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Hypergeometric–type solutions for Coulomb self–energy model of uniformly charged corona-like cylinder

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Abstract The article aims at studying hypergeometric–type mathematical techniques based on extension of the mathematical model occuring in description of the Coulomb self-energy of a uniformly charged three-dimensional cylinder. The associated *crossed term* integral is investigated and solved by computational series built by hypergeometric–type terms for different values of parameters involved.

Keywords Gaussian hypergeometric function · generalized hypergeometric function · Kampé de Fériet hypergeometric function of two variables · Bessel function of the first kind · Appell function F_4 · integral form · closed form.

Mathematics Subject Classification (2000) 33C10, 33C20, 33C65, 33C75, 33E05

1 Introduction

Application of different methods to solve a given problem may potentially result in interesting transformations and identities that otherwise are not so obvious. The aim of this work is to derive various presentations in terms of

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hypergeometric–type functions for an integral expression that is often encountered in studies of electrostatics. To this effect, we adopt various methods to solve the problem of the Coulomb self-energy of a uniformly charged hollow cylinder with inner radius, R_1 and outer radius, R_2.

Uniformly charged regular bodies have always been of great interest to physics as well as to several other scientific disciplines. They are encountered in many studies that require the knowledge of the electrostatic field/potential created by a uniformly charged body. Unfortunately, exact expressions are available only when the uniformly charged body is highly regular/symmetric. Descriptions of the electrostatic properties of several uniformly charged bodies with various symmetries (spherical, cylindrical, etc.) are widely reported in the literature [1–10]. Among these cases, one standard problem often encountered in electrostatics is that of the Coulomb self-energy of the system. This is not a simple problem as demonstrated by the case of a uniformly charged wire with finite length [11–15]. The three-dimensional (3D) version of finite wire is a uniformly solid charged cylinder and its self energy has been obtained. The aim of this work is to extend the calculations to the more challenging model of a hollow cylinder. To this effect, we first obtain expressions for the quantity of interest by applying standard integration techniques. We then implement suitable transformations involving various special functions that allow us to obtain compact representations of the final result.

The paper is organized as follows: In Section 2 we present the problem under consideration; in Section 3 the problem is solved for suitably small positive values of the input parameters, while in Section 4 the solution is derived for general parameter space. In Section 5 we discuss the results and present some concluding remarks.

2 The Problem

Let us consider a 3D cylindrical domain D as depicted in Fig. 1. The use of a cylindrical system of coordinates is obviously mandatory. It is assumed that this domain contains a total amount of charge, Q which is uniformly spread. Hence, the uniform surface charge density in this case is: $\rho(\mathbf{r}) = \rho_0 = Q/V$, where $V = \pi\,(R_2^2 - R_1^2)\,L$. The total electrostatic energy arising from the Coulomb interaction between all infinitesimal charges between the two coaxial cylinders, namely, the electrostatic Coulomb self-energy of the body is given by

$$U = \frac{k_e}{2}\int_D d\mathbf{r}\int_D d\mathbf{r'}\,\frac{\rho(\mathbf{r})\rho(\mathbf{r'})}{|\mathbf{r}-\mathbf{r'}|} = \frac{k_e\,\rho_0^2}{2}\int_D d\mathbf{r}\int_D d\mathbf{r'}\,\frac{1}{|\mathbf{r}-\mathbf{r'}|}, \qquad (1)$$

where k_e is Coulomb's electric constant and $\mathbf{r}$ ($\mathbf{r}'$) are 3D position vectors.

Computations are greatly simplified if one expands $|\mathbf{r} - \mathbf{r}'|^{-1}$ using the following formula (see [16, p. 565] or [17, p. 140]):

$$\frac{1}{|\mathbf{r}_1-\mathbf{r}_2|} = \sum_{m\in\mathbb{Z}}\int_0^\infty \mathrm{d}k\,\mathrm{e}^{\mathrm{i}\,m\,(\varphi_1-\varphi_2)}\,J_m(k\,\rho_1)\,J_m(k\,\rho_2)\,\mathrm{e}^{-k|z_1-z_2|} \qquad (2)$$

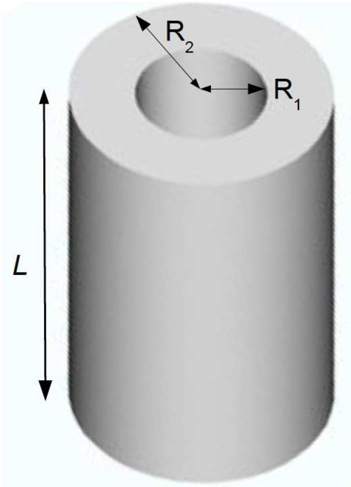

Fig. 1 Outlook of the original problem. A total charge Q is uniformly deposited between radii R_1 and R_2, with a height (or lenght) equal to L. See text for details.

where $J_m(x)$ stands for the Bessel functions of the first kind of integral order m.

