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Inferring Directed Graphs for Networks from Corrupt Data-Streams*

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Abstract— **The structure of a complex networked system can be modeled as a graph with nodes representing the agents and the links describing a notion of dynamic coupling between them. Data-driven methods to identify such influence pathways is central to many application domains. However, such dynamically related data-streams originating at different sources are prone to corruption caused by asynchronous time-stamps, packet drops and noise. In this article, we provide a tight characterization of the connectivity structure of the agents that can be constructed based solely on measured data streams that are corrupted. A necessary and sufficient condition that delineates the effects of corruption on a set of nodes is obtained. Here, the generative system that yields the data admits nonlinear dynamic influences between agents and can involve feedback loops. Directed information based concepts are utilized in conjunction with tools from graphical models theory to arrive at the results.**

I. Introduction

Many complex systems lend themselves to effective modeling described by a network of interacting agents. Such modeling is prevalent in many areas such as climate science [1], biological systems [2], quantitative finance [3] and in many engineered systems like the Internet of Things [4]. In these systems, identification of influence pathways and determining the topology of the underlying interaction network is of significant interest. In many scenarios such as the power grid, metabolic pathways in cells and financial markets it is impractical, impossible or impermissible to externally influence the system. Here network structure identification has to be achieved via passive means.

Often, the measurements in such large systems are not immune to effects of noise, asynchronous sensor clocks and packet drops [5]. When dealing with problems of identifying structural and functional connectivity of a large network, there is a pressing need to rigorously study such uncertainties and address detrimental effects of corrupt data-streams on network reconstruction.

Authors in [6] leveraged multivariate Wiener filters to reconstruct the undirected topology of the generative network model. With assumptions of perfect measurements, and linear time invariant (LTI) interactions, it is established that the multivariate Wiener filter can recover the connectivity structure without recovering the directionality of the influence.

For a network of interacting agents with nonlinear, dynamic dependencies and strictly causal interactions, the authors in [7] proposed the use of directed information to determine the directed structure of the network. Sufficient conditions to recover the directed structure are provided. More recently, [8] defined and used information transfer to determine underlying causal interactions in a power network. Here too it is assumed that the data-streams are ideal with no distortions.

Despite its significance, little is known on the effects of measurement uncertainties on network reconstruction. Recently, in [9] focusing on networks with linear time-invariant interactions, authors provided locality characterization of spurious links that can appear due to data-corruption. An important insight obtained is that the spurious links if present are restricted to the neighborhood of the perturbed node. In [9], the analysis is restricted to LTI systems. Moreover, the directions of the spurious interactions is not determined.

In this article, we consider the more challenging case in which the underlying dynamics generating the data is allowed to be non-linear. and admits feedback loops. Here, the interactions are assumed to be strictly causal. We emphasize on the determination of the directions of spurious links that can arise when inferring network structure from corrupt data-streams. The knowledge of the directionality of spurious effects can lead to a better assessment of the quality of reconstruction of mutual influences, and can aid, elimination of spurious links using complimentary methods. Moreover, results can be used to determine smart stationing of high fidelity sensors and superior communication resources to limit the detection of spurious interactions. The results characterizing the corrupted reconstruction are, surprisingly, direct and non-conservative; given the difficulties that typically result in the analysis of nonlinear stochastic networked system.

Section II-A defines few terms that will be extensively used in the article. Section II-B presents the class of generative models that has been considered to effectively abstract complex networks. Next, Section III highlights corruption models that are of practical relevance. The main results and methods to identify the network structure are discussed in Section IV. Section V verifies the theoretical predictions by providing simulation results. Finally, a conclusion is provided in Section VI.

II. Preliminaries

Notations:
Y denotes a vector with y_i being i^{th} element of Y.
$z[\cdot]$ denotes a sequence and $z^{(t)}$ denotes the sequence $z[0], z[1], \dots z[t]$.
P_X represents the probability density function of a random variable X.
$X \perp\!\!\!\perp Y$ denotes that the random variables X and Y are independent.

