Network Restructuring Control for Conic Invariance with Application to Neural Networks

Matthew F. Singh and ShiNung Ching

Abstract— **Recent advances in the study of artificial and biological neural networks support the power of dynamic representations–computing with information stored as nontrivial limit-sets rather than fixed-point attractors. Understanding and manipulating these computations in nonlinear networks requires a theory of control for abstract objective functions. Towards this end, we consider two properties of limit-sets: their topological dimension and orientation (covariance) in phase space and combine these abstract properties into a single well-defined objective: conic control-invariant sets in the derivative space (i.e., the vector field). Real-world applications, such as neural-medicine, constrain which control laws are feasible with less-invasive controllers being preferable. To this end, we derive a feedback control-law for conic invariance which corresponds to constrained restructuring of the network connections as might occur with pharmacological intervention (as opposed to a physically separate control unit). We demonstrate the ease and efficacy of the technique in controlling the orientation and dimension of limit sets in high-dimensional neural networks.**

I. INTRODUCTION

The study and control of brain and brain-like networks is significant for two reasons. First, recurrent neural networks enable universal function approximation so that results gained in terms of neural forms may be employed upon neural approximations of general systems [1], [2]. For this reason, a substantial literature now concerns using neural networks to learn control laws for potentially non-neural systems [3], [4]. Secondly, the study of neural control is important in its own right for applications ranging from neural medicine such as transcranial stimulation to the control of artificial neuromorphic circuits (e.g. [5]). However, large scale neural systems also pose major challenges for control systems analyses as the desired objectives – for example, modulating the 'cognitive state' of an individual – may be far less concrete than those arising in conventional control problems.

Large-scale neural computations associated with cognition are increasingly being identified in two ways: either in terms of their 'functional connectivity' (the correlation pattern between neural populations [6]–[8]), or in terms of spectral coupling patterns which involve periodic behavior in at least a subset of the population (e.g. [9]). Thus, control objectives based upon cognition may involve the generation of periodic orbits with a specific orientation or potentially chaotic orbits with a desired correlation structure. These aims stand in contrast with most existing control approaches that involve stabilization of a fixed point, following extrinsically specified trajectories, and/or rendering very simple manifolds in phase space attractive (hyperplanes, facets, and simple full-synchronization manifolds). The objective of the current work is to identify closed-loop control strategies for neural systems to reach limit sets satisfying criteria in terms of topological dimension (i.e. static, periodic, or chaotic), and covariance. We will demonstrate that these goals may be condensed into a single objective of attractivity for certain geometric cones embedded in the derivative phase space (i.e., the vector field). To identify closed-loop control strategies for this objective (cone invariance) we employ a new technique involving projective geometry on a system's set of attainable Jacobian matrices. This technique reduces an implicit problem over the set of all possible Jacobians to evaluating an explicit problem at a finite number of points. In addition, all results depend only upon the maximal slope of the nonlinear (transfer) function linking neurons rather than its explicit form. Thus, results are amenable to uncertainty regarding the explicit input-output transformation being performed by network components.

II. PROBLEM STATEMENT

We seek to determine a feedback control strategy for recurrent neural systems which will satisfy two objectives: 1) an upper bound on the topological dimension of limit sets (n_ω) and 2) a specific orientation of limit sets in phase space as approximated by the covariance matrix of limiting trajectories. We define a neural control-system as a dynamical systems of the following form over $\mathbb{R}^{\mathrm{n}}$:

$$\dot{x} = \hat{W}\psi(x) - Ax + c + Bu(x) \quad (1)$$

The variable of interest, x, is the vector of neural activation with each element corresponding to one neuron. The parameters $\hat{W}$ and A are square matrices, c is a vector and ψ is a vector of C^1, monotone, univariate functions whose derivatives span a bounded interval: $\operatorname{Im}\psi_i' = [0, \operatorname{limsup}(\psi_i')]$. To ensure bounded-input-bounded-output (BIBO) properties of the system we require that $-A$ is a Hurwitz matrix (i.e. all eigenvalues of A have strictly positive real part). This formulation corresponds to a neural system with control input matrix B and an instantaneous feedback function $u(x)$: $\mathbb{R}^{\mathrm{n}} \to \mathbb{R}^{\mathrm{n}}$. At present, we consider a more restricted form in

Corresponding Author: MS is with the Departments of Neuroscience, Psychology, and Electrical and Systems Engineering at Washington University in St. Louis, USA, f.singh@wustl.edu

SC is with the Department of Electrical and Systems Engineering and Biomedical Engineering, at Washington University in St. Louis, USA, shinung@wustl.edu

MS was funded by NSF-DGE-1143954 from the US National Science Foundation. SC holds a Career Award at the Scientific Interface from the Burroughs-Wellcome Fund. Portions of this work were supported by AFOSR 15RT0189, NSF ECCS 1509342, NSF CMMI 1537015 and 1653589, from the US Air Force Office of Scientific Research and the US National Science Foundation, respectively.