Now, we shall introduce a set of dimensionless parameters that will describe the shape of two concentric cylinders such that $R_1 < r < R_2$). Let us define $c \equiv L/R_1$, $d \equiv L/R_2$ and $\gamma \equiv R_1/R_2 = d/c \in [0, 1]$.

The total electrostatic self-energy of two uniformly charged concentric cylinders such that charge is spatially distributed only inside $R_1 < r < R_2$ and $z \in [0, L]$ is given –after replacing relation (2) into (1) and integrating, by the expression

$$U = 4\pi^2 \, k_e \rho_0^2 \int_0^\infty \mathrm{d}k \left[\int_{R_1}^{R_2} \mathrm{d}\rho \, \rho \, J_0(k\,\rho) \right]^2 \frac{\mathrm{e}^{-kL} - 1 + kL}{k^2} \, . \tag{3}$$

The integral being squared is equal to

$$\frac{1}{k} \Big[R_2 \, J_1(k\,R_2) - R_1 \, J_1(k\,R_1) \Big] \, .$$

Thus, by developing the square in the previous expressions, we shall end up with three contributions, namely,

$$\frac{1}{k^2} \Big[R_2^2 \, J_1^2(k\,R_2) + R_1^2 \, J_1^2(k\,R_1) - 2\,R_1\,R_2 \, J_1(k\,R_1) \, J_1(k\,R_2) \Big] \, ,$$

that shall be integrated altogether with $k^{-2}\left(\mathrm{e}^{-kL} - 1 + kL\right)$ over the variable k. It is apparent from the previous relation that the two first contributions are identical as far as the integration over k is concerned. Physically, the two first terms will give rise to the total self-energy of two uniformly charged cylinders with radii R_2 and R_1, respectively. The last term, which we shall call from here onwards the *crossed term*, is responsible for subtracting the right amount of energy to two cylinders in order to account only for the charge present between them. As we shall see later on, this contribution, which could *a priori* looks not more complex than the other two, will indeed constitute a mathematical challenge.

Taking into account the following identity [1]

$$\left[\frac{J_1(x)}{x}\right]^2 = \frac{1}{4}\,{}_1F_2\left(\frac{3}{2}; 2, 3; -x^2\right),$$

and the following two results

$$\int_0^\infty \mathrm{d}t\, t^{-2}\,{}_1F_2\left(\frac{3}{2}; 2, 3; -t^2\right)\left(\mathrm{e}^{-ct} + ct\right) = c\left[\ln 2c - \frac{3}{4}\right] - \frac{1}{4c}\,{}_4F_3\left(\frac{1}{2}, 1, 1, \frac{5}{2}; 2, 3, 4; -\frac{4}{c^2}\right),$$

and

$$\int_0^\infty \mathrm{d}t\, t^{-2}\,{}_1F_2\left(\frac{3}{2}; 2, 3; -t^2\right) = -\frac{128}{45\pi},$$

we define the function $u \mapsto \mathscr{G}(u)$ by

$$\mathscr{G}(u) := \frac{1}{u^2}\left(u\left[\ln 2u - \frac{3}{4}\right] - \frac{1}{4u}\,{}_4F_3\left(\frac{1}{2}, 1, 1, \frac{5}{2}; 2, 3, 4; -\frac{4}{u^2}\right) + \frac{128}{45\pi}\right).$$

Finally, the total energy U (3), in units of $\frac{k_e\, Q^2}{R_2}$, can be splitted into three contributions, reads as follows

$$\frac{\mathscr{G}(c) + \gamma^3\,\mathscr{G}(c/\gamma)}{(1-\gamma^2)^2} - \frac{8\,\gamma}{(1-\gamma^2)^2 c^2}\int_0^\infty \mathrm{d}t\, t^{-4}\, J_1(\gamma\, t)\, J_1(t)\big(\mathrm{e}^{-ct} - 1 + ct\big). \quad (4)$$

Thus, we have a final expression for the total self-energy depending only on two parameters. The crossed term involving the product of two Bessel functions of the first kind is actually more involved that it may appear. In point of fact, its analytic value is extremely involved, and shall be considered in the following Sections using hypergeometric–type functions.

3 The Crossed Term Integral

In this section we give a closed form of the so–called *crossed term integral* which we derived for some appropriately small values of the parameters involved. During the derivation procedure we make use of the Appell hypergeometric function F_4, while the final result is expressed in terms of the Kampé de Fériet generalized hypergeometric function of two variables.

[1] The displays [21, p. 2307, Eqs. (C.6-9)] contain the redundantly written ${}_2F_3(\frac{3}{2}, 2; 2, 2, 3; z)$. We will write this hypergeometric term *here and in what follows* in the reduced correct form ${}_1F_2(\frac{3}{2}; 2, 3; z)$.