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$i \to j$ indicates an arc or edge from node i to node j in a directed graph.
$i - j$ denotes $i \to j$ or $j \to i$ or both.
$\mathbb{E}[\cdot]$ denotes the expectation operator.

A. Definitions

In this section few graph theory notions that will be used in the article are discussed [10].

Definition 1 (Directed Graphs): A *directed graph* G is a pair (V, A) where V is a set of vertices or nodes and A is a set of edges given by ordered pairs (v_i, v_j) where $v_i, v_j \in V$. If $(v_i, v_j) \in A$, then we say that there is an edge from v_i to v_j. Also denoted as $v_i \to v_j$.

Definition 2 (Trail and Chain): Nodes $v_1, v_2, \ldots, v_k \in V$ forms a *trail* in G if for every $i = 1, 2, \ldots, k-1$ we have $v_i - v_{i+1}$. If all the edges along the trail have the same orientation, then the trail is called a *chain*.

Definition 3 (Children, Parents and Descendants): Given a directed graph $G = (V, A)$ and a node $j \in V$, the children of j are defined as $\mathcal{C}(j) := \{i | j \to i \in A\}$ and the parents of j as $\mathcal{P}(j) := \{i | i \to j \in A\}$. If there is a chain from i to j then j is called a *descendant* of i

Definition 4 (Collider): A node v_k is a *collider* in G if there are two other nodes v_i, v_j such that $v_i \to v_k \leftarrow v_i$ holds.

Definition 5 (Active Trail): A trail $v_1 - v_2 - \cdots - v_n$ in G is *active* given a set of nodes Z if one of the following statements holds for every triple $v_{m-1} - v_m - v_{m+1}$ along the trail:

a) If v_m is not a collider, then $v_m \notin Z$.
b) If v_m is a collider, then v_m or one of its descendants is in Z.

Definition 6 (d-separation): Let X, Y and Z be a set of nodes in G. X and Y are *d-separated* by Z in G if there is no active trail between any $x \in X$ and any $y \in Y$ given Z. It is denoted as d-sep (X, Z, Y).

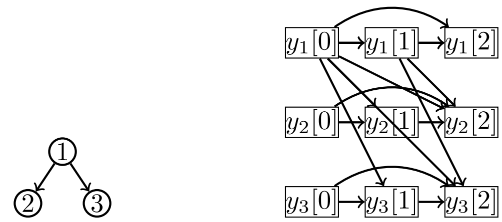
Definition 7 (Directed Cycle): A *directed cycle* from a node v_i to v_i in G has the form $v_i \to w_1 \to \cdots \to w_k \to v_i$ for some sequence of nodes $\{w_n\}_{n=1}^{k}$ in G.

Definition 8 (Directed Acyclic Graph): A directed graph with no directed cycles is called a *directed acyclic graph* (DAG).

Definition 9 (Faithful Bayesian Network): Suppose $G = (V, A)$ is a DAG whose N nodes represent random variables $y_1, \ldots, y_N$. G is called a *Bayesian Network* if for any three subsets X, Y and Z of V, d-sep(X, Z, Y) implies X is independent of Y given Z. G is called a *Faithful Bayesian network* (BN) if for any three subsets X, Y and Z of V, it holds that X and Y are independent given Z, if and only if d-sep(X, Z, Y) is true.

B. Generative Model

Here the *generative model* that is assumed to generate the measured data is described. Consider N agents that interact over a network. Let Y denote the set of all random process $\{y_1, \ldots, y_N\}$ with a parent set $\mathcal{P}(i)$ defined for $i = 1, \ldots, N$. The *generative model* for y_i is described by the structural relationship:



(a) Generative Graph G (b) DBN G' for 3 time slices

Fig. 1: This figure shows 1a generative graph, 1b its associated DBN for 3 time slices.

$$y_i[t] = f_i \left(y_i^{(t-1)}, \bigcup_{j \in \mathcal{P}(i)} y_j^{(t-1)}, e_i[t] \right) \tag{1}$$

where f_i's are arbitrary functions. To each agent we associate a discrete time sequence $y_i[\cdot]$ and another sequence $e_i[\cdot]$. The process $e_i[\cdot]$ is considered innate to agent i and thus e_i and e_j are independent for $i \neq j$. It is also assumed that $e_i[\cdot]$ is independent across time. All discrete time sequences have a finite horizon assumed to be T. The structural description of (1) induces a *generative graph* $G = (V, A)$ formed by identifying the set of vertices, V, with random processes y_i and the set of directed links, A, obtained by introducing a directed link from every element in the parent set $\mathcal{P}(i)$ of agent i to i.

