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STOCHASTIC SETUP-COST INVENTORY MODEL WITH BACKORDERS AND QUASICONVEX COST FUNCTIONS

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This paper studies a periodic-review single-commodity setup-cost inventory model with backorders and holding/backlog costs satisfying quasiconvexity assumptions. We show that the Markov decision process for this inventory model satisfies the assumptions that lead to the validity of optimality equations for discounted and average-cost problems and to the existence of optimal (s, S) policies. In particular, we prove the equicontinuity of the family of discounted value functions and the convergence of optimal discounted lower thresholds to the optimal average-cost lower threshold for some sequence of discount factors converging to 1. If an arbitrary nonnegative amount of inventory can be ordered, we establish stronger convergence properties: (i) the optimal discounted lower thresholds converge to optimal average-cost lower threshold; and (ii) the discounted relative value functions converge to average-cost relative value function. These convergence results previously were known only for subsequences of discount factors even for problems with convex holding/backlog costs. The results of this paper also hold for problems with fixed lead times.

Keywords: average-cost optimality equations, inventory control, relative value functions, (s, S) policies

1. INTRODUCTION

In this paper, we study a periodic-review single-commodity setup-cost inventory model with backorders and holding/backlog costs satisfying quasiconvexity assumptions. We show that the Markov decision process for this inventory model satisfies the assumptions that lead to the validity of optimality equations for discounted and average-cost problems and to the existence of optimal (s, S) policies. In particular, we prove the equicontinuity of the family of discounted value functions and the convergence of optimal discounted lower thresholds to the optimal average-cost lower threshold for some sequence of discount factors converging to 1. If an arbitrary nonnegative amount of inventory can be ordered, we establish stronger convergence properties: (i) the optimal discounted lower thresholds s_α converge to an optimal average-cost lower threshold s; and (ii) the discounted relative value functions converge to an average-cost relative value function. These convergence results previously were known only for subsequences of discount factors even for problems with convex holding/backlog costs. The results of this paper hold for problems with deterministic positive lead times.

For problems with convex holding/backlog cost functions, Scarf [22] introduced the concept of K-convexity to prove the optimality of (s, S) policies for finite-horizon problems with continuous demand and convex holding/backlog costs. Zabel [27] indicated some gaps in Scarf [22] and corrected them. References [2–6,11–13,18,22,24,26] deal with convex or linear holding/backlog cost functions. Iglehart [18] extended Scarf's [22] results to infinite-horizon problems with continuous demand. Veinott and Wagner [26] proved the optimality of (s, S) policies for both finite-horizon and infinite-horizon problems with discrete demand. Beyer and Sethi [3] completed the missing proofs in Iglehart [18] and Veinott and Wagner [26]. Chen and Simchi-Levi [4,5] studied coordinating inventory control and pricing problems and proved the optimality of (s, S) policies without assuming that the demand is discrete or continuous. Under certain assumptions, their results imply the optimality of (s, S) policies for problems without pricing. Beyer et al. [2] and Huh et al. [17] studied problems with parameters depending on exogenous factors modeled by a Markov chain. Additional references can be found in monographs by Porteus [20] and Zipkin [29].

The analysis of periodic-review inventory models is based on the theory of Markov Decision Processes (MDPs). However, most of inventory control papers use only basic facts from the MDP theory, and the corresponding general results had been unavailable for a long time. Feinberg et al. [7] developed the results on MDPs with Borel state spaces, possibly noncompact action sets, and possibly unbounded one-step costs. Discrete-time periodic-review inventory control problems are particular examples of such MDPs; see Feinberg [6] for details. Feinberg and Lewis [11] obtained additional convergence results for convergence of optimal actions for MDPs and established the optimality of (s, S) policies for inventory control p roblems as well as other results. Feinberg and Liang [12] provided descriptions of optimal policies for all possible values of discount factors (for some parameters, optimal (s, S) policies may not exist for discounted and finite-horizon problems). Feinberg and Liang [13] proved that discrete-time periodic-review inventory models with backorders and convex holding/backlog costs satisfy the equicontinuity assumption, and this implies several additional properties of optimal average-cost policies including the validity of average-cost optimality equations (ACOEs).

Veinott [25] studied the nonstationary setup-cost inventory model with a fixed lead time, backorders, and holding/backlog costs satisfying quasiconvexity assumptions. Veinott [25] proved the optimality of (s, S) policies for finite-horizon problems and also provided bounds on the values of the optimal thresholds s and S. Zheng [28] proved the optimality of (s, S) policies for models with quasiconvex cost functions and discrete demand under both discounted and average cost criteria by constructing a solution to the optimality equations.

In this paper, we consider the infinite-horizon stationary inventory model with holding/backlog costs satisfying quasiconvexity assumptions. These quasiconvexity assumptions are introduced by Veinott [25] for finite-horizon nonstationary models. Zheng [28] and Chen and Simchi-Levi [5] considered a slightly stronger quasiconvexity assumption for infinite-horizon stationary models. For inventory model with holding/backlog costs satisfying quasiconvexity assumptions, this paper establishes convergence properties of optimal discounted thresholds for discounted problems to the corresponding thresholds for average-cost problems. Some of the results are new even for problems with convex holding/backlog costs. While convergence of optimal thresholds and relative discounted value functions was known only for subsequences of discount factors (see [2,11,13,17]), here we show that convergence of lower thresholds and discounted value functions takes place for all discount factors tending to 1.

The rest of the paper is organized in the following way. Section 2 describes the setup-cost inventory model and introduces the assumptions used in this paper. Section 3 establishes the optimality of (s_α, S_α) policies for the infinite-horizon problem with the discount factor α. Section 4 verifies average-cost optimality assumptions and the equicontinuity conditions for discounted relative value functions. Section 5 establishes the validity of ACOEs for the inventory model and the optimality of (s, S) policies under the average cost criterion. Section 6 establishes the convergence of discounted optimal lower thresholds s_α when the discount factor α converges to 1, to the average-cost optimal lower threshold s. Section 7 establishes the convergence of discounted relative value functions, when the discount factor converges to 1. Section 8 presents a reduction from the inventory model with constant lead times to the model without lead times using Veinott's [25] approach. The proofs of all the lemmas and corollaries in this paper are presented in Appendices.

2. SETUP-COST INVENTORY MODEL WITH BACKORDERS: DEFINITIONS AND ASSUMPTIONS

Let $\mathbb{R}$ denote the real line, $\mathbb{Z}$ denote the set of all integers, $\mathbb{R}_+ := [0, +\infty)$ and $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$. Consider the stochastic periodic-review setup-cost inventory model with backorders. At times $t = 0, 1, \ldots$, a decision-maker views the current inventory of a single commodity and makes an ordering decision. Assuming zero lead times, the products are immediately available to meet demand. The cost of ordering is incurred at the time of delivery of the order. Demand is then realized, the decision-maker views the remaining inventory, and the process continues. The unmet demand is backlogged. The demand and the order quantity are assumed to be nonnegative. The objective is to minimize the infinite-horizon expected total discounted cost for discount factor $\alpha \in (0, 1)$ and long-run average cost per unit time for $\alpha = 1$. The inventory model is defined by the following parameters:

1. $K > 0$ is a fixed ordering cost;
2. $\bar{c} > 0$ is the per unit ordering cost;
3. $\{D_t, t = 1, 2, \ldots\}$ is a sequence of i.i.d. nonnegative finite random variables representing the demand at periods $0, 1, \ldots$. We assume that $\mathbb{E}[D] < +\infty$ and $P(D > 0) > 0$, where D is a random variable with the same distribution as D_1;
4. $h(x)$ is the holding/backlog cost per period if the inventory level is x. Assume that: (i) the function $\mathbb{E}[h(x - D)]$ is finite and continuous for all $x \in \mathbb{X}$; and (ii) $\mathbb{E}[h(x - D)] \to +\infty$ as $|x| \to +\infty$.

Without loss of generality, assume that the function $\mathbb{E}[h(x - D)]$ is nonnegative. The assumption $P(D > 0) > 0$ avoids the trivial case when there is no demand.

Now we formulate an MDP for this inventory model. The state and action spaces can be either (i) $\mathbb{X} = \mathbb{R}$ and $\mathbb{A} = \mathbb{R}_+$; or (ii) $\mathbb{X} = \mathbb{Z}$ and $\mathbb{A} = \mathbb{N}_0$, if the demand D takes only integer values and only integer orders are allowed.

The dynamics of the system are defined by the equation

$$x_{t+1} = x_t + a_t - D_{t+1}, \quad t = 0, 1, 2, \ldots, \tag{2.1}$$

where x_t and a_t denote the current inventory level and the ordered amount at period t, respectively. The transition probability $q(dx_{t+1}|x_t, a_t)$ for the MDP defined by the stochastic equation (2.1) is

$$q(B|x_t, a_t) = P(x_t + a_t - D_{t+1} \in B) \tag{2.2}$$

for each measurable subset B of $\mathbb{R}$. The one-step expected cost is

$$c(x,a) := K\mathbf{1}_{\{a>0\}} + \bar{c}a + \mathbb{E}[h(x+a-D)], \quad (x,a) \in \mathbb{X} \times \mathbb{A}, \tag{2.3}$$

where $\mathbf{1}_B$ is an indicator of the event B.

Let $H_t = (\mathbb{X} \times \mathbb{A})^t \times \mathbb{X}$ be the set of histories for $t = 0, 1, \ldots$. Let Π be the set of all policies. A (randomized) decision rule at period $t = 0, 1, \ldots$ is a regular transition probability $\pi_t : H_t \mapsto \mathbb{A}$, that is, (i) $\pi_t(\cdot|\mathbf{h}_t)$ is a probability distribution on $\mathbb{A}$, where $\mathbf{h}_t = (x_0, a_0, x_1, \ldots, a_{t-1}, x_t)$, and (ii) for any measurable subset $B \subset \mathbb{A}$, the function $\pi_t(B|\cdot)$ is measurable on H_t. A policy π is a sequence $(\pi_0, \pi_1, \ldots)$ of decision rules. Moreover, π is called non-randomized if each probability measure $\pi_t(\cdot|\mathbf{h}_t)$ is concentrated at one point. A non-randomized policy is called stationary if all decisions depend only on the current state. According to the Ionescu Tulcea theorem (see Hernández-Lerma and Lasserre [16, p. 178]), given the initial state x, a policy π defines the probability distribution P_x^π on the set of all trajectories $H_{+\infty} = (\mathbb{X} \times \mathbb{A})^{+\infty}$. We denote by $\mathbb{E}_x^\pi$ the expectation with respect to P_x^π.

For a finite-horizon $N = 0, 1, \ldots$, let us define the expected total discounted costs

$$v_{N,\alpha}^\pi(x) := \mathbb{E}_x^\pi\left[\sum_{t=0}^{N-1} \alpha^t c(x_t, a_t)\right], \quad x \in \mathbb{X}, \tag{2.4}$$

where $\alpha \in [0,1]$ is the discount factor and $v_{0,\alpha}^\pi(x) = 0$, $x \in \mathbb{X}$. When $N = +\infty$ and $\alpha \in [0,1)$, (2.4) defines the infinite-horizon expected total discounted cost denoted by $v_\alpha^\pi(x)$. Let $v_\alpha(x) := \inf_{\pi\in\Pi} v_\alpha^\pi(x)$, $x \in \mathbb{X}$. A policy π is called optimal for the respective criterion with discount factor α if $v_{N,\alpha}^\pi(x) = v_{N,\alpha}(x)$ or $v_\alpha^\pi(x) = v_\alpha(x)$ for all $x \in \mathbb{X}$.

The *average cost per unit time* is defined as

$$w^\pi(x) := \limsup_{N\to+\infty} \frac{1}{N} v_{N,1}^\pi(x), \quad x \in \mathbb{X}. \tag{2.5}$$

Define the optimal value function $w^{\mathrm{ac}}(x) := \inf_{\pi\in\Pi} w^\pi(x)$, $x \in \mathbb{X}$. A policy π is called average-cost optimal if $w^\pi(x) = w^{\mathrm{ac}}(x)$ for all $x \in \mathbb{X}$.

Recall the definition of quasiconvex functions.

Definition 2.1: *A function f is quasiconvex on a convex set $X \subset \mathbb{R}$, if for all x, $y \in X$, and* $0 \le \lambda \le 1$

$$f(\lambda x + (1-\lambda)y) \le \max\{f(x), f(y)\}.$$

For $\alpha \in (0,1]$, let us define

$$h_\alpha(x) := h(x) + (1-\alpha)\bar{c}x + \bar{c}\mathbb{E}[D], \quad x \in \mathbb{X}. \tag{2.6}$$

Note that since $\mathbb{E}[h(x-D)] \to +\infty$ as $x \to +\infty$ and $(1-\alpha)\bar{c} \ge 0$ for all $\alpha \in (0,1]$, the function $\mathbb{E}[h_\alpha(x-D)] = \mathbb{E}[h(x-D)] + (1-\alpha)\bar{c}x + \alpha\bar{c}\mathbb{E}[D]$ tends to $+\infty$ as $x \to +\infty$ for all $\alpha \in [0,1]$. In addition, for $\alpha \in (0,1]$ the function $\mathbb{E}[h_\alpha(x-D)]$ is continuous on $\mathbb{X}$ because the functions $\mathbb{E}[h(x-D)]$ and $(1-\alpha)\bar{c}x$ are continuous on $\mathbb{X}$.

Consider the following assumptions on the quasiconvexity or convexity of the cost function.

Assumption 1: *There exists* $\alpha^* \in [0,1)$ *such that for all* $\alpha \in (\alpha^*, 1]$*:*

(i) *The function* $\mathbb{E}[h_\alpha(x-D)]$ *is quasiconvex; and*
(ii) $\lim_{x\to-\infty} \mathbb{E}[h_\alpha(x-D)] > K + \inf_{x\in\mathbb{X}}\{\mathbb{E}[h_\alpha(x-D)]\}$.

Assumption 2: *The function* $h(\cdot)$ *is convex on* $\mathbb{X}$.

For the discounted criterion, consider the following assumption, which is weaker than Assumption 1. Assumption 1 is used for the convergence of discounted-cost problems to the average-cost problem.

Assumption 3: *For a given* $\alpha \in (0,1]$ *assume that:*

(i) *the function* $\mathbb{E}[h_\alpha(x-D)]$ *is quasiconvex; and*
(ii) $\lim_{x\to-\infty} \mathbb{E}[h_\alpha(x-D)] > K + \inf_{x\in\mathbb{X}}\{\mathbb{E}[h_\alpha(x-D)]\}$.

We recall that Veinott [25] considered quasiconvexity assumptions for finite-horizon nonstationary problems. Being applied to stationary infinite-horizon problems, the corresponding assumption is Assumption 3. For stationary infinite-horizon models and discrete demands, Zheng [28] used a slightly stronger assumption, which is Assumption 3 with inequality (ii) replaced with $\lim_{x\to-\infty} \mathbb{E}[h_\alpha(x-D)] = +\infty$.

For $\alpha \in [0,1]$, if

$$\lim_{x\to-\infty} \mathbb{E}[h_\alpha(x-D)] > \inf_{x\in\mathbb{X}} \mathbb{E}[h_\alpha(x-D)], \tag{2.7}$$

then we define

$$x_\alpha^{\min} := \min\left\{\operatorname*{argmin}_{x\in\mathbb{X}}\{\mathbb{E}[h_\alpha(x-D)]\}\right\}. \tag{2.8}$$

Since the function $\mathbb{E}[h_\alpha(x-D)]$ is continuous, $\mathbb{E}[h_\alpha(x-D)] \to +\infty$ as $x \to +\infty$ and (2.7) imply that $|x_\alpha^{\min}| < +\infty$.

The following assumption is used to establish the convergence of the discounted optimal lower thresholds and relative value functions in Sections 6 and 7, respectively.

Assumption 4: *For a given* $\alpha \in (0,1]$*, the function* $\mathbb{E}[h_\alpha(x-D)]$ *is strictly decreasing on* $(-\infty, x_\alpha^{\min}]$*, where* $x_\alpha^{\min}$ *is defined in* (2.8).

We state the relationships between these assumptions in the following two lemmas.

Lemma 2.2: *Assumption 2 implies the validity of Assumption 1 with*

$$\alpha^* \in [\max\left\{1 + \lim_{x\to-\infty} \frac{h(x)}{\bar{c}x}, 0\right\}, 1) \tag{2.9}$$

and the validity of Assumption 4 for all $\alpha \in (\alpha^*, 1]$.

Lemma 2.3: *Assumption 1 implies Assumption 3 for* $\alpha \in (\alpha^*, 1]$.

3. SETUP-COST INVENTORY MODEL WITH DISCOUNTED COSTS

This section establishes the existence of optimal (s_α, S_α) policies for the problems with discounted costs stated in Theorem 3.6. We start this section by verifying the weak continuity of the transition probability q defined in (2.2) and the $\mathbb{K}$-inf-compactness of the one-step cost function c defined in (2.3). These properties stated in Assumption $\mathbf{W}^*$, imply the validity of optimality equations and the convergence of value iterations for problems with discounted costs; see Feinberg et al. [7, Theorem 4].

Recall that a function $f : U \mapsto \mathbb{R} \cup \{+\infty\}$, where U is a subset of a metric space $\mathbb{U}$, is called inf-compact, if for every $\lambda \in \mathbb{R}$ the level set $\{u \in \mathbb{U} : f(u) \le \lambda\}$ is compact.

Definition 3.1 Feinberg et al. [8, Definition 1.1], Feinberg [6, Definition 2.1]: *A function $f : \mathbb{X} \times \mathbb{A} \mapsto \bar{\mathbb{R}}$ is called $\mathbb{K}$-inf-compact, if for every nonempty compact subset $\mathbb{K}$ of $\mathbb{X}$, the function $f : \mathbb{K} \times \mathbb{A} \mapsto \bar{\mathbb{R}}$ is inf-compact.*

It is known for discounted MDPs that if the one-step cost function c and transition probability q satisfy the Assumption $\mathbf{W}^*$ below, then it is possible to write the optimality equations for the finite-horizon and infinite-horizon problems, these equations define the sets of stationary and Markov optimal policies for infinite and finite horizons, respectively, $v_\alpha(x) = \lim_{N\to+\infty} v_{N,\alpha}(x)$ for all $x \in \mathbb{X}$, and the functions $v_{N,\alpha}$, $N = 1, 2, \ldots$, and v_α are lower semicontinuous; see Feinberg et al. [7, Theorems 3, 4].

Assumption $\mathbf{W}^*$ (Feinberg et al. [7], Feinberg and Lewis [11], or Feinberg [6]):

(i) *The function c is $\mathbb{K}$-inf-compact and bounded below, and*

(ii) *the transition probability $q(\cdot|x, a)$ is weakly continuous in $(x, a) \in \mathbb{X} \times \mathbb{A}$, that is, for every bounded continuous function $f : \mathbb{X} \mapsto \mathbb{R}$, the function $\tilde{f}(x, a) := \int_{\mathbb{X}} f(y)q(dy|x, a)$ is continuous on $\mathbb{X} \times \mathbb{A}$.*

Theorem 3.2: *The inventory model satisfies Assumption $\mathbf{W}^*$, and the one-step cost function c is inf-compact.*

Proof: Since the function $\mathbb{E}[h(x - D)]$ is continuous and tends to $+\infty$ as $|x| \to +\infty$, the proof of Theorem 3.2 follows from the same arguments as in Feinberg and Lewis [11, Theorem 5.3(i)] and Feinberg [6, p. 22] ■

According to Feinberg and Lewis [11], since Assumption $\mathbf{W}^*$ holds for the MDP corresponding to the described inventory model, the optimality equations for the total discounted costs can be written as

$$v_{t+1,\alpha}(x) = \min\left\{\min_{a\ge 0}[K + G_{t,\alpha}(x + a)], G_{t,\alpha}(x)\right\} - \bar{c}x, \quad t = 0, 1, 2, \ldots, x \in \mathbb{X}, \quad \textbf{(3.1)}$$

$$v_\alpha(x) = \min\left\{\min_{a\ge 0}[K + G_\alpha(x + a)], G_\alpha(x)\right\} - \bar{c}x, \quad x \in \mathbb{X}, \quad \textbf{(3.2)}$$

where

$$G_{t,\alpha}(x) := \bar{c}x + \mathbb{E}[h(x - D)] + \alpha\mathbb{E}[v_{t,\alpha}(x - D)], \quad t = 0, 1, 2, \ldots, \ x \in \mathbb{X}, \quad \textbf{(3.3)}$$

$$G_\alpha(x) := \bar{c}x + \mathbb{E}[h(x - D)] + \alpha\mathbb{E}[v_\alpha(x - D)], \quad x \in \mathbb{X}, \quad \textbf{(3.4)}$$

and $v_{0,\alpha}(x) = 0$ for all $x \in \mathbb{X}$. Let $G := G_{t,\alpha}$, $t = 1, 2, \ldots$, or $G := G_\alpha$. Then $a^* = 0$ is an optimal action defined by (3.1) or (3.2) if $G(x) \le K + G(x + a)$ for all $a > 0$. Also, $a^* > 0$

is an optimal action of the same equation if $G(x) \geq K + G(x+a)$ for some $a > 0$ and $G(x+a^*) = \min_{a>0} G(x,a)$. According to Feinberg et al. [7, Theorem 2], for a finite (infinite) horizon problem there exists an optimal Markov (stationary) policy and the set of optimal Markov (stationary) policies is defined by the set of optimal actions that achieve the minimum in (3.1) ((3.2)). The following lemma states properties of the value functions.