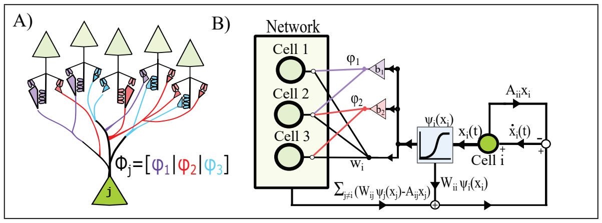

Fig. 1. Schematic of control based upon network reorganization. A) In a physiological setting, the control corresponds to manipulating the efficacy of post-synaptic connections. For pharmacological applications the constraints correspond to common chemical receptors (colors) between synapses as formalized in the constraint matrix for the pre-synaptic cell Φ_j. Feedback is not pictured in this panel. B) A traditional block-diagram of the system contains both linear and nonlinear feedback to the plant (neuron). Control is implemented by manipulating the gains for a set of amplifiers (b_1 and b_2). In the text the corresponding notation is $[b_i]_1, [b_i]_2$

which the control-rule is a linear operation on each neuron's output and parameterized as a set of constant vectors $\{b_i\}$ transformed by the corresponding $\{\Phi_i\}$:

$$Bu = \sum_i^n \Phi_i b_i \psi_i(x_i) \tag{2}$$

Here, each Φ_i is a $n \times m_i$ matrix with $m_i \leq n$. This control scheme corresponds to control by altering the connection network structure of the existing system (rewiring) rather than inserting new components into the network (i.e. a new node to implement feedback, Fig. 1). The constraints Φ_i correspond to constraints on which network structures are allowable. Intuitively, each column of Φ_i corresponds to a potential output channel for the i^{th} neuron and the role of the control vector b_i is to distribute weight among potential output channels in order to achieve the control objective. Condensing this feedback law with the existing system leads to:

$$\dot{x} = (\hat{W} + H)\psi(x) - Ax + c \tag{3}$$

$$H = [\Phi_1 b_1 \; |\Phi_2 b_2 \; | \; ...\Phi_n b_n] \tag{4}$$

This formulation is particularly relevant for pharmacological control of brain circuits for which the main method of control is altering the connection (synaptic) strengths between neurons with constraints based upon the anatomical connectivity and dependencies due to common chemical compositions between synapses (Fig. 1 A). Unlike conventional control problems the main challenge for abstract objectives relates to how the objective is phrased. In the current case we consider a desired topological dimension and covariance structure for limit sets. Neither of these goals are easily framed in terms of classic control strategies–in particular the subset of phase space which satisfies these objectives is not fixed, but rather a function of the vector field, so closed-loop control in the form of (2) generates a 'moving-target' problem: changing the control to stabilize a set also changes whether that set satisfies the objectives. The main theoretical innovation of our approach is that we re-frame objectives in terms of the closed-loop's system's derivatives. We associate our objectives with a fixed set in the derivative space: the negative quadratic cone generated by nonsingular matrix P:

$$C^-(P) := \{x \in \mathbb{R}^n \, | x^T P x < 0\}. \tag{5}$$

Without loss of generality we assume that P is symmetric, $C^-(P) = C^-(P + P^T)$. New advances in the theory of monotone dynamical systems have demonstrated that, under certain conditions, the topological dimension of all limit sets contained in P will be less than or equal to the number of P's negative eigenvalues (referred to as the rank of $C^-(P)$). This condition is referred to as $C^-(P)$-cooperativity:

Definition 1 ($C^-(P)$-cooperativity): We will say that the system $\dot{x} = f(x)$ is $C^-(P)$-cooperative if there exists a scalar-valued function $\lambda : \mathbb{R}^n \to \mathbb{R}$ such that:

$$P\,F'(x) + (P\,F'(x))^T + \lambda(x)P < 0 \quad \forall x \in \mathbb{R}^n \tag{6}$$

Throughout we use the notation ($<$) and ($\leq$) to denote the matrix ordering ($A < B \Rightarrow (B - A)$ is pos. def.) and similarly for $\leq$ and positive semi-definiteness.

$C^-(P)$-cooperativity was first introduced by Sanchez [10],[11] to generalize Poincaré-Bendixson properties to high-dimensional systems. This result was derived from the theory of monotone systems [12] and has since been extended by Feng and colleagues [13]. The power of $C^-(P)$-cooperativity is that it simultaneously limits the manner in which a system may evolve in terms of state-space orientation and the dimensionality of limit sets.

Theorem 2.1 ([10],[13]): Suppose that the system $\dot{x} = f(x)$ is $C^-(P)$-cooperative for some nonsingular, symmetric matrix P possessing precisely k negative eigenvalues. Then:

1) $\dot{x}(t) \in \text{cl}(C^-(P)) \implies \dot{x}(T) \in \text{Int}(C^-(P)) \;\; \forall T > t$
2) $If \;\; \dot{x}(t) \in \text{cl}(C^-(P)) \; then \; \omega(x) \; is \; topologically \; conjugate \; to \; an \; invariant \; set \; of \; a \; Lipschitz \; vector \; field \; in \; \mathbb{R}^k$

The notation $\omega(x)$ denotes the ω-limit set for a point $x \in \mathbb{R}^n$ with the implied vector field. Previous work has demonstrated that conic-invariance for the derivative time-series is a powerful method to discriminate (decode) brain states [14]. The cones used to classify brain states are based upon the covariance matrices. Namely, for a set of elliptical statistical distributions with covariance matrices ($\{\Sigma_i\}$) and zero mean, the Bayes-optimal conic classification rule is:

$$\text{class}(x) := \arg\min_i \frac{x^T \Sigma_i^{-1} x}{\det|\Sigma_i^{-1}|^{1/n}} \tag{7}$$