For an illustration, consider the dynamics of a generative model described by:

$$\begin{aligned} y_1[t] &= f_1(y_1^{(t-1)}, e_1[t]), \\ y_2[t] &= f_2(y_1^{(t-1)}, y_2^{(t-1)}, e_2[t]), \\ y_3[t] &= f_3(y_1^{(t-1)}, y_3^{(t-1)}, e_3[t]) \end{aligned}$$

Its associated generative graph is shown in Fig. 1(a). Note that for all i in $\{1, 2, 3\}$, $i \to i$ is not shown.

Dynamic Bayesian Network (DBN)

Let $G = (V, A)$ be a generative graph. Let y_i be as defined in (1) for all $i \in V$. Consider the graph $G' = (V', A')$ where $V' = \left(\bigcup_{\substack{i \in V \\ t \in \{0,1,\ldots T\}}} y_i[t] \right)$ and $A' = \bigcup_{\substack{i \in V \\ t \in \{0,1,\ldots T\}}} \left(\bigcup_{\substack{j \in \mathcal{P}(i) \cup \{i\} \\ k \in \{0,1,\ldots t-1\}}} y_j[k] \to y_i[t] \right)$.
The joint distribution of $Y^{(T)}$ is given by:

$$P_{Y^{(T)}} = P_{y_1[0]} \ldots P_{y_N[0]} \prod_{t=1}^{T} \prod_{i=1}^{N} P_{y_i[t] | \mathcal{P}(y_i[t])} \tag{2}$$

where the parents of $y_i[t]$ are obtained from G'. It can be shown that G' is the Bayesian network for the random variables $\{y_i[t] : t = 0, 1, 2, \ldots, T, \ i = 1, 2, \ldots, N\}$ and is considered the Dynamic Bayesian Network for $\{y_i : i = 1, 2, \ldots, N\}$. See Fig. 1(b) for an illustrative example.

III. Uncertainty Description

In this section we provide a description for how uncertainty affects the time-series y_i. In particular, we provide a discussion on how the dynamic Bayesian network associated with the measured data-streams gets altered.

A. General Perturbation Models

Consider i^{th} node in a generative graph and it's associated unperturbed time-series be y_i. The corrupt data-stream u_i associated with i is assumed to follow:

$$u_i[t] = g_i(y_i^{(t)}, u_i^{(t-1)}, \zeta_i[\cdot]). \tag{3}$$

where u_i can depend dynamically on y_i and ζ_i represents a stochastic description of the corruption.

Assumption 1: Suppose ζ_i is a random process in the perturbation model (3). It is assumed that ζ_i is independent across time.

We highlight three important perturbation models that are practically relevant. See [9] for more details.

Temporal Uncertainty: Consider a node i in a generative graph G. Suppose n is the true clock index. Suppose node i measures a noisy clock index which is given by a random process $\zeta_i[n]$ described by:

$$\zeta_i[n] = \begin{cases} n + k_1, & \text{with probability } p_i \\ n + k_2, & \text{with probability } (1 - p_i). \end{cases}$$

Therefore the corruption model from (3) takes the form:

$$u_i[n] = y_i[\zeta_i[n]]. \tag{4}$$

Measurement Noise: Suppose the data-stream y_i associated with node i is corrupted with uncorrelated measurement noise $\zeta_i[\cdot]$, which is a stochastic process and is independent of $y_i[\cdot]$. Here the perturbation model is given by:

$$u_i[n] = y_i[n] + \zeta_i[n]. \tag{5}$$

Packet Drops: The measurement $u_i[n]$ corresponding to a ideal measurement $y_i[n]$ packet reception at time n can be stochastically modeled as:

$$u_i[n] = \begin{cases} y_i[n], & \text{with probability } p_i \\ u_i[n-1], & \text{with probability } (1 - p_i). \end{cases} \tag{6}$$