LEMMA 3.3: *For $x \leq y$ and $t = 1, 2, \ldots$*

$$v_{t,\alpha}(x) + \bar{c}x \leq v_{t,\alpha}(y) + \bar{c}y + K, \tag{3.5}$$

$$v_a(x) + \bar{c}x \leq v_a(y) + \bar{c}y + K, \tag{3.6}$$

$$G_{t,\alpha}(y) - G_{t,\alpha}(x) \geq \mathbb{E}[h_\alpha(y-D)] - \mathbb{E}[h_\alpha(x-D)] - \alpha K, \tag{3.7}$$

$$G_\alpha(y) - G_\alpha(x) \geq \mathbb{E}[h_\alpha(y-D)] - \mathbb{E}[h_\alpha(x-D)] - \alpha K. \tag{3.8}$$

The properties of the value functions stated in Lemma 3.3 imply that it is possible to consider smaller action sets.

LEMMA 3.4: *Let Assumption 3 hold for some $\alpha \in (0,1)$. If a^* is an optimal action defined by (3.1) or (3.2) for some $x \in \mathbb{X}$, then $a^* \in [0, \max\{S^*_\alpha - x, 0\}]$, where*

$$S^*_\alpha := \inf\{x > x^{\min}_\alpha : \mathbb{E}[h_\alpha(x-D)] \geq K + \mathbb{E}[h_\alpha(x^{\min}_\alpha - D)]. \tag{3.9}$$

Therefore, without loss of generality, it is possible to reduce the action sets $A(x)$ to the action sets $\tilde{A}(x) = A(x) \cap [0, \max\{S^*_\alpha - x, 0\}]$, $x \in \mathbb{X}$.

Recall the definition of (s,S) policies. Suppose $f(x)$ is a lower semicontinuous function such that $\liminf_{|x|\to+\infty} f(x) > K + \inf_{x\in\mathbb{X}} f(x)$. Let

$$S \in \underset{x\in\mathbb{X}}{\operatorname{argmin}}\{f(x)\}, \tag{3.10}$$

$$s = \inf\{x \leq S : f(x) \leq K + f(S)\}. \tag{3.11}$$

DEFINITION 3.5: *Let s_t and S_t be real numbers such that $s_t \leq S_t$, $t = 0, 1, \ldots$. A policy is called an (s_t, S_t) policy at step t if it orders up to the level S_t, if $x_t < s_t$, and does not order, if $x_t \geq s_t$. A Markov policy is called an (s_t, S_t) policy if it is an (s_t, S_t) policy at all steps $t = 0, 1, \ldots$. A policy is called an (s,S) policy if it is stationary and it is an (s,S) policy at all steps $t = 0, 1, \ldots$.*

In this section, we consider Assumption 3, which guarantees the optimality of (s_α, S_α) policies for infinite-horizon problems with the discount factor α, as this is stated in the following theorem, the proof of which is delayed until later in this section.

THEOREM 3.6: *Let Assumption 3 hold for some $\alpha \in (0,1)$. For the infinite-horizon problem, there exists an optimal (s_α, S_α) policy, where S_α and s_α are real numbers such that S_α satisfies (3.10) and s_α is defined in (3.11) with $f(x) := G_\alpha(x)$, $x \in \mathbb{X}$.*

Remark 3.7: Under slightly stronger assumptions, this theorem is proved by Zheng [28] for inventory models with integer demands and integer orders. Under Assumption 2 and some other technical assumptions, this conclusion also follows from Chen and Simchi-Levi [5]. Under Assumption 2, Theorem 3.6 is proved in Feinberg and Liang [12, Theorem 4.4] with $\alpha \in (\alpha^*, 1)$ for α^* defined in (2.9). In addition, the structure of optimal policies is described in Feinberg and Liang [12, Theorem 4.4] for all $\alpha \in [0,1)$. However, under Assumption 3 from this paper, if $\alpha \in [0,\alpha^*)$, then the structure of optimal policies is currently not clear.

To prove the optimality of (s_α, S_α) policies, we first consider the same inventory model with a terminal cost $-\bar{c}x$, that is, each unit of stock left over can be discarded with the return of $\bar{c}$ and each unit of backlogged demand is satisfied at the cost $\bar{c}$. By using Lemma 3.4, for all $x \in \mathbb{X}$ we reduce the action sets $A(x)$ to the action sets $\tilde{A}(x)$ defined in the lemma. For the model with terminal costs, the one-step cost function is the same as the original problem and the expected total discounted cost is

$$\tilde{v}^\pi_{N,\alpha}(x) := \mathbb{E}^\pi_x\left[\sum_{t=0}^{N-1}\alpha^t c(x_t,a_t) - \alpha^N \bar{c}x_N\right], \quad x \in \mathbb{X}.$$

In view of Lemma 3.4, the function $\tilde{v}^\pi_{N,\alpha}$ is well-defined because $\mathbb{E}^\pi_x[x_N] \le \max\{S^*_\alpha, x\}$ for each $N = 1,2,\ldots$. Then we transform the problem into the one with $\bar{c} = 0$ and follow the induction proofs in Veinott [25] to establish properties for $\tilde{v}_\alpha$. Then, we shall also show that $\tilde{v}_\alpha = v_\alpha$.

The finite-horizon discounted cost optimality equations for the inventory model with terminal costs $-\bar{c}x$ are the same as (3.1) with $v_{0,\alpha}(x) = 0$, $v_{t,\alpha}$, and $G_{t,\alpha}$ replaced with $\tilde{v}_{0,\alpha}(x) = -\bar{c}x$, $\tilde{v}_{t,\alpha}$, and $\tilde{G}_{t,\alpha}$ for $t = 1,2,\ldots$.

Then, following Veinott [25], we transform the model with the positive unit ordering cost $\bar{c}$ and terminal cost $-\bar{c}x$ into the model with zero unit and terminal costs and holding/backlog costs h_α defined in (2.6). The one-step cost function for the new model is

$$c_\alpha(x,a) = K\mathbf{1}_{\{a>0\}} + \mathbb{E}[h_\alpha(x+a-D)] \qquad \mathbf{(3.12)}$$

and the expected total discounted cost is

$$\bar{v}^\pi_{N,\alpha}(x) := \mathbb{E}^\pi_x\left[\sum_{t=0}^{N-1}\alpha^t c_\alpha(x_t,a_t)\right], \quad x \in \mathbb{X}.$$

Since the function $\mathbb{E}[h_\alpha(x-D)]$ is quasiconvex, $\lim_{x\to-\infty}\mathbb{E}[h_\alpha(x-D)] > K + \inf_{x\in\mathbb{X}} \mathbb{E}[h_\alpha(x-D)]$, and $\lim_{x\to+\infty}\mathbb{E}[h_\alpha(x-D)] = +\infty$, the function $\mathbb{E}[h_\alpha(x-D)]$ is bounded below. Therefore, c_α is bounded below and the new model satisfies Assumption W*. The optimality equations for the new model are

$$\bar{v}_{t+1,\alpha}(x) = \min\left\{\min_{a\ge 0}[K + \bar{G}_{t,\alpha}(x+a)], \bar{G}_{t,\alpha}(x)\right\}, \quad t = 0,1,2,\ldots,\ x \in \mathbb{X}, \qquad \mathbf{(3.13)}$$

$$\bar{v}_\alpha(x) = \min\left\{\min_{a\ge 0}[K + \bar{G}_\alpha(x+a)], \bar{G}_\alpha(x)\right\}, \quad x \in \mathbb{X}, \qquad \mathbf{(3.14)}$$

where

$$\bar{G}_{t,\alpha}(x) = \mathbb{E}[h_\alpha(x-D)] + \alpha\mathbb{E}[\bar{v}_{t,\alpha}(x-D)], \quad t = 0,1,2,\ldots,\ x \in \mathbb{X}, \qquad \mathbf{(3.15)}$$

$$\bar{G}_\alpha(x) = \mathbb{E}[h_\alpha(x-D)] + \alpha\mathbb{E}[\bar{v}_\alpha(x-D)], \quad x \in \mathbb{X}, \qquad \mathbf{(3.16)}$$

and $\bar{v}_{0,\alpha}(x) = \tilde{v}_{0,\alpha}(x) + \bar{c}x = 0$ for all $x \in \mathbb{X}$. The arguments in the proof of Lemma 3.4 imply that $\min_{a>0}$ can be replaced with $\min_{a\in\tilde{A}(x)}$ in formulae (3.13) and (3.14).

It is easy to see by induction that

$$\bar{v}_{t,\alpha}(x) = \tilde{v}_{t,\alpha}(x) + \bar{c}x, \quad x \in \mathbb{X}, \tag{3.17}$$

$$\bar{G}_{t,\alpha}(x) = \tilde{G}_{t,\alpha}(x), \quad t = 0,1,2,\ldots, \ x \in \mathbb{X}. \tag{3.18}$$

Since the validity of Assumption W* for the model with zero unit cost implies that $\bar{v}_{t,\alpha} \to \bar{v}_\alpha$ as $t \to +\infty$, in view of (3.17) and (3.18), we can define

$$\tilde{v}_\alpha(x) := \lim_{t\to+\infty} \tilde{v}_{t,\alpha}(x) = \lim_{t\to+\infty} \bar{v}_{t,\alpha}(x) - \bar{c}x = \bar{v}_\alpha(x) - \bar{c}x. \quad x \in \mathbb{X}. \tag{3.19}$$

In view of (3.18), the finite-horizon model with terminal costs $-\bar{c}x$ and the finite-horizon model with zero unit and terminal costs have the same sets of optimal actions for the same state-time pairs. In addition, (3.14) implies that

$$\tilde{v}_\alpha(x) = \min\left\{\min_{a\geq 0}[K + \tilde{G}_\alpha(x+a)], \tilde{G}_\alpha(x)\right\} - \bar{c}x, \tag{3.20}$$

where, in view of (3.16),

$$\tilde{G}_\alpha(x) = \bar{G}_\alpha(x) = \bar{c}x + \mathbb{E}[h(x-D)] + \alpha\mathbb{E}[\tilde{v}_\alpha(x-D)], \quad x \in \mathbb{X}. \tag{3.21}$$

Now, we extend the properties of finite-horizon value functions $\bar{v}_{t,\alpha}$ and $\bar{G}_{t,\alpha}$, $t = 0,1,2,\ldots$, stated in Veinott [25, Lemmas 1 and 2] to infinite-horizon value functions $\bar{v}_\alpha$ and $\bar{G}_\alpha$.

LEMMA 3.8: *For $x \leq y$ and $t = 1,2,\ldots$*

$$\bar{v}_{t,\alpha}(x) \leq \bar{v}_{t,\alpha}(y) + K, \tag{3.22}$$

$$\bar{v}_\alpha(x) \leq \bar{v}_\alpha(y) + K, \tag{3.23}$$

$$\bar{G}_{t,\alpha}(y) - \bar{G}_{t,\alpha}(x) \geq \mathbb{E}[h_\alpha(y-D)] - \mathbb{E}[h_\alpha(x-D)] - \alpha K, \tag{3.24}$$

$$\bar{G}_\alpha(y) - \bar{G}_\alpha(x) \geq \mathbb{E}[h_\alpha(y-D)] - \mathbb{E}[h_\alpha(x-D)] - \alpha K. \tag{3.25}$$

LEMMA 3.9: *Let Assumption 3 hold for some $\alpha \in (0,1)$. Then for $t = 0,1,\ldots$ and $x \leq y \leq x_\alpha^{\min}$, where $x_\alpha^{\min}$ is defined in (2.8),*

$$\bar{G}_{t,\alpha}(y) - \bar{G}_{t,\alpha}(x) \leq 0, \tag{3.26}$$

$$\bar{v}_{t,\alpha}(y) - \bar{v}_{t,\alpha}(x) \leq 0, \tag{3.27}$$

$$\bar{v}_\alpha(y) - \bar{v}_\alpha(x) \leq 0, \tag{3.28}$$

$$\bar{G}_\alpha(y) - \bar{G}_\alpha(x) \leq 0. \tag{3.29}$$

THEOREM 3.10: *Let Assumption 3 hold for some $\alpha \in (0,1)$. For the inventory model with zero unit and terminal costs, the following statements hold:*

(i) *For an N-horizon problem, where $N = 1,2,\ldots$, there exists a Markov optimal $(s_{t,\alpha}, S_{t,\alpha})_{t=0,1,2,\ldots,N-1}$ policy, where $S_{t,\alpha}$ and $s_{t,\alpha}$ are real numbers such that $S_{t,\alpha}$ satisfies (3.10) and $s_{t,\alpha}$ is defined in (3.11) with $f(x) := \bar{G}_{N-t-1,\alpha}(x)$, $t = 0,1,2,\ldots,N-1$, $x \in \mathbb{X}$. In addition, the functions $\bar{v}_{t,\alpha}$ and $\bar{G}_{t,\alpha}$, $t = 0,1,\ldots,N$, are continuous on $\mathbb{X}$;*

(ii) *For an infinite-horizon problem, there exists a stationary optimal (s_α, S_α) policy, where S_α and s_α are real numbers such that S_α satisfies (3.10) and s_α is defined* *in* (3.11) *with* $f(x) := \bar{G}_\alpha(x)$, $x \in \mathbb{X}$. *In addition, the functions* $\bar{v}_\alpha$ *and* $\bar{G}_\alpha$ *are continuous on* $\mathbb{X}$*;*

(iii) *For all* $s_{t,\alpha}$, $S_{t,\alpha}$, $t = 0, 1, 2, \ldots,$ *and* S_α *defined in (i) and (ii),*

$$s_{t,\alpha} \le x_\alpha^{\min} \le S_{t,\alpha} \le S_\alpha^* \quad \textit{and} \quad s_\alpha \le x_\alpha^{\min} \le S_\alpha \le S_\alpha^*, \tag{3.30}$$

where S_α^* *is defined in* (3.9).

Proof of Theorem 3.10: (i) Consider $N = 1, 2, \ldots$ and $t = 0, 1, 2, \ldots, N-1$. According to Theorem 3.2, Assumption W*. Therefore, in view of Feinberg et al. [7, Theorem 2], the functions $\bar{v}_{N-t-1,\alpha}$ and $\bar{G}_{N-t-1,\alpha}$ are lower semicontinuous functions. In view of Lemma 3.9, the function $\bar{G}_{N-t-1,\alpha}(x)$ is nonincreasing on $(-\infty, x_\alpha^{\min}]$, where $x_\alpha^{\min}$ is defined in (2.8). In view of (3.15), $\bar{G}_{N-t-1,\alpha}(x) \ge \bar{c}x \to +\infty$ as $x \to +\infty$. Therefore, the function $\bar{G}_{N-t-1,\alpha}$ is inf-compact; see the definition of inf-compact functions in the paragraph preceding Definition 3.1. In view of (3.15) and (3.27), $\bar{G}_{N-t-1,\alpha}(x) \ge \mathbb{E}[h_\alpha(x-D)] + \alpha\mathbb{E}[\bar{v}_{N-t-1,\alpha}(x_\alpha^{\min} - D)]$ for all $x \le x_\alpha^{\min}$. Therefore,

$$\begin{aligned}\liminf_{x\to-\infty} \bar{G}_{N-t-1,\alpha}(x) &\ge \lim_{x\to-\infty} \mathbb{E}[h_\alpha(x-D)] + \alpha\mathbb{E}[\bar{v}_{N-t-1,\alpha}(x_\alpha^{\min} - D)]\\ &> K + \mathbb{E}[h_\alpha(x_\alpha^{\min} - D)] + \alpha\mathbb{E}[\bar{v}_{N-t-1,\alpha}(x_\alpha^{\min} - D)]\\ &= K + \bar{G}_{N-t-1,\alpha}(x_\alpha^{\min}) \ge K + \inf_{x\in\mathbb{X}} \bar{G}_{N-t-1,\alpha}(x),\end{aligned} \tag{3.31}$$

where the first inequality follows from the inequality in the previous sentence, the second inequality follows from Assumption 3, the equality follows from the definition of the function $\bar{G}_{N-t-1,\alpha}$ in (3.16), and the last inequality is straightforward.
Let $S_{t,\alpha}$ satisfy (3.10) and $s_{t,\alpha}$ be defined in (3.11) with $f := \bar{G}_{N-t-1,\alpha}$. The lower semicontinuity of $\bar{G}_{N-t-1,\alpha}(x)$ implies that

$$\bar{G}_{N-t-1,\alpha}(s_{t,\alpha}) \le \bar{G}_{N-t-1,\alpha}(S_{t,\alpha}) + K. \tag{3.32}$$

Since the function $\bar{G}_{N-t-1,\alpha}(x)$ is nonincreasing on $(-\infty, x_\alpha^{\min}]$,

$$S_{t,\alpha} \ge x_\alpha^{\min}. \tag{3.33}$$

To prove the optimality of $(s_{t,\alpha}, S_{t,\alpha})$ policies, we consider three cases: (1) $x \ge x_\alpha^{\min}$; (2) $s_{t,\alpha} \le x \le x_\alpha^{\min}$; and (3) $x < s_{t,\alpha}$. (1) In view of Lemma 3.8, for $x_\alpha^{\min} \le x < y$

$$\bar{G}_{N-t-1,\alpha}(y) + K - \bar{G}_{N-t-1,\alpha}(x) \ge \mathbb{E}[h_\alpha(y-D)] - \mathbb{E}[h_\alpha(x-D)] + K - \alpha K > 0, \tag{3.34}$$

where the first inequality follows from (3.24) and the second one holds because the function $\mathbb{E}[h_\alpha(x-D)]$ is nondecreasing on $[x_\alpha^{\min}, +\infty)$ and $K - \alpha K > 0$. Therefore, the action $a = 0$ is optimal for $x \ge x_\alpha^{\min}$. In addition, (3.34) implies that $\bar{G}_{N-t-1,\alpha}(x) < \bar{G}_{N-t-1,\alpha}(S_{t,\alpha}) + K$ for all $x \in [x_\alpha^{\min}, S_{t,\alpha}]$, which implies

$$s_{t,\alpha} \le x_\alpha^{\min}. \tag{3.35}$$

(2) For $s_{t,\alpha} \le x \le x_\alpha^{\min}$,

$$\bar{G}_{N-t-1,\alpha}(x) \le \bar{G}_{N-t-1,\alpha}(s_{t,\alpha}) \le K + \bar{G}_{N-t-1,\alpha}(S_{t,\alpha}) = K + \min_{y\in\mathbb{X}} \bar{G}_{N-t-1,\alpha}(y),$$

where the first inequality follows from (3.26) and the second one follows from (3.32). Therefore, the action $a = 0$ is optimal for $s_{t,\alpha} \le x \le x_\alpha^{\min}$.

(3) For $x < s_{t,\alpha}$

$$\bar{G}_{N-t-1,\alpha}(x) > K + \bar{G}_{N-t-1,\alpha}(S_{t,\alpha}) = K + \min_{y\in\mathbb{X}} \bar{G}_{N-t-1,\alpha}(y),$$

where the inequality follows from the definition of $s_{t,\alpha}$ in (3.11) with $f := \bar{G}_{N-t-1,\alpha}$. Therefore, the action $a = S_{t,\alpha} - x$ is optimal for $x < s_{t,\alpha}$. Thus, for N-horizon problem the $(s_{t,\alpha}, S_{t,\alpha})_{t=0,1,2,\ldots,N-1}$ policy is optimal.

Now, we prove that the functions $\bar{v}_{t,\alpha}$ and $\bar{G}_{t,\alpha}$, $t = 0, 1, \ldots, N$, are continuous on $\mathbb{X}$. Observe that $\bar{v}_{0,\alpha}(x) = 0$, $x \in \mathbb{X}$, and $\bar{G}_{0,\alpha}(x) = \mathbb{E}[h_\alpha(x - D)]$, then the functions $\bar{v}_{0,\alpha}$ and $\bar{G}_{0,\alpha}$ are continuous on $\mathbb{X}$. Since $\mathbb{E}[h_\alpha(x - D)]$ is continuous, the same arguments as the proof of case (ii) from Feinberg and Liang [12, Theorem 5.2] with $v_{t,\alpha}$ and $G_{t,\alpha}$ replaced with $\bar{v}_{t,\alpha} - \bar{c}x$ and $\bar{G}_{t,\alpha}$ imply that the functions $\bar{v}_{t,\alpha}$ and $\bar{G}_{t,\alpha}$, $t = 1, 2, \ldots, N$, are continuous on $\mathbb{X}$.

(ii) In view of Lemmas 3.8 and 3.9, $\bar{G}_\alpha$ and $\bar{v}_\alpha$ satisfy the same properties as $\bar{G}_{t,\alpha}$ and $\bar{v}_{t,\alpha}$. Therefore, statement (ii) follows from the same arguments as those in the proof of (i) with $\bar{G}_{N-t-1,\alpha}$, $\bar{v}_{N-t-1,\alpha}$, $s_{t,\alpha}$, and $S_{t,\alpha}$ replaced with $\bar{G}_\alpha$, $\bar{v}_\alpha$, s_α, and S_α respectively. The continuity of the function $\bar{v}_\alpha$ and $\bar{G}_\alpha$ follows from the same arguments of Feinberg and Liang [12, Theorem 5.3] with v_α and G_α replaced with $\bar{v}_\alpha - \bar{c}x$ and $\bar{G}_\alpha$.