Consider the integral

$$I(a,b,c) := \int_0^\infty \mathrm{d}t\, t^{-4} J_1(a\,t) J_1(b\,t) \left(\mathrm{e}^{-ct} - 1 + ct\right) ,$$

where all three parameters all positive. Recall that by equating $a = \gamma$ and $b + 1$ we recover the special case considered in (4). By substituting $tc^{-1} \mapsto t$ we get

$$\begin{aligned} I(a,b,c) &= c^3 \int_0^\infty \mathrm{d}t\, t^{-4} J_1(\alpha t) J_1(\beta t) \left(\mathrm{e}^{-t} - 1 + t\right) \\ &= c^3 I\left(\frac{a}{c}, \frac{b}{c}, 1\right) =: c^3 H(\alpha, \beta)\,, \end{aligned} \tag{5}$$

where the shorthand $\alpha = ac^{-1}, \beta = bc^{-1}$ is used. Bearing in mind that

$$t^{-2}\left(\mathrm{e}^{-t} - 1 + t\right) = \int_0^1 \mathrm{d}s\,(1-s)\,\mathrm{e}^{-ts}\,,$$

by changing the order of integration we conclude

$$H(\alpha,\beta) = \int_0^1 \mathrm{d}s\,(1-s) \int_0^\infty \mathrm{d}t\, t^{-2} J_1(\alpha t) J_1(\beta t)\,\mathrm{e}^{-st} =: \int_0^1 \mathrm{d}s\,(1-s)\mathscr{I}_s\,. \tag{6}$$

Next, we express the product of Bessel functions as a double series in the following way: [2]

$$\begin{aligned} J_1(\alpha t) J_1(\beta t) &= t^{-2} \sum_{m\geq 0} \frac{(-1)^m \left(\frac{\alpha t}{2}\right)^{2m+1}}{\Gamma(m+2)\, m!} \sum_{n\geq 0} \frac{(-1)^n \left(\frac{\beta t}{2}\right)^{2n+1}}{\Gamma(n+2)\, n!} \\ &= \frac{\alpha\beta}{4} \sum_{m,n\geq 0} \frac{(-1)^{m+n} \alpha^{2m} \beta^{2n}}{4^{m+n}\,(2)_m (2)_n\, m!\, n!}\, t^{2(m+n)}\,, \end{aligned}$$

where the Pochhammer symbol notation

$$(a)_n = a(a+1)\cdots(a+n-1)$$

was used.

Thus, the inner integral appearing in (6) becomes

$$\begin{aligned} \mathscr{I}_s &= \int_0^\infty t^{-2} J_1(\alpha t) J_1(\beta t)\,\mathrm{e}^{-st}\mathrm{d}t \\ &= \frac{\alpha\beta}{4} \sum_{m,n\geq 0} \frac{(-1)^{m+n} \alpha^{2m} \beta^{2n}}{4^{m+n}\,(2)_m (2)_n\, m!\, n!} \int_0^\infty t^{2(m+n)}\,\mathrm{e}^{-st}\mathrm{d}t \\ &= \frac{\alpha\beta}{4s} \sum_{m,n\geq 0} \frac{(-1)^{m+n} \alpha^{2m} \beta^{2n}\, \Gamma(2m+2n+1)}{(4s^2)^{m+n}\,(2)_m (2)_n\, m!\, n!}\,. \end{aligned}$$

[2] The case $\alpha = \beta$ is known, see e.g. [18, p. 360, Eq. 9.1.4].

Applying here Legendre's duplication formula

$$\sqrt{\pi}\, \Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\tfrac{1}{2}), \qquad z>0\,,$$

we have

$$\mathscr{I}_s = \frac{\alpha\beta}{4\,s} \sum_{m,n\geq 0} \frac{(-1)^{m+n}(\frac{1}{2})_{m+n}\,(1)_{m+n}\,(\alpha s^{-1})^{2m}(\beta s^{-1})^{2n}}{(2)_m(2)_n\,m!\,n!}.$$

Recall the series representation of Appell's hypergeometric function of two variables [26, Eq. 16.13.4]

$$F_4(u,v;w,w';x,y) = \sum_{m,n\geq 0} \frac{(u)_{m+n}(v)_{m+n}}{(w)_m(w')_n}\frac{x^m}{m!}\frac{y^n}{n!}, \qquad \sqrt{|x|}+\sqrt{|y|}<1\,,$$

by which we deduce

$$\mathscr{I}_s = \frac{\alpha\beta}{4\,s} F_4\left(\frac{1}{2},1;2,2;-\left(\frac{\alpha}{s}\right)^2,-\left(\frac{\beta}{s}\right)^2\right)\,.$$