B. Perturbed Dynamic Bayesian Network

Let $G' = (V', A')$ be the associated dynamic Bayesian network for a generative graph $G = (V, A)$. Suppose $Z \subset V$ is the set of perturbed nodes with perturbation model described in (3). Denote the measured data-streams by $U = \{u_1, \ldots, u_N\}$. If node i is uncorrupted, then $u_i = y_i$. Otherwise, u_i is related to y_i via (3). Consider the graph $G'_Z = (V'_Z, A'_Z)$ where $V'_Z = U \cup \left(\bigcup_{\substack{k \in Z \\ t \in \{0,1,\ldots T\}}} y_k[t] \right)$ and $A'_Z = A' \cup$

$$\left(\bigcup_{\substack{k \in Z \\ i \in \{0,1,\ldots t\}}} y_k[i] \to u_k[t] \right) \cup \left(\bigcup_{\substack{k \in Z \\ i \in \{0,1,\ldots t-1\}}} u_k[i] \to u_k[t] \right)$$

for all $t \in \{0, 1, 2, \ldots, T\}$. Note that the vertex set V'_Z consists of all measurements (given by the set U) and also the uncorrupted versions y_k of the corrupted versions u_k for $k \in Z$.

Consider the set of random variables, $W = \{y_i[t] : i \in \{1, 2, \ldots, N\} \text{ and } t \in \{0, 1, 2, 3, \ldots, T\}\} \cup \{u_i[t] : i \in \{1, 2, \ldots, N\} \text{ and } t \in \{0, 1, 2, 3, \ldots, T\}\}$. The joint distribution P_W is given by:

$$P_W = \left(\prod_{i \in V} P_{u_i[0]} \right) \cdot \left(\prod_{j \in Z} P_{y_j[0]} \right) \cdot \left(\prod_{t=1}^{T} \prod_{i=1}^{N} P_{u_i[t] | \mathcal{P}(u_i[t])} \right) \cdot \left(\prod_{t=1}^{T} \prod_{j \in Z} P_{y_j[t] | \mathcal{P}(y_j[t])} \right) \tag{7}$$

where the parents of $u_i[t], y_j[t]$ are obtained from G'_Z. G'_Z is the Bayesian Network for the random variables W and is considered as the perturbed DBN (PDBN) associated with $U \cup Y$.

Fig 2. shows an example of a perturbed DBN corresponding to the generative graph in Fig. 1(a) for 3 time slices when node 1 data-streams are corrupt.

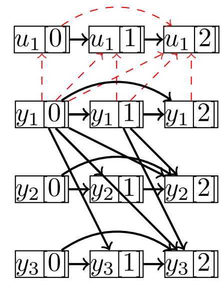


Fig. 2: Perturbed DBN G'_Z for 3 time slices when node 1 is corrupt.

IV. Network Inference using Directed Information

In this section we discuss how the structure of a generative graph can be recovered using directed information measures. First, we focus on learning from ideal measurements.

A. Inference from Ideal Data-Streams

Consider a generative graph G with N nodes and let Y denote the collection of N random processes. The authors in [7] defined and applied directed information (DI) in a network of of dynamically interacting agents, to determine if a process causally influences another. A slightly modified definition of DI as defined in [7] is:

Definition 10 (Directed Information): The directed information (DI) from node y_j to y_i is given by:

$$I(y_j \to y_i \parallel Y_{\overline{ij}}) = \mathbb{E}\left[\log \frac{P_{y_i \| y_j, Y_{\overline{ij}}}}{P_{y_i \| Y_{\overline{ij}}}} \right] \tag{8}$$

where $P_{y_i \| Y_{\bar{i}\bar{j}}} = \prod_{t=1}^{T} P_{y_i[t] | y_i^{(t-1)}, Y_{\bar{i}\bar{j}}^{(t-1)}}$ and $Y_{\bar{i}\bar{j}} = Y \setminus \{y_i, y_j\}$.