(iii) In view of (3.9), $\lim_{x\to+\infty} \mathbb{E}[h_\alpha(x - D)] = +\infty$ and Assumption 3 imply that $|S^*_\alpha| < +\infty$ and for $x > S^*_\alpha$

$$\mathbb{E}[h_\alpha(x - D)] \geq \mathbb{E}[h_\alpha(S^*_\alpha - D)] \geq K + \mathbb{E}[h_\alpha(x^{\min}_\alpha - D)]. \tag{3.36}$$

Therefore, (3.24) and (3.36) imply that, for $t = 0, 1, 2, \ldots$ and $x > S^*_\alpha$,

$$\bar{G}_{t,\alpha}(x) - \bar{G}_{t,\alpha}(x^{\min}_\alpha) \geq \mathbb{E}[h_\alpha(x - D)] - \mathbb{E}[h_\alpha(x^{\min}_\alpha - D)] - \alpha K \geq K - \alpha K > 0. \tag{3.37}$$

Thus, for $t = 0, 1, 2, \ldots$ and $x > S^*_\alpha$,

$$\bar{G}_{t,\alpha}(x) > \bar{G}_{t,\alpha}(x^{\min}_\alpha) \geq \min_{x\in\mathbb{X}} \bar{G}_{t,\alpha}(x). \tag{3.38}$$

Therefore, if $x \in \operatorname{argmin}\{\bar{G}_{t,\alpha}(x)\}$, $t = 0, 1, 2, \ldots$, then $x \leq S^*_\alpha$. Thus, $S_{t,\alpha} \leq S^*_\alpha$, $t = 0, 1, 2, \ldots$. In addition, the same arguments with (3.24) and $\bar{G}_{t,\alpha}$ replaced with (3.25) and $\bar{G}_\alpha$ imply that $S_\alpha \leq S^*_\alpha$.

Furthermore, (3.33) and (3.35) imply that $s_{t,\alpha} \leq x^{\min}_\alpha \leq S_{t,\alpha}$ and the same arguments as those before (3.33) and (3.35) being applied to infinite-horizon problem with $\bar{G}_{N-t-1,\alpha}$ replaced with $\bar{G}_\alpha$ imply that $s_\alpha \leq x^{\min}_\alpha \leq S_\alpha$. Hence, (3.30) holds. ■

Lemma 3.11: *Let Assumption 3 hold for some $\alpha \in (0, 1)$. Then*

$$\bar{v}_\alpha(x) - \bar{c}x = \tilde{v}_\alpha(x) = v_\alpha(x) \geq 0, \quad x \in \mathbb{X}. \tag{3.39}$$

In addition, (3.39) implies that $G_\alpha(x) = \bar{G}_\alpha(x)$, $x \in \mathbb{X}$.

Proof of Theorem 3.6: Theorem 3.6 follows from Theorem 3.10(ii) and Lemma 3.11 because equations (3.2) and (3.14) are equivalent, and they define the same optimal (s_α, S_α) policies. ■

Remark 3.12: Note that $(s_{t,\alpha}, S_{t,\alpha})_{t=0,1,2,\ldots,N-1}$ policies are optimal for $N = 1, 2, \ldots$ horizon inventory models with terminal costs $-\bar{c}x$ (see Theorem 3.10(i)), they may not be optimal for finite-horizon inventory models without terminal costs (see Example 1). However, Theorem 3.6 states that there exists an optimal (s_α, S_α) policy for infinite-horizon discounted cost inventory models.

Example 1: Consider the inventory model without terminal costs defined by the following parameters: fixed ordering cost $K = 1$, per unit ordering cost $\bar{c} = 1$, deterministic demand $D = 1$, holding/backlog cost function $h(x) = 0.5|x|$, and the discount factor $\alpha = 0.75$. Since $\mathbb{E}[h_\alpha(x - D)] = 0.5|x - 1| + 0.25x + 0.75$, the function $\mathbb{E}[h_\alpha(x - D)]$ is convex and hence quasiconvex. In addition, $\lim_{x\to-\infty} \mathbb{E}[h_\alpha(x - D)] = +\infty > K + \inf_{x\in\mathbb{X}} \mathbb{E}[h_\alpha(x - D)]$. Therefore, Assumption 3 holds. For the single-period problem, the policy that does not order is optimal, because the cost incurred if nothing is ordered, is $0.5|x - 1|$ and the cost incurred if $a > 0$ units are ordered, is $1 + a + 0.5|x + a - 1| = 1 + 0.5(a + |-a| + |x + a - 1|) \geq 1 + 0.5(a + |-a + x + a - 1|) > 0.5|x - 1|$.

4. VERIFICATION OF AVERAGE-COST OPTIMALITY ASSUMPTIONS FOR THE SETUP-COST INVENTORY MODEL

In this section, we show that, in addition to Assumption W*, under Assumption 1, the setup-cost inventory model satisfies Assumption B introduced by Schäl [23]. This implies the validity of average-cost optimality inequalities (ACOIs) and the existence of stationary optimal policies; see Feinberg et al. [7, Theorem 4]. In addition, we show that, under Assumption 1 the inventory model satisfies the equicontinuity condition from Feinberg and Liang [13, Theorem 3.2], which implies the validity of the ACOE for the inventory model.

As in Schäl [23] and Feinberg et al. [7], define

$$\begin{aligned} m_\alpha &:= \inf_{x\in\mathbb{X}} v_\alpha(x), \quad u_\alpha(x) := v_\alpha(x) - m_\alpha, \\ \underline{w} &:= \liminf_{\alpha\uparrow 1} (1-\alpha) m_\alpha, \quad \bar{w} := \limsup_{\alpha\uparrow 1} (1-\alpha) m_\alpha. \end{aligned} \tag{4.1}$$

The function u_α is called the discounted relative value function. Consider the following assumption in addition to Assumption W*.

ASSUMPTION B:

(i) $w^* := \inf_{x\in\mathbb{X}} w^{\mathrm{ac}}(x) < +\infty$, *and*

(ii) $\sup_{\alpha\in[0,1)} u_\alpha(x) < +\infty$, $x \in \mathbb{X}$.

As follows from Schäl [23, Lemma 1.2(a)], Assumption B(i) implies that $m_\alpha < +\infty$ for all $\alpha \in [0, 1)$. Thus, all the quantities in (4.1) are defined. According to Feinberg et al. [7, Theorems 3, 4], if Assumptions **W*** and **B** hold, then $\underline{w} = \bar{w}$ and therefore,

$$\lim_{\alpha\uparrow 1} (1-\alpha) m_\alpha = \underline{w} = \bar{w}. \tag{4.2}$$

Define the following function on $\mathbb{X}$ for the sequence $\{\alpha_n \uparrow 1\}_{n=1,2,\ldots}$:

$$\tilde{u}(x) := \liminf_{n\to+\infty, y\to x} u_{\alpha_n}(y). \tag{4.3}$$

In words, $\tilde{u}(x)$ is the largest number such that $\tilde{u}(x) \leq \liminf_{n\to+\infty} u_{\alpha_n}(y_n)$ for all sequences $\{y_n \to x\}$. Since $u_\alpha(x)$ is nonnegative by definition, $\tilde{u}(x)$ is also nonnegative. The function

$\tilde{u}$, defined in (4.3) for a sequence $\{\alpha_n \uparrow 1\}_{n=1,2,\ldots}$ of nonnegative discount factors, is called an average-cost relative value function.

If Assumptions **W*** and **B** hold, then Feinberg et al. [7, Corollary 2] implies the validity of ACOIs and

$$w^{\phi}(x) = \underline{w} = \lim_{\alpha \uparrow 1}(1-\alpha)v_\alpha(x) = \bar{w} = w^*, \quad x \in \mathbb{X}, \tag{4.4}$$

where $w^{\phi}(x)$ is defined in (2.5). Furthermore, let us define $w := \underline{w}$; see (4.2) and (4.4) for other equalities for w.

Consider the renewal counting process

$$\mathbf{N}(t) := \sup\{n = 0, 1, \ldots\ : \mathbf{S}_n \leq t\}, \tag{4.5}$$

where $t \in \mathbb{R}_+$, $\mathbf{S}_0 := 0$, and

$$\mathbf{S}_n := \sum_{j=1}^{n} D_j, \quad n = 1, 2, \ldots\ . \tag{4.6}$$

Observe that since $P(D > 0) > 0$, $\mathbb{E}[\mathbf{N}(t)] < +\infty$, $t \in \mathbb{R}_+$; see Resnick [21, Theorem 3.3.1]. For $x \in \mathbb{X}$ and $y \geq 0$ define

$$E_y(x) := \mathbb{E}[h(x - \mathbf{S}_{\mathbf{N}(y)+1})]. \tag{4.7}$$

Since $x - y \leq x - \mathbf{S}_{\mathbf{N}(y)} \leq x$ and the function $\mathbb{E}[h(x-D)]$ is quasiconvex,

$$E_y(x) = \mathbb{E}[h(x - \mathbf{S}_{\mathbf{N}(y)} - D)] \leq \max\{\mathbb{E}[h(x-y-D)], \mathbb{E}[h(x-D)]\} < +\infty. \tag{4.8}$$

THEOREM 4.1: *Let Assumption 1 hold. The inventory model satisfies Assumption B.*

PROOF: Assumption B(i) follows from the same arguments in the first paragraph of the proof in Feinberg and Lewis [11, Proposition 6.3].

The inf-compactness of the function $c : \mathbb{X} \times \mathbb{A} \mapsto \mathbb{R}$ and the validity of Assumption W* imply that for each $\alpha \in [0, 1)$ the function v_α is inf-compact (Feinberg and Lewis [10, Proposition 3.1(iv)]), and therefore the set

$$X_\alpha := \{x \in \mathbb{X} : v_\alpha(x) = m_\alpha\}, \tag{4.9}$$

where m_α is defined in (4.1), is nonempty and compact. Furthermore, the validity of Assumption B(i) implies that there is a compact subset $\mathcal{K}$ of $\mathbb{X}$ such that $\mathbb{X}_\alpha \subset \mathcal{K}$ for all $\alpha \in [0, 1)$; see Feinberg et al. [7, Theorem 6]. Following Feinberg and Lewis [11], consider a bounded interval $[x_L^*, x_U^*] \subset \mathbb{X}$ such that

$$X_\alpha \subset [x_L^*, x_U^*] \quad \text{for all } \alpha \in [0, 1). \tag{4.10}$$

Consider an arbitrary $\alpha \in [0, 1)$ and a state x_α such that $v_\alpha(x_\alpha) = m_\alpha$, where m_α is defined in (4.1). In view of (4.10), the inequalities $x_L^* \leq x_\alpha \leq x_U^*$ hold.

Let

$$E(x) := \mathbb{E}[h(x-D)] + E_{x - x_L^*}(x) < +\infty, \tag{4.11}$$

where the function $E_y(x)$ is defined in (4.7) and its finiteness is stated in (4.8). For $x_t = x - \mathbf{S}_t$, $t = 1, \ldots, \mathbf{N}(x - x_L^*) + 1$,

$$\mathbb{E}[h(x_t)] \leq \mathbb{E}[h(x-D)] + \mathbb{E}[h(x - \mathbf{S}_{\mathbf{N}(x-x_L^*)+1})] = E(x), \tag{4.12}$$

where the inequality holds because the function $\mathbb{E}[h(x-D)]$ is quasiconvex and $x - \mathbf{S}_{\mathbf{N}(x-x_L^*)+1} = x - \mathbf{S}_{\mathbf{N}(x-x_L^*)} - D \leq x_t = x_{t-1} - D \leq x - D$ for $t = 1, \ldots, \mathbf{N}(x - x_L^*) + 1$.

By considering the same policy σ and following the arguments thereafter as in the proof in Feinberg and Lewis [11, Proposition 6.3] with the equation (6.14) there replaced with (4.12), we obtain the validity of Assumption **B**. ■

Now we establish the boundedness and the equicontinuity of the discounted relative value functions u_α defined in (4.1). Consider

$$U(x) := \begin{cases} K + \bar{c}(x_U^* - x), & \text{if } x < x_L^*, \\ K + \bar{c}(x_U^* - x_L^*) + (E(x) + \bar{c}\mathbb{E}[D])(1 + \mathbb{E}[\mathbf{N}(x - x_L^*)]), & \text{if } x \geq x_L^*, \end{cases} \qquad \textbf{(4.13)}$$

where the real numbers x_L^* and x_U^* are defined in (4.10) and the function $E(x)$ is defined in (4.11).

LEMMA 4.2: *Let Assumption 1 hold. The following statements hold for all* $\alpha \in [0, 1)$*:*

(i) $u_\alpha(x) \leq U(x) < +\infty$ *for all* $x \in \mathbb{X}$*;*
(ii) *If* $x_*, x \in \mathbb{X}$ *and* $x_* \leq x$*, then* $C(x_*, x) := \sup_{y\in[x_*,x]} U(y) < +\infty$*;*
(iii) $\mathbb{E}[U(x - D)] < +\infty$ *for all* $x \in \mathbb{X}$*.*

PROOF: The proof of this lemma is identical to the proof in Feinberg and Liang [13, Lemma 4.6]. ■

The following theorem is proved in Feinberg and Lewis [11, Theorem 6.10(iii)] under Assumption 2. The proof there remains correct under the weaker Assumption 1.

THEOREM 4.3: *Let Assumption 1 hold. For each nonnegative discount factor* $\alpha \in (\alpha^*, 1)$*, consider an optimal* (s'_α, S'_α) *policy for the discounted criterion with the discount factor* α*. Let* $\{\alpha_n \uparrow 1\}_{n=1,2,\ldots}$ *be a sequence of negative numbers with* $\alpha_1 > \alpha^*$*. Every sequence* $\{(s'_{\alpha_n}, S'_{\alpha_n})\}_{n=1,2,\ldots}$ *is bounded, and each of its limit points* (s^*, S^*) *defines an average-cost optimal* (s^*, S^*) *policy. Furthermore, this policy satisfies the optimality inequality*

$$w + \tilde{u}(x) \geq \min\left\{\min_{a\geq 0}[K + H(x + a)], H(x)\right\} - \bar{c}x, \qquad \textbf{(4.14)}$$

where

$$H(x) := \bar{c}x + \mathbb{E}[h(x - D)] + \mathbb{E}[\tilde{u}(x - D)], \qquad \textbf{(4.15)}$$

where the function $\tilde{u}$ *is defined in (4.3) for an arbitrary subsequence* $\{\alpha_{n_k}\}_{k=1,2,\ldots}$ *of* $\{\alpha_n\}_{n=1,2,\ldots}$ *satisfying* $(s^*, S^*) = \lim_{k\to+\infty}(s'_{\alpha_{n_k}}, S'_{\alpha_{n_k}})$*.*

Recall the following definition of equicontinuity.

DEFINITION 4.4: *A family* $\mathcal{H}$ *of real-valued functions on a metric space* X *is called equicontinuous at the point* $x \in X$ *if for each* $\varepsilon > 0$ *there exists an open set* G *containing* x *such that*

$$|h(y) - h(x)| < \varepsilon \quad \text{for all } y \in G \text{ and for all } h \in \mathcal{H}.$$

The family of functions $\mathcal{H}$ *is called equicontinuous (on* X*) if it is equicontinuous at all* $x \in X$*.*

Consider the following assumption on the discounted relative value functions.

Assumption EC (Feinberg and Liang [13]): *There exists a sequence* $\{\alpha_n \uparrow 1\}_{n=1,2,\ldots}$ *of nonnegative discount factors such that*

(i) *the family of functions* $\{u_{\alpha_n}\}_{n=1,2,\ldots}$ *is equicontinuous, and*

(ii) *there exists a nonnegative measurable function* $U(x)$, $x \in \mathbb{X}$, *such that* $U(x) \geq u_{\alpha_n}(x)$, $n = 1, 2, \ldots,$ *and* $\int_{\mathbb{X}} U(y)q(dy|x,a) < +\infty$ *for all* $x \in \mathbb{X}$ *and* $a \in \mathbb{A}$.

The following theorem provides sufficient conditions for the existence of a stationary policy ϕ and a function $\tilde{u}(\cdot)$ satisfying the ACOEs.

Theorem 4.5 (Feinberg and Liang [13, Theorem 3.2]): *Let Assumptions* ***W**** *and* ***B*** *hold. Consider a sequence* $\{\alpha_n \uparrow 1\}_{n=1,2,\ldots}$ *of nonnegative discount factors. If Assumption* ***EC*** *is satisfied for the sequence* $\{\alpha_n\}_{n=1,2,\ldots}$, *then the following statements hold.*

(i) *There exists a subsequence* $\{\alpha_{n_k}\}_{k=1,2,\ldots}$ *of* $\{\alpha_n\}_{n=1,2,\ldots}$ *such that* $\{u_{\alpha_{n_k}}(x)\}$ *converges pointwise to* $\tilde{u}(x)$, $x \in \mathbb{X}$, *where* $\tilde{u}(x)$ *is defined in* (4.3) *for the subsequence* $\{\alpha_{n_k}\}_{k=1,2,\ldots}$, *and the convergence is uniform on each compact subset of* $\mathbb{X}$. *In addition, the function* $\tilde{u}(x)$ *is continuous.*

(ii) *There exists a stationary policy* ϕ *satisfying the ACOE with the nonnegative function* $\tilde{u}$ *defined for the sequence* $\{\alpha_{n_k}\}_{k=1,2,\ldots}$ *mentioned in statement (i), that is, for all* $x \in \mathbb{X}$,

$$w + \tilde{u}(x) = c(x, \phi(x)) + \int_{\mathbb{X}} \tilde{u}(y)q(dy|x,\phi(x)) = \min_{a \in \mathbb{A}} [c(x,a) + \int_{\mathbb{X}} \tilde{u}(y)q(dy|x,a)], \tag{4.16}$$

and every stationary policy satisfying (4.16) *is average-cost optimal.*

The following theorem shows that the equicontinuity conditions stated in Theorem 4.5 holds for the inventory model with holding/backlog costs satisfying quasiconvexity assumptions.

Theorem 4.6: *Let Assumption* 1 *hold. Then for each* $\beta \in (\alpha^*, 1)$, *the family of functions* $\{u_\alpha\}_{\alpha \in [\beta,1)}$ *is equicontinuous on* $\mathbb{X}$.

Proof: Consider $\beta \in (\alpha^*, 1)$. According to Theorem 3.6, since the (s_α, S_α) policies are optimal, the arguments provided to prove formula (4.38) in Feinberg and Liang [13, Theorem 4.9(a)] imply that the set $\{s_\alpha\}_{\alpha \in [\beta,1)}$ is bounded. Therefore, there exist constants $b > 0$ and $\delta_0 > 0$ such that

$$-b \leq s_\alpha - \delta_0 < s_\alpha + \delta_0 \leq b, \quad \alpha \in [\beta, 1). \tag{4.17}$$

Consider $x^* \in \mathbb{X}$ and $\varepsilon > 0$. Let $M := \max\{b, x^*\} + 1$. To prove that the family $\{u_\alpha\}_{\alpha \in [\beta,1)}$ is equicontinuous at x^*, we show that there exists $\delta^* > 0$ such that, if $y^* \in \mathbb{X}$ satisfies $|x^* - y^*| < \delta^*$, then for all $\alpha \in [\beta, 1)$

$$|u_\alpha(x^*) - u_\alpha(y^*)| < \varepsilon. \tag{4.18}$$

We consider the following three cases: (1) $x^* < s_\alpha$ and $y^* < s_\alpha$; (2) either $x^* \leq s_\alpha \leq y^*$ or $y^* \leq s_\alpha \leq x^*$; and (3) $x^* > s_\alpha$ and $y^* > s_\alpha$. In each case, we will prove the validity of (4.18).