Obviously $\mathscr{I}_s$ is not integrable with respect intervals which contain origin as endpoint, we transform F_4 by the formula [26, Eq. 16.16.10]

$$\begin{aligned} &F_4(u,v;w,w';x,y) \\ &\quad = \frac{\Gamma(w')\Gamma(v-u)}{\Gamma(w'-u)\Gamma(v)}(-y)^{-u}\,F_4\left(u,u-w'+1;w,u-v+1;\frac{x}{y},\frac{1}{y}\right) \\ &\qquad + \frac{\Gamma(w')\Gamma(u-v)}{\Gamma(w'-v)\Gamma(u)}(-y)^{-v}F_4\left(v,v-w'+1;w,v-u+1;\frac{x}{y},\frac{1}{y}\right), \end{aligned}$$

getting

$$\begin{aligned} \mathscr{I}_s &= \frac{\alpha\beta}{4\,s}\left\{\frac{\Gamma(2)\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})\Gamma(1)}\sqrt{\frac{s^2}{\beta^2}}\,F_4\left(\frac{1}{2},-\frac{1}{2};2,\frac{1}{2};\left(\frac{\alpha}{\beta}\right)^2,-\left(\frac{s}{\beta}\right)^2\right)\right. \\ &\qquad \left. + \frac{\Gamma(2)\Gamma(-\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(1)}\,\frac{s^2}{\beta^2}\,F_4\left(1,0;2,\frac{3}{2};\left(\frac{\alpha}{\beta}\right)^2,-\left(\frac{s}{\beta}\right)^2\right)\right\} \\ &= \frac{\alpha}{2}\left\{\operatorname{sgn}(s)\,F_4\left(\frac{1}{2},-\frac{1}{2};2,\frac{1}{2};\left(\frac{\alpha}{\beta}\right)^2,-\left(\frac{s}{\beta}\right)^2\right)-\frac{s}{\beta}\right\}. \end{aligned} \tag{7}$$

Indeed, since the conventional $(0)_n = \delta_{0n}$, where δ_{ts} stands for the Kronecker symbol, the second F_4 term obviously reduces to 1.

Now, including $\mathscr{I}_s$ from (7) into (6) and having in mind that $s>0$, we have

$$H(\alpha,\beta) = \frac{\alpha}{2}\int_0^1 \mathrm{d}s\,(1-s)\left\{F_4\left(\frac{1}{2},-\frac{1}{2};2,\frac{1}{2};\left(\frac{\alpha}{\beta}\right)^2,-\left(\frac{y}{\beta}\right)^2\right)-\frac{s}{\beta}\right\}. \tag{8}$$

The *Kampé de Fériet generalized hypergeometric function of two variables* defined by the double–series [19] in a notation given e.g. by Srivastava and Panda [29, p. 423, Eq. (26)]

$$F^{p:q;k}_{l:m;n}\Big[\begin{matrix}(a_p):\,(b_q);\,(c_k)\\(\alpha_l):(\beta_m);(\gamma_n)\end{matrix}\Big|\,x,y\Big] = \sum_{r,h\geq 0}\frac{\prod\limits_{j=1}^{p}(a_j)_{r+h}\prod\limits_{j=1}^{q}(b_j)_r\prod\limits_{j=1}^{k}(c_j)_h}{\prod\limits_{j=1}^{l}(\alpha_j)_{r+h}\prod\limits_{j=1}^{m}(\beta_j)_r\prod\limits_{j=1}^{n}(\gamma_j)_h}\,\frac{x^r}{r!}\,\frac{y^h}{h!}\,,$$

which converges [27] when

(i) $p+q<l+m+1,\ p+k<l+n+1,\ \max\{|x|,|y|\}<\infty,$ or

(ii) $p+q=l+m+1,\ p+k=l+n+1$ and

$$\begin{cases}|x|^{\frac{1}{p-l}}+|y|^{\frac{1}{p-l}}<1, & l<p\\ \max\{|x|,|y|\}<1, & l>p\end{cases}.$$

Routine calculation gives the value of the first integral in (8):

$$\begin{aligned}
&\int_0^1 \mathrm{d}s\,(1-s)F_4\Big(\frac12,-\frac12;2,\frac12;\Big(\frac{\alpha}{\beta}\Big)^2,-\Big(\frac{s}{\beta}\Big)^2\Big)\\
&\quad=\frac14\sum_{m,n\geq0}\frac{(\frac12)_{m+n}(-\frac12)_{m+n}}{(2)_m(\frac12)_n\,(n+1)}\left(\frac{1}{n+\frac12}-\frac{1}{n+\frac32}\right)\frac{(\alpha\beta^{-1})^{2m}(-\beta^{-2})^n}{m!\,n!}\\
&\quad=\frac14\sum_{m,n\geq0}\frac{(\frac12)_{m+n}(-\frac12)_{m+n}\,\Gamma(n+1)\,\Gamma(n+\frac12)}{(2)_m(\frac32)_n(2)_n\,\Gamma(n+2)\,\Gamma(n+\frac52)}\,\frac{(\alpha\beta^{-1})^{2m}(-\beta^{-2})^n}{m!\,n!}\\
&\quad=\frac13\sum_{m,n\geq0}\frac{(\frac12)_{m+n}(-\frac12)_{m+n}(\frac12)_n(1)_n}{(2)_m(\frac32)_n(2)_n(\frac52)_n}\,\frac{(\alpha\beta^{-1})^{2m}(-\beta^{-2})^n}{m!\,n!}\\
&\quad=\frac13\,F^{2:-;2}_{-:1;3}\Big[\begin{matrix}\frac12,-\frac12 & : - & ; \frac12,1\\ - & : 2 & ; \frac32,2,\frac52\end{matrix}\Big|\,\frac{\alpha^2}{\beta^2},\ -\frac{1}{\beta^2}\Big],
\end{aligned}$$

