ ring spectrum E such that E^* is torsion-free, together with a complex orientation u defined by a map $MX_1 \to E$. (We will continue to write $\mathbb{G}$ for the associated formal group law.) In this section we will analyze the obstruction to p-local maps from MX_p.

As in Section 5, the space of extensions of the complex orientation to an E_∞ ring map $MX_p \to E$ is the space of orientations of the Thom spectrum over F_p. The homotopy pushout diagram for F_p from equation (2) expresses the map $F_p \to B_1 \to \mathrm{Pic}(E)$ as a coherently commutative diagram

$$\begin{tikzcd} B(\Sigma_p \wr U(1)) \arrow[r] \arrow[d] & BU(p) \arrow[d, "\gamma_p"] \\ B\Sigma_p \arrow[r, "\rho"'] & \mathrm{Pic}(E). \end{tikzcd}$$

From diagram (1), the map $BU(p) \to \mathrm{Pic}(E)$ classifies the Thom spectrum $M_E\gamma_p = E \wedge M\gamma_p$ associated to the tautological bundle γ_p of $BU(p)$, while the map $B\Sigma_p \to \mathrm{Pic}(E)$ classifies the bundle $M_E\rho$ associated to the regular representation $\rho\colon \Sigma_p \to U(p)$. An orientation of the resulting Thom spectrum over F_p exists if and only if there are orientations of $M_E\gamma_p$ and $M_E\rho$ whose restrictions to $B(\Sigma_p \wr U(1))$ agree.

Proposition 30. *There exists an orientation of the Thom spectrum over F_p if and only if the orientations $t_u(\rho)$ and $t_u(\gamma_p)$ have the same restriction to $B(\Sigma_p \wr U(1))$, or equivalently if*

$$t_u(\rho) \circ \mathbb{P}_p^{\wedge_E}(\gamma_1) = t_u(\mathbb{P}_p(\gamma_1)).$$

Proof. The "if" direction is clear. In the other direction, we start by assuming that we have some pair of orientations whose restrictions agree.

Any orientation of $M_E\rho$ is of the form $a \cdot t_u(\rho)$ for some $a \in E^0(B\Sigma_p)^\times$, and similarly any orientation of $M_E\gamma_p$ is of the form $b \cdot t_u(\gamma_p)$ for some $b \in E^0(BU(p))^\times$. These restrict to $a \cdot t_u(\rho) \circ \mathbb{P}_p^{\wedge_E}(\gamma_1)$ and $b \cdot t_u(\mathbb{P}_p(\gamma_1))$ respectively.

By Corollary 20, the restrictions of these orientations to $BU(1)^p$ are $\epsilon(a) \cdot t_u(\gamma_1^{\boxplus p})$ and $b \cdot t_u(\gamma_1^{\boxplus b})$, where $\epsilon(a)$ is the natural restriction of a to $(E^0)^\times$ and we identify b with its image under the injection $E^0(BU(p)) \to E^0(BU(1)^p)$. Similarly, the restrictions of these orientations to $B\Sigma_p$ are $a \cdot t_u(\rho)$ and $\epsilon(b) \cdot t_u(\rho)$ respectively.

For these to be equal as needed, both a and b must be in the image of $(E^0)^\times$ and equal. Changing the orientation by multiplying by a^{-1} then gives the desired result. $\square$

Combining this with Corollary 20 and Corollary 29, we find the following.

Proposition 31. *There exists an orientation of the Thom spectrum over F_p if and only if the orientations $t_u(\rho)$ and $t_u(\gamma_p)$ define the same generating class after first restricting to $B\Sigma_p \times BU(1)$ and then tensoring over $F^*(B\Sigma_p)$ with $F^*(B\Sigma_p)/I_{tr}$.*

We recall for the following that, if E^* is torsion-free, we have an isomorphism

$$E^*(B\Sigma_p) \cong E^*(BC_p)^{\mathbb{F}_p^\times} = (E^* [\![z]\!] / [p]_{\mathbb{G}}(z))^{\mathbb{F}_p^\times},$$

where the action of $\mathbb{F}_p^\times$ is by $z \mapsto [i]_{\mathbb{G}}[z]$.

Theorem 32. *If E is an E_∞ ring spectrum such that E^* is p-local and torsion-free, an E_∞ orientation $MX_1 \to E$ extends to an E_∞ orientation $MX_p \to E$ if and only if the power operation Ψ_u satisfies the Ando criterion: we must have*

$$\Psi_u(c_1) = \prod_{i=0}^{p-1} (c_1 +_{\mathbb{G}} [i]_{\mathbb{G}}(z))$$

in the ring $E^{2pk}(\mathrm{Th}(\rho \boxtimes \gamma_1))/I_{tr}$.

Proof. It is necessary and sufficient, by Proposition 31, to know that the restrictions of $t_u(\rho) \circ \mathbb{P}_p^{\wedge_E}(\gamma_1)$ and $t_u(\mathbb{P}_p(\gamma_1))$ to this target ring are equal. By definition, the former restricts to $\Psi_u(t_u\gamma_1) = \Psi_u(c_1)$. The latter restricts to $t_u(\rho \boxtimes \gamma_1)$, and so this formula for the Thom class follows from the splitting principle. $\square$

10 Cohomology monomorphisms for E-theory

In this section we will show the following result, which is closely related to Proposition 26 when $X = BU(1)_+$.