The following theorem was proved in [7] that specifies a necessary and sufficient condition to detect a presence of link in the generative graph.

***Theorem** 1: A directed edge from j to i exists in the directed graph G if and only if $I(y_j \to y_i \parallel Y_{\bar{i}\bar{j}}) > 0$.*

Note that DI is always non-negative. So, if there is no directed edge from j to i in G, then we must have that $I(y_j \to y_i \parallel Y_{\bar{i}\bar{j}}) = 0$.

B. Main Result: Spurious Links from Data Corruption

In this section, we will describe how data uncertainty will lead to non-zero directed information between links that are absent in the original graph. Our main result gives a precise characterization of the possible spurious links. In order to present the main result, some definitions are required.

Definition 11 (Perturbed Graph): Let $G = (V, A)$ be a generative graph. Suppose $Z \subset V$ is the set of perturbed nodes with each perturbation model is as described in (3). The perturbed graph $G_Z = (V, A_Z)$ is the graph such that there is an edge $i \to j \in A_Z$ if and only if there is a trail $i = v_1 - v_2 - \cdots - v_{k-1} - v_k = j$ in G such that the following conditions hold:

i) If $j \notin Z$, then $v_{k-1} \to j \in A$.
ii) For $m \in \{2, 3, \ldots, k-1\}$, if $v_{m-1} \to v_m \leftarrow v_{m+1}$, and $v_m \notin Z$, then $v_{m+1} \in Z$.
iii) If v_m is an intermediate node which is not a collider, then $v_m \in Z$.

***Remark** 1:* Note that any trail that does not violate one of the conditions guarantees that $i \to j \in A_Z$. For example, if $i \to j \in A$ then $i \to j \in A_Z$. Indeed, if $j \notin Z$ then $i \to j \in A_Z$ by condition i) we have that $i \to j \in A_Z$. On the other hand, if $j \in Z$, then there is no corresponding condition. So vacuously, none of the conditions are violated, and so we have that $i \to j \in A_Z$.

Another important case is that if $i \leftarrow j \in A$ and $j \in Z$, then $i \to j \in A_Z$. As above, this corresponds to none of the conditions above, so vacuously, none can be violated.

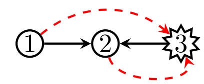
Similary, the trail could have colliders of the form $v_{m-1} \to v_m \leftarrow v_{m+1}$ with $v_m \in Z$.

Definition 12 (Spurious Links): Let $G = (V, A)$ be a generative graph, $Z \subset V$ be the set of perturbed nodes and $G_Z = (V, A_Z)$ be the perturbed graph. Spurious links are those links $i \to j \in A_Z$ that do not belong to A.

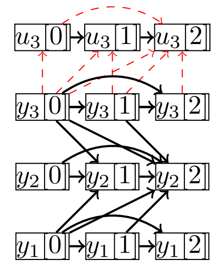
Before presenting the main result, we present an example to understand the intuition behind the presence of spurious links.

***Example** 1:* Consider a generative graph as shown in Figure 3a. Suppose node 3 is subject to data-corruption. The perturbed graph G_Z, is constructed as defined in definition 11 and is shown in figure 3b. The corresponding perturbed DBN, G'_Z, is shown for 3 time steps in figure 3c. We will reason out the presence and absence of an edge in G_Z by identifying the presence and absence of active trails in the perturbed DBN. Consider $1 \to 3 \in A_Z$. There is a trail

(a) G corresponding to ideal Measurements Y



(b) G_Z corresponding to unreliable Measurements U.



(c) Perturbed dynamic Bayesian network for 3 time steps

Fig. 3: This figure illustrates the proof of Theorem 2

$u_1[0] = y_1[0] \to u_2[1] = y_2[1] \leftarrow y_3[0] \to y_3[1] \to u_3[2] \in A'_Z$. Note that the collider $u_2[1]$ is observed. Therefore, the trail is active given $\{u_3^{(1)}, u_2^{(1)}\}$.

Take the edge $2 \to 3 \in A_Z$. There exists a trail $u_2[1] = y_2[1] \leftarrow y_3[0] \to u_3[2]$ in G'_Z. Note that the node $y_3[0]$ is not a collider and is not observed. Thus, the trail is active given $\{u_3^{(1)}, u_1^{(1)}\}$.