Conditions (ii) imply that the convergence region is $|\alpha|+1<|\beta|$, which reduces to $a+c<b$. Finally, it follows:

$$H(\alpha,\beta)=\frac{\alpha}{6}\left\{F^{2:-;2}_{-:1;3}\Big[\begin{matrix}\frac12,-\frac12 & : - & ; \frac12,1\\ - & : 2 & ; \frac32,2,\frac52\end{matrix}\Big|\,\frac{\alpha^2}{\beta^2},\ -\frac{1}{\beta^2}\Big]-\frac{1}{2\beta}\right\}.$$

By this we have proved the following result.

Proposition 1 *Assume that $a,b,c>0$. Then for all $a+c<b$ we have*

$$I(a,b,c)=\frac{ac^2}{6}\left\{F^{2:-;2}_{-:1;3}\Big[\begin{matrix}\frac12,-\frac12 & : - & ; \frac12,1\\ - & : 2 & ; \frac32,2,\frac52\end{matrix}\Big|\,\frac{a^2}{b^2},\ -\frac{c^2}{b^2}\Big]-\frac{c}{2b}\right\}. \tag{9}$$

Obviously, our physical model which describes *via* the crossed term integral occurring in Coulomb self–energy of a uniformly charged cylinder is a special case of (9). Now we devote the following lines to these specifications and the asymptotic of the getting result for small parameter $\gamma = R_1/R_2 \in (0,1)$.

Corollary 1 *Let $\gamma \in (0,1)$, $c > 0$ where $\gamma + c < 1$. Then the crossed term integral takes the form*

$$I(\gamma,1,c) = \frac{\gamma c^2}{6}\left\{ F^{2:-;2}_{-:1;3}\left[\begin{matrix} \frac12, -\frac12 & : - & ; \frac12, 1 \\ - & : 2 & ; \frac32, 2, \frac52 \end{matrix} \,\Big|\, \gamma^2,\ -c^2 \right] - \frac{c}{2} \right\}.$$

Our special attention deserves the asymptotic of this expression. Thus,

$$\begin{aligned} I(\gamma,1,c) &= -\frac{\gamma c^3}{12} + \frac{\gamma c^2}{4} \sum_{m,n\geq 0} \frac{(\frac12)_{m+n}(-\frac12)_{m+n}(1)_n}{(2)_m(\frac32)_n(2)_n} \frac{\gamma^{2m}(-c^2)^n}{m!\ n!} \\ &= \frac{c^2}{4}\left\{ {}_3F_2\left(\begin{matrix} \frac12, -\frac12, 1 \\ \frac32, 2 \end{matrix} \,\Big|\, -c^2 \right) - \frac{c}{3} \right\}\gamma - \frac{c^2}{32}\, {}_3F_2\left(\begin{matrix} \frac12, -\frac12, 1 \\ \frac32, 2 \end{matrix} \,\Big|\, -c^2 \right) \gamma^3 \\ &\quad - \frac{c^2}{128}\, {}_3F_2\left(\begin{matrix} \frac12, -\frac12, 1 \\ \frac32, 2 \end{matrix} \,\Big|\, -c^2 \right) \gamma^5 + \mathscr{O}(\gamma^7), \end{aligned}$$

since the Maclaurin series of $I(\gamma,1,c)$ is given by the Kampé de Fériet function. Here we can write explicitly

$${}_3F_2\left(\begin{matrix} \frac12, -\frac12, 1 \\ \frac32, 2 \end{matrix} \,\Big|\, -c^2 \right) = \frac{1}{3c^2}\left\{ 2 - (2+c^2)\sqrt{1+c^2} - 3c\,\sinh^{-1}(c) \right\}.$$

Moreover, the related Maclaurin series reads

$${}_3F_2\left(\begin{matrix} \frac12, -\frac12, 1 \\ \frac32, 2 \end{matrix} \,\Big|\, -c^2 \right) = 1 + \frac{c^2}{12} + \frac{c^4}{120} + \frac{c^6}{448} + \frac{c^8}{1152} + \mathscr{O}(c^{10}).$$