In the two-class case the classification boundary is equivalently describable by the sign of a quadratic form $x^T \tilde{P} x$:

$$\tilde{P} := \Sigma_1^{-1}\det|\Sigma_1|^{1/n} - \Sigma_2^{-1}\det|\Sigma_2|^{1/n} \tag{8}$$

so brain states defined in terms of covariance (i.e. 'functional connectivity' [6]) may be separated based upon conic-invariance for the derivative time-series (using the derivative covariances). This ability stems from the 'balancing' properties of derivatives which dictates that the mean derivative over a suitably long interval is always zero. As such, the direction of individual derivative vectors indicates which covariance distribution they were generated from and conversely, by controlling the direction of derivative vectors, we may (heuristically) control the resultant covariance structure. In the current framework, we only consider deterministic systems. The notion of covariance is thus in the context of the steady-state distribution for a given limit-set and thus describes geometric orientation [14] rather than statistical dependencies. One limitation of the current work in regards to covariance is that the objective is based upon a contrast (i.e. moving between two covariance structures) which is less parsimonious than being based upon the covariance objective alone. Likewise, the result concerning topological dimension of limit sets provides an upper limit equal to the number of P's negative eigenvalues but not an exact number. However, all theorems and derivations are for a general matrix P with no consideration for how it was chosen.

III. MAIN RESULTS

Our general aim is to ensure $C^-(P)$-cooperativity for a target matrix P. In particular, we aim to determine the set of control vectors which minimize the maximal eigenvalue of equation (6) and to determine computationally-tractable necessary and sufficient conditions to determine whether a closed loop condition satisfies $C^-(P)$-cooperativity. As the inequality, as stated, involves an infinite set of negative-definite programming problems (one for each $F'(x)$), reducing the problem without loss of generality is non-trivial. The main results are Proposition 3.1 which derives the set of control vectors which optimize an eigenvalue condition for $C^-(P)$-cooperativity, Proposition 3.2 which gives equivalent conditions for $C^-(P)$-cooperativity, and Proposition 3.3 which applies this controller and further reduces the complexity of evaluating sufficient conditions. We first present the propositions and delay their proof until the end.

As the $C^-(P)$-cooperativity conditions must hold over all Jacobians, the control choice which is most likely to satisfy inequality (6) is the one which minimizes the maximum eigenvalue over all Jacobians. However, this highly nonlinear relationship involves a very large set of matrices and may prove computationally intractable. Instead, we seek to minimize the maximal eigenvalue of each nonlinear component's contribution to the overall equation. Explicitly, we minimize

$$\hat{Q}(\{b_i\}) := \sum_j \lambda_{max}\Bigg(\text{sgn}(\Sigma)L^T([\hat{W}]_j+\Phi_j b_j)([L^{-T}]_j^T)+ \big(\text{sgn}(\Sigma)L^T([\hat{W}]_j+\Phi_j b_j)([L^{-T}]_j^T)\big)^T\Bigg) \quad (9)$$

with the notation $[X]_i$ denoting the i^{th} column of matrix X. We additionally add a minor condition which ensures that a minimum exists. In Lemma 1 we consider the alternative case to this condition ($\forall j, [\hat{W}]_j \notin \text{Im}[\Phi_j]$) in which the maximal eigenvalue may be made arbitrarily small but not zero.

Proposition 3.1 (Control Vectors): Suppose that $\forall j$, $[\hat{W}]_j \notin \text{Im}[\Phi_j]$ and each $\Phi_j^T\Phi_j$ invertible. Suppose that $J := \text{sgn}(\Sigma)L^T$ is invertible. Then the set of vectors which uniquely minimize $\hat{Q}$ (as defined in (9)) are:

$$b_i = -\beta_i\Bigg(\frac{\|(I-\Theta_i)J[\hat{W}]_i\|}{\sqrt{\|[L^{-1}]_i\|^2 - \|\Theta_i[L^{-1}]_i\|^2}}[L^{-1}]_i + J[\hat{W}]_i\Bigg),$$
$$\Theta_i := J\Phi_i\beta \;\; \beta_i := ([J\Phi_i]^T[J\Phi_i])^{-1}[J\Phi_i]^T \quad (10)$$

This proposition follows immediately from Lemma 6.1, whose proof is provided in the Appendix. We now provide the main proposition which reduces the computational load of inequality (6) in giving necessary and sufficient conditions for $C^-(P)$-cooperativity in the closed loop system. We give this Proposition in its most general form (i.e. without specifying a closed-loop control) as it is equally relevant to more complex control schemes than we currently consider.

Proposition 3.2: Consider a system of the form $\dot{x} = W\psi(x) - Ax + c$ with parameters defined as in (1) and a non-semi-definite, symmetric matrix $P \in \mathbb{S}_n$. Denote the surface of the k-sphere as $\partial\mathcal{S}^k$. Let ψ'_{max} denote the vector: $(\psi'_{max})_i := max\{\frac{\partial\psi_i}{\partial x_i}\}$ and let $\mathcal{B}^n$ denote the set of n-dimensional binary vectors: $\{0,1\}^n$. Denote the eigenvalue decomposition of P (ordered greatest to least) as

$$P = U\begin{bmatrix}\Sigma^+ & 0\\ 0 & -\Sigma^-\end{bmatrix}U^H, \quad L := U\begin{bmatrix}\sqrt{\Sigma^+} & 0\\ 0 & \sqrt{\Sigma^-}\end{bmatrix}. \quad (11)$$

Hence, $P = L(\text{sgn}(\Sigma))L^T = LJ$ with Σ being the diagonal matrix of P's eigenvalues. Let r, s denote the number of positive and negative eigenvalues of P, respectively and define the matrix function $M(x) : \mathbb{R}^n \to \mathbb{S}_n =$

$$J(W(\psi'_{max}\circ x)-A)L^{-T} + (J(W(\psi'_{max}\circ x)-A)L^{-T})^T = M(x) := \begin{bmatrix}R(x)_{r\times r} & T(x)_{r\times s}\\ T(x)^T & K(x)_{s\times s}\end{bmatrix}. \quad (12)$$