The edges $3 \to 1$ and $2 \to 1$ are absent in G_Z. Ideally, we look for a trail from $u_3^{(t-1)}$ to $u_1[t] = y_1[t]$ that is active given $\{u_1^{(t-1)}, u_2^{(t-1)}\}$ and a trail from $u_2^{(t-1)}$ to $u_1[t] = y_1[t]$ that is active given $\{u_1^{(t-1)}, u_3^{(t-1)}\}$. Note that every trail from $u_3^{(t-1)}$ and $u_2^{(t-1)}$ to $u_1[t]$ traverses through a node in $u_1^{(t-1)}$ which is in the observed set and this holds for all t. This blocks the information flow along the trail. Therefore, all these trails are inactive.

The following theorem is our main result. It states that the potential spurious links are precisely characterized by the perturbed graph.

***Theorem** 2: Consider a generative graph $G = (V, A)$ consisting of N nodes. Let $Z = \{v_1, \ldots, v_n\} \subset V$ be the set of n perturbed nodes where each perturbation is described by* (3). *Let $U = \{u_1, \ldots, u_N\}$ be the measured data-streams. There is a directed edge from i to j in perturbed graph, $G_Z = (V, A_Z)$, if and only if $I(u_i \to u_j \parallel U_{\bar{j}\bar{i}}) > 0$.*

Proof sketch: Let Y denote data-streams corresponding to G such that each $y_i[t]$ is described by (1) and $G' = (V', A')$ be its associated dynamic Bayesian network (DBN). The DBN associated with U is the perturbed DBN $G'_Z = (V'_Z, A'_Z)$. The perturbed graph is $G_Z = (V, A_Z)$.

Note that faithfulness assumption ensures that $I(u_i \to u_j \parallel U_{\bar{j}\bar{i}}) > 0$ if and only if there is an active trail in the perturbed DBN, G'_Z, between $u_i^{(t-1)}$ and $u_j[t]$ given $\{u_j^{(t-1)}, U_{\bar{j}\bar{i}}^{(t-1)}\}$, for some t. Thus, to prove the theorem, it suffices to show that such an active trail exists if and only if $i \to j \in A_Z$, where A_Z is the set of links in the perturbed graph. Due to space constraints we omit the full proof here. However, we provide a proof sketch that is motivated and illustrated through example 1.

($\Rightarrow$) Suppose $i \to j$ is in A_Z. Then, there is a trail described by $i = v_1 - v_2 - \cdots - v_k = j$ in G, satisfying conditions in Definition 14. Depending on whether one of i, j or both are perturbed, we have 4 cases of finding trails in G'_Z between $u_i^{(t-1)}$ and $u_j[t]$, that are active given $\{u_j^{(t-1)}, U_{\bar{j}\bar{i}}^{(t-1)}\}$. Note that these trails could be active or inactive depending whether they contain subtrails with colliders or not.

($\Leftarrow$) We will prove the contrapositive statement that if $i \to j \notin A_Z$, then no corresponding active trail exists in G'_Z. Note that if $i \to j \notin A_Z$, then either there is no trail from i to j in G, or every trail from i to j in G violates at least one of the conditions of Definition 11. Then, consider these cases separately to show that no corresponding active trail exists in G'_Z. We emphasize here that to prove this direction there is no need to assume faithfulness. We consider all corresponding trails in G'_Z and show that they are not active. ■

V. Simulation Results

To verify the predictions of Theorem 2, we estimate directed information rates (DIR), which are DI estimates averaged along the sequence length till the horizon. We use the methods proposed in [11] to compute DIR. For both the networks, the horizon length is chosen as 10^4. The DIR estimates are then averaged over 50 trials. We consider binary valued processes that admit a finite alphabet of $\{0, 1\}$.