4 Hypergeometric–Type Solution for General Parameter Space

In this section we treat the crossed term integral $I(a,b,c)$ by a method which leads to final expression valid for all positive $c > 0$ and $\gamma \in (0,1)$ occurring in the model $I(\gamma,1,c)$. In this purpose recalling (5) we concentrate to

$$\begin{aligned} H(\alpha,\beta) &= \int_0^1 \mathrm{d}s\,(1-s)\int_0^\infty t^{-2} J_1(\alpha t) J_1(\beta t)\,\mathrm{e}^{-st}\mathrm{d}t \\ &= \int_0^1 \mathrm{d}s\,(1-s)\int_0^\infty \mathrm{d}t\, t^{-2} \sum_{n\geq 0} \frac{(-1)^n(\alpha t/2)^{2n+1}}{(2)_n\, n!}\, J_1(\beta t)\,\mathrm{e}^{-st} \\ &= \frac{\alpha}{2}\int_0^1 \mathrm{d}s\,(1-s)\sum_{n\geq 0}\frac{(-1)^n(\alpha/2)^{2n}}{(2)_n\, n!} \int_0^\infty \mathrm{e}^{-st}\, t^{2n-1} J_1(\beta t)\,\mathrm{d}t\,. \end{aligned}$$

The Laplace–Mellin transform formula [25, p. 246, Eq. (2)]

$$\int_0^\infty \mathrm{e}^{-pt} t^\mu J_\nu(\lambda t)\mathrm{d}t = \frac{(\lambda/2)^\nu\, \Gamma(\mu+\nu+1)}{r^{\mu+\nu+1}\, \Gamma(\nu+1)}\, {}_2F_1\Big(\frac{\mu+\nu+1}{2}, \frac{\nu-\mu}{2}; \nu{+}1; \frac{\lambda^2}{r^2}\Big)\,,$$

under constraint $\Re(\mu{+}\nu) > -1$, $\Re(p) > |\Im(\lambda)|$, $|\lambda/r| < 1$, $r = \sqrt{p^2+\lambda^2}$. The inner t–integral $H(\alpha,\beta)$ satisfies all conditions upon the parameters, hence by the Legendre's duplication formula it follows

$$H(\alpha,\beta) = \frac{\alpha\beta}{4} \sum_{n\ge0} \frac{(-1)^n (\alpha)^{2n} (1/2)_n\, (1)_n}{(2)_n\, n!} \times \int_0^1 \mathrm{d}s\, \frac{(1-s)}{(\beta^2+s^2)^{n+1/2}}\, {}_2F_1\Big(n+1/2, 1-n; 2; \frac{\beta^2}{\beta^2+s^2}\Big).$$

In turn, the hypergeometric function one reduces to a polynomial of degree $n-1$, that is

$$H(\alpha,\beta) = \frac{\alpha\beta}{4} \sum_{n\ge0} \sum_{k=0}^{n-1} \frac{(-1)^n (1/2)_n (1)_n (n+1/2)_k (1-n)_k\, \alpha^{2n}\, \beta^{2k}}{(2)_n\, (2)_k\, n!\, k!} \times \int_0^1 \mathrm{d}s\, \frac{1-s}{(\beta^2+s^2)^{n+k+1/2}}\,.$$

The Pochhammer symbol's identity $(1/2)_n\, (n{+}1/2)_k = (1/2)_{n+k}$ encompasses

$$H(\alpha,\beta) = \frac{\alpha\beta}{4} \sum_{n\ge0} \sum_{k=0}^{n-1} \frac{(-1)^n (1/2)_{n+k} (1-n)_k\, \alpha^{2n} \beta^{2k}}{(2)_n\, (2)_k\, n!\, k!} \int_0^1 \frac{\mathrm{d}s\, (1-s)}{(\beta^2+s^2)^{n+k+1/2}}\,.$$

At this point we check the parameter space which ensures the convergence issue of the series $H(\alpha,\beta)$. Since

$$|H(\alpha,\beta)| \le \frac{\alpha\beta}{4} \sum_{n\ge0} \frac{\alpha^{2n}}{(2)_n\, n!} \left| \sum_{k=0}^{n-1} \frac{(1/2)_{n+k} (1-n)_k\, \beta^{2k}}{(2)_k\, k!} \int_0^1 \frac{\mathrm{d}s}{(\beta^2+s^2)^{n+k+1/2}} \right|\,, \tag{10}$$

and for m enough large there holds true

$$\int_0^1 \mathrm{d}s\, \frac{1}{(\beta^2+s^2)^{m+1/2}} = \frac{1}{\beta^{2m}\sqrt{2m}} \int_0^{\frac{\sqrt{2m}}{\beta}} \mathrm{d}s \left(1+\frac{s^2}{2m}\right)^{-m-1/2}$$
$$\sim \frac{1}{\beta^{2m}\sqrt{2m}} \int_0^{\frac{\sqrt{2m}}{\beta}} \mathrm{d}s\, \mathrm{e}^{-s^2/2} = \frac{\sqrt{\pi}}{\beta^{2m}\sqrt{m}}\, \Phi\Big(\frac{\sqrt{2m}}{\beta}\Big) \sim \frac{\sqrt{\pi}}{2\,\beta^{2m}\sqrt{m}}\,,$$

where

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_0^z \mathrm{d}x\, \mathrm{e}^{-x^2/2}$$

