# "Let's See" – Students Play *Vector Unknown*, An Inquiry-Oriented Linear Algebra Digital Game

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*Abstract: The results we report are a product of the first iteration of a design-based study that uses a game, Vector Unknown, to support students in learning about vector equations in both algebraic and geometric contexts. While playing the game, students employed various numeric and geometric strategies that reflect differing levels of mathematical sophistication. Additionally, results indicate that students developed connections between the algebraic and geometric contexts during gameplay. The game's design was a collaborative effort between mathematics educators and computer scientists and was based on a framework that integrates inquiry-oriented instruction and inquiry-based learning (IO/IBL), game-based learning (GBL), and realistic mathematics education (RME).*

*Keywords:* Linear Algebra, Inquiry-Oriented Instruction, Game-Based Learning, Realistic Mathematics Education, Digital Game

Student results in Linear Algebra courses and the extent of students' struggles in the course are at times surprising to mathematicians and instructors. Making an enthusiastic case for the importance of linear algebra, Tucker (1993) states that Linear Algebra's "theory is so well structured and comprehensive, yet requires limited mathematical prerequisites" (p. 3). In addition, he states "Linear Algebra is … appealing because it is so powerful yet simple" (p. 4).

The limited number of prerequisites and the simplicity described by Tucker often does not translate into ease for students (Britton, 2009; Dogan, 2017; Dorier & Sierpinska, 2001; Hannah, 2016; Hillel, 2000; Stewart, 2018; Wawro, Sweeney, & Rabin, 2011). Typically, a single course of linear algebra is offered or required in undergraduate education, a situation that presents additional challenges. Tucker acknowledges and describes that "the challenge is to find a middle ground blending vector spaces and matrix methods and at a level that does not scare off the users and yet smooths the transition for mathematics majors to advanced courses" (p. 8).

The inquiry-oriented linear algebra (IOLA) curriculum was created based on principles of RME to guide students through differing levels of activity and reflection and to leverage their intuitive knowledge in the development of more formal mathematics (Andrews-Larson, Wawro, & Zandieh, 2017; Wawro, Rasmussen, Zandieh, Sweeney, & Larson, 2012; Zandieh, Wawro, & Rasmussen, 2017). Specifically, the curriculum includes a unit known as the Magic Carpet Ride (MCR) sequence that aims to support students in learning the concepts of span and linear independence (Wawro et al., 2012). Expanding IOLA and MCR into the realm of GBL, the promotion of learning by using digital games, may prove to be a productive way to support students' learning of basic linear algebra concepts. Studies show a clear relation between games and learning (Gee, 2003), especially when thoughtful learning theories are incorporated into the design of games (Gee, 2005; Gresalfi & Barnes, 2015; Williams-Pierce, 2016). The combination of these various perspectives resulted in the development of the game *Vector Unknown*.

## Theoretical Framework Utilized for Game Design

The MCR task sequence, which follows RME design principles, aligns well with the structure of game design supported by GBL. Zandieh, Plaxco, Williams-Pierce, and Amresh (2018) developed a framework aligning aspects of GBL with RME and IO/IBL instruction. In considering the three perspectives, Zandieh et al (2018) focused on four aspects of design and implementation: structure of task sequence, nature of task sequence, students' role, and teachers' role. Drawing on specific recommendations from the literature the authors identified similarities along each of these four dimensions for each of the three perspectives. Consider, for instance, the structure and nature of task sequences. Gee (2003) states "Good games operate at the outer and growing edge of a player's competence, remaining challenging, but do-able ... [therefore] they are often also pleasantly frustrating, which is a very motivating state for human beings". Similarly, Rasmussen & Kwon (2007) articulate a perspective for Inquiry-Oriented instruction when they suggest that "challenging tasks, often situated in realistic situations, serve as the starting point for students' mathematical inquiry"; they also assert that students should solve novel problems. Further, Laursen, Hassi, Kogan, & Weston (2014) state that "IBL methods invite students to work out ill-structured but meaningful problems". Our research team has drawn on the design principles of GBL and IO/IBL to convert the first task of the MCR sequence (an RME-based task) to produce the game *Vector Unknown*.

## *Vector Unknown* Gameplay

Gameplay currently consists of five levels and data from Levels 1, 2, and 5 was analyzed. The goal is to guide the rabbit to the basket; a sample screen is displayed in Figure 1. The player moves the rabbit by dragging up to two vectors from the Vector Selection area into the Vector Equation. Adjusting the scalars in front of the vectors in the Vector Equation generates a geometric representation (Predicted Path) of the linear combination. When the player has made selections and presses GO, the rabbit moves along each component vector until it reaches the sum of the rabbit's location and the outcome of the vector equation. The mathematical notation for the move is recorded in the Log.