(1) The optimality of (s_α, S_α) policy implies that $v(x) = \bar{c}(s_\alpha - x) + v_\alpha(s_\alpha)$ if $x \le s_\alpha$, Then $u_\alpha(x^*) - u_\alpha(y^*) = \bar{c}(y^* - x^*)$. Therefore, if $|x^* - y^*| < \delta_1 := \varepsilon/4\bar{c}$, then

$$|u_\alpha(x^*) - u_\alpha(y^*)| = \bar{c}|x^* - y^*| < \frac{\varepsilon}{4}. \tag{4.19}$$

(2) In this case, we first prove that there exists $\delta_2 > 0$ such that for $x \in [s_\alpha, s_\alpha + \delta_2]$

$$|u_\alpha(x) - u_\alpha(s_\alpha)| < \frac{\epsilon}{4}. \tag{4.20}$$

Let $\tilde{h}(x) := \mathbb{E}[h(x - D)]$. For $x \ge s_\alpha$

$$v_\alpha(x) = \tilde{h}(x) + \alpha\mathbb{E}[v_\alpha(x - D)] \tag{4.21}$$

and

$$\begin{aligned}\mathbb{E}[v_\alpha(x - D)] &= P(D \ge x - s_\alpha)\mathbb{E}[\bar{c}(s_\alpha - x + D)|D \ge x - s_\alpha] \\ &\quad + P(0 < D < x - s_\alpha)\mathbb{E}[v_\alpha(x - D)|0 < D < x - s_\alpha] + P(D = 0)v_\alpha(x).\end{aligned} \tag{4.22}$$

Formulae (4.21) and (4.22) imply

$$\begin{aligned}[1 - \alpha P(D = 0)]v_\alpha(x) &= \tilde{h}(x) + \alpha(P(D \ge x - s_\alpha)\mathbb{E}[\bar{c}(s_\alpha - x + D)|D \ge x - s_\alpha] \\ &\quad + P(0 < D < x - s_\alpha)\mathbb{E}[v_\alpha(x - D)|0 < D < x - s_\alpha]).\end{aligned} \tag{4.23}$$

Therefore, since $u_\alpha(y_1) - u_\alpha(y_2) = v_\alpha(y_1) - v_\alpha(y_2)$ for all $y_1, y_2 \in \mathbb{X}$, for $x \in [s_\alpha, s_\alpha + \delta_1]$

$$\begin{aligned}&[1 - \alpha P(D = 0)]|u_\alpha(x) - u_\alpha(s_\alpha)| = [1 - \alpha P(D = 0)]|v_\alpha(x) - v_\alpha(s_\alpha)| \\ &\quad = \Big|\tilde{h}(x) - \tilde{h}(s_\alpha) + \alpha P(D \ge x - s_\alpha)\bar{c}(s_\alpha - x) \\ &\qquad + \alpha P(0 < D < x - s_\alpha)\mathbb{E}[u_\alpha(x - D) - u_\alpha(s_\alpha - D)|0 < D < x - s_\alpha]\Big| \\ &\quad \le |\tilde{h}(x) - \tilde{h}(s_\alpha)| + \bar{c}(x - s_\alpha) + 2P(0 < D < x - s_\alpha)C(-b, b),\end{aligned} \tag{4.24}$$

where the nonnegative function C is defined in Lemma 4.2. Let us define $Q_1 := (1 - P(D = 0))^{-1}$, and $Q_2(x, s_\alpha) := P(0 < D < x - s_\alpha)$. Recall that $P(D > 0) > 0$, which is equivalent to $P(D = 0) < 1$. Since $(1 - \alpha P(D = 0))^{-1} \le Q_1$, formula (4.24) implies that

$$|u_\alpha(x) - u_\alpha(s_\alpha)| \le Q_1(|\tilde{h}(x) - \tilde{h}(s_\alpha)| + \bar{c}(x - s_\alpha) + 2Q_2(x, s_\alpha)C(-b, b)). \tag{4.25}$$

Since the function $\tilde{h}$ is uniformly continuous on the interval $[-b, b]$, all three summands in the right-hand side of the last equations converge uniformly in α to 0 as $x \downarrow s_\alpha$. Therefore, there exists $\delta_2 \in (0, \delta_0)$ such that (4.20) holds for all $x \in [s_\alpha, s_\alpha + \delta_2]$.

For $x \le s_\alpha \le y$ satisfying $|x - y| < \delta_3 := \min\{\delta_1, \delta_2\}$,

$$|u_\alpha(x) - u_\alpha(y)| \le |u_\alpha(x) - u_\alpha(s_\alpha)| + |u_\alpha(s_\alpha) - u_\alpha(y)| < \frac{\epsilon}{2}, \tag{4.26}$$

where the first inequality is the triangle property and the second one follows from (4.19) and (4.20). Therefore, in this case, (4.26) imply that, if $|x^* - y^*| < \delta_3$, then (4.18) holds.

(3) Consider $y^* < M$. Let $z_1 := \min\{x^*, y^*\}$ and $z_2 := \max\{x^*, y^*\}$. Then $s_\alpha < z_1 \leq z_2 < M$. The optimality of (s_α, S_α) policy implies that

$$\begin{aligned}
|u_\alpha(x^*) - u_\alpha(y^*)| &= |u_\alpha(z_1) - u_\alpha(z_2)| = |v_\alpha(z_1) - v_\alpha(z_2)| \\
&= \Bigg| \mathbb{E}\Bigg[\sum_{j=1}^{\mathbf{N}(z_1 - s_\alpha)+1} \alpha^{j-1}(\tilde{h}(z_1 - \mathbf{S}_{j-1}) - \tilde{h}(z_2 - \mathbf{S}_{j-1})) \\
&\quad + \alpha^{\mathbf{N}(z_1 - s_\alpha)+1}(v_\alpha(z_1 - \mathbf{S}_{\mathbf{N}(z_1 - s_\alpha)+1}) - v_\alpha(z_2 - \mathbf{S}_{\mathbf{N}(z_1 - s_\alpha)+1}))\Bigg]\Bigg| \\
&\leq \mathbb{E}\Bigg[\sum_{j=1}^{\mathbf{N}(M+b)+1} |\tilde{h}(z_1 - \mathbf{S}_{j-1}) - \tilde{h}(z_2 - \mathbf{S}_{j-1})| \Bigg] \\
&\quad + \mathbb{E}[|u_\alpha(z_1 - \mathbf{S}_{\mathbf{N}(z_1 - s_\alpha)+1}) - u_\alpha(z_2 - \mathbf{S}_{\mathbf{N}(z_1 - s_\alpha)+1})|], \qquad \textbf{(4.27)}
\end{aligned}$$

where the inequality follows from the standard properties of expectations and absolute values, $z_1 - s_\alpha \leq M + b$, and $\alpha < 1$. Recall that the function $\tilde{h}(x)$ is continuous and finite. Therefore, the function $\tilde{h}$ is uniformly continuous on the closed interval $[-(M + 2b), M]$. In addition, Assumption 1 implies that the function $\tilde{h}$ is quasiconvex.

In view of (4.26), if $|z_1 - z_2| < \delta_3$, then with probability 1

$$|u_\alpha(z_1 - \mathbf{S}_{\mathbf{N}(z_1 - s_\alpha)+1}) - u_\alpha(z_2 - \mathbf{S}_{\mathbf{N}(z_1 - s_\alpha)+1})| < \frac{\epsilon}{2},$$

and therefore

$$\mathbb{E}[|u_\alpha(z_1 - \mathbf{S}_{\mathbf{N}(z_1 - s_\alpha)+1}) - u_\alpha(z_2 - \mathbf{S}_{\mathbf{N}(z_1 - s_\alpha)+1})|] < \frac{\epsilon}{2}. \qquad \textbf{(4.28)}$$

Now, we estimate the first term in right-hand side of the inequality in (4.27). Since $-(M + 2b) < x - \mathbf{S}_{j-1} < M$ for all $x \in [-b, M]$ and $j = 1, 2, \ldots, \mathbf{N}(M + b) + 1$, the nonnegativity and quasiconvexity of $\tilde{h}$ imply that

$$0 \leq \tilde{h}(x - \mathbf{S}_{j-1}) \leq \max\{\tilde{h}(-(M + 2b)), \tilde{h}(M)\}, \quad x \in (-b, M). \qquad \textbf{(4.29)}$$

Since $-b < z_1 \leq z_2 < M$

$$\begin{aligned}
&\mathbb{E}\Bigg[\sum_{j=1}^{\mathbf{N}(M+b)+1} |\tilde{h}(z_1 - \mathbf{S}_{j-1}) - \tilde{h}(z_2 - \mathbf{S}_{j-1})| \Bigg] \\
&\quad \leq \mathbb{E}\Bigg[\sum_{j=1}^{\mathbf{N}(M+b)+1} \max\{\tilde{h}(-(M + 2b)), \tilde{h}(M)\} \Bigg] \\
&\quad \leq \mathbb{E}[\mathbf{N}(M + b) + 1] \max\{\tilde{h}(-(M + 2b)), \tilde{h}(M)\} < +\infty, \qquad \textbf{(4.30)}
\end{aligned}$$

where the first inequality follows from (4.29), the second one follows from Wald's identity, and the last one follows from the finiteness of the function $\tilde{h}$. Therefore,

$$\lim_{z\to z_2} \mathbb{E}\left[\sum_{j=1}^{\mathbf{N}(M+b)+1} |\tilde{h}(z-\mathbf{S}_{j-1})-\tilde{h}(z_2-\mathbf{S}_{j-1})|\right]$$
$$= \mathbb{E}\left[\sum_{j=1}^{\mathbf{N}(M+b)+1} \lim_{z\to z_2} |\tilde{h}(z-\mathbf{S}_{j-1})-\tilde{h}(z_2-\mathbf{S}_{j-1})|\right] = 0, \quad \textbf{(4.31)}$$

where the first equality follows from (4.30) and Lebesgue's dominated convergence theorem, and the second one follows from the continuity of $\tilde{h}$. In view of (4.31), there exists $\delta_4 > 0$ such that, if $|z_1 - z_2| < \delta_4$, then

$$\mathbb{E}\left[\sum_{j=1}^{\mathbf{N}(M+b)+1} |\tilde{h}(z_1-\mathbf{S}_{j-1})-\tilde{h}(z_2-\mathbf{S}_{j-1})|\right] \le \frac{\varepsilon}{2}. \quad \textbf{(4.32)}$$

In view of (4.27), (4.28), and (4.32), for $z_1, z_2 \ge s_\alpha$ satisfying $|z_1 - z_2| < \delta_5 := \min\{\delta_3, \delta_4\}$

$$|u_\alpha(z_1) - u_\alpha(z_2)| < \varepsilon, \quad \textbf{(4.33)}$$

which is equivalent to that, in this case, if $|x^* - y^*| < \delta_5$, then (4.18) holds.

Hence, the cases (1-3) imply that (4.18) holds with $\delta^* := \min\{\delta_1, \delta_3, \delta_5\}$. ■

THEOREM 4.7: *Let Assumption 1 hold. Then for $\alpha \in (\alpha^*, 1)$, the functions v_α and G_α are continuous on $\mathbb{X}$.*

Proof of Theorem 4.7: According to Theorem 3.6, there exists an optimal (s_α, S_α) optimal policy for the infinite-horizon problem. In addition, Theorem 4.6 implies that the function $v_\alpha(x) = u_\alpha(x) + m_\alpha$ is continuous on $\mathbb{X}$. Therefore, since the function $\mathbb{E}[h(x - D)]$ is continuous, the same arguments in the proof of Feinberg and Liang [12, Theorem 5.3], starting from the definition of the function g_α there, imply that the function G_α is continuous on $\mathbb{X}$. ■

5. SETUP-COST INVENTORY MODEL: AVERAGE COSTS PER UNIT TIME

As follows from Chen and Simchi-Levi [5], an average-cost optimal (s, S) policy exists if Assumption 3 holds for $\alpha = 1$. In this section, we study approximations of average-cost optimal (s, S) policies by discount-cost optimal (s_α, S_α) policies as the discount factor tend to 1. The following theorem establishes the convergence of discounted-cost optimality equations to the ACOEs for the described inventory model and the optimality of (s, S) policies under the average cost criterion under Assumption 1.

THEOREM 5.1: *Let Assumption 1 hold. For every sequence $\{\alpha_n \uparrow 1\}_{n=1,2,\ldots}$ of nonnegative discount factors with $\alpha_1 > \alpha^*$, there exist a subsequence $\{\alpha_{n_k}\}_{k=1,2,\ldots}$ of $\{\alpha_n\}_{n=1,2,\ldots}$, a stationary policy φ, and a function $\tilde{u}$ defined in (4.3) for the subsequence $\{\alpha_{n_k}\}_{k=1,2,\ldots}$*

such that for all $x \in \mathbb{X}$

$$\begin{aligned} w + \tilde{u}(x) &= K\mathbf{1}_{\{\varphi(x)>0\}} + H(x+\varphi(x)) - \bar{c}x \\ &= \min\left\{\min_{a\geq 0}[K + H(x+a)], H(x)\right\} - \bar{c}x, \end{aligned} \tag{5.1}$$

where the function H *is defined in* (4.15)*. In addition, the functions* $\tilde{u}$ *and* H *are continuous and inf-compact, and a stationary optimal policy* φ *satisfying* (5.1) *can be selected as an* (s^*, S^*) *policy described in Theorem* 4.3*. It also can be selected as an* (s, S) *policy with the real numbers* S *and* s *satisfying* (3.10) *and defined in* (3.11) *respectively for* $f(x) = H(x)$*,* $x \in \mathbb{X}$*.*

Remark 5.2: The relations between the function $\tilde{u}$ in ACOE (5.1) and the solutions to the ACOE constructed by Chen and Simchi-Levi [5] are currently not clear.

To prove Theorem 5.1, we first establish several properties of the average-cost relative value function. Recall that $x_1^{\min}$ is defined in (2.8).

LEMMA 5.3: *Let Assumption* 1 *hold. Consider the function* $\tilde{u}$ *defined in* (4.3) *for a sequence* $\{\alpha_n\}_{n=1,2,\ldots}$ *such that* $\alpha_n \uparrow 1$ *and* $\alpha_1 > \alpha^*$*. Then the following statements hold:*

(i) *For* $x \leq y$

$$\tilde{u}(x) + \bar{c}x \leq \tilde{u}(y) + \bar{c}y + K, \tag{5.2}$$

$$H(y) - H(x) \geq \mathbb{E}[h(y-D)] - \mathbb{E}[h(x-D)] - \alpha K. \tag{5.3}$$

(ii) *For* $x \leq y \leq x_1^{\min}$

$$\tilde{u}(y) + \bar{c}y - \tilde{u}(x) - \bar{c}x \leq 0, \tag{5.4}$$

$$H(y) - H(x) \leq 0. \tag{5.5}$$

Proof of Theorem 5.1: The proof of this theorem is identical to the proof in Feinberg and Liang [13, Theorem 4.5] with the following changes: (i) Lemmas 4.6 and 4.7 from [13] should be replaced with Lemma 4.2 and Theorem 4.6 from this paper; and (ii) the proof of the K-convexity of the functions u and H and the optimality of (s, S) policy under the average cost criterion should be replaced with the following arguments. Consider cases (1-3) in the proof of Theorem 3.10(i) with $G_{N-t-1,\alpha}$, h_α and $x_\alpha^{\min}$ replaced with H, h, and $x_1^{\min}$, respectively. Then Lemma 5.3 implies that there exists an optimal (s, S) policy, with the real numbers S and s satisfying (3.10) and defined in (3.11) for $f := H$. ■

Furthermore, the continuity of average-cost relative value functions implies the following corollary.

COROLLARY 5.4: *Let Assumption* 1 *hold, the state space* $\mathbb{X} = \mathbb{R}$*, and the action space* $\mathbb{A} = \mathbb{R}_+$*. For the* (s, S) *policy defined in Theorems* 5.1*, consider the stationary policy* φ *coinciding with this policy at all* $x \in \mathbb{X}$*, except* $x = s$*, and with* $\varphi(s) = S - s$*. Then the stationary policy* φ *also satisfies the optimality equation* (5.1)*, and it is therefore average-cost optimal.*

6. CONVERGENCE OF OPTIMAL LOWER THRESHOLDS S_α

This section establishes convergence of discounted optimal lower thresholds $s_\alpha \to s$ as $\alpha \uparrow 1$, where s the average-cost optimal lower threshold (stated in Theorem 5.1) for the inventory model with holding/backlog costs satisfying quasiconvexity assumptions. In this and the following sections, we assume that the state space $\mathbb{X} = \mathbb{R}$ and the action sets $\mathbb{A} = A(x) = \mathbb{R}_+$ for all $x \in \mathbb{X}$. This means that an arbitrary nonnegative amount of inventory can be ordered at any state.

The quasiconvexity of $\mathbb{E}[h(x-D)]$ assumed in Assumption 1 implies that the function $\mathbb{E}[h(x-D)]$ is nonincreasing on $(-\infty, x_1^{\min})$, where $x_1^{\min}$ is defined in (2.8). The stronger Assumption 4 is used in this section and Section 7. The following theorem establishes convergence of the discounted optimal lower thresholds s_α when the discount factor α converges to 1.

THEOREM 6.1: *Let Assumption 1 hold and for $\alpha = 1$ Assumption 4 hold. Then the limit*

$$s_1 := \lim_{\alpha \uparrow 1} s_\alpha \tag{6.1}$$

exists and $s_1 \le x_1^{\min}$.

Remark 6.2: As shown in Corollary 7.5, if Assumption 1 holds and for $\alpha = 1$ Assumption 4 holds, then all the sequences $\{\alpha_n \uparrow 1\}_{n=1,2,\ldots}$ define the same functions $\tilde{u}$ and H in (5.1), and, according to Theorem 7.6, there exists a unique threshold s, for which there is an (s, S) policy satisfying the ACOE (5.1). However, if $\mathbb{X} = \mathbb{Z}$, as this takes place for problems with discrete commodity, there may be multiple average-cost optimal thresholds s, as shown in Example 2.

Example 2: Consider the inventory model with $\mathbb{X} = \mathbb{A} = \mathbb{Z}$, $K = 124$, $\bar{c} = 0$, $P(D = 1) = 1 - P(D = 0) = 0.417$, and

$$h(x) = \begin{cases} -0.11x & \text{if } x < 0, \\ 0 & \text{if } 0 \le x \le 100, \\ 1.05(x - 100) & \text{if } x > 100. \end{cases}$$

Let us consider $S = 100$, $s = -5$, and $\tilde{s} = -4$. Straightforward calculations based on Chen and Simchi-Levi [5, Lemma 2] imply that the average costs per unit time for the (s, S) and $(\tilde{s}, S)$ policies are equal to 20.584. Johnson [19, Theorem 5.2] implies that at least one of these policies is optimal. Therefore, there are multiple average-cost optimal thresholds s.

Before the proof of Theorem 6.1, we first state several auxiliary facts. Consider the infinite-horizon value function $\bar{v}_\alpha$ for the model with zero unit and terminal costs. According to Lemma 3.11, $\bar{v}_\alpha(x) - \bar{c}x = v_\alpha(x)$, $x \in \mathbb{X}$. For $x \in \mathbb{X}$, define

$$\bar{m}_\alpha := \min_{x \in \mathbb{X}} \{\bar{v}_\alpha(x)\} \quad \text{and} \quad \bar{u}_\alpha(x) := \bar{v}_\alpha(x) - \bar{m}_\alpha. \tag{6.2}$$

If there exists an α-discount optimal (s_α, S_α) policy, then (3.14) can be written as

$$\bar{v}_\alpha(x) = \begin{cases} G_\alpha(x) & \text{if } x \ge s_\alpha, \\ K + G_\alpha(S_\alpha) & \text{if } x < s_\alpha, \end{cases} \tag{6.3}$$

which implies that

$$\bar{m}_\alpha = \min_{x\in\mathbb{X}}\{G_\alpha(x)\} = G_\alpha(S_\alpha). \tag{6.4}$$

Consider $x_\alpha \in X_\alpha$, where X_α is defined in (4.10). For $\alpha \in [0, 1)$

$$\bar{m}_\alpha \le \bar{v}_\alpha(x_\alpha) = m_\alpha + \bar{c}x_\alpha \le m_\alpha + \bar{c}x_U^*, \tag{6.5}$$

where x_U^* is defined in (4.10). In view of (6.3), the continuity of $\bar{v}_\alpha(x)$ implies that $\bar{v}_\alpha(x) = \bar{v}_\alpha(s_\alpha)$ for all $x \le s_\alpha$. Therefore,

$$\bar{m}_\alpha = \inf_{x\ge s_\alpha} \bar{v}_\alpha(x) = \inf_{x\ge s_\alpha} \{v_\alpha(x) + \bar{c}x\} \ge \inf_{x\ge s_\alpha} \{v_\alpha(x) + \bar{c}s_\alpha\} \ge m_\alpha + \bar{c}s_\alpha, \tag{6.6}$$

where the first inequality holds because $x \ge s_\alpha$ and the last one follows from $m_\alpha = \inf_x v_\alpha(x)$. Then (6.5) and (6.6) imply

$$m_\alpha + \bar{c}s_\alpha \le \bar{m}_\alpha \le m_\alpha + \bar{c}x_U^*. \tag{6.7}$$

For $\alpha \in (\alpha^*, 1)$ define the set of all possible optimal discounted lower thresholds

$$\mathcal{G}_\alpha := \{x \in [s_\alpha, S_\alpha] : G_\alpha(y) \ge K + G_\alpha(S_\alpha) \text{ for all } y \le x\}, \tag{6.8}$$

where S_α satisfies (3.10) and s_α is defined in (3.11) with $f := G_\alpha$. Note that $s_\alpha \in \mathcal{G}_\alpha$ and $y \ge s_\alpha$ for all $y \in \mathcal{G}_\alpha$.

Remark 6.3: The set $\mathcal{G}_\alpha$ is not empty if $\mathbb{X} = \mathbb{R}$ because the function G_α is continuous (see Theorem 4.7) and $\lim_{x\to-\infty} G_\alpha(x) > K + G_\alpha(S_\alpha)$. If $\mathbb{X} = \mathbb{Z}$, as this takes place for problems with discrete commodity, it is possible that $\mathcal{G}_\alpha$ is an empty set.

The following three lemmas state the relations between parameters defined in this section.