denotes the Laplace function, the main term a_n, say, of the right-hand-side series (10) behaves like

$$\begin{aligned} a_n &\sim \frac{\sqrt{\pi}\,(1/2)_n}{2\,(2)_n\, n!} \left| \sum_{k=0}^{n-1} \frac{(1/2+n)_k (1-n)_k}{(2)_k\, \sqrt{n+k}\, k!} \right| \Big(\frac{\alpha}{\beta}\Big)^{2n} \\ &\le \frac{\sqrt{\pi}\,(1/2)_n}{2\,(2)_n\, n!} \left| \sum_{k=0}^{n-1} \frac{(1/2+n)_k (1-n)_k}{(2)_k\, k!} \right| \Big(\frac{\alpha}{\beta}\Big)^{2n} \\ &= \frac{\sqrt{\pi}\,(1/2)_n (-1/2)_n}{(2)_n\, [n!]^2} \Big(\frac{\alpha}{\beta}\Big)^{2n}, \qquad n\to\infty. \end{aligned} \tag{11}$$

It is not hard to see that $\sqrt[n]{a_n} \to 0$ with growing n. Consequently by the Cauchy-Hadamard theorem $H(\alpha,\beta)$ converges for all $\alpha/\beta > 0$, that is *a fortiori* for all positive a, b, c as the derived bound (11) is uniform with respect to the input parameter c.

Now, routine algebra and Euler hypergeometric transformation formula [18, p. 559, Eq. 15.3.3.] give us

$$\begin{aligned} \int_0^1 \mathrm{d}s\, \frac{1}{(\beta^2+s^2)^{m+1/2}} &= \beta^{-2m-1}\, {}_2F_1\big(1/2, m+1/2; 3/2; -\beta^{-2}\big) \\ &= \frac{1}{\beta^2\,(1+\beta^2)^{m-1/2}}\, {}_2F_1\big(1, 1-m; 3/2; -\beta^{-2}\big)\,, \end{aligned}$$

which turns out to be a hypergeometric polynomial of degree $m-1$ in $-\beta^{-2}$, while

$$\int_0^1 \mathrm{d}s\, \frac{s}{(\beta^2+s^2)^{m+1/2}} = \frac{\beta^{1-2m} - (\beta^2+1)^{1/2-m}}{2m-1}\,.$$

Using the obvious transformation

$$\frac{1}{2(n+k)-1} = -\frac{(-1/2)_{n+k}}{(1/2)_{n+k}}\,,$$

and putting $m = n+k$ in the last two expressions, we conclude

$$\begin{aligned} H(\alpha,\beta) = \frac{\alpha\sqrt{1+\beta^2}}{4\beta} &\sum_{n\ge 0}\sum_{k=0}^{n-1} \frac{(-1)^n (1/2)_{n+k} (1-n)_k\, \alpha^{2n}\beta^{2k}}{(2)_n\,(2)_k\, n!\, k!\, (1+\beta^2)^{n+k}} \\ &\times {}_2F_1\big(1, 1-n-k; 3/2; -\beta^{-2}\big) \\ &+ \frac{\alpha\beta^2}{4} \sum_{n\ge 0}\sum_{k=0}^{n-1} \frac{(-1)^n (-1/2)_{n+k} (1-n)_k}{(2)_n\,(2)_k\, n!\, k!} \Big(\frac{\alpha^2}{\beta^2}\Big)^n \\ &- \frac{\alpha\beta\sqrt{1+\beta^2}}{4} \sum_{n\ge 0}\sum_{k=0}^{n-1} \frac{(-1)^n (-1/2)_{n+k} (1-n)_k\, \alpha^{2n}\beta^{2k}}{(2)_n\,(2)_k\, n!\, k!\, (1+\beta^2)^{n+k}}\,. \end{aligned}$$

Hence, we deduce

$$\begin{aligned} I(a,b,c) = {} & \frac{ac^2\sqrt{b^2+c^2}}{4b}\sum_{n\geq 0}\sum_{k=0}^{n-1}\frac{(-1)^n(1/2)_{n+k}(1-n)_k\,a^{2n}b^{2k}}{(2)_n\,(2)_k\,n!\,k!(b^2+c^2)^{n+k}} \\ & \times {}_2F_1\Big(1,1-n-k;\frac{3}{2};-\Big(\frac{c}{b}\Big)^2\Big) \\ & + \frac{ab^2}{4}\sum_{n\geq 0}\sum_{k=0}^{n-1}\frac{(-1)^n(-1/2)_{n+k}(1-n)_k}{(2)_n\,(2)_k\,n!\,k!}\Big(\frac{a^2}{b^2}\Big)^n \\ & - \frac{ab\sqrt{b^2+c^2}}{4}\sum_{n\geq 0}\sum_{k=0}^{n-1}\frac{(-1)^n(-1/2)_{n+k}(1-n)_k\,a^{2n}b^{2k}}{(2)_n\,(2)_k\,n!\,k!\,(b^2+c^2)^{n+k}} \qquad (12) \\ =: {} & S_1+S_2-S_3. \end{aligned}$$