We use $a_1 \circ a_2$ to denote the diagonal matrix formed by element-wise multiplication of a_1, a_2. Then the system is $C^-(P)$-cooperative (i.e. satisfies (6)) if and only if the following implications hold $\forall x \in \mathcal{B}^n, y \in \partial\mathcal{S}^r, z \in \partial\mathcal{S}^s$:

$$(I_s - yy^T)(Tz + Ry) = (I_s - zz^T)(T^Ty + Kz) = 0 \quad (13)$$
$$\implies \begin{bmatrix}y\\ z\end{bmatrix}^T M(x)\begin{bmatrix}y\\ z\end{bmatrix} < 0. \quad (14)$$

This proposition reduces the relevant domain for $(P\,F')$ from all possible Jacobians for the system into the finite set of binary vectors. Moreover the conditions for each Jacobian are reduced from a negative-definite feasibility problem into evaluating a quadratic form over what is almost-surely a finite number of points. This relation can be equivalently stated as linearly coupled eigenvalue problems in two variables.

Corollary 2.1: the system is $C^-(P)$-cooperative (i.e. satisfies (6)) if and only if the following implications hold $\forall x \in \mathcal{B}^n, y \in \partial\mathcal{S}^r, z \in \partial\mathcal{S}^s$ with T, K, R as in (12).

$$Tz + Ry = \lambda_1 y,\; T^Ty + Kz = \lambda_2 z \implies \lambda_1 + \lambda_2 < 0 \quad (15)$$

In the following proposition 3.3 we provide sufficient (but not necessary) conditions for this inequality to be fulfilled based upon a computationally-trivial eigenvalue relation in the current context of control by network restructuring. All control laws admissable under this framework are equivalent to changing the network connection weights of the base system (equation (2)). We now present the main result.

Proposition 3.3: Consider the system described in Proposition 3.2, the closed-loop system as defined in equations (1), (2) and control vectors $\{b_i\}$ (10). Define the matrix Ω:

$$\chi_i := J(\hat{W}_i + \Phi_i b_i), \lambda_i := [L^{-T}]_i^T\chi_i + \|[L^{-T}]_i\|\|\chi_i\|$$
$$v_i := [L^{-T}]_i\|\chi_i\| + \chi_i\|[L^{-T}]_i\|, \Omega = \sum_i \lambda_i v_i v_i^T \hat{\psi}'_i \quad (16)$$

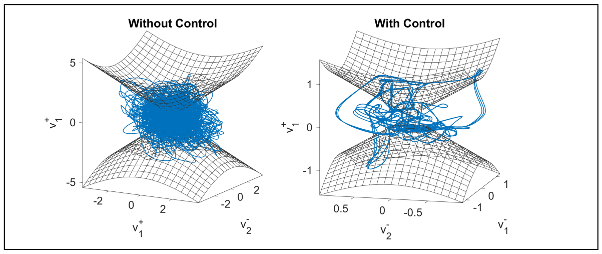

Fig. 2. Low-dimensional visualization of simulated derivative limit sets. Trajectories are plotted for the derivative ($\dot{x}(t)$) projected down into the largest positive eigenspace (denoted v^+) and the two largest negative eigenspaces (denoted v^-). Left) without control the system generates highly chaotic limit sets while the control (Right) generates complex periodic orbits within the desired space (the "outside" of the mesh). Due to the low-dimensional projection there is minor visual overlap between the projected cone and the projected trajectories despite invariance in the full-dimensional space. The cone has been rotated for visualization so that the "orientation" corresponds to the azimuth

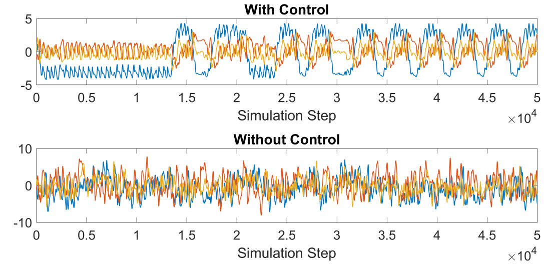

Fig. 3. Time series of the original simulated time series ($x(t)$) corresponding to Figure 2. Top) With the identified control law, neural activity converges to a complex periodic orbit. Bottom) In the absence of control, the dynamics are far less ordered. Colors correspond to each eigenspace (blue and red are negative, yellow is positive)

with $\hat{\psi}'_i = [\psi'_{max}]_i$. Define the block matrix G:

$$G = \Omega - JAL^{-T} - (JAL^{-T})^T = \begin{bmatrix} R_{r\times r} & T_{r\times s} \\ T^T & K_{s\times s} \end{bmatrix} \quad (17)$$

If there exists a positive definite matrix $\hat{S}_{r\times r}$ such that

$$\lambda_{max}(R+\hat{S}) + \lambda_{max}(K + T^T\hat{S}^{-1}T) < 0 \quad (18)$$

or a positive definite $\tilde{S}_{s\times s}$ such that

$$\lambda_{max}(R+T\tilde{S}^{-1}T^T) + \lambda_{max}(K+\tilde{S}) < 0 \quad (19)$$

Then the closed loop system is $C^-(P)$-cooperative.