LEMMA 6.4: *If Assumption 1 holds, then, for all $\alpha \in (\alpha^*, 1)$ and $y \in \mathcal{G}_\alpha$,*

$$(1-\alpha)(\bar{m}_\alpha + K) = \mathbb{E}[h_\alpha(y - D)]. \tag{6.9}$$

LEMMA 6.5: *If Assumption 1 holds, then $y \le x_\alpha^{\min} \le x_1^{\min}$ for all $\alpha \in (\alpha^*, 1)$ and $y \in \mathcal{G}_\alpha$.*

LEMMA 6.6: *If Assumption 1 holds, then*

$$\lim_{\alpha\uparrow 1}(1-\alpha)\bar{m}_\alpha = \lim_{\alpha\uparrow 1}\mathbb{E}[h_\alpha(s_\alpha - D)] = w. \tag{6.10}$$

Proof of Theorem 6.1: The proof is by contradiction. According to Theorem 4.3, for $\alpha_n \uparrow 1$, $n = 1, 2, \ldots$, with $\alpha_1 > \alpha^*$, every sequence $\{(s_{\alpha_n}, S_{\alpha_n})\}_{n=1,2,\ldots}$ is bounded. Consider two real numbers $s^{(1)} < s^{(2)}$ such that there exist two sequences $\{\alpha_n\}_{n=1,2,\ldots}$ and $\{\tilde{\alpha}_n\}_{n=1,2,\ldots}$ satisfying $\lim_{n\to+\infty} s_{\alpha_n} = s^{(1)}$ and $\lim_{n\to+\infty} s_{\tilde{\alpha}_n} = s^{(2)}$.

Since the function $\mathbb{E}[h(x-D)]$ is continuous,

$$\lim_{n\to+\infty}\mathbb{E}[h(s_{\alpha_n}-D)]=\mathbb{E}[h(s^{(1)}-D)]. \tag{6.11}$$

Therefore,

$$\begin{aligned}\lim_{n\to+\infty}\mathbb{E}[h_{\alpha_n}(s_{\alpha_n}-D)]&=\lim_{n\to+\infty}\left\{\mathbb{E}[h(s_{\alpha_n}-D)]+(1-\alpha_n)\bar{c}s_{\alpha_n}+\alpha_n\bar{c}\mathbb{E}[D]\right\}\\&=\mathbb{E}[h(s^{(1)}-D)]+\bar{c}\mathbb{E}[D],\end{aligned} \tag{6.12}$$

where the second equality follows from (6.11) and $s_{\alpha_n}\to s^{(1)}\in\mathbb{R}$ as $\alpha_n\uparrow 1$. According to Lemma 6.6, $\mathbb{E}[h(s^{(1)}-D)]=w-\bar{c}\mathbb{E}[D]$. By the same arguments with α_n replaced with $\tilde{\alpha}_n$, the formula $\mathbb{E}[h(s^{(2)}-D)]=w-\bar{c}\mathbb{E}[D]$ holds. Therefore,

$$\mathbb{E}[h(s^{(1)}-D)]=\mathbb{E}[h(s^{(2)}-D)]. \tag{6.13}$$

According to Lemma 6.5, $s_\alpha\le x_\alpha^{\min}$ for all $\alpha\in(\alpha^*,1)$. Therefore

$$s^{(2)}=\lim_{n\to+\infty}s_{\tilde{\alpha}_n}\le\liminf_{n\to+\infty}x_{\tilde{\alpha}_n}^{\min}\le x_1^{\min}, \tag{6.14}$$

where the last inequality follows from Lemma 6.5. Since $s^{(1)}<s^{(2)}\le x_1^{\min}$, Assumption 4 implies that

$$\mathbb{E}[h(s^{(1)}-D)]>\mathbb{E}[h(s^{(2)}-D)], \tag{6.15}$$

which contradicts (6.13). Thus, the limit $\lim_{\alpha\uparrow 1}s_\alpha$ exists and (6.14) implies that $s_1\le x_1^{\min}$. ∎

The following theorem establishes the uniqueness of possible optimal lower thresholds for the inventory model with convex cost functions under the discounted criterion.

THEOREM 6.7: *Let Assumption 1 hold and for some $\alpha\in(\alpha^*,1)$ Assumption 4 hold. Then $\mathcal{G}_\alpha=\{s_\alpha\}$, where $\mathcal{G}_\alpha$ and s_α are defined in (6.8) and (3.11) with $f:=G_\alpha$, respectively.*

PROOF: Recall that $s_\alpha\in\mathcal{G}_\alpha$ and $y\ge s_\alpha$ for all $y\in\mathcal{G}_\alpha$. The proof is by contradiction. Assume that there exists $y_1\in\mathcal{G}_\alpha$ such that $y_1>s_\alpha$. According to Lemma 6.4,

$$\mathbb{E}[h_\alpha(y_1-D)]=(1-\alpha)(\bar{m}_\alpha+K)=\mathbb{E}[h_\alpha(s_\alpha-D)].$$

Since Assumption 4 holds for the discount factor α, $x_\alpha^{\min}<y_1$, where $x_\alpha^{\min}$ is defined in (2.8). However, according to Lemma 6.5, $y_1\le x_\alpha^{\min}$, which implies that $y_1\le x_\alpha^{\min}<y_1$. Therefore, $\mathcal{G}_\alpha=\{s_\alpha\}$. ∎

7. CONVERGENCE OF DISCOUNTED RELATIVE VALUE FUNCTIONS

This section establishes convergence of discounted relative value functions to the average-cost relative value function for the setup-cost inventory model when the discount factor tends to 1. This is a stronger result than the convergence for a subsequence that follows from Theorem 5.1. We recall that in this section it is assumed that $\mathbb{X}=\mathbb{R}$ and $\mathbb{A}=A(x)=\mathbb{R}_+$ for all $x\in\mathbb{X}$.

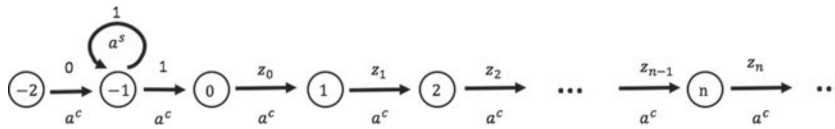

FIGURE 1. MDP described in Example 3.

Let us define

$$u(x) := \liminf_{\alpha\uparrow 1, y\to x} u_\alpha(y). \tag{7.1}$$

According to Feinberg et el. [7, Theorems 3, 4], the ACOI holds for the relative value functions $\tilde{u}$ and u defined in (4.3) and (7.1), respectively.

The following theorem states the convergence of discounted relative value functions, when the discount factor converges to 1, to the average-cost relative value function u.

THEOREM 7.1: *Let Assumption 1 hold and for $\alpha = 1$ Assumption 4 hold. Then,*

$$\lim_{\alpha\uparrow 1} u_\alpha(x) = u(x), \quad x \in \mathbb{X}, \tag{7.2}$$

and the function u is continuous.

In particular, Theorem 7.1 implies that the function $\tilde{u}$ defined in (4.3) is the same for every particular sequence $\{\alpha_n \uparrow 1\}_{n=1,2,\dots}$. The following example demonstrates that this is not true in general for MDPs under Assumptions W* and B.

Example 3: Consider an MDP with state space $\mathbb{X} = \{-2,-1,0,1,2,\dots\}$ and action space $\mathbb{A} = \{a^s, a^c\}$, where the action a^s stands for ‘stop’ and the action c stands for ‘continue”; see Figure 1. Let $A(-1) = \mathbb{A}$ and $A(n) = \{a^c\}$ for $n \in \mathbb{X}\backslash\{-1\}$. The transition probabilities are $P(-1|-1,a^s) = 1$ and $P(n+1|n,a^c) = 1$ for $n \in \mathbb{X}$. The costs are $c(-2,a^c) = 0$, $c(-1,a) = 1$ for $a \in \mathbb{A}$, and $c(n,a^c) = z_n^{(1)}$ for $n = 0,1,\dots$, where $z_n^{(1)}$ is defined as

$$z_n^{(1)} = \begin{cases} z_0 + 1, & \text{if } n = 0, \\ z_n - z_{n-1} + 1, & \text{if } n = 1,2,\dots, \end{cases} \tag{7.3}$$

where the sequence z_n is taken from Bishop et al. [1, Equation (11)]:

$$z_n = \begin{cases} 1, & \text{if } D(2k-1) \le n < D(2k), \quad k = 1,2,\dots, \\ 0, & \text{otherwise}, \end{cases}$$

where $D(k) := \sum_{i=1}^{k} i!$, $k = 1,2,\dots$. For the sequence $\{z_n\}_{n=0,1,\dots}$, define the function

$$f(\alpha) := (1-\alpha)\sum_{i=0}^{\infty} z_i \alpha^i, \quad \alpha \in [0,1). \tag{7.4}$$

As shown in the proof of Lemma 7.2 in Appendix E, the relative value function

$$u_\alpha(n) = \begin{cases} 0, & \text{if } n = -2, \\ 1, & \text{if } n = -1, \\ f(\alpha) + 1, & \text{if } n = 0, \\ (1-\alpha)\sum_{i=0}^{\infty} z_{n+i}\alpha^i - z_{n-1} + 1, & \text{if } n = 1, 2, \dots . \end{cases} \tag{7.5}$$

According to Bishop et al. [1, Proposition 1], $\liminf_{\alpha\uparrow 1} f(\alpha) = 0$ and $\limsup_{\alpha\uparrow 1} f(\alpha) = 1$. Hence, $\liminf_{\alpha\uparrow 1} u_\alpha(0) = 1$ and $\limsup_{\alpha\uparrow 1} u_\alpha(0) = 2$, that is, in this example there exist multiple relative value functions $\tilde{u}$ defined in (4.3).

Lemma 7.2: *The MDP described in Example 3 satisfies Assumptions* $\boldsymbol{W^*}$ *and* $\boldsymbol{B}$*, where the discrete metric* $d(x, y) = \boldsymbol{1}_{\{x=y\}}$ *is considered on* $\mathbb{X}$ *and* $\mathbb{A}$*.*

Before the proof of Theorem 7.1, we first state several properties of the functions $\bar{u}_\alpha$ defined in (6.2). If there exists an α-discounted optimal (s_α, S_α) optimal policy, then (6.3) implies that

$$\bar{u}_\alpha(x) = \begin{cases} G_\alpha(x) - \bar{m}_\alpha, & \text{if } x \geq s_\alpha, \\ K, & \text{if } x < s_\alpha. \end{cases} \tag{7.6}$$

Lemma 7.3: *If Assumption 1 holds, then,*

(i) *for each* $\beta \in (\alpha^*, 1)$ *the family of functions* $\{\bar{u}_\alpha\}_{\alpha\in[\beta,1)}$ *is equicontinuous on* $\mathbb{X}$*;*
(ii) $\sup_{\alpha\in(\alpha^*,1)} \bar{u}_\alpha(x) < +\infty$ *for all* $x \in \mathbb{X}$*.*

Lemma 7.4: *Let Assumption 1 hold and for* $\alpha = 1$ *Assumption 4 hold. Then there exists the limit*

$$\bar{u}(x) := \lim_{\alpha\uparrow 1} \bar{u}_\alpha(x), \quad x \in \mathbb{X}, \tag{7.7}$$

where the function $\bar{u}$ *is continuous on* $\mathbb{X}$*.*

In view of (4.1), (3.39), and (6.2),

$$u_\alpha(x) = \bar{u}_\alpha(x) + \bar{m}_\alpha - m_\alpha - \bar{c}x, \quad x \in \mathbb{X}. \tag{7.8}$$

Proof of Theorem 7.1: The theorem follows from the following two statements:

(i) there exists the limit $u^*(x) := \lim_{\alpha\uparrow 1} u_\alpha(x)$, $x \in \mathbb{X}$, and the function u^* is continuous on $\mathbb{X}$; and
(ii) $u^*(x) = \underline{u}(x) := \liminf_{\alpha\uparrow 1, y\to x} u_\alpha(x)$ for all $x \in \mathbb{X}$.

Let us prove statements (i) and (ii). (i) We show that (1) there exists the limit $u^*(s_1) := \lim_{\alpha\uparrow 1} u_\alpha(s_1)$, where s_1 is defined in (6.1); and (2) the limit exists for all $x \in \mathbb{X}$.

(1) Consider $x_\alpha \in X_\alpha$, $\alpha \in [0, 1)$, where X_α is defined in (4.9), and any given $\beta \in (\alpha^*, 1)$. In view of (4.10), since $X_\alpha \subset [x_L^*, x_U^*]$ for all $\alpha \in [0, 1)$, for every sequence $\{\alpha_n \uparrow 1\}_{n=1,2,\dots}$, there exists a subsequence $\{\alpha_{n_k} \uparrow 1\}_{k=1,2,\dots}$ of the sequence $\{\alpha_n \uparrow 1\}_{n=1,2,\dots}$ such that $\alpha_{n_1} \geq \beta$ and $x_{\alpha_{n_k}} \to x^*$ as $k \to +\infty$ for some $x^* \in [x_L^*, x_U^*]$.

Consider $\varepsilon > 0$. Since the family of functions $\{u_\alpha\}_{\alpha\in[\beta,1)}$ is equicontinuous (see Theorem 4.6), there exists an integer $M(\varepsilon) > 0$ such that for all $k \geq M(\varepsilon)$

$$|u_{\alpha_{n_k}}(x_{\alpha_{n_k}}) - u_{\alpha_{n_k}}(x^*)| < \varepsilon. \tag{7.9}$$

Since $u_{\alpha_{n_k}}(x_{\alpha_{n_k}}) = 0$ for all $k = 1, 2, \ldots$, (7.9) implies that for $k \geq M(\varepsilon)$

$$|u_{\alpha_{n_k}}(x^*)| < \varepsilon. \tag{7.10}$$

Therefore, (7.10) implies that

$$\lim_{k\to+\infty} u_{\alpha_{n_k}}(x^*) = 0. \tag{7.11}$$

Since the function $u_{\alpha_{n_k}}$ is nonnegative, (7.10) implies that for $k \geq M(\varepsilon)$

$$u_{\alpha_{n_k}}(x^*) < u_{\alpha_{n_k}}(x) + \varepsilon, \quad x \in \mathbb{X}. \tag{7.12}$$

Then (7.12) and (7.8) imply that for $k \geq M(\varepsilon)$

$$\bar{u}_{\alpha_{n_k}}(x^*) - \bar{c}x^* < \bar{u}_{\alpha_{n_k}}(x) - \bar{c}x + \varepsilon, \quad x \in \mathbb{X}. \tag{7.13}$$

By taking the limit of both sides of (7.13) as $k \to +\infty$, Lemma 7.4 implies that

$$\bar{u}(x^*) - \bar{c}x^* \leq \bar{u}(x) - \bar{c}x + \varepsilon, \quad x \in \mathbb{X}. \tag{7.14}$$

Since ε can be chosen arbitrarily, (7.14) implies that

$$\bar{u}(x^*) - \bar{c}x^* = \min_{x\in\mathbb{X}}\{\bar{u}(x) - \bar{c}x\}. \tag{7.15}$$

Let $M_{\bar{u}} := \bar{u}(s_1) - \bar{c}s_1 - \min_{x\in\mathbb{X}}\{\bar{u}(x) - \bar{c}x\}$. Then

$$\begin{aligned}\lim_{k\to+\infty} u_{\alpha_{n_k}}(s_1) - u_{\alpha_{n_k}}(x^*) &= \lim_{k\to+\infty} \bar{u}_{\alpha_{n_k}}(s_1) - \bar{c}s_1 - [\bar{u}_{\alpha_{n_k}}(x^*) - \bar{c}x^*] \\ = \bar{u}(s_1) - \bar{c}s_1 - [\bar{u}(x^*) - \bar{c}x^*] &= \bar{u}(s_1) - \bar{c}s_1 - \min_{x\in\mathbb{X}}\{\bar{u}(x) - \bar{c}x\} = M_{\bar{u}},\end{aligned} \tag{7.16}$$

where the first equality follows from (7.8), the second one follows from Lemma 7.4, and the third one follows from (7.15). In view of (7.11) and (7.16),

$$\lim_{k\to+\infty} u_{\alpha_{n_k}}(s_1) = M_{\bar{u}}. \tag{7.17}$$

Thus, for every sequence $\{\alpha_n \uparrow 1\}_{n=1,2,\ldots}$ there exists a subsequence $\{\alpha_{n_k}\}_{k=1,2,\ldots}$ such that (7.17) holds. Therefore, $\lim_{n\to+\infty} u_{\alpha_n}(s_1) = M_{\bar{u}}$ for every sequence $\{\alpha_n \uparrow 1\}_{n=1,2,\ldots}$, which is equivalent to

$$u^*(s_1) := \lim_{\alpha\uparrow 1} u_\alpha(s_1) = M_{\bar{u}}. \tag{7.18}$$

(2) Now we prove that there exists the limit $u^*(x) := \lim_{\alpha\uparrow 1} u(x)$ for $x \in \mathbb{X}$. For $x \in \mathbb{X}$

$$\begin{aligned}\lim_{\alpha\uparrow 1} u_\alpha(x) - u_\alpha(s_1) &= \lim_{\alpha\uparrow 1} \bar{u}_\alpha(x) - \bar{c}x - [\bar{u}_\alpha(s_1) - \bar{c}s_1] \\ &= \bar{u}(x) - \bar{c}x - [\bar{u}(s_1) - \bar{c}s_1],\end{aligned} \tag{7.19}$$

where the first equality follows from (7.8) and the second one follows from Lemma 7.4. Therefore, (7.18) and (7.19) imply that there exists the limit

$$u^*(x) := \lim_{\alpha\uparrow 1} u_\alpha(x) = M_{\bar{u}} + \bar{u}(x) - \bar{c}x - [\bar{u}(s_1) - \bar{c}s_1], \quad x \in \mathbb{X}. \tag{7.20}$$

Furthermore, since the family of functions $\{u_\alpha\}_{\alpha\in[\beta,1)}$ is equicontinuous and Assumption **B** holds, Ascoli's theorem (Hernández-Lerma and Lasserre [16, p. 96]) implies that the function u^* is continuous.

(ii) Consider sequences $\{\alpha_n \uparrow 1\}_{n=1,2,\ldots}$ and $\{y_n \to x\}_{n=1,2,\ldots}$ such that $\alpha_1 > \alpha^*$ and $\lim_{n\to+\infty} u_{\alpha_n}(y_n) = \liminf_{\alpha\uparrow 1, y\to x} u_\alpha(y)$. Then,

$$\liminf_{\alpha\uparrow 1, y\to x} u_\alpha(y) \le \liminf_{n\to+\infty, y\to x} u_{\alpha_n}(y) \le \lim_{n\to+\infty} u_{\alpha_n}(y_n) = \liminf_{\alpha\uparrow 1, y\to x} u_\alpha(x),$$

which implies that

$$\liminf_{\alpha\uparrow 1, y\to x} u_\alpha(y) = \liminf_{n\to+\infty, y\to x} u_{\alpha_n}(y). \tag{7.21}$$

According to Feinberg and Liang [13, Lemma 3.3], since $\lim_{n\to+\infty} u_{\alpha_n}(x) = u^*(x)$, Theorems 4.1 and 4.6 imply that

$$\liminf_{n\to+\infty, y\to x} u_{\alpha_n}(y) = \lim_{n\to+\infty} u_{\alpha_n}(x) = u^*(x). \tag{7.22}$$

Therefore, (7.21) and (7.22) imply that $u := \liminf_{\alpha\uparrow 1, y\to x} u_\alpha(y) = u^*$. This completes the proof. ■

Theorem 7.1 implies that (4.15) can be written as

$$H(x) := \bar{c}x + \mathbb{E}[h(x-D)] + \mathbb{E}[u(x-D)]. \tag{7.23}$$

COROLLARY 7.5: *Let Assumption 1 and for $\alpha = 1$ Assumption 4 hold. Consider the function u defined in (7.2). Then the conclusions of Theorem 5.1 hold with $\tilde{u} = u$ and, in particular, the lower threshold s^*, for the optimal (s^*, S^*) optimal policy whose existence is stated in Theorem 5.1, can be chosen as s_1 defined in (6.1).*

Define the set of all possible optimal average-cost lower thresholds

$$\mathcal{G} := \{x \in [s, S] : H(y) \ge K + H(S) \text{ for all } y \le x\}, \tag{7.24}$$

where $S = \min\{\operatorname{argmin}_x\{H(x)\}\}$ and s is defined in (3.11) with $f := H$ Note that $s \in \mathcal{G}$ and $y \ge s$ for all $y \in \mathcal{G}$

The following theorem establishes the uniqueness of the optimal lower threshold satisfying the optimality equations for the inventory model with holding/backlog costs satisfying quasiconvexity assumptions under the average cost criterion.