The second sum one easily transforms into

$$S_2 = \frac{ab^2}{4}\sum_{n\geq 0}\sum_{k=0}^{n-1}\frac{(-1)^n(-1/2)_n}{(2)_n\,n!}\Big(\frac{a^2}{b^2}\Big)^n {}_2F_1\Big(-\frac{1}{2}+n,1-n;2;1\Big).$$

In turn, since

$${}_2F_1\Big(-\frac{1}{2}+n,1-n;2;1\Big) = \frac{2(-1)^n(-3/2)_n}{3\,(1)_n},$$

the closed form becomes

$$S_2 = \frac{ab^2}{6}\sum_{n\geq 0}\frac{(-1/2)_n(-3/2)_n}{(1)_n(2)_n\,n!}\Big(\frac{a^2}{b^2}\Big)^n = \frac{ab^2}{6}\,{}_2F_2\Big(-\frac{1}{2},-\frac{3}{2};1,2;\frac{a^2}{b^2}\Big).$$

The third sum in (12) one rewrites into

$$\begin{aligned} S_3 = {} & \frac{ab\sqrt{b^2+c^2}}{4}\sum_{n\geq 0}\sum_{k=0}^{n-1}\frac{(-1)^n(-1/2)_n}{(2)_n\,n!}\Big(\frac{a^2}{b^2+c^2}\Big)^n \\ & \times {}_2F_1\Big(-\frac{1}{2}+n,1-n;2;\frac{a^2}{b^2+c^2}\Big). \end{aligned}$$

As the convergence issues have solved previously by (11) collecting all these values we deduce the final result.

Proposition 2 *Let $a,b,c>0$. Then we have*

$$\begin{aligned} I(a,b,c) = {} & \frac{ac^2\sqrt{b^2+c^2}}{4b}\sum_{n\geq 0}\sum_{k=0}^{n-1}\frac{(-1)^n(1/2)_{n+k}(1-n)_k\,a^{2n}b^{2k}}{(2)_n\,(2)_k\,n!\,k!(b^2+c^2)^{n+k}} \\ & \times {}_2F_1\Big(1,1-n-k;\frac{3}{2};-\Big(\frac{c}{b}\Big)^2\Big) \\ & + \frac{ab^2}{6}\,{}_2F_2\Big(-\frac{1}{2},-\frac{3}{2};1,2;\frac{a^2}{b^2}\Big) \end{aligned}$$

$$- \frac{ab\sqrt{b^2+c^2}}{4} \sum_{n\geq 0} \sum_{k=0}^{n-1} \frac{(-1)^n(-1/2)_n}{(2)_n\, n!} \Big(\frac{a^2}{b^2+c^2} \Big)^n$$
$$\times\, {}_2F_1\Big(-\frac{1}{2}+n, 1-n; 2; \frac{a^2}{b^2+c^2} \Big).$$

Corollary 2 *For all* $c > 0, \gamma \in (0,1)$ *we have*

$$I(\gamma,1,c) = \frac{\gamma c^2\sqrt{1+c^2}}{4} \sum_{n\geq 0} \sum_{k=0}^{n-1} \frac{(1/2)_{n+k}(1-n)_k\,(-\gamma^2)^n}{(2)_n\,(2)_k\, n!\, k!(1+c^2)^{n+k}}$$
$$\times\, {}_2F_1\Big(1, 1-n-k; \frac{3}{2}; -c^2 \Big)$$
$$+ \frac{\gamma}{6}\, {}_2F_2\Big(-\frac{1}{2}, -\frac{3}{2}; 1, 2; \gamma^2 \Big)$$
$$- \frac{\gamma\sqrt{1+c^2}}{4} \sum_{n\geq 0} \sum_{k=0}^{n-1} \frac{(-1/2)_n}{(2)_n\, n!} \Big(\frac{-\gamma^2}{1+c^2} \Big)^n$$
$$\times\, {}_2F_1\Big(-\frac{1}{2}+n, 1-n; 2; \frac{\gamma^2}{1+c^2} \Big).$$

5 Conclusions

The study of the Coulomb self-energy of two finite coaxial cylinders whose space between them has been filled some dielectric material capable of supporting a total charge Q has been performed. A previous study consisting of a single cylinder provided a closed formula for the total energy. Surprisingly enough, the fact of "emptying" the cylinder leads to new, exciting mathematical integrals, which can be tackled analytically up to some extend. It is, therefore, this precise study that has brought special functions of hypergeometric –type to our attention, of special significance being the Kampé de Fériet hypergeometric function one of two variables. To the best of our knowledge, this function - the *crossed term integral* - rarely appears or it does not at all when considering systems amenable to be studied by means of mathematical physics. Finally, we draw the interested reader's attention to the recent article [31] in which similar integrals have been resolved in terms of elliptic integrals of the first and second kind.

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