IV. Simulations

To illustrate the power of the approach we simulated a recurrent neural network with 100 neurons and a rank 25 control-input matrix common to all cells (i.e. $\Phi_1 = \Phi_2 \ldots$). Elements of Φ were drawn from a standard normal distribution. Connection weights were drawn from the truncated distribution: $W_{i,j} \sim 10\ \tanh(\mathcal{N}(0,1)^3)$. We set the linear ($A$) term equal to a random positive diagonal matrix with elements drawn from: $A_{i,i} \sim 10 + 10|\mathcal{N}(0,1)|$. Values for c were drawn from $c \sim \mathcal{N}(0, 1/3)$. The nonlinear functions ψ_i were all $\tanh$ functions. The P matrix was randomly generated as well. We first generated a matrix $\hat{P}_{i,j} \sim \mathcal{N}(0,1)$ and made it positive definite through $\tilde{P} = \hat{P}\hat{P}^T$. The final matrix ($P$) was generated by negating the second and third largest eigenvalues of $\tilde{P}$. We simulated identical initial conditions for both the base and closed-loop network using Euler's method (dt=.002, 50,000 steps) and the controller as derived in Proposition 3.1. Since P was designed to have precisely two negative eigenvalues, a successful controller would generate either fixed points or periodic-orbits for any initial conditions entering the cone. Results demonstrate that this is what happened (Fig. 2,3).

V. Conclusion

We have introduced an approach for formulating and solving closed-loop control problems with abstract objectives such as the topological dimension and orientation of limit sets. We have shown that such problems may be associated with invariant cones in the derivative space. At present, we emphasize neuromorphic models and implement control by (constrained) restructuring of the network connection matrix. Our control objectives are particularly relevent for neural medicine as the covariance/orientation of brain activity differentiates among cognitive states as well as neural pathologies (e.g. [7],[9],[14]). Pharmacological interventions generally involve modifying specific sets of synaptic connections such as those utilizing a common neurotransmitter. By equating the control matrices Φ_i with binding efficacy, and the elements of each b_i with a drug's dosage, drug effects can be modeled as restructuring control. In non-biological contexts, this control problem corresponds to deciding fixed gains for subcircuits connecting neuromorphic units (Fig. 1). Future applications may benefit from also penalizing magnitudes of control solutions ('control vectors') so as to minimize side-effects in the pharmaceutical case, or power consumption in the case of neuromorphic networks.

VI. Proofs

A. Proof of Proposition 3.1

Proposition 3.1 follows immediately from applying the control conditions to the following Lemma (6.1).

Lemma 6.1: Consider a vector $c \in \mathbb{R}^n, c \neq 0$ a matrix $\Phi \in \mathbb{M}^{n,m\leq n}$ with $\Phi^T\Phi$ invertible and a nonzero vector $q \in \mathbb{R}^n$. Consider the maximal eigenvalue (λ_{max}) of

$$Q(\mu) := \lambda_{max}\big((c+\Phi\mu)q + ((c+\Phi\mu)q)^T\big). \quad (20)$$

If $\exists v$ with $\Phi v = q$ then μ may be chosen so that λ_{max} is arbitrarily close to zero ($\forall \epsilon > 0,\ \exists \varepsilon > 0 | \mu = -\varepsilon v \implies \lambda_{max} < \epsilon$). Otherwise (20) has a global minimum at

$$\mu = -\beta\left(\frac{\|(I-\Phi\beta)c\|}{\sqrt{\|q\|^2 - \|\Phi\beta q\|^2}}q + c\right). \quad (21)$$

Proof: To begin, we consider the eigenvalue decomposition of matrices formed by the symmetrized sum. For brevity we exclude the derivation. Let $p, q \in \mathbb{R}^n$ be nonzero vectors. Then $pq^T + qp^T$ has eigenvalues and corresponding eigenvectors (up to scaling):

$$\lambda = \{< p,q > + \|p\|\|q\|, < p,q > - \|p\|\|q\|\}$$
$$v = \{\|q\|p + \|p\|q, \|q\|p - \|p\|q\} \quad (22)$$

All other eigenvalues are equal to zero. By this decomposition the maximal eigenvalue is therefore $< c + \Phi\mu, q >$

$+||c+\Phi\mu|| ||q||$ which is always nonnegative. For the first case ($\exists v|\Phi v = q$) we choose $\mu = -\varepsilon v$ ($\Phi\mu = -\varepsilon q$) and consider the limit as ε approaches infinity:

$$\lim_{\varepsilon\to\infty} < c-\epsilon q, q > +||c-\epsilon q|| ||q||$$
$$= c^T q + \left(\sqrt{\varepsilon^2\|q\|^2 - 2\varepsilon c^T q + \|c\|^2} - \varepsilon\|q\|\right)\|q\| \quad (23)$$

We then factor $\varepsilon\|q\|$ from the square root so that the remaining $\|c\|^2$ term inside of the square root vanishes.

$$\lim_{\varepsilon\to\infty}\bigg(-1+\sqrt{1-2\frac{< c,q >}{\varepsilon\|q\|^2}+\frac{\|c\|^2}{\varepsilon^2\|q\|^2}}\bigg)\varepsilon\|q\|^2$$
$$= \varepsilon\|q\|^2\bigg(-1+\sqrt{\Big(1-\frac{< c,q >}{\varepsilon\|q\|^2}\Big)^2}\bigg) \quad (24)$$

Re-substituting into the original eigenvalue definition produces a limiting eigenvalue of zero which satisfies the first conclusion. Otherwise, we have that $q \notin \text{Im}(\Phi)$ which enables us to show the existence of global minima.