THEOREM 7.6: *Let Assumption 1 hold and for $\alpha = 1$ Assumption 4 hold. Then, $\mathcal{G} = \{s_1\}$, and therefore $s = s_1$, where s_1 is defined in (6.1).*

PROOF: Consider $\mathcal{G}$ and S defined in (7.24). Recall that $s \in \mathcal{G}$ and $y \ge s$, where s is defined in (3.11) with $f := H$. According to Theorem 5.1 and Corollary 7.5, for $y \in \mathcal{G}$

$$w + u(x) + \bar{c}x = \begin{cases} K + H(S), & \text{if } x \le y, \\ H(x), & \text{if } x \ge y, \end{cases} \tag{7.25}$$

which implies that for $x \le y$

$$\begin{aligned} H(x) &= \bar{c}x + \mathbb{E}[h(x-D)] + \mathbb{E}[u(x-D)] \\ &= K + \mathbb{E}[h(x-D)] + H(S) + \bar{c}\mathbb{E}[D] - w. \end{aligned} \tag{7.26}$$

Since $H(y) = K + H(S)$ for $y \in \mathcal{G}$, in view of (7.26),

$$\mathbb{E}[h(y-D)] = w - \bar{c}\mathbb{E}[D], \quad y \in \mathcal{G}. \tag{7.27}$$

The rest of the proof is by contradiction. Assume that there exists $y_1 \in \mathcal{G}$ such that $y_1 > s$. Then (7.27) implies that $\mathbb{E}[h(y_1 - D)] = \mathbb{E}[h(s-D)]$. Therefore, Assumption 4 implies that $x_1^{\min} < y_1$, where $x_1^{\min}$ is defined in (2.8). Since $S = \min\{\operatorname{argmin}_x\{H(x)\}\}$, (7.25) implies that for $x < S$

$$w + u(x) + \bar{c}x > H(S). \tag{7.28}$$

Therefore, for $x < S$

$$\begin{aligned} H(y_1) &= K + H(S) = K + \bar{c}S + \mathbb{E}[h(S - D)] + \mathbb{E}[u(x - D)] \\ &> K + \mathbb{E}[h(x_1^{\min} - D)] + H(S) + \bar{c}\mathbb{E}[D] - w, \end{aligned} \tag{7.29}$$

where the first equality holds because $y_1 \in \mathcal{G}$, the second one follows from (7.23), and the inequality holds because $\mathbb{E}[h(S - D)] \geq \mathbb{E}[h(x_1^{\min} - D)]$ and (7.28). Since $y \in \mathcal{G}$ and $x_1^{\min} < y_1$, $H(x_1^{\min}) \geq H(y_1)$. In view of (7.26),

$$H(y_1) \leq H(x_1^{\min}) = K + \mathbb{E}[h(x_1^{\min} - D)] + H(S) + \bar{c}\mathbb{E}[D] - w,$$

which contradicts (7.29). Then, $\mathcal{G} = \{s\}$. In addition, Corollary 7.5 implies that $s_1 \in \mathcal{G}$, where s_1 is defined in (6.1). Therefore, $s = s_1$ and $\mathcal{G} = \{s_1\}$. ■

The following corollary states that all the results of this paper hold for inventory models with convex holding/backlog costs.

COROLLARY 7.7: *The conclusions of all the lemmas, theorems, and corollaries in Sections 6 and 7 hold under Assumption 2.*

8. VEINOTT'S REDUCTION OF PROBLEMS WITH BACKORDERS AND POSITIVE LEAD TIMES TO PROBLEMS WITHOUT LEAD TIMES

In this section, we explain, by using the technique introduced without formal proofs by Veinott [25] for finite-horizon problems with continuous demand, that the infinite-horizon inventory model with positive lead times and backorders can be reduced to the model without lead times. Therefore, the results of this paper, Feinberg and Lewis [11], and Feinberg and Liang [12,13] also hold for the inventory model with positive lead times. For inventory model with positive lead times, we also provide a formal formulation of the MDP with transformed state space.

Consider the inventory model defined in Section 2. Instead of assuming zero lead times, assume that the fixed lead time is $L \in \mathbb{N} := \{1, 2, \ldots\}$, that is, an order placed at the beginning of time t will be delivered at the beginning of time $t + L$. In addition, let $h^L(x)$ be the holding/backlog cost per period if the inventory level is x. We define

$$h^*(x) := \mathbb{E}[h^L(x - \sum_{i=1}^{L} D_i)]. \tag{8.1}$$

For the inventory model, the dynamics of the system are defined by the equation

$$x_{t+1} = x_t + a_{t-L} - D_{t+1}, \quad t = 0, 1, 2, \ldots, \tag{8.2}$$

where x_t and a_t are the current inventory level before replenishment and the ordered amount at period t. Equation (8.2) means that a decision-maker observes at the end of the period t the history $\mathbf{h}_t$, places an order of amount a_t, which will be delivered in L periods (that is, at the end of the period $t+L$), and the demand occurred during the period $t+1$ is D_{t+1}. In addition, at period t, the one-step cost is

$$\tilde{c}(\mathbf{h}_t^L, a_t) := K\mathbf{1}_{\{a_{t-L}>0\}} + \bar{c}a_{t-L} + \mathbb{E}[h^L(x_t + a_{t-L} - D_{t+1})], \quad t = 0, 1, \ldots, \tag{8.3}$$

where $\mathbf{h}_t^L = (a_{-L}, a_{-L+1}, \ldots, a_{-1}, x_0, a_0, \ldots, a_{t-1}, x_t)$ is the history at period t.

As usual, consider the set of possible trajectories $\mathbf{h}_{+\infty}^L = (\mathbf{h}_t^L, a_t, x_{t+1}, a_{t+1}, \ldots)$. An arbitrary policy is a regular probability distribution $\pi(da_t|\mathbf{h}_t^L)$, $t = 0, 1, \ldots$, on $\mathbb{R}_+$. It defines the transition probability for $\mathbf{h}_t^L$ to $(\mathbf{h}_t^L, a_t)$. The transition probability for $(\mathbf{h}_t^L, a_t)$ can be defined by (8.2). Therefore, given the initial state $\mathbf{h}_0^L = (a_{-L}, a_{-L+1}, \ldots, a_{-1}, x_0)$, a policy π defines, in view of the Ionescu Tulcea theorem, the probability distribution $P^\pi_{\mathbf{h}_0^L}$ on the set of trajectories. We denote by $\mathbb{E}^\pi_{\mathbf{h}_0^L}$ the expectation with respect to $P^\pi_{\mathbf{h}_0^L}$.

For a finite-horizon $N = 1, 2, \ldots$ the expected total discounted cost is

$$\tilde{v}^\pi_{N,\alpha}(\mathbf{h}_0^L) := \mathbb{E}^\pi_{\mathbf{h}_0^L}\left[\sum_{t=0}^{N-1} \alpha^t \tilde{c}(x_t, a_t)\right] = \mathbb{E}^\pi_{\mathbf{h}_0^L}\left[\sum_{t=0}^{L-1} \alpha^t \tilde{c}(x_t, a_t) + \alpha^L \sum_{t=0}^{N-1} \alpha^t \tilde{c}(x_{t+L}, a_{t+L})\right], \tag{8.4}$$

where $\alpha \in [0,1]$ is the discount factor and $\tilde{v}^\pi_{0,\alpha}(\mathbf{h}_0^L) = 0$. When $N = +\infty$ and $\alpha \in [0,1)$, (8.4) defines the infinite-horizon expected total discounted cost denoted by $\tilde{v}^\pi_\alpha(\mathbf{h}_0^L)$. The *average cost per unit time* is defined as $\tilde{w}^\pi(h_0) := \limsup_{N\to+\infty} 1/N \tilde{v}^\pi_{N,1}(\mathbf{h}_0^L)$.

Let us define

$$y_t := x_t + \sum_{i=1}^{L} a_{t-i} = x_{t+L} + \sum_{i=1}^{L} D_{t+i}, \quad t = 0, 1, \ldots, \tag{8.5}$$

where y_t is the sum of the current inventory level and the outstanding orders at the end of period t. Since the distribution of x_{t+L} is determined by y_t, in view of (8.2), we show that it is possible to make the decision a_t only based on the quantity y_t.

Let us construct an MDP with state space $\mathbb{Y} = \mathbb{R}$ (or $\mathbb{Y} = \mathbb{Z}$) with states y_t defined in (8.5). The actions are the amount of orders that can be placed at each period t; $A(y) = \mathbb{A} = \mathbb{R}_+$ (or $A(y) = \mathbb{A} = \mathbb{N}_0$) for all $y \in \mathbb{Y}$. In view of (8.2) and (8.5), the dynamics of the system are defined by the equation

$$y_{t+1} = y_t + a_t - D_{t+1}, \quad t = 0, 1, 2, \ldots . \tag{8.6}$$

The transition probabilities for the MDP corresponding to (8.6) is

$$q^*(B|y_t, a_t) = P(y_t + a_t - D_{t+1} \in B), \tag{8.7}$$

for each measurable subset B of $\mathbb{Y}$. Let the one-step cost be

$$c^*(y, a) := K\mathbf{1}_{\{a>0\}} + \bar{c}a + \mathbb{E}[h^*(y + a - D)]. \tag{8.8}$$

As was noticed by Veinott [25], the state space $\mathbb{Y}$, action space $\mathbb{A}$, action sets $A(\cdot)$, transition probabilities (8.7), and costs (8.8) define the same MDP as for in the problem without lead time with the only difference that the holding/backlog cost function h is substituted with the function h^*. In addition, though the amount of inventory x_{t+L} at time $(t+L)$ is not known at time t, the distribution of x_{t+L} is known because $x_{t+L} \sim y_t - \sum_{l=1}^{L} D^{(l)}$, where $D^{(1)}, \ldots, D^{(L)}$ are i.i.d. random variables with $D^{(l)} \sim D$, $l = 1, 2, \ldots, L$.

Since the actual amount of inventory level x_{t+L} is unknown at time t, when the amount a_t is ordered, this problem can be modeled as a Partially Observable MDP. According to the current available theory (see Hernández-Lerma [15, Chapter 4], Feinberg et al. [9], and references therein), such models can be reduced to the MDPs whose states are probability distributions of x_{t+L} known at time t, which is the distribution of $y_t - \sum_{l=1}^{L} D^{(l)}$. The value of y_t defines this distribution. This relation implies that optimal policies for the MDP introduced by Veinott [25] with state space $\mathbb{Y}$, action space $\mathbb{A}$, and transition probabilities (8.7), and costs (8.8) define the optimal actions at epoch $t = 0, 1, 2, \ldots$.

THEOREM 8.1: *Consider the problem with the lead time $L = 1, 2, \ldots$. Then the MDP $\{\mathbb{Y}, \mathbb{A}, q^*, c^*\}$ coincides the MDP $\{\mathbb{X}, \mathbb{A}, q, c\}$ with the function h substituted with h^*. Therefore, the conclusions of theorems in this paper hold for the problems with the lead time $L = 1, 2, \ldots$, if the holding/backlog cost function h^* satisfies the conditions assumed for the function h in the corresponding statements.*

PROOF: Since $y_t \in \mathbb{X}$ and the actions are the same for these two models, we need to verify only the correspondence for transition probabilities and costs. If $h = h^*$, then formulae (2.3) and (8.8) coincide with $x_t = y_t$. The transition probabilities q^* defined in (8.7) also coincides with (2.2). Observe that it is easy to show that

$$\tilde{v}^{\pi}_{N,\alpha}(\mathbf{h}_0^L) = f(\mathbf{h}_0^L) + \alpha^L v^{\pi}_{N,\alpha}(y_0), \tag{8.9}$$

where $f(\mathbf{h}_0^L) := \sum_{t=0}^{L-1} \alpha^t \mathbb{E}[\tilde{c}(x_0 + \sum_{i=0}^{t-1} a_{-L+i} - \sum_{i=0}^{t-1} D_{t+i}, a_t)]$. ■

For the problems with convex holding/backlog cost function h^L, the function h^* is also convex and $\mathbb{E}[h^*(x - D)] \to +\infty$ as $|x| \to +\infty$. We also need the additional assumption that $\mathbb{E}[h^*(x - D)] < +\infty$ for all $x \in \mathbb{X}$. Then the results in this paper formulated under Assumption 2 and the results in Feinberg and Lewis [11] and Feinberg and Liang [12,13] hold for the problems with the lead time $L = 1, 2, \ldots$.

Remark 8.2: Note that the assumption on the finiteness of the function $\mathbb{E}[h^*(x - D)]$ is necessary for the problems with convex holding/backlog costs. Consider the lead time $L = 1$, the holding/backlog cost function

$$h^L(x) := \begin{cases} x + \dfrac{e^2}{5}, & \text{if } x \geq 0, \\ \dfrac{e^{-x+2}}{(x-2)^2 + 1}, & \text{if } x \leq 0, \end{cases}$$

and the random variable D is exponential distribution with the density function $f_D(x) = e^{-x}$, if $x > 0$, and $f_D(x) = 0$, otherwise. Then the random variable $\mathbf{S}_2$ follows the Erlang distribution with density function $f_{\mathbf{S}_2}(x) = xe^{-x}$, if $x > 0$, and $f_{\mathbf{S}_2}(x) = 0$, otherwise. Observe that the function h^L is continuous and nonnegative. Some calculations show that the function h^L is convex on $\mathbb{R}$ and $\mathbb{E}[h^L(x - D)] < +\infty$ for all $x \in \mathbb{R}$. However, $\mathbb{E}[h^*(0 - D)] = \mathbb{E}[h^L(0 - \mathbf{S}_2)] = +\infty$.

Remark 8.3: The reduction discussed in this section does not hold for the inventory model with lost-sales. For such model with lead time $L > 0$, the dynamics of the system are defined by the equation

$$x_{t+1} = (x_t + a_{t-L} - D_{t+1})^+ := \max\{x_t + a_{t-L} - D_{t+1}, 0\}, \quad t = 0, 1, 2, \ldots .$$

Consider the transformation similar to the one defined in (8.5). Then $x_{t+L} = y_t - \sum_{i=1}^{L} \tilde{D}_{t+i}$, where $\tilde{D}_j := \min\{D_j, x_{j-1} + a_{j-L-1}\}$, $j = t+1, t+2, \ldots, t+L$. Since the distribution of x_{t+L} does not depend solely on the information available at time t, the reduction does not hold. Indeed, the structure of the optimal policies may depend on the lead times. In particular, if the lead times are large, then the constant-order policy performs nearly optimally; see Goldberg et al. [14].

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APPENDIX A. PROOFS TO SECTION 2

Proof of Lemma 2.2: Since $\mathbb{E}[h(x-D)] \to +\infty$ as $x \to -\infty$,

$$\limsup_{x\to-\infty} h(x) = +\infty. \tag{A.1}$$

To see this note that if $\limsup_{x\to-\infty} h(x) < +\infty$, then there exist real numbers M_1, $M_2 > 0$ such that $h(x) \le M_1$ for $x \le -M_2$. Since D is a nonnegative random variable, $\mathbb{E}[h(x-D)] \le M_1$ for $x \le -M_2$ and $\limsup_{x\to-\infty} \mathbb{E}[h(x-D)] \le M_1 < +\infty$. This contradicts the assumption that $\mathbb{E}[h(x-D)] \to +\infty$ as $x \to -\infty$.

Since the function h is convex, the function $\mathbb{E}[h(x-D)]$ is convex. Therefore, in view of (2.6), the function $\mathbb{E}[h_\alpha(x-D)]$ is convex for all $\alpha \in [0,1]$. Since every convex function is quasiconvex, the function $\mathbb{E}[h_\alpha(x-D)]$ is quasiconvex for all $\alpha \in [0,1]$.

Since the function h is convex on $\mathbb{X}$, it is continuous. Therefore, (A.1) implies $\lim_{x\to-\infty} h(x) = +\infty$. As explained in Feinberg and Liang [12, Equations (2.3), (4.1)], the convexity of the function h imply that $1 + \lim_{x\to-\infty} h(x)/\bar{c}x < 1$.

Consider $\alpha^* \in [\max\{1 + \lim_{x\to-\infty} h(x)/\bar{c}x, 0\}, 1)$. For $\alpha \in (\alpha^*, 1]$, since the function $h_\alpha(x) = \bar{c}x(h(x)/\bar{c}x + 1 - \alpha)$ tends to $+\infty$ as $x \to -\infty$,

$$\lim_{x\to-\infty} \mathbb{E}[h_\alpha(x-D)] = +\infty, \quad \alpha \in (\alpha^*, 1]. \tag{A.2}$$

Therefore, the convexity of the function $\mathbb{E}[h_\alpha(x-D)]$ implies that $\lim_{x\to-\infty} \mathbb{E}[h_\alpha(x-D)] > K + \inf_{x\in\mathbb{X}} \mathbb{E}[h_\alpha(x-D)]$ for all $\alpha \in (\alpha^*, 1]$. Hence, Assumption 1 holds with $\alpha^* \in [\max\{1 + \lim_{x\to-\infty} h(x)/\bar{c}x, 0\}, 1)$. In view of (A.2), the convexity of the function $\mathbb{E}[h_\alpha(x-D)]$ implies that Assumption 4 holds for all $\alpha \in (\alpha^*, 1]$. ■

Proof of Lemma 2.3: It is straightforward that Assumption 1 implies Assumption 3 for $\alpha \in (\alpha^*, 1)$. In addition, since $\mathbb{E}[h_\alpha(x-D)] \to \mathbb{E}[h(x-D)]$ as $\alpha \uparrow 1$ for all $x \in \mathbb{X}$, the quasiconvexity of the function $\mathbb{E}[h_\alpha(x-D)]$ implies that the function $\mathbb{E}[h(x-D)]$ is quasiconvex. Since $\inf_{x\in\mathbb{X}} \mathbb{E}[h(x-D)] < +\infty$ and $\mathbb{E}[h(x-D)] = \mathbb{E}[h_\alpha(x-D)] - (1-\alpha)\bar{c}x \to +\infty$ as $x \to -\infty$ for each $\alpha \in (\alpha^*, 1)$, Assumption 3 holds for $\alpha = 1$. ■

APPENDIX B. PROOFS TO SECTION 3

Proof of Lemma 3.3: In view of (3.1), for $x \le y$ and $t = 1, 2, \ldots$

$$\begin{aligned} v_{t,\alpha}(x) + \bar{c}x &= \min\left\{\min_{a\ge 0}\{K + G_{t-1,\alpha}(x+a)\}, G_{t-1,\alpha}(x)\right\} \le \min_{a\ge 0}\{K + G_{t-1,\alpha}(x+a)\} \\ &\le K + \min_{a\ge y-x} G_{t-1,\alpha}(x+a) = K + \min_{a\ge 0} G_{t-1,\alpha}(y+a) \\ &\le K + \min\left\{\min_{a\ge 0}\{K + G_{t-1,\alpha}(y+a)\}, G_{t-1,\alpha}(y)\right\} = K + v_{t,\alpha}(y) + \bar{c}y, \end{aligned}$$

where the second inequality follows from $y - x \ge 0$. Furthermore, (3.2) and the same arguments imply (3.6).

In view of (3.3), for $x \le y$ and $t = 1, 2, \ldots$

$$\begin{aligned} G_{t,\alpha}(y) - G_{t,\alpha}(x) &= \mathbb{E}[h_\alpha(y-D)] - \mathbb{E}[h_\alpha(x-D)] \\ &\quad + \alpha\mathbb{E}[v_{t,\alpha}(y-D) + \bar{c}(y-D) - v_{t,\alpha}(x-D) - \bar{c}(x-D)] \\ &\ge \mathbb{E}[h_\alpha(y-D)] - \mathbb{E}[h_\alpha(x-D)] - \alpha K, \end{aligned}$$

where the inequality follows from (3.5). Furthermore, (3.4) and the same arguments imply (3.8). ∎

Proof of Lemma 3.4: In view of (3.9), $S^*_\alpha \ge x^{\min}_\alpha$. Since $\lim_{x\to+\infty}\mathbb{E}[h_\alpha(x-D)] = +\infty$ and Assumption 3 holds, $|S^*_\alpha| < +\infty$ and for $x > S^*_\alpha$

$$\mathbb{E}[h_\alpha(x-D)] \ge \mathbb{E}[h_\alpha(S^*_\alpha - D)] \ge K + \mathbb{E}[h_\alpha(x^{\min}_\alpha - D)]. \tag{B.1}$$

Consider $x^* \in \mathbb{X}$. Let the function $G := G_{t,\alpha}$, $t = 1, 2, \ldots$, or $G := G_\alpha$ and a^* be an optimal action defined by (3.1) or (3.2) for x^*. Then consider the following two cases: (i) $x^* \le x^{\min}_\alpha$; and (ii) $x^* > x^{\min}_\alpha$.

(i) Lemma 3.3 and (B.1) imply that for $x > S^*_\alpha \ge x^{\min}_\alpha$

$$G(x) - G(x^{\min}_\alpha) \ge \mathbb{E}[h_\alpha(x-D)] - \mathbb{E}[h_\alpha(x^{\min}_\alpha - D)] - \alpha K \ge K - \alpha K > 0. \tag{B.2}$$

Therefore, for $x > S^*_\alpha \ge x^{\min}_\alpha$

$$G(x) > G(x^{\min}_\alpha) \ge \min_{x\in\mathbb{X}} G(x). \tag{B.3}$$

Then, for $x^* \le x^{\min}_\alpha$, (B.3) implies that $a^* \in [0, S^*_\alpha - x^*]$.

