This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TCNS.2019.2913556, IEEE Transactions on Control of Network Systems

Comment on 'Detecting Topology Variations in Networks of Linear Dynamical Systems'

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Abstract—The detectability of topology variations in dynamical networks is studied in [1]. This note points out that Proposition 2 and Theorem 4 of [1], which provide network-theoretic conditions for discerning topology variations, require a stronger assumption than controllability of the subsystem dynamics.

The article [1] presents conditions under which topology variations in a network of homogeneous linear subsystems can be detected, using measurements of the network's natural response. The conditions are developed by first characterizing discernibility of the natural responses of a nominal and modified linear system for different initial states (Lemma 1, Corollary 1, and Proposition 1), and then applying this result to the dynamical-network model of interest (Proposition 2 and following results). Specifically, Proposition 2 distills discernability for the network model into a condition phrased entirely in terms of the network's topology, specifically the spectrum of the Laplacian matrix associated with the network's graph, along with controllability of the subsystem model. This result is then used to specify conditions for detectability of topology variations (Theorems 1 and 2). Theorem 4 extends the discernibility analysis to a setting with output rather than full-state measurements.

This note points out that the condition for discernibility in Proposition 2 is not sufficient, and also that the set of indiscernable states identified in the ensuing discussion may include additional vector directions. The proof of Proposition 2 rests on the property that the eigenvectors (and generalized eigenvectors) of the network transition matrix Φ are always Kronecker products of the eigenvectors of L and $A-\alpha B$, where $\alpha \in \text{spec}(L)$ (see equation (24) in [1]). However, this is only necessarily true when the eigenvalues of $A-\alpha_i B$ corresponding to different $\alpha_i \in \text{spec}(L)$ are mutually distinct. Otherwise, if different matrices $A-\alpha_i B$ share eigenvalues, the eigenvectors of Φ may be linear combinations of such Kronecker-product vectors. In this case, as shown in the next example, discernibility and consequently detection of topology variations cannot always be distilled to a topological condition even when the subsystem is controllable. Hence, the condition in Proposition 2 is insufficient.

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Per the notation in [1], we consider an example with the following parameters:

$$A=\begin{bmatrix}7&0&0\\0&0&1\\1&0&1\end{bmatrix},\ B=\begin{bmatrix}1&1&-1\\0&-1&1\\0&0&0\end{bmatrix},$$

$$L=\begin{bmatrix}2&-1&-1&0\\-1&2&-1&0\\-1&-1&3&-1\\0&0&-1&1\end{bmatrix},\ \overline{L}=\begin{bmatrix}1&-1&0&0\\-1&2&-1&0\\0&-1&2&-1\\0&0&-1&1\end{bmatrix}.$$

We notice that the pair (A,B) is controllable, and that the Laplacian matrices L and $\overline{L}$ correspond to networks which differ by a single link. The eigenvalues of L are $\alpha=(0,1,3,4)$, while the eigenvalues of $\overline{L}$ are $\alpha=(0,.59,2,3.4)$. The two Laplacian matrices thus have only one eigenvalue in common, at $\alpha=0$; the right eigenvectors of the two matrices associated with this eigenvalue are also identical, specifically the vector with all unity entries. From Proposition 2 and the following development, the non-null indiscernible states of the network model should be a three-dimensional space, corresponding to the synchronized states of the model. Indeed, the transition matrices $\Phi=I_4\otimes A-L\otimes B$ and $\overline{\Phi}=I_4\otimes A-\overline{L}\otimes B$ are seen to have common eigenvalues at $(0,1,7)$ whose corresponding eigenvectors are identical, and specify the synchronous manifold. However, the matrices Φ and $\overline{\Phi}$ also share an eigenvalue at 1 whose algebraic multiplicity is 4. Further, any vector x of the form

$$x=\begin{bmatrix}a\\b\\c\\d\end{bmatrix}\otimes\begin{bmatrix}0\\1\\1\end{bmatrix} \tag{1}$$

is seen to be an eigenvector of both Φ and $\overline{\Phi}$ associated with the eigenvalue 1. Thus, the non-null indiscernible states form a six-dimensional space, consisting of the synchronous states as well as states of the form (1). This disagrees with Proposition 2 and the ensuing discussion in [1].