1) Existence of a Global Minimum (Sketch): We consider two cases. First, suppose that there is a $\kappa \geq 0|(c+\kappa q) \in span\{\Phi\}$. Then we may choose μ so that $c + \Phi\mu = -\kappa qq^T$ for which $\lambda_{max} = 0$ is the global minimum. Therefore, suppose the opposite, that no such κ exists. In this case, there is no μ such that $\lambda_{max} = Q(\mu)= 0$ (Cauchy-Schwartz inequality), so a global minimum exists iff. there is no sequence $\{\mu_j, ..\} \subseteq span\{\Phi\}$ for which $Q(\mu_j)$ approaches zero. Suppose such a sequence does exist. We know that $Q(\mu) = 0$ iff.$\exists\varepsilon \leq 0|c+\Phi\mu = \varepsilon q$ (Cauchy-Schwartz inequality) so the sequence must limit onto the negative span of q. To complete the proof, show that this sequence approaches zero iff. $q \in span\{\Phi\}$ which violates the hypothesis.

2) Minimizing the Maximum Eigenvalue: Since Q is smooth and we have proved the existence of a global minimum, we can simply find that value using the gradient:

$$\nabla_\mu \lambda_{max} = \Phi^T q + \|q\|\frac{\Phi^T\Phi\mu + \Phi^T c}{\|\Phi\mu + c\|} = 0. \quad (25)$$

We define $\alpha := \|\Phi\mu + c\|$ and $\beta := (\Phi^T\Phi)^{-1}\Phi^T$ for brevity. Solving for μ produces the relation $\mu = -\beta((\alpha q)/\|q\| + c)$. We then resubstitute μ to solve for α:

$$\alpha^2 = \bigg\| -\frac{\alpha}{\|q\|}\Phi\beta q + (I - \Phi\beta)c\bigg\|^2. \quad (26)$$

Since $\Phi\beta$ is idempotent and symmetric, $(\Phi\beta)^T(I-\Phi\beta) = 0$, leaving a simple quadratic equation:

$$\alpha^2 = \frac{\alpha^2}{\|q\|^2}\|\Phi\beta q\|^2 + \|(I-\Phi\beta)c\|^2 \quad (27)$$

Solving for α and substituting in μ completes the proof. ■

B. Proof of Proposition 3.2

Proof: The proof consists of three main parts: 1) showing it is necessary and sufficient to evaluate equation (6) at the boundary points of each transfer function (0 and lim sup(ψ_i')), 2) showing that equation (6) holds if and only if $w^T P w = 0 \implies w^T\Gamma(x)w < 0$ and 3) reducing the latter condition's domain in w to only the critical points.

1) Reduction from Jacobians to Vertices: : For the first part we note that both the set of Jacobians F' and $P\,F'$ form polytopes in matrix space: affine transformations of hypercubes with the edge orientations described by the columns $P[W_i]$ and translated by the constant PA. The same polytope geometry holds for the symmetrization $P\,F' + (P\,F')^T$ within the symmetrized matrix space. Define the set:

$$\Lambda_P := \{N \in \mathbb{M}^{nxn} | \exists\lambda \in \mathbb{R} \ \ (N + N^T + \lambda P) \leq 0\} \quad (28)$$

We use $\mathbb{M}^{nxn}$ to denote the set of nxn matrices. Clearly Λ_P is a convex cone and inequality (6) is satisfied iff. $Im(PF') \subseteq \Lambda_P$. Since $\mathbb{R}^n$ is a complete metric space and ΛP is closed we have Im($P\,F'$) $\subseteq \Lambda_P$ iff. cl(Im($P\,F'$)) $\subseteq \Lambda_P$ with cl denoting the closure. We now use the Lemma:

Lemma 6.2: Consider a compact, convex set $S \subseteq \mathbb{R}^n$ and a polytope $\mathcal{P} \subset \mathbb{R}^n$ with vertices $\mathcal{V}$. Then $\mathcal{P} \subseteq S$ iff. $\mathcal{V} \subseteq S$.

Proof: Since S is a compact set in $\mathbb{R}^n$ and $\mathcal{P}$ is closed, $\mathcal{P} \subseteq S$ iff. $\partial\mathcal{P} \subseteq S$. The boundary of $\mathcal{P}$ consists of its faces $\{\mathcal{F}\}$ and each face is equivalent to the convex hull of the vertices it contains. By the definition of the convex hull $\mathcal{F}_i$ is contained in a convex set S iff the associated vertices are in S. We conclude that $\mathcal{P} \subseteq S$ iff. $\mathcal{V} \subseteq S$. ■

By applying Lemma 6.2 to the containment of $\Gamma(x)$ within Λ_P we thereby reduce this problem to the containment of the vertices $\mathcal{V}_F$. By factoring out the maximal slope in equation (12), the vertices correspond to the binary sets of length n. We now reduce containment within Λ_P into an explicit form.

2) Definite-Programming Feasability: : Using the projection orthogonal to P in matrix space (Π_P) we determine that for a symmetric matrix ($M \in \mathbb{S}_n$), $M \in \Lambda_P$ iff. $\Pi_P M \in \Pi_P\mathbb{S}_n^-$ with $\mathbb{S}^-$ denoting the set of symmetric negative semidefinite matrices which forms a closed, convex, full, and self-dual cone. We now use a second lemma:

Lemma 6.3: Consider a closed, convex, full and self-dual cone C in a real-valued Hilbert space V of dimension $n \in \mathbb{N}$. Denote the associated extremal rays of C as Ext(C) with the convex hull conv(Ext(C)) $= C$. Consider a vector $u \in V$ and denote the projection orthogonal to u as Π_u. Then $(\Pi_u C)^* \cap \Pi_u V = \Pi_u C$. Moreover for any vector $v \in \Pi_u V$, $v \in \Pi_u C$ iff. $\langle v, \Pi_u b\rangle \geq 0 \ \ \forall b \in$ Ext(C).