(ii) For $x^{\min}_\alpha \le x < y$

$$G(y) - G(x) + K \ge \mathbb{E}[h_\alpha(y-D)] - \mathbb{E}[h_\alpha(x-D)] - \alpha K + K > 0, \tag{B.4}$$

where the first inequality follows from Lemma 3.3 and the second one holds because the function $\mathbb{E}[h_\alpha(x-D)]$ is nondecreasing on $[x^{\min}_\alpha, +\infty)$ and $K - \alpha K > 0$. For $x^* > x^{\min}_\alpha$, it follows from (B.4) that $G(x^*) < G(y) + K$ for all $y > x^*$, which implies that $a^* = 0$. Therefore, cases (i) and (ii) imply that $a^* \in [0, \max\{S^*_\alpha - x^*, 0\}]$. ∎

Proof of Lemma 3.8: In view of (3.13), for $x \le y$ and $t = 1, 2, \ldots$

$$\begin{aligned}\bar v_{t,\alpha}(x) &= \min\left\{\min_{a\ge 0}\{K + \bar G_{t-1,\alpha}(x+a)\}, \bar G_{t-1,\alpha}(x)\right\} \le \min_{a\ge 0}\{K + \bar G_{t-1,\alpha}(x+a)\}\\ &\le K + \min_{a\ge y-x} \bar G_{t-1,\alpha}(x+a) = K + \min_{a\ge 0} \bar G_{t-1,\alpha}(y+a)\\ &\le K + \min\left\{\min_{a\ge 0}\{K + \bar G_{t-1,\alpha}(y+a)\}, \bar G_{t-1,\alpha}(y)\right\} = K + \bar v_{t,\alpha}(y),\end{aligned}$$

where the second inequality follows from $y - x \ge 0$. Furthermore, (3.14) and the same arguments imply (3.23).

In view of (3.15), for $x \le y$ and $t = 1, 2, \ldots$

$$\begin{aligned}\bar G_{t,\alpha}(y) - \bar G_{t,\alpha}(x) &= \mathbb{E}[h_\alpha(y-D)] - \mathbb{E}[h_\alpha(x-D)] + \alpha\mathbb{E}[\bar v_{t,\alpha}(y-D) - \bar v_{t,\alpha}(x-D)]\\ &\le \mathbb{E}[h_\alpha(y-D)] - \mathbb{E}[h_\alpha(x-D)] + \alpha K,\end{aligned}$$

where the inequality follows from (3.22). Furthermore, (3.16) and the same arguments imply (3.25). ■

Proof of Lemma 3.9: The proof is by induction on t. For $t = 0$, (3.27) holds because $\bar v_{0,\alpha}(x) = 0$, $x \in \mathbb{X}$, and (3.26) follows from $\bar G_{0,\alpha}(x) = \mathbb{E}[h_\alpha(x-D)]$, $x \in \mathbb{X}$, and $|x_\alpha^{\min}| < +\infty$, which is true in view of Assumption 3. To complete the induction arguments, assume that (3.26) holds for $t = k \in \{0, 1, 2, \ldots\}$. Then for $x \le y \le x_\alpha^{\min}$

$$\begin{aligned}\bar v_{k+1,\alpha}(x) &= \min\left\{\bar G_{k,\alpha}(x), \min_{a\ge 0}\{K + \bar G_{k,\alpha}(x+a)\}\right\}\\ &= \min\left\{\bar G_{k,\alpha}(x), \min_{a\ge 0}\{K + \bar G_{k,\alpha}(x+a)\}\right\}\\ &\ge \min\left\{\bar G_{k,\alpha}(y), \min_{0\le a<y-x}\{K + \bar G_{k,\alpha}(x+a)\}, \min_{a\ge y-x}\{K + \bar G_{k,\alpha}(x+a)\}\right\}\\ &\ge \min\left\{\bar G_{k,\alpha}(y), K + \bar G_{k,\alpha}(y), \min_{a\ge 0}\{K + \bar G_{k,\alpha}(y+a)\}\right\}\\ &\ge \min\left\{\bar G_{k,\alpha}(y), \min_{a\ge 0}\{K + \bar G_{k,\alpha}(y+a)\}\right\} = \bar v_{k+1,\alpha}(y),\end{aligned}$$

where the first and last equalities follow from (3.13), the first two inequalities follow from (3.26), and the last inequality follows from $K > 0$. Thus, (3.27) holds for $t = k + 1$. In addition, for $x \le y \le x_\alpha^{\min}$

$$\begin{aligned}&\bar G_{k+1,\alpha}(y) - \bar G_{k+1,\alpha}(x)\\ &\quad = \mathbb{E}[h_\alpha(y-D)] - \mathbb{E}[h_\alpha(x-D)] + \alpha\mathbb{E}[\bar v_{k+1,\alpha}(y-D) - \bar v_{k+1,\alpha}(x-D)] \le 0,\end{aligned}$$

where the equality follows from (3.15) and the inequality holds because the function $\mathbb{E}[h_\alpha(x-D)]$ is nonincreasing on $(-\infty, x_\alpha^{\min}]$ and (3.27).

Since $\bar v_{t,\alpha} \to \bar v_\alpha$ as $t \to +\infty$, (3.27) implies (3.28). In addition, for $x \le y \le x_\alpha^{\min}$

$$\bar G_\alpha(y) - \bar G_\alpha(x) = \mathbb{E}[h_\alpha(y-D)] - \mathbb{E}[h_\alpha(x-D)] + \alpha\mathbb{E}[\bar v_\alpha(y-D) - \bar v_\alpha(x-D)] \le 0,$$

where the equality follows from (3.16) and the inequality holds since the function $\mathbb{E}[h_\alpha(x-D)]$ is nonincreasing on $(-\infty, x_\alpha^{\min}]$ and (3.28). ■

Proof of Lemma 3.11: We first prove that

$$\bar{v}_\alpha(x) - \bar{c}x = \tilde{v}_\alpha(x) \geq v_\alpha(x) \geq 0, \quad x \in \mathbb{X}. \tag{B.5}$$

Note that $v_\alpha(x) \geq 0$ for all $x \in \mathbb{X}$, because all costs in the original inventory model are nonnegative. Theorem 3.10(i), (3.17), and (3.18) imply that for $N = 1, 2, \ldots$

$$\tilde{v}_{N,\alpha}(x) = \tilde{v}^{\phi^N}_{N,\alpha}(x), \quad x \in \mathbb{X}, \tag{B.6}$$

where the policy ϕ^N is the $(s_{t,\alpha}, S_{t,\alpha})_{t=0,1,2,\ldots,N-1}$ policy defined in Theorem 3.10(i). Therefore,

$$\begin{aligned}\tilde{v}_{N,\alpha}(x) &= \tilde{v}^{\phi^N}_{N,\alpha}(x) = \mathbb{E}^{\phi^N}_x\left[\sum_{t=0}^{N-1} c(x_t, a_t) - \alpha^N \bar{c}x_N\right] \\ &= v^{\phi^N}_{N,\alpha}(x) - \alpha^N \bar{c}\mathbb{E}^{\phi^N}_x[x_N] \geq v_{N,\alpha}(x) - \alpha^N \bar{c}\max\{x, S^*_\alpha\}, \quad x \in \mathbb{X},\end{aligned} \tag{B.7}$$

where the last inequality holds because $v^{\phi^N}_{N,\alpha}(x) \geq v_{N,\alpha}(x)$, $x \in \mathbb{X}$, and for all $N = 1, 2, \ldots$ Theorem 3.10 (iii) implies that $\mathbb{E}^{\phi^N}_x[x_N] \leq \max\{x, S^*_\alpha\}$. Hence,

$$\bar{v}_\alpha(x) - \bar{c}x = \tilde{v}_\alpha(x) = \lim_{N\to+\infty} \tilde{v}_{N,\alpha}(x) \geq \lim_{N\to+\infty} v_{N,\alpha}(x) = v_\alpha(x) \geq 0,$$

where the first two equalities follow from (3.19), the first inequality follows from (B.7) and $\lim_{N\to+\infty} \alpha^N \bar{c}\max\{x, S^*_\alpha\} = 0$ for each $x \in \mathbb{X}$. Therefore, (B.5) holds.

To prove Lemma 3.11, it remains to prove that $v_\alpha(x) \geq \tilde{v}_\alpha(x)$, $x \in \mathbb{X}$. Observe that for $t = 0, 1, 2, \ldots$ and $\pi \in \Pi$

$$x_t \geq x_0 - \mathbf{S}_t \quad \text{and} \quad \mathbb{E}^\pi_{x_0}[x_t] \geq x_0 - t\mathbb{E}[D],$$

where $\mathbf{S}_t$ is defined in (4.6). Then for $N = 1, 2, \ldots$

$$\begin{aligned}\tilde{v}_{N,\alpha}(x) &\leq \tilde{v}^\pi_{N,\alpha}(x) = \mathbb{E}^\pi_x\left[\sum_{t=0}^{N-1} c(x_t, a_t) - \alpha^N \bar{c}x_N\right] = v^\pi_{N,\alpha}(x) - \alpha^N \bar{c}\mathbb{E}^\pi_x[x_N] \\ &\leq v^\pi_{N,\alpha}(x) - \alpha^N \bar{c}(x - N\mathbb{E}[D]), \quad x \in \mathbb{X}.\end{aligned} \tag{B.8}$$

Observe that $\lim_{N\to+\infty} \alpha^N \bar{c}(x - N\mathbb{E}[D]) = 0$ for each $x \in \mathbb{X}$. Thus, by taking the limits as $N \to +\infty$ of both sides of (B.8), $\tilde{v}_\alpha(x) \leq v^\pi_\alpha(x)$ for all $\pi \in \Pi$, which implies that $\tilde{v}_\alpha(x) \leq v_\alpha(x)$, $x \in \mathbb{X}$. Hence, $\tilde{v}_\alpha(x) = v_\alpha(x) = \bar{v}_\alpha(x) - \bar{c}x$, $x \in \mathbb{X}$. ■

APPENDIX C. PROOFS TO SECTION 5

Proof of Lemma 5.3: (i) In view of (3.23) and Lemma 3.11, (5.2) holds because $u_\alpha(x) - u_\alpha(y) = v_\alpha(x) - v_\alpha(y) = \bar{v}_\alpha(x) - \bar{c}x - (\bar{v}_\alpha(y) - \bar{c}y)$ and the function $u_{\alpha_n}(x)$ converges pointwise to $\tilde{u}(x)$ as $n \to +\infty$. For $x \leq y$

$$\begin{aligned}H(y) - H(x) &= \mathbb{E}[h(y - D)] + \alpha\mathbb{E}[\tilde{u}(y - D) + \bar{c}(y - D)] \\ &\quad - \mathbb{E}[h(x - D)] - \alpha\mathbb{E}[\tilde{u}(x - D) + \bar{c}(x - D)] \\ &\leq \mathbb{E}[h_\alpha(y - D)] - \mathbb{E}[h_\alpha(x - D)] + \alpha K,\end{aligned}$$

where the equality follows from (4.15) and the inequality follows from (5.2).

(ii) We first show that for $1 \geq \alpha \geq \beta > \alpha^*$

$$x_\alpha^{\min} \geq x_\beta^{\min}. \tag{C.1}$$

To verify this inequality, consider $1 \geq \alpha \geq \beta > \alpha^*$. Then, for $x < x_\beta^{\min}$

$$\begin{aligned}
&\mathbb{E}[h_\alpha(x-D)] - \mathbb{E}[h_\alpha(x_\beta^{\min}-D)] \\
&\quad = \mathbb{E}[h(x-D)] + \mathbb{E}[h(x_\beta^{\min}-D)] + (1-\alpha)\bar{c}(x-x_\beta^{\min}) \\
&\quad > \mathbb{E}[h(x-D)] + \mathbb{E}[h(x_\beta^{\min}-D)] + (1-\beta)\bar{c}(x-x_\beta^{\min}) \\
&\quad = \mathbb{E}[h_\beta(x-D)] - \mathbb{E}[h_\beta(x_\beta^{\min}-D)] > 0,
\end{aligned} \tag{C.2}$$

where the equalities follow from (2.6) and the inequality holds because $1-\alpha < 1-\beta$ and $\bar{c}(x-x_\beta^{\min}) < 0$. If $x_\alpha^{\min} < x_\beta^{\min}$, then (C.2) with $x = x_\alpha^{\min}$ implies that

$$\mathbb{E}[h_\alpha(x_\alpha^{\min}-D)] - \mathbb{E}[h_\alpha(x_\beta^{\min}-D)] > 0. \tag{C.3}$$

However, the definition of $x_\alpha^{\min}$ in (2.8) implies that $\mathbb{E}[h_\alpha(x_\alpha^{\min}-D)] - \mathbb{E}[h_\alpha(x_\beta^{\min}-D)] \leq 0$, which contradicts (C.3). Therefore, (C.1) holds.

Now, we prove that $x_\alpha^{\min} \uparrow x_1^{\min}$ as $\alpha \uparrow 1$. Consider a fixed discount factor $\beta \in (\alpha^*, 1)$. In view of (C.1), since $x_\beta^{\min} \leq x_\alpha^{\min} \leq x_1^{\min}$ for all $\alpha \in (\beta, 1)$ and $|x_\beta^{\min}|, |x_1^{\min}| < +\infty$, the monotone convergence theorem implies that there exists $x_{1,*}^{\min} \in [x_\beta^{\min}, x_1^{\min}]$ such that $x_\alpha^{\min} \uparrow x_{1,*}^{\min}$ as $\alpha \uparrow 1$. In addition, for $\alpha \in (\beta, 1)$

$$\begin{aligned}
0 &\leq \mathbb{E}[h_1(x_\alpha^{\min}-D)] - \mathbb{E}[h_1(x_1^{\min}-D)] \\
&= \mathbb{E}[h_\alpha(x_\alpha^{\min}-D)] - (1-\alpha)\bar{c}(x_\alpha^{\min}-\mathbb{E}[D]) - \mathbb{E}[h_1(x_1^{\min}-D)] \\
&\leq \mathbb{E}[h_\alpha(x_1^{\min}-D)] - (1-\alpha)\bar{c}(x_\alpha^{\min}-\mathbb{E}[D]) - \mathbb{E}[h_1(x_1^{\min}-D)] \\
&= (1-\alpha)\bar{c}(x_1^{\min}-x_\alpha^{\min}) \leq (1-\alpha)\bar{c}(x_1^{\min}-x_\beta^{\min}),
\end{aligned} \tag{C.4}$$

where the first two inequalities follow from the definition of $x_1^{\min}$ and $x_\alpha^{\min}$ in (2.8), the first equality holds because $\mathbb{E}[h_1(x_\alpha^{\min}-D)] = \mathbb{E}[h_\alpha(x_\alpha^{\min}-D)] - (1-\alpha)\bar{c}(x_\alpha^{\min}-\mathbb{E}[D])$, the second equality holds because $\mathbb{E}[h_\alpha(x_1^{\min}-D)] - \mathbb{E}[h_1(x_1^{\min}-D)] = (1-\alpha)\bar{c}(x_1^{\min}-\mathbb{E}[D])$, and the last inequality follows from $x_\beta^{\min} \leq x_\alpha^{\min}$ and $(1-\alpha)\bar{c} > 0$ for $\alpha \in (\beta, 1)$. Observe that $(1-\alpha)\bar{c}(x_1^{\min}-x_\beta^{\min}) \to 0$ as $\alpha \uparrow 1$. Then (C.4) implies that $\lim_{\alpha\uparrow 1} \mathbb{E}[h_1(x_\alpha^{\min}-D)] - \mathbb{E}[h_1(x_1^{\min}-D)] = 0$. Therefore, the continuity of the function $\mathbb{E}[h_1(x-D)]$ implies that

$$\mathbb{E}[h_1(x_{1,*}^{\min}-D)] = \mathbb{E}[h_1(x_1^{\min}-D)]. \tag{C.5}$$

Recall that $x_{1,*}^{\min} \in [x_\beta^{\min}, x_1^{\min}]$. If $x_{1,*}^{\min} < x_1^{\min}$, then Assumption 1 and (2.8) imply that $\mathbb{E}[h_1(x_{1,*}^{\min}-D)] > \mathbb{E}[h_1(x_1^{\min}-D)]$, which contradicts (C.5). Therefore, $x_{1,*}^{\min} = x_1^{\min}$ and $x_\alpha^{\min} \uparrow x_1^{\min}$ as $\alpha \uparrow 1$.

In view of (3.28) and Lemma 3.11, (5.4) holds because $u_\alpha(x) - u_\alpha(y) = v_\alpha(x) - v_\alpha(y) = \bar{v}_\alpha(x) - \bar{c}x - (\bar{v}_\alpha(y) - \bar{c}y)$, the function $u_{\alpha_n}(x)$ converges pointwise to $\tilde{u}(x)$ as $n \to +\infty$, and $x_\alpha^{\min} \uparrow x_1^{\min}$ as $\alpha \uparrow 1$. For $x \leq y \leq x_1^{\min}$

$$\begin{aligned}
H(y) - H(x) &= \mathbb{E}[h(y-D)] - \mathbb{E}[h(x-D)] \\
&\quad + \alpha\mathbb{E}[\tilde{u}(y-D) + \bar{c}(y-D) - \tilde{u}(x-D) - \bar{c}(x-D)] \leq 0,
\end{aligned}$$

where the equality follows from (4.15) and the inequality follows from that the function $\mathbb{E}[h(x-D)]$ is nonincreasing on $(-\infty, x_1^{\min}]$ and (5.4). ■

Proof of Corollary 5.4: The proof of the optimality of (s, S) policies is based on the fact that $K + H(S) < H(x)$, if $x < s$, and $K + H(S) \geq H(x)$, if $x \geq s$. Since the function H is continuous, we have that $K + H(S) = H(s)$. Thus both actions are optimal at the state s. ■

APPENDIX D. PROOFS TO SECTION 6

Proof of Lemma 6.4: According to (6.3), $\bar{v}_\alpha(x) = K + G_\alpha(S_\alpha) = K + \bar{m}_\alpha$ for $x \le y$. In view of (3.16), (6.8), and Lemma 3.11,

$$K + \bar{m}_\alpha = G_\alpha(y) = \mathbb{E}[h_\alpha(y-D)] + \alpha\mathbb{E}[\bar{v}_\alpha(y-D)] = \mathbb{E}[h_\alpha(y-D)] + \alpha(K+\bar{m}_\alpha),$$

which implies (6.9). ∎

Proof of Lemma 6.5: Observe that the second inequality in Lemma 6.5 follows from (C.1). The following proof is by contradiction. Assume that there exist $\alpha \in (\alpha^*, 1)$ and $y \in \mathcal{G}_\alpha$ such that $y > x_\alpha^{\min}$. According to (6.3), $\bar{v}_\alpha(x) = K + \bar{m}_\alpha$ for $x \le y$. Therefore, (3.16) and Lemma 3.11 imply that for $x \le y$

$$G_\alpha(x) = \mathbb{E}[h_\alpha(x-D)] + \alpha(K+\bar{m}_\alpha). \tag{D.1}$$

The definition (2.8) of $x_\alpha^{\min}$ and (D.1) imply that $G_\alpha(x_\alpha^{\min}) \le G_\alpha(y)$. According to the definition of s_α in (3.11), $G_\alpha(x) \ge G_\alpha(y)$ for $x \le y$. Therefore, $G_\alpha(x_\alpha^{\min}) = G_\alpha(y)$, which implies that

$$\begin{aligned} &\mathbb{E}[h_\alpha(x_\alpha^{\min}-D)] + \alpha(K+\bar{m}_\alpha) = G_\alpha(x_\alpha^{\min}) = G_\alpha(y) = K + G_\alpha(S_\alpha) \\ &\quad = K + \mathbb{E}[h_\alpha(S_\alpha - D)] + \alpha\mathbb{E}[\bar{v}_\alpha(S_\alpha - D)] > \mathbb{E}[h_\alpha(x_\alpha^{\min}-D)] + \alpha(K+\bar{m}_\alpha), \end{aligned} \tag{D.2}$$

where the first equality follows from (D.1), the last equality follows from (3.16) and Lemma 3.11, and the inequality follows from $K > \alpha K$ and the definition of $x_\alpha^{\min}$ and $\bar{m}_\alpha$. The contradiction in (D.2) implies that $y \le x_\alpha^{\min}$ for all $y \in \mathcal{G}_\alpha$. ∎

Proof of Lemma 6.6: According to equation (4.17), for any given $\beta \in (\alpha^*, 1)$, there exists a constant $b > 0$ such that $s_\alpha \in (-b, b)$ for all $\alpha \in [\beta, 1)$. In view of (4.2), since b and x_U^* are real numbers, where x_U^* is defined in (4.10),

$$\lim_{\alpha\uparrow 1}(1-\alpha)(m_\alpha - \bar{c}b) = \lim_{\alpha\uparrow 1}(1-\alpha)(m_\alpha + \bar{c}x_U^*) = w. \tag{D.3}$$

Therefore, since $s_\alpha > -b$ for all $\alpha \in [\beta, 1)$, (6.7) and (D.3) imply that $\lim_{\alpha\uparrow 1}(1-\alpha)\bar{m}_\alpha = w$. Therefore, in view of Lemma 6.4, $\lim_{\alpha\uparrow 1}\mathbb{E}[h_\alpha(s_\alpha - D)] = w$. ∎

APPENDIX E. PROOFS TO SECTION 7

Proof of Lemma 7.2: We first verify the validity of Assumption W*. It is obvious that the nonnegative cost function c is $\mathbb{K}$-inf-compact and the transition probabilities are weakly continuous. Thus, Assumption W* holds and a stationary discount cost optimal policy exists for every $\alpha \in [0,1)$. To verify the validity of Assumption B, we calculate the relative value function u_α.