A. Single node Perturbation

Consider a network consisting of 2 nodes with a common child as shown in Fig. 4a). The true generative model is described by:

$$
\begin{aligned}
y_1[t] &= e_1[t], \\
y_2[t] &= y_1[t-1] + y_3[t-1] + e_2[t], \\
y_3[t] &= e_3[t]
\end{aligned}
$$

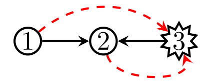
where $e_1[t] \sim$ Bernouilli(0.7), $e_2[t] \sim$ Bernouilli(0.4) and $e_3[t] \sim$ Bernouilli(0.6) and '+' is logical 'OR' operation.

The perturbation considered here is the time-origin uncertainty at node 2. The corruption model takes the form:

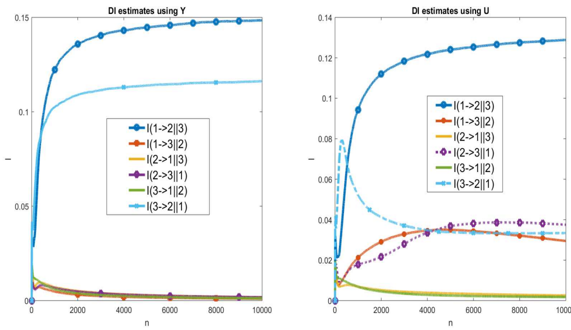
$$
u_2[t] = \begin{cases} y_2[t-2], & \text{with probability } 0.5 \\ y_2[t], & \text{with probability } 0.5. \end{cases}
$$

The perturbed graph predicted by Theorem 2 is shown in Fig. 4b). The DIR estimates from ideal (Y) and unreliable measurements (U) are shown in Fig. 4c). We observe non-zero DIR estimates and add edges to G_Z respectively. In particular, note the substantial rise in $I(u_1 \to u_3 \parallel u_2)$ and in $I(u_2 \to u_3 \parallel u_1)$. This indicates the presence of spurious links $1 \to 3$ and $2 \to 3$ in the inferred perturbed graph.

(a) Ideal Measurements Y



(b) Unreliable Measurements U.



(c) Comparison of directed information estimates between perfect measurements and corrupted data-streams. DIR I is shown along X-axis and the sample length n is along Y-axis.

Fig. 4: This figure shows how unreliable measurements at node 3 can result in spuriously inferring a direct dynamic influence of node 1 on the third and a spurious influence of node 2 on node 3.

B. Multiple Perturbation

Consider a network of 6 nodes as shown in Fig. 5a). The true generative model is given by:

$$
\begin{aligned}
y_1[t] &= e_1[t], \\
y_2[t] &= y_1[t-1] + e_2[t], \\
y_3[t] &= y_2[t-1] + e_3[t], \\
y_4[t] &= e_4[t], \\
y_5[t] &= (y_2[t-1] + y_4[t-1]) \cdot e_5[t], \\
y_6[t] &= y_5[t-1] + e_6[t]
\end{aligned}
$$

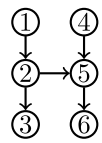
where $e_1[t] \sim$ Bernouilli(0.55), $e_2[t] \sim$ Bernouilli(0.5), $e_3[t] \sim$ Bernouilli(0.2), $e_4[t] \sim$ Bernouilli(0.4), $e_5[t] \sim$ and $e_6[t] \sim$ Bernouilli(0.3)and '+' is logical 'OR' operation while '$\cdot$' is logical 'AND' operation. The perturbations considered here are time-origin uncertainties at nodes 2 and 5. The corruption models takes the form:

$$
u_2[t] = \begin{cases} y_2[t-2], & \text{with probability } 0.5 \\ y_2[t], & \text{with probability } 0.5. \end{cases}
$$

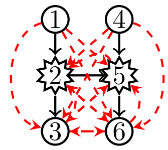
and,

$$
u_5[t] = \begin{cases} y_5[t-2], & \text{with probability } 0.5 \\ y_5[t], & \text{with probability } 0.5. \end{cases}
$$