Proposition 33. *Let E be a p-local, complex orientable E_∞ ring spectrum whose coefficient ring has no p-torsion, and let X be a based space with p-fold smash power $X^{(p)}$ such that E_*X is a direct sum of (unshifted) copies of E_*. Then the restriction map*

$$E^*(X^{(p)}_{hC_p}) \to E^*(X^{(p)}) \times E^*((BC_p)_+ \wedge X)$$

is a monomorphism.

For instance, this is satisfied when E is Morava E-theory and X is of finite type with $\mathbb{Z}_{(p)}$-homology concentrated in even degrees. This allows us to remove the finite type hypothesis from the cohomology theory in [BMMS86, VIII.7.3] so that it applies to Morava E-theory (e.g. see [And95, 4.4.2], [AHS04, proof of 6.1]). We would like to thank Eric Peterson and Nathaniel Stapleton for bringing this to our attention.

Proof. Write $M = F(\Sigma^\infty X^{(p)}, E)$ for the function spectrum, which has an action of C_p from the source, and $N = F(\Sigma^\infty X, E)$, with the trivial action of C_p. The homotopy fixed-point map

$$M^{hC_p} \to M,$$

on homotopy groups, becomes the map $E^*(X^{(p)}_{hC_p}) \to E^*(X^{(p)})$. On the other hand, the map

$$M^{hC_p} \to N^{hC_p}$$

becomes the map $E^*(X^{(p)}_{hC_p}) \to E^*((BC_p)_+ \wedge X)$. We want to prove that these are jointly monomorphisms.

By the assumptions on X, we have that $E \wedge X \simeq \bigvee_\alpha E$ as E-modules, and so there is a C_p-equivariant equivalence $E \wedge X^{(p)} \cong (E \wedge X)^{\wedge_E p}$. Using the decomposition of $E \wedge X$ into a wedge of copies of E, this decomposes C_p-equivariantly as an E-module into a C_p-fixed component and a C_p-free component:

$$(E \wedge X)^{\wedge}_{E(p)} \cong \left(\bigvee_{\alpha} E\right) \vee \left(\bigvee_{\beta} (C_p)_+ \wedge E\right).$$

The spectrum M is E-dual to this, and we calculate

$$\pi_* M^{hC_p} \cong \left(\prod_{\alpha} E_* [\![x]\!] / [p]_F(x)\right) \times \prod_{\beta} E_*.$$

We find that the map $M^{hC_p} \to M$ is a monomorphism on the right-hand factor. On the left-hand factor coming from the terms with trivial action, it becomes a product of projection maps

$$\prod_{\alpha} E_* [\![x]\!] / [p]_F(x) \to \prod_{\alpha} E_*$$

which send x to zero. The kernel of this consists precisely of the multiples of x. Therefore, to finish the proof, we simply need to show that the multiples of x map monomorphically into the homotopy of N^{hC_p}.

We now consider, for any finite subcomplex $Y \subset X$, the natural diagram of homotopy fixed-point and Tate spectra:

$$\begin{tikzcd}
F(X^{(p)}, E)^{hC_p} \arrow[r] \arrow[d] & F(X, E)^{hC_p} \arrow[d] \\
F(X^{(p)}, E)^{tC_p} \arrow[r] \arrow[d] & F(X, E)^{tC_p} \arrow[d] \\
F(Y^{(p)}, E)^{tC_p} \arrow[r] & F(Y, E)^{tC_p}.
\end{tikzcd}$$

The upper left-hand map is the localization

$$\left(\prod_{\alpha} E_* [\![x]\!] / [p]_F(x)\right) \times \prod_{\beta} E_* \to x^{-1} \prod_{\alpha} E_* [\![x]\!] / [p]_F(x),$$

which is a monomorphism on the multiples of x.

The bottom map is a natural transformation of functors in Y, and is evidently an equivalence when Y is S^0. Both of these functors take cofiber sequences of based spaces Y to fiber sequences of spectra, as follows. For a cofiber sequence $Y' \to Y \to Y''$, the smash power of Y has an equivariant filtration whose k'th associated graded consists of smash products of k copies of Y' and $(p-k)$ copies of $\Sigma Y''$; the terms other than $k=0$ and $k=p$ are acted on freely by C_p and do not contribute to the Tate spectrum. Therefore, the bottom map is an equivalence for any finite complex Y.

Given our space X, we observe that

$$\operatorname*{colim}_{Y} E_*Y \cong E_*X \cong \bigoplus_{\alpha} E_*$$

as Y ranges over the filtered system of finite subcomplexes of X. Therefore, for any index α the corresponding generator of E_*X lifts to E_*Y for some such Y. Any nonzero element in $\pi_*F(X^{(p)},E)^{tC_p}$ then has nonzero restriction to $\pi_*F(Y^{(p)},E)^{tC_p}$ for some Y, and hence (since the bottom map is an isomorphism) has nonzero image in $\pi_*F(X,E)^{tC_p}$. Thus, the center map is injective.

Since the center map is injective and the upper-left map is injective on multiples of x, the top map in the diagram is also injective on multiples of x as desired. □

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