Let us calculate the value functions v_α for $\alpha \in [0,1)$. Since there is only one action at $n = 0, 1, \ldots,$ the infinite-horizon value function

$$v_\alpha(n) = \sum_{i=0}^{\infty} z_{n+i}^{(1)}\alpha^i, \quad n = 0, 1, \ldots\ . \tag{E.1}$$

Therefore, for $n = 0$, (E.1) implies

$$\begin{aligned} v_\alpha(0) &= \sum_{i=0}^{\infty} z_i^{(1)}\alpha^i = z_0 + \sum_{i=1}^{\infty}(z_i - z_{i-1})\alpha^i + \sum_{i=0}^{\infty}\alpha^i \\ &= \sum_{i=0}^{\infty} z_i\alpha^i - \alpha\sum_{i=0}^{\infty} z_i\alpha^i + \frac{1}{1-\alpha} = f(\alpha) + \frac{1}{1-\alpha}, \end{aligned} \tag{E.2}$$

where the second equality follows from (7.3), the third equality is straightforward, and the last equality follows from (7.4). Furthermore, (E.1) implies that for $n = 1, 2, \dots$

$$v_\alpha(n) = \sum_{i=0}^{\infty}(z_{n+i} - z_{n+i-1} + 1)\alpha^i = (1-\alpha)\sum_{i=0}^{\infty} z_{n+i}\alpha^i - z_{n-1} + \frac{1}{1-\alpha}. \qquad \textbf{(E.3)}$$

There are only two stationary policies for this problem: φ^1 with $\varphi^1(-1) = a^s$ and φ^2 with $\varphi^2(-1) = a^c$. Observe that $v_\alpha^{\varphi^1}(-1) = 1/(1-\alpha)$ and $v_\alpha^{\varphi^2}(-1) = 1 + \alpha v_\alpha(0) = 1/(1-\alpha) + \alpha f(\alpha)$, where $f(\alpha) > 0$. Therefore,

$$v_\alpha(-1) = \min\left\{\frac{1}{1-\alpha}, \frac{1}{1-\alpha} + \alpha f(\alpha)\right\} = \frac{1}{1-\alpha}. \qquad \textbf{(E.4)}$$

Formulae (E.2) and (E.4) imply

$$v_\alpha(-2) = \alpha v_\alpha(-1) = \frac{\alpha}{1-\alpha} \le v_\alpha(-1) \le v_\alpha(0). \qquad \textbf{(E.5)}$$

In view of (E.3) and (E.5), for $n = 1, 2, \dots$

$$v_\alpha(n) \ge -1 + \frac{1}{1-\alpha} = v_\alpha(-2), \qquad \textbf{(E.6)}$$

where the inequality follows from $z_n \in \{0, 1\}$, $n = 0, 1, \dots$. (E.5) and (E.6) imply

$$m_\alpha := \inf_{x\in\mathbb{X}} v_\alpha(x) = v_\alpha(-2) = \frac{\alpha}{1-\alpha}, \qquad \textbf{(E.7)}$$

Thus, (E.2)–(E.5) and (E.7) imply (7.5).

Note that $w^* \le w^{\varphi^{(\infty)}}(-1) = 1 < \infty$. Then to complete the proof of the validity of Assumption B, we need to prove that $\sup_{\alpha\in[0,1)} u_\alpha(n) < \infty$ for $n \in \mathbb{X}$. Since $0 \le z_n \le 1$, $n = 0, 1, \dots$, (7.5) implies that $0 \le u_\alpha(n) \le 1 + (1-\alpha)\sum_{i=0}^{\infty}\alpha^i = 2$ for $n \in \mathbb{X}$ and $\alpha \in [0, 1)$. This completes the proof. ■

Proof of Lemma 7.3: (i) For all $\alpha \in [0, 1)$

$$\begin{aligned} |\bar u_\alpha(x) - \bar u_\alpha(y)| &= |\bar v_\alpha(x) - \bar v_\alpha(y)| = |v_\alpha(x) - v_\alpha(y) + \bar c(x-y)| \\ &= |u_\alpha(x) - u_\alpha(y) + \bar c(x-y)| \le |u_\alpha(x) - u_\alpha(y)| + \bar c|x-y|, \end{aligned} \qquad \textbf{(E.8)}$$

where the first equality follows from (6.2), the second one follows from Lemma 3.11, and the third one follows from (4.1).

Consider $\varepsilon > 0$. For each $\beta \in (\alpha^*, 1)$, since the family of functions $\{u_\alpha\}_{\alpha\in[\beta,1)}$ is equicontinuous (see Theorem 4.6), there exists $\delta > 0$ such that $|u_\alpha(x) - u_\alpha(y)| < \varepsilon/2$ for all $|x-y| < \delta$ and $\alpha \in [\beta, 1)$. Therefore, for $|x-y| < \delta_1 := \min\{\delta, \varepsilon/2\bar c\}$, $\bar c|x-y| < \varepsilon/2$ and (E.8) implies that $|\bar u_\alpha(x) - \bar u_\alpha(y)| \le \varepsilon$ for $|x-y| < \delta_1$ and $\alpha \in [\beta, 1)$. Thus, the family of functions $\{\bar u_\alpha\}_{\alpha\in[\beta,1)}$ is equicontinuous.

(ii) Consider $x \in \mathbb{X}$. For all $\alpha \in (\alpha^*, 1)$

$$\begin{aligned} \bar u_\alpha(x) - u_\alpha(x) &\le |\bar u_\alpha(x) - u_\alpha(x)| \\ &= |\bar v_\alpha(x) - v_\alpha(x) - (\bar m_\alpha - m_\alpha)| = |\bar c x - (\bar m_\alpha - m_\alpha)| \\ &\le \bar c|x| + |\bar m_\alpha - m_\alpha| \le \bar c(|x| + |s_\alpha| + |x_U^*|) \le \bar c(|x| + b + |x_U^*|), \end{aligned} \qquad \textbf{(E.9)}$$

where the last two inequalities follow from (6.7) and Theorem 6.1, respectively.

According to Theorem 4.1, since Assumption **B** holds, $\sup_{\alpha\in(\alpha^*,1)} u_\alpha(x) < +\infty$. Therefore, (E.9) implies that

$$\sup_{\alpha\in(\alpha^*,1)} \bar{u}_\alpha(x) \le \bar{c}(|x| + b + |x_U^*|) + \sup_{\alpha\in(\alpha^*,1)} u_\alpha(x) < +\infty, \quad x \in \mathbb{X}.$$

■

Proof of Lemma 7.4: Consider any given $\beta \in (\alpha^*, 1)$ and s_1 defined in (6.1). According to (4.17), there exists $b > 0$ such that $s_\alpha \in [-b, b]$ for all $\alpha \in [\beta, 1)$. The proof of this lemma consists of the following steps: (i) we show that if $x < s_1$, then $\lim_{\alpha\uparrow 1} \bar{u}_\alpha(x) = K$; (ii) we show that (7.7) holds for $x \in [s_1, +\infty)$ except at most countably infinite many points $x \in \mathcal{D}$ (see the definition of the set $\mathcal{D}$ in (E.11)) using Lebesgue's dominated convergence theorem; and (iii) we show that the continuity of the limiting function of every convergent sequence $\{\bar{u}_{\alpha_n}\}_{n=1,2,\ldots}$ implies that (7.7) holds for $x \in \mathcal{D}$ and establish the continuity of the function $\bar{u}$.

(i) For $x < s_1$, according to Theorem 6.1, there exists $\hat{\alpha} > \beta$ such that $x < s_\alpha$ for all $\alpha \in [\hat{\alpha}, 1)$. Then (7.6) implies that $\bar{u}_\alpha(x) = K$ for all $\alpha \in [\hat{\alpha}, 1)$. Therefore,

$$\lim_{\alpha\uparrow 1} \bar{u}_\alpha(x) = K, \quad x < s_1. \tag{E.10}$$

(ii) Recall the renewal counting process $\mathbf{N}(\cdot)$ defined in (4.5) and $\mathbf{S}_n$ defined in (4.6). Consider the sets

$$\mathcal{D}_n := \{x \in \mathbb{R} : \text{ distribution function } P(\mathbf{S}_n \le x) \text{ is discontinuous}\}.$$

Therefore, each set $\mathcal{D}_n$, $n = 0, 1, \ldots$, is at most countably infinite. Let

$$\mathcal{D} = \{s_1\} \cup \{x > s_1 : x = s_1 + y, \quad y \in \cup_{n=0}^{+\infty} \mathcal{D}_n\} \quad \text{and} \quad \mathcal{C} = [s_1, +\infty)\backslash\mathcal{D}. \tag{E.11}$$

Hence, $\mathcal{D}$ is also at most countably infinite. In addition, $P(\mathbf{S}_n \le x - s_1)$ is continuous at $x - s_1$ and for $n = 0, 1, \ldots$ and $x \in \mathcal{C}$

$$\lim_{\alpha\uparrow 1} P(\mathbf{S}_n \le x - s_\alpha) = P(\mathbf{S}_n \le x - s_1). \tag{E.12}$$

Then we will show that (7.7) holds for $x \in \mathcal{C}$. Consider $x \in \mathcal{C}$. According to Theorem 6.1, there exists $\hat{\alpha} > \beta$ such that $s_\alpha < x$ for all $\alpha \in [\hat{\alpha}, 1)$. Therefore, in view of (3.16), (6.2), (6.4), and (7.6), for all $\alpha \in [\hat{\alpha}, 1)$

$$\bar{u}_\alpha(x) = \big(\mathbb{E}[h_\alpha(x - D)] - (1-\alpha)\bar{m}_\alpha\big) + \alpha\mathbb{E}[\bar{u}_\alpha(x - D)]. \tag{E.13}$$

Using the same arguments as in (4.27) and $\bar{u}_\alpha(x - \mathbf{S}_{\mathbf{N}(x-s_\alpha)+1}) = K$, which follows from (7.6), (E.13) implies that

$$\bar{u}_\alpha(x) = \mathbb{E}\left[\sum_{j=1}^{\mathbf{N}(x-s_\alpha)+1} \alpha^{j-1}(\tilde{h}_\alpha(x - \mathbf{S}_{j-1}) - (1-\alpha)\bar{m}_\alpha)\right] + \mathbb{E}[\alpha^{\mathbf{N}(x-s_\alpha)+1}K], \tag{E.14}$$

where $\tilde{h}_\alpha(x) := \mathbb{E}[h_\alpha(x - D)]$, $x \in \mathbb{X}$. Then it suffices to prove that the two expectations in (E.14) converge as the discount factor $\alpha \uparrow 1$. We start with the second one in (E.14). For $\alpha \in [\beta, 1)$

$$1 \ge \mathbb{E}[\alpha^{\mathbf{N}(x-s_\alpha)+1}] \ge \mathbb{E}[\alpha^{\mathbf{N}(x+b)+1}] \ge \alpha^{\mathbb{E}[\mathbf{N}(x+b)+1]}, \tag{E.15}$$

where the first inequality follows from $\alpha < 1$, the second one follows from $s_\alpha \ge -b$ and $\alpha < 1$, and the last one follows from Jensen's inequality. Since $P(D > 0) > 0$, $\mathbb{E}[\mathbf{N}(x + b) + 1] <$

$+\infty$, which implies that $\lim_{\alpha\uparrow 1}\alpha^{\mathbb{E}[\mathbf{N}(x+b)+1]}\to 1$. In view of (E.15),

$$\lim_{\alpha\uparrow 1}\mathbb{E}[\alpha^{\mathbf{N}(x-s_\alpha)+1}K]=K. \tag{E.16}$$

Now we show that the first expectation in (E.14) converges as the discount factor $\alpha\uparrow 1$. Note that the first expectation in (E.14) can be written as

$$\begin{aligned}
&\mathbb{E}\left[\sum_{j=1}^{\mathbf{N}(x-s_\alpha)+1}\alpha^{j-1}(\tilde{h}_\alpha(x-\mathbf{S}_{j-1})-(1-\alpha)\bar{m}_\alpha)\right]\\
&=\sum_{i=0}^{+\infty}\sum_{j=1}^{i+1}\alpha^{j-1}\mathbb{E}[(\tilde{h}_\alpha(x-\mathbf{S}_{j-1})-(1-\alpha)\bar{m}_\alpha)\mathbf{1}_{\{\mathbf{N}(x-s_\alpha)=i\}}]\\
&=\sum_{j=0}^{+\infty}\sum_{i=j}^{+\infty}\alpha^{j}\mathbb{E}[(\tilde{h}_\alpha(x-\mathbf{S}_{j})-(1-\alpha)\bar{m}_\alpha)\mathbf{1}_{\{\mathbf{N}(x-s_\alpha)=i\}}]\\
&=\sum_{j=0}^{+\infty}\alpha^{j}\mathbb{E}[(\tilde{h}_\alpha(x-\mathbf{S}_{j})-(1-\alpha)\bar{m}_\alpha)\mathbf{1}_{\{\mathbf{N}(x-s_\alpha)\ge j\}}]\\
&=\sum_{j=0}^{+\infty}\alpha^{j}\mathbb{E}[(\tilde{h}_\alpha(x-\mathbf{S}_{j})-(1-\alpha)\bar{m}_\alpha)\mathbf{1}_{\{\mathbf{S}_j\le x-s_\alpha\}}],
\end{aligned} \tag{E.17}$$

where the first and the third equalities are straightforward, the second one changes the order of summation, and its validity follows from the nonnegativity of $h(\cdot)$ and the finiteness of $\bar{u}_\alpha(x)$, and the last one holds because $\{\mathbf{N}(t)\ge n\}=\{\mathbf{S}_n\le t\}$.
Then we construct a finite upper bound of the sum in (E.17). Since the function $\tilde{h}_\alpha(x)$ is quasiconvex and $s_\alpha>-b$, for $j=0,1,\ldots$

$$\alpha^j\mathbb{E}[\tilde{h}_\alpha(x-\mathbf{S}_j)\mathbf{1}_{\{\mathbf{S}_j\le x-s_\alpha\}}]\le\mathbb{E}[h_\alpha(x-D)]+\mathbb{E}[h_\alpha(-b-D)]<+\infty. \tag{E.18}$$

Since $P(D>0)>0$, there exists a constant $\Delta_D>0$ such that $P(D>\Delta_D)>0$. Let

$$\tilde{D}=\begin{cases}0 & \text{if } D<\Delta_D,\\ \Delta_D & \text{otherwise.}\end{cases}$$

Then $\mathbb{E}[\tilde{D}]=\Delta_D P(D\ge\Delta_D)>0$ and $Var(\tilde{D})=\Delta_D^2P(D\ge\Delta_D)(1-P(D\ge\Delta_D))<+\infty$. Define $\tilde{\mathbf{S}}_0=0$ and $\tilde{\mathbf{S}}_n=\sum_{i=1}^n\tilde{D}$, $n=1,2,\ldots$. Therefore, $P(\mathbf{S}_n\le x)\le P(\tilde{\mathbf{S}}_n\le x)$ for all $x\in\mathbb{R}$ and $n=0,1,\ldots$. Since $\mathbb{E}[\tilde{D}]>0$, there exists $N_1>0$ such that $n\mathbb{E}[\tilde{D}]>x+b$ for all $n\ge N_1$. Let $\Delta(n):=n\mathbb{E}[\tilde{D}]-(x+b)>0$. Hence, for $n\ge N_1$

$$\begin{aligned}
&\mathbb{E}[\mathbf{1}_{\{\mathbf{S}_n\le x-s_\alpha\}}]=P(\mathbf{S}_n\le x-s_\alpha)\le P(\tilde{\mathbf{S}}_n\le x-s_\alpha)\le P(\tilde{\mathbf{S}}_n\le x+b)\\
&\quad=P(\tilde{\mathbf{S}}_n-n\mathbb{E}[\tilde{D}]\le x+b-n\mathbb{E}[\tilde{D}])\le P(|\tilde{\mathbf{S}}_n-n\mathbb{E}[\tilde{D}]|\ge\Delta(n))\le\frac{Var(\tilde{D})}{\Delta^2(n)},
\end{aligned} \tag{E.19}$$

where the last inequality follows from Chebyshev's inequality. In addition, according to Lemma 6.6, there exists $M_1>0$ such that $|(1-\alpha)\bar{m}_\alpha|\le M_1$ for all $\alpha\in[\beta,1)$. Therefore, in view of (E.18) and (E.19),

$$\sum_{j=0}^{+\infty}\left|\alpha^j\mathbb{E}[(\tilde{h}_\alpha(x-\mathbf{S}_j)-(1-\alpha)\bar{m}_\alpha)\mathbf{1}_{\{\mathbf{S}_j\le x-s_\alpha\}}]\right|$$
$$\le(\mathbb{E}[h_\alpha(x-D)]+\mathbb{E}[h_\alpha(-b-D)]+M_1)\sum_{j=0}^{+\infty}\mathbb{E}[\mathbf{1}_{\{\mathbf{S}_j\le x-s_\alpha\}}] \tag{E.20}$$
$$\le(\mathbb{E}[h_\alpha(x-D)]+\mathbb{E}[h_\alpha(-b-D)]+M_1)(N_1+\sum_{j=N_1}^{+\infty}\frac{Var(\tilde{D})}{\Delta^2(j)})<+\infty.$$

Then

$$\lim_{\alpha\uparrow 1}\mathbb{E}\left[\sum_{j=1}^{\mathbf{N}(x-s_\alpha)+1}\alpha^{j-1}(\tilde{h}_\alpha(x-\mathbf{S}_{j-1})-(1-\alpha)\bar{m}_\alpha)\right]$$
$$=\lim_{\alpha\uparrow 1}\sum_{j=0}^{+\infty}\alpha^j\mathbb{E}[(\tilde{h}_\alpha(x-\mathbf{S}_j)-(1-\alpha)\bar{m}_\alpha)\mathbf{1}_{\{\mathbf{S}_j\le x-s_\alpha\}}] \tag{E.21}$$
$$=\sum_{j=0}^{+\infty}\mathbb{E}[(\tilde{h}_1(x-\mathbf{S}_j)-w)\mathbf{1}_{\{\mathbf{S}_j\le x-s_1\}}]=\mathbb{E}\left[\sum_{j=1}^{\mathbf{N}(x-s_1)+1}(\tilde{h}_1(x-\mathbf{S}_{j-1})-w)\right],$$

where the first equality follows from (E.17), the second one follows from Theorem 6.1, Lemma 6.6, (E.12), (E.20), and Lebesgue's dominated convergence theorem, and the last one follows from $\{\mathbf{S}_j\le x-s_1\}=\{\mathbf{N}(x-s_1)\ge j\}$. In view of (E.14), (E.16), and (E.21),

$$\lim_{\alpha\uparrow 1}\bar{u}_\alpha(x)=\mathbb{E}\left[\sum_{j=1}^{\mathbf{N}(x-s_1)+1}(\tilde{h}_1(x-\mathbf{S}_{j-1})-w)\right]+K,\quad x\in\mathcal{C}. \tag{E.22}$$

Thus, (7.7) is proved for $x\in\mathbb{X}\backslash\mathcal{D}$.

(iii) It remains to prove (7.7) for $x\in\mathcal{D}$ and continuity of the function $\bar{u}$. In view of Lemma 7.3 and Ascoli's theorem, there exist a sequence $\{\alpha_n\uparrow 1\}_{n=1,2,\dots}$ with $\alpha_1\ge\beta$ and a continuous function $\bar{u}^*$ such that $\lim_{n\to+\infty}\bar{u}_{\alpha_n}(x)=\bar{u}^*(x)$, $x\in\mathbb{X}$.

Assume that (7.7) does not hold for some $x\in\mathcal{D}$. Then there exists a sequence $\{\gamma_n\uparrow 1\}_{n=1,2,\dots}$ with $\gamma_1>\beta$ such that $\lim_{n\to+\infty}u_{\gamma_n}(x)\ne\bar{u}^*(x)$. According to Lemma 7.3 and Ascoli's theorem, there exist a subsequence $\{\gamma_{n_k}\}_{k=1,2,\dots}$ of $\{\gamma_n\}_{n=1,2,\dots}$ and a continuous function $\bar{u}'$ such that $\lim_{k\to+\infty}\bar{u}_{\gamma_{n_k}}(x)=\bar{u}'(x)$, $x\in\mathbb{X}$. Then $\bar{u}'(x)\ne\bar{u}^*(x)$. However, $\bar{u}'$ and $\bar{u}^*$ are two continuous functions on $\mathbb{R}$ that coincide outside of a countable set. This implies that $\bar{u}'=\bar{u}^*$. Therefore, (7.7) holds for $x\in\mathbb{X}$ and the continuity of the function $\bar{u}$ follows from Ascoli's theorem. ■

Proof of Corollary 7.5: This corollary follows from Theorems 5.1, 6.1 and 7.1. ■

Proof of Corollary 7.7: This corollary follows directly from Lemma 2.2. ■