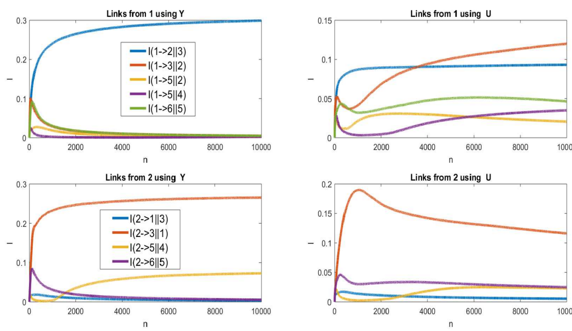
The perturbed graph predicted by Theorem 2 is shown in figure 5b). The DIR estimates from ideal (Y) and unreliable measurements (U) are shown in figures 5c) and 6. We observe non-zero DIR estimates and add edges to G_Z respectively. For clarity of visualization, only non-zero DIR estimates that would be predicted by Theorem 2 are shown.



(a) True generative graph.



(b) Network inferred from corrupt data-streams at nodes 2 and 5



(c) Comparison of directed information rate (DIR) estimates for links from nodes 1 and 2, between ideal data-streams Y and uncertain measurements U. DIR I is shown along X-axis and the sample length n is along Y-axis.

Fig. 5: 5a shows true generative graph. 5c depicts DIR estimates to detect links from nodes 1 and 2 using ideal measurements Y and when there is corruption at nodes 2 and 5. It can be clearly observed that there are lot of spurious links detected.

***Remark** 2:* Though the systems considered in the simulations are not faithful, the necessary and sufficient conditions of Theorem 2 are verified to be true. However, recall that to arrive at the assertion: if $I(u_i \to u_j \parallel U_{\overline{ji}}) > 0$ then $i \to j \in A_Z$, in the proof of Theorem 2 faithfulness was not assumed.

VI. Conclusion and Future Work

We studied the problem of inferring directed graphs for a large class of networks that admit non-linear and strictly causal interactions between several agents. Particularly, we established that inferring from corrupt data-streams can result in spurious edges and we precisely characterized the directionality of such spurious edges. Simulation results were provided to verify the theoretical predictions. Motivated by remark 2, it is imperative to analyze conditions under which the faithfulness assumptions can be relaxed. This will allow a larger class of networks and perturbation models to be studied and moreover, to formulate necessary and sufficient conditions to detect the presence of links in the inferred graph.

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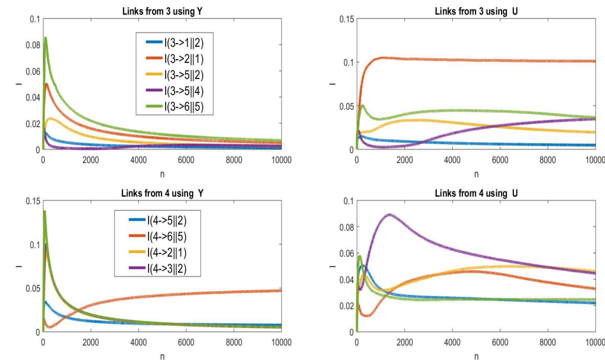
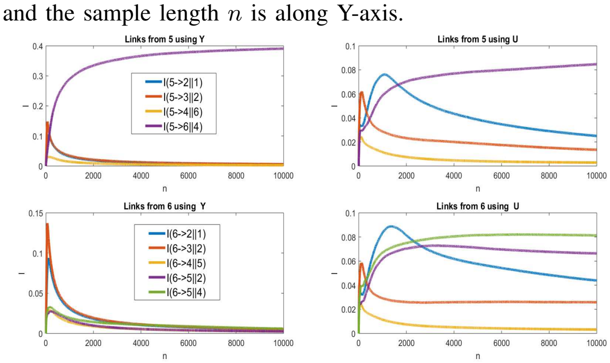


(a) A comparison of DIR estimates to detect links from nodes 3 and 4 using ideal measurements and when there is corruption at nodes 2 and 5. DIR I is shown along X-axis and the sample length n is along Y-axis.



(b) A comparison of DIR estimates to detect links from nodes 5 and 6 using ideal measurements and when there is corruption at nodes 2 and 5 is shown. DIR I is shown along X-axis and the sample length n is along Y-axis.

Fig. 6: DI estimates to detect links from nodes 3,4,5 and 6.

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