Proof: Part(1): Since Π_u is an orthogonal projection, $\Pi_u v = v - \langle v, u\rangle u/\|u\| \forall v \in V$. Thus the dual cone $(\Pi_u C)^*$ is the set $\{x \in V | \langle x, b - \langle u, b\rangle u/\|u\|\rangle \geq 0 \ \forall b \in C\}$ and the intersection $(\Pi_u C)^* \cap \Pi_u V = \Pi_u C$ is the set $\{x \in \Pi_u V | \langle x, b\rangle \geq 0 \ \forall b \in C\}$. As C is self-dual $\langle x, b\rangle \geq 0, \forall b \in C$ iff. $x \in C$. Thus $x \in (\Pi_u C)^* \cap \Pi_u V$ iff. $(x \in \Pi_u V) \cap C = \Pi_u C$. We conclude $(\Pi_u C)^* \cap \Pi_u V = \Pi_u C$.
Part(2): As C is equal to the convex hull of its extremal rays $b \in C^*$ implies that b may be decomposed into a positive linear combination of extremal rays. Since the extremal rays are contained in $C = C^*$ it is necessary that $\langle x, b\rangle \geq 0 \ \ \forall b \in$ Ext(C) for $x \in C^* = C$. As the outer product of x with any element $b \in C$ may be written as a positive linear combination of outer-products with extremal rays we have that it is also sufficient. Thus, $x \in C$ iff. $\langle x, b\rangle \geq 0$ $\forall b \in$ Ext(C). Moreover the fact that C is self-dual implies $x \in \Pi_u V \cap C$ iff. $x \in \Pi_u V \cap C^*$. We combine this result with the previous statement regarding extremal rays to rewrite the second (right) condition as $x \in \Pi_u V$ and $\langle x, b\rangle \geq 0$

$\forall b \in \text{Ext}(C)$. As the projection is orthogonal we may also apply it to the second term in the inner product. Hence, for $x \in \Pi_u V$, $x \in \Pi_u C$ iff. $\langle x, \Pi_u b\rangle \geq 0 \ \forall b \in \text{Ext}(C)$. ■

By applying Lemma 6.3 we determine that $M \in \mathbb{S}_n$ is contained in Λ_P iff. $\langle \Pi_P M, \Pi_P u\rangle \geq 0 \forall u \in \text{Ext}(\mathbb{S}_n^-)$. The extremal rays of $\mathbb{S}^-$ consist of the sets of negative dyadic/outer products: $\text{Ext}(\mathbb{S}_n^-) = \{-yy^T | y \in \mathbb{R}^n\}$ with the "dimensions" of $\mathbb{S}_n^-$ implied to be nxn throughout (by "dimension" of a matrix space we mean the row/column numbers of matrices for which it is composed, not the rank of the actual vector space). Substituting we have $\langle \Pi_P M, \Pi_P yy^T\rangle = \langle PM, \Pi_P yy^T\rangle \leq 0 \quad \forall y \in \mathbb{R}^n$. As $\Pi_P yy^T = \{yy^T | \langle P, yy^T\rangle = y^T P y = 0\}$ and $\langle \Pi_P A, \Pi_P B\rangle = \langle A, \Pi_P B\rangle$ the previous condition is satisfied iff. $y^T P y = 0$ implies $\langle M, yy^T\rangle = y^T M y \leq 0$. Thus the following equivalency holds for Λ_P:

$$\Lambda_P \equiv \{N \in \mathbb{M}^{n\times n} | \forall w \in \mathbb{R}^n \backslash \{0\}, \\ w^T P w = 0 \Rightarrow w^T N w \leq 0\} \quad (29)$$

Shedding the closure of Λ_P by making the inequality strict ($w^T N w < 0$) ensures definiteness.

Part 3: Reduction to Extrema: For this section we demonstrate that the condition $w^T P w = 0$ is equivalent to the conditions in equation (13). Consider the previously described eigenvalue decomposition $P = L(\text{sgn}(\Sigma))L^T = LJ$ with the eigenvalues ordered from greatest to least. Clearly, $w^T P F' w < 0$ iff. $(L^T w)^T (JF' L^{-T}(L^T w) < 0$. Denote the indices corresponding to P's positive eigenvalues as $\{r\}$ and those for the negative eigenvalues as $\{s\}$. We use r as shorthand for the number of elements in $\{r\}$ and similarly for s and $\{s\}$. Consider a vector (w) and its transformation $[yz]^T = L^T w$ with $y = [L^T w]_{\{r\}}$ and $z = [L^T w]_{\{s\}}$. Then the statement $y^T P y = 0$ is equivalent to $\|y\| = \|z\|$. The statement $y^T P y = 0 \implies y^T P F'(x) y = 0$ is therefore equivalent to $V_x(y,z) < 0, \forall y, x \neq 0)$ with

$$V_x(y,z) \quad := \quad \begin{bmatrix} y/\|y\| \\ z/\|z\| \end{bmatrix}^T J F'(x) L^{-T} \begin{bmatrix} y/\|y\| \\ z/\|z\| \end{bmatrix} \quad (30)$$