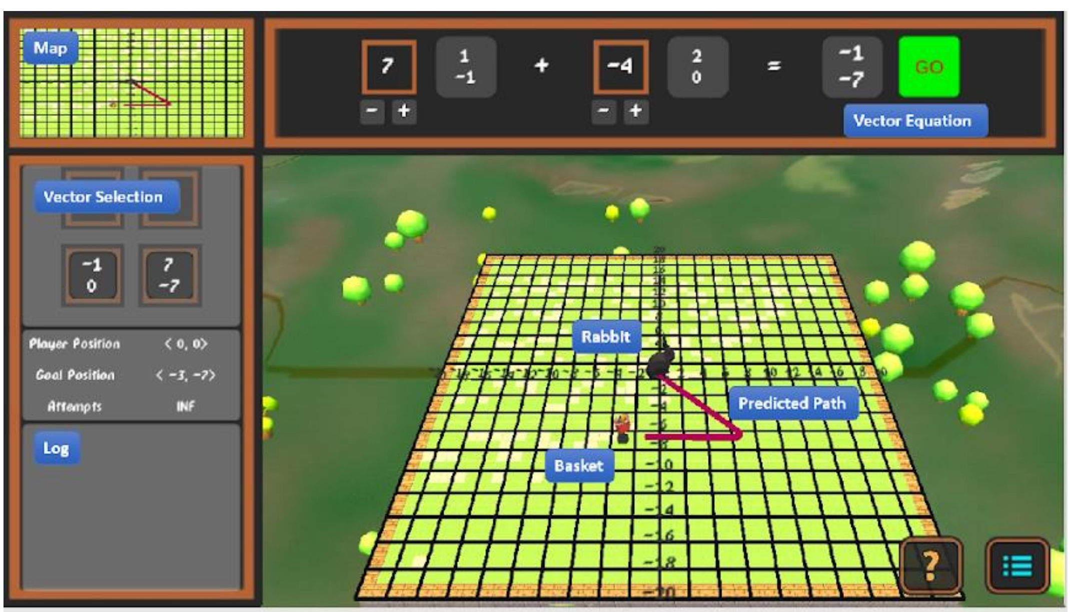


*Figure 1. Sample Screen*

The game controls reflect common mathematical notation for a vector equation. Scalars were constrained to integers and can be adjusted using plus and minus controls to encourage players 1) to make connections between numerical scalar adjustment and the corresponding change in geometry, and 2) to explore the idea of span. Each level includes a pair of linearly independent vectors along with a scalar multiple of each of the vectors. Level 2 excludes the Predicted Path provided in Level 1, requiring the player to visualize the path on their own or to find the solution using numerical methods. Level 5 includes the Predicted Path from Level 1, but the player must collect one to three keys on the board prior to approaching the basket; this requires the player to consider travel from a point other than the origin.

## Research Questions

This report presents some findings from the first iteration of a design-based research study (Cobb, Confrey, DiSessa, Lehrer, & Schauble, 2003) and will focus on answering the following questions:

1. What are students' strategies for completing the game *Vector Unknown*?
2. How do students' strategies vary according to their level of experience with linear algebra?

## Methodology and Participants

This project is a collaborative effort of three public institutions: 1) a comprehensive Research I university in the southwestern United States, 2) a multi-purpose regional university in the southeastern United States, and 3) a comprehensive Research I university in the southeastern United States. Eleven clinical interviews were conducted across the three participating universities. Each interview lasted approximately one hour, during which participants were asked to complete three levels of the *Vector Unknown* digital game. As needed, the interviewer provided help on how to navigate the game's screens and use the controls. Interviewers asked scripted questions along with impromptu follow-up questions. Impromptu questions were asked to further clarify and explore the participants' thinking about gameplay as well as any mathematical insights or strategies the participant developed during gameplay.

Participants were diverse with five students identifying as white, five students identifying as black, and one student identifying as Asian; five of the participants were males, and six were female. The students were selected to have a broad range of experience with linear algebra. Participants included Math, Biology, Computer Science, Education, and Engineering majors. The research team reviewed the interviews for strategies used in completing the game, and selected three research subjects to highlight differences in level of expertise in linear algebra. One student had never taken a linear algebra course, one was enrolled in linear algebra, and one had completed a linear algebra course a few months prior.

## Preliminary Results

### Case Study 1: Gwen - Limited Exposure to Linear Algebra

Gwen has a degree in psychology and will be taking linear algebra in preparation for graduate school. She had no experience with linear algebra prior to playing the game. Her strategy for Level 1 consisted of a trial-and-error approach with vectors and scalars selected at random. Before long Gwen began to realize that the vector equation allows for two vectors to be used simultaneously and attempted to decipher what the scalars did: "I'm trying to figure out what the orange square has to do […] is it 2 times [<0, -2>] to get me 0 over -4?" Gwen completed the Level 1 even though she "had no idea what I just did".

Gwen completed Level 1 again to gain a better understanding of what allowed her to complete the level. On her second attempt, she focused more on the numbers that would get the rabbit to the basket. Despite her numeric approach, Gwen described her strategy as "mindlessly clicking" until the trajectory path showed the correct combination of vectors and scalars. When asked to explain what happened, she responded:

> the little numbers in the orange square are […] multiplying by the numbers given. […] I guess it's what can I multiply in each of these areas to—hold on. […] I'm trying to figure out what I can multiply to get 0 on the x-axis or the numerator while at the same time getting from 0 to 12 on the denominator.