In the example above, the matrix

$$A-\alpha B=\begin{bmatrix}7-\alpha&-\alpha&\alpha\\0&\alpha&1-\alpha\\1&0&1\end{bmatrix}$$

can be seen to have an eigenvalue at 1 with corresponding right eigenvector $\begin{bmatrix}0&1&1\end{bmatrix}^\top$, for any complex α. Thus, we immediately recover that Φ has an eigenvalue at 1 with multiplicity equal to the number of nodes, and further any vector of the form (1) is a right eigenvector associated with the eigenvalue at 1. By the same argument, $\overline{\Phi}$ also has the same eigenvalue-eigenvector pairs, and the additional indiscernible states are clarified.

The example suggests that, under only the assumption that (A, B) is controllable, in general the indiscernible states cannot be determined solely based on the topology of the network. Thus, any type of topology variation – including link and node disconnection – can be indiscernible for some initial states outside the synchronous manifold.

As discussed above, the omission may be corrected by replacing, in the statement of Proposition 2, controllability of (A, B) with the stronger assumption that the eigenvalues of $A - \alpha_i B$ corresponding to different eigenvalues α_i are distinct, specifically

$$\begin{aligned} &\mathrm{spec}(A - \alpha_i B) \cap \mathrm{spec}(A - \alpha_j B) = \emptyset \\ &\qquad\qquad \forall \alpha_i, \alpha_j \in \mathrm{spec}(L) \text{ with } \alpha_i \neq \alpha_j \\ &\mathrm{spec}(A - \overline{\alpha}_i B) \cap \mathrm{spec}(A - \overline{\alpha}_j B) = \emptyset \\ &\qquad\qquad \forall \overline{\alpha}_i, \overline{\alpha}_j \in \mathrm{spec}(\overline{L}) \text{ with } \overline{\alpha}_i \neq \overline{\alpha}_j \end{aligned} \tag{2}$$

For the same reason, the above condition should be added as an assumption also in the ensuing results in the statement of Theorem 4 in [1], which is concerned with discernibility from output rather than state measurements.

A more complete characterization of the set of discernable states can be obtained by either pursuing a full eigenvector analysis of the dynamical-network model (see [2], [3]), or perhaps by exploiting the concept of a network-invariant mode [4]. We also note that the subtlety in the eigenvector analysis of the dynamical-network model discussed here has led to errors in the controllability analysis of the model (e.g. [5]).

The gap between controllability of (A, B) and condition (2) can be studied analytically. As an example, an intuitive control-theoretic interpretation can be given when B has rank one (as it happens when the network arises from the interconnection of single-input single-output systems).

In this case, B can be decomposed as the outer vector product $B = b\,c^\top$ for suitable vectors b and c. Then, controllability of (A, B) is equivalent to controllability of (A, b). Conversely, condition (2) is satisfied if and only if (A, b) is controllable and, in addition, $(A, c^\top)$ is observable. In fact, in this case we have

$$\det(\lambda I - A + \alpha B) = p(\lambda) + \alpha\, q(\lambda)$$

for any α, where $p(\lambda) = \det(\lambda I - A)$ and $q(\lambda) = c^\top \mathrm{Adj}(\lambda I - A)\, b$ with Adj matrix adjoint. Notice that the polynomials $p(\lambda)$ and $q(\lambda)$ are coprime if and only if controllability of (A, b) and observability of $(A, c^\top)$ hold. Further, when the two polynomials $p(\lambda)$ and $q(\lambda)$ are coprime, for different α_i and α_j the eigenvalues of $A - \alpha_i B$ and $A - \alpha_j B$ are always distinct because $\alpha_i = -p(\lambda)/q(\lambda)$ and $\alpha_j = -p(\lambda)/q(\lambda)$ cannot be simultaneously true.

Hence, it follows from the above arguments that, in this case, controllability of (A, B) and condition (2) are jointly satisfied for generic choices of A and B.

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