The notation V_x here is meant to denote that V is considered for a fixed value of x as opposed to denoting a partial derivative. Without loss of generality, we suppose that both y and z have unit length, hence they both inhabit closed sets, namely the surface of the r-sphere ($\partial\mathcal{S}^r$) and the s-sphere ($\partial\mathcal{S}^s$), respectively. We denote the matrix function $M(x) = (JF'(x)L^{-T} + (JF'(x)L^{-T})^T$. We denote the block matrix form, for a fixed value of x as M(x) as defined in Equation (12). Clearly $V(y,z)$ is smooth for each choice of x and obeys $4\lambda_{min}(M) \leq V \leq 4\lambda_{max}(M)$. Since this mapping is C^1 for the entire domain and maps a compact (Heine-Borel Theorem), connected set onto a bounded domain, it possesses a global maximum which is attained at some point with $\nabla V = 0$ within respect to the restricted domain. Thus inequality (30) is satisfied over the entire domain for y, z if and only if it is satisfied for all points with $\nabla V = 0$. Therefore, we evaluate the gradient of V with respect to y:

$$\frac{\partial V}{\partial y} = \frac{2}{\|y\|^2}\left[\frac{\|y\|^2 Ry - (y^T R y)y}{\|y\|} + \frac{\|y\|^2 Tz - (y^T T z)y}{\|z\|}\right] \\ = 2(I - yy^T)(Tz + Ry) \quad (31)$$

and similarly for z. We then set $\nabla V = 0$. ■

C. Proposition 3.3

Proof: (Sketch) This corollary follows from Proposition 3.2. Weyl's inequality [15] ensures that the nonlinear term's spectrum is less than that of Ω. Case 1: split G as below. Case 2) add $\tilde{S}$ to block [2,2] instead of $\hat{S}$ to [1,1].

$$G = \begin{bmatrix} R + \hat{S} & 0 \\ 0 & K + T^T \hat{S}^{-1} T \end{bmatrix} - \begin{bmatrix} \hat{S} & T \\ T^T & T^T \hat{S}^{-1} T \end{bmatrix} \quad (32)$$

The right matrix is positive semidefinite by Haynsworth inertia additivity ([16]), and the hypothesis ensure the left's quadratic term is negative as $\|y\| = \|z\|$. ■

REFERENCES

[1] S. Chen, S. A. Billings, and P. M. Grant. Non-linear system identification using neural networks, International Journal of Control, vol. 51, no. 6, pp. 1191-1214, 1990.

[2] E. B. Kosmatopoulos, M. M. Polycarpou, M. A. Christodoulou, and P.A. Ioannou, High-order neural network structures for identification of dynamical systems, IEEE TNN, vol. 6, no. 2, pp. 422-431, 1995.

[3] A. G. Barto, R. S. Sutton, and C. W. Anderson, Neuronlike adaptive elements that can solve difficult learning control problems, IEEE Tran. on Syst., Man, and Cyb., vol. SMC-13, no. 5, pp. 834-846, 1983.

[4] T. Zhang, S. S. Ge, and C. C. Hang, Adaptive neural network control for strict-feedback nonlinear systems using backstepping design, Automatica, vol. 36, pp. 1835-1846, 2000.

[5] S. Still, K. Hepp, and R.J. Douglas, Neuromorphic walking gait control, IEEE TNN, vol. 17, no. 2, pp. 496-508, March 2006.

[6] B. Biswal, F. Z. Yetkin, V. M. Haughton, and J. S. Hyde, Functional connectivity of the motor cortex of resting human brain using echo-planar mri, Mag. Res. in Med., vol. 34, no. 4, pp. 537-541, Oct. 1995.

[7] R. L. Buckner, J. Sepulcre, T. Talukdar, F. M. Krienen, H. Liu, T. Hedden, J. R. Andrews-Hanna, R. A. Sperling, and K. A. Johnson, Cortical hubs revealed by intrinsic functional connectivity: Mapping, assessment of stability, and relation to alzheimer's disease, The Journal of Neuroscience, vol. 29, no. 6, pp. 1860-1873, February 2009.

[8] M. Kafashan, S. Ching, and B. J. A. Palanca, Sevoflurane alters spatiotemporal functional connectivity motifs that link resting-state networks during wakefulness, Front. in Neural Circuits, vol 10, 2016.

[9] M. A. Belluscio, K. Mizuseki, R. Schmidt, R. Kempter and G. Buzsáki, Cross-frequency phase–phase coupling between theta and gamma oscillations in the hippocampus, Journal of Neuroscience, vol. 32, no. 2, pp. 423-435, January 2012.

[10] L. A. Sanchez, Cones of rank 2 and the poincare-bendixson property for a new class of monotone systems, Journal of Differential Equations, vol. 246, pp. 1978-1990, 2009.

[11] L. A. Sanchez, Existence of periodic orbits for high-dimensional autonomous systems, Journal of Mathematical Analysis and Applications, vol. 363, no. 2, pp. 409-418, 2010.

[12] R.A. Smith, Existence of periodic orbits of autonomous ordinary differential equations, Proceedings of the Royal Society of Edinburgh Section A: Mathematics, vol. 85, no. (1-2), pp. 153-172, 1980.

[13] L. Feng, Y. Wang, and J. Wu, Semiflows "monotone with respect to high-rank cones" on a banach space, SIAM Journal on Mathematical Analysis, vol. 49, no. 1, pp. 142-161, 2017.

[14] M. F. Singh, T. S. Braver, and ShiNung Ching, Geometric Classification of brain network dynamics via conic derivative discriminants, Journal of Neuroscience Methods, vol. 308, pp. 88-105, 2018.

[15] H. Weyl, Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen, Math. Ann., 71, pp. 441-479, 1912.

[16] E. V. Haynsworth, Determination of the inertia of a partitioned Hermitian matrix, Linear Alg. and its Appl., vol. 1, pp. 73-81, 1968.