For Level 2 Gwen was more numeric than in her approach to solving the previous level: "I'm not even looking at the position of the rabbit going to the basket. I was just trying to throw in numbers until I got to the position". Level 5 contained one key before the basket unlocked. Gwen immediately selected the vector <-4, -6> from the Vector Selection and scaled it by -1 to reach the key at <4, 6>. Although Gwen had the Predicted Path, she was less dependent on it on Level 5 than on Level 1. In summary, Gwen's guess-and-check numerical strategies evolved during gameplay, and her comments seemed indicative of a growing understanding of the vector equation.

**Case Study 2: Andrew – Enrolled in a non-IOLA Linear Algebra Course**

Andrew, a senior biology and computation science major who had completed three post-secondary mathematics courses, was enrolled in linear algebra. He focused on making the vector equation yield the goal position. Only after he completed Level 5, where he had to move to the key before moving to the basket, did he begin to direct his attention to the graph. He focused so completely on the equation that he initially noticed no difference in Level 1 and Level 2. However, after he had completed Level 5 and went back to Levels 1 and 2, he noticed that Level 2 does not "show me where it would take me". He mentioned using trial-and-error and intuition and seemed to have strong number sense that allowed him to complete each level quickly.

Andrew's more inquisitive nature came out while talking about scalar multiples as illustrated by the following dialogue; his geometric conceptions seemed to be emerging.

*Interviewer*: Do you notice anything special about those vectors?

*Andrew*: About the -3 and the 9 and 3? [indicating <-3, -1> and <9, 3>] Well, one of them is both negative and one of them is both positive, and also they are multiples of each other. …

*Interviewer*: So where could you get on the board with just those two vectors?

*Andrew*: Um…can I try and see? [interviewer concurs]

*Andrew*: Alright, let's see! [Andrew moves the scalar multiples to vector equation, scales them up and down, and notes that the bunny was moving along the same line.]

*Andrew*: Alright! [nods and points] Ok, so now I see kinda what it's doing. […] if you add to this one or take away from it [referring to increasing and decreasing one of the scalars], it's still on that same line. Likewise with this one. And since this is the multiple of that one, that means that this is the dependent one on that vector.

*Interviewer*: You used the word linearly dependent. What does linearly dependent mean to you?

*Andrew*: So as far as I've learned in my linear algebra class, it means that, basically kinda like what I just said. […] it just means that if you multiply the independent vector by some scalar 1, 2, 3, whatever, -1, -2, you will be able to get that other vector, basically. [Andrew continuously clicks the mouse to change the scalars.]

Ultimately, Andrew began to make connections between the numeric and graphical ideas of linear dependence despite his strong systematic use of numerical strategies.

**Case Study 3: Lauren - Completed an IOLA Linear Algebra Course**

Lauren was a junior applied mathematics major who had completed six post-secondary mathematics courses, including linear algebra in Fall 2017. Lauren took on the conscious role of game tester and teacher during her interview. She volunteered information about aspects of gameplay that she liked and did not like without being prompted by the interviewer. This perspective precipitated in gameplay that was less focused on reaching the goal during each level and more focused on discovering how adjusting aspects of the vector equation and pressing GO resulted in different movements of the rabbit as illustrated by Figure 2.

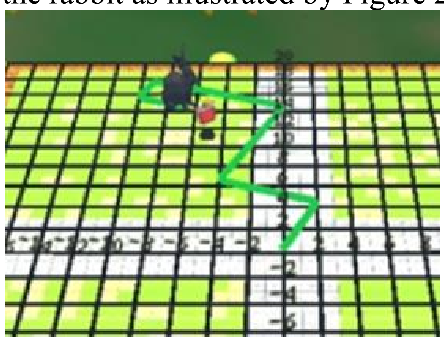

*Figure 2. Sample Path for Lauren in Level 2*

Lauren's playful nature resulted in an explanation of why two linearly independent vectors span $\mathbb{R}^2$:

> [She chooses two linearly independent vectors.] This diagonal line stretches on forever [points to Vector 1] and this diagonal stretches on forever [points to Vector 2]. However much you multiply that vector, and they start wherever you add them […] And you can start anywhere along this by shifting it [points to Vector 1]. And so you can cover the entire board by starting with this vector [points to Vector 2] anywhere along this vector [Vector 1].

In brief, Lauren used a playful geometric approach and gave indications that she was beginning to conceptualize the idea of span.

## Preliminary Conclusion/Questions for Audience

Preliminary analysis of the data reveals that students used a variety of strategies which evolved during gameplay and resulted in mathematical realizations. What are some suggestions for expanding the game to help teach span and linear independence? How could this game be incorporated into a linear algebra course? What instructional sequences in linear algebra could be translated into a level of the game?

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