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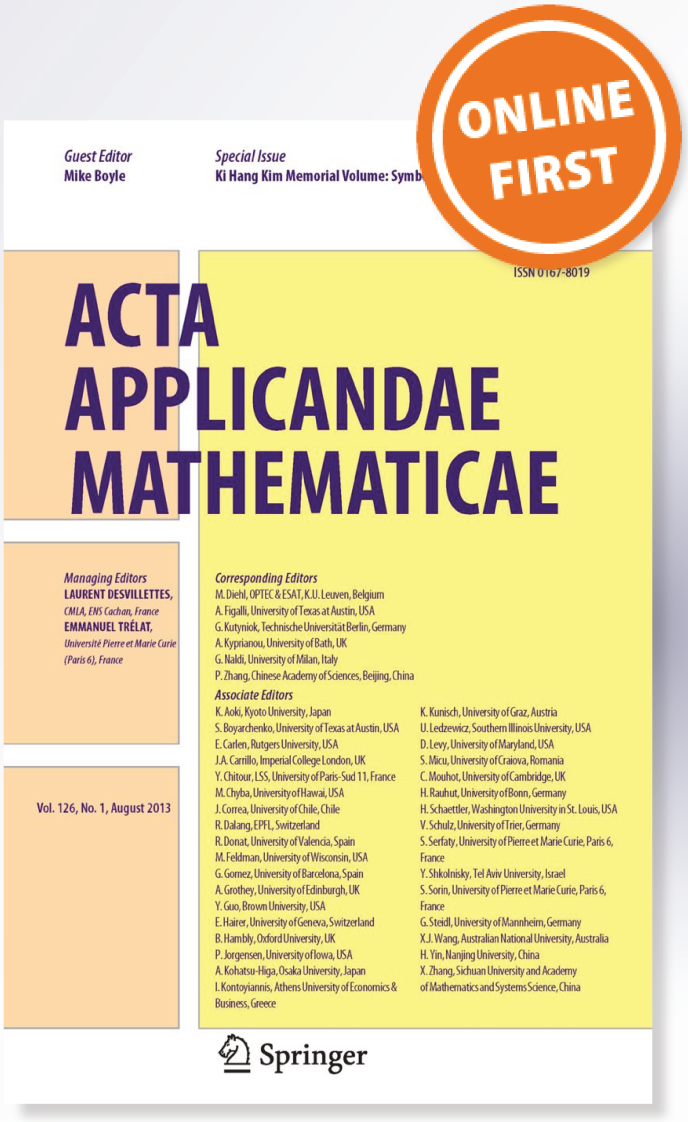

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Quasiperiodic Dynamics and Magnetoresistance in Normal Metals

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Abstract In this article we give a brief survey on the physics and mathematics of the phenomenon of conductivity in metals under a strong magnetic field.

Keywords Magnetoresistance · Quasiperiodic functions · Poisson brackets · Stereographic maps

1 Quasiperiodic Functions

The canonical projection $\pi_k : \mathbb{R}^k \to \mathbb{T}^k \simeq \mathbb{R}^k/\mathbb{Z}^k$ determines a canonical linear embedding $C^\infty(\mathbb{T}^k) \hookrightarrow C^\infty(\mathbb{R}^k)$ defined by $F \mapsto \pi_k^* F = F \circ \pi_k$ whose image is the set of *periodic* functions on $\mathbb{R}^k$.

Quasiperiodic (QP) functions are produced by the following small perturbation of the procedure above: they are the elements of the union of all embeddings $C^\infty(\mathbb{T}^k) \hookrightarrow C^\infty(\mathbb{R}^k)$ defined by $F \mapsto F_\psi = (\pi_n \circ \psi)^* F$, where ψ is an affine embedding of $\mathbb{R}^k$ into $\mathbb{R}^n$, $k < n$. The smallest n for which a QP function f admits such decomposition is the number of quasiperiods of f. For instance, $f(x) = \cos(2\pi x) + \cos(\sqrt{2}\, 2\pi x)$ is a QP function in one variable and two quasiperiods. We say that ψ is *rational* when $\overline{\pi_n(\psi(\mathbb{R}^k))} = \mathbb{T}^k$ and *fully irrational* when $\overline{\pi_n(\psi(\ell))} = \mathbb{T}^n$ for every straight line $\ell \subset \mathbb{R}^k$. When ψ_1, ψ_2 differ just by a constant, namely $\psi_1(x) = \psi_2(x) + y_0$ for some $y_0 \in \mathbb{R}^n$ and all $x \in \mathbb{R}^k$, their images are parallel k-planes and we say that they are *siblings*.

Quasiperiodic functions were introduced independently in the mathematical literature by P. Bohl [7] at the end of XIX century and E. Esclangon [20], who introduced the terminology, about ten years later. Their first systematic study, though, within the theory of almost

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periodic functions, is due to H. Bohr in 1926 [8]. Unlike QP functions in one variable, whose role in completely integrable Hamiltonian systems was known for a long time, the first concrete application of QP multi-variable functions was found only in the Seventies, by S.P. Novikov, as solutions of integrable PDEs (see [18] for several other important cases). In this article we will discuss in some detail a different case, whose relation with quasiperiodicity was also noticed by Novikov, namely the conductivity in a normal metal under a strong magnetic field.

In the Fifties I.M. Lifshitz, M.Ya. Azbel and M.I. Kaganov, from the Kharkov-Moscow school of solid state physics, in order to find a model able to describe several phenomena explained until then through somewhat artificial assumptions, developed a theory of conductivity in metals under a strong magnetic field based solely on the semiclassical model [29–31]. An essential feature of this model is that quasi-electrons are treated as classical particles except for the (fundamental!) difference that their quasi-momenta belong to $\mathbb{T}^3$ rather than $\mathbb{R}^3$.

Recall that a magnetic field in $\mathbb{R}^3$ can be seen as a closed 2-form $\mathbf{B}(x) = B_{ij}(x)dx^i \wedge dx^j$ whose effect on the dynamics of a particle with charge e is deforming the standard symplectic form $\omega_0 = dx^i \wedge dp_i$ on the phase space $T^*\mathbb{R}^3$ into

$$\omega_{\mathbf{B}} = \omega_0 + \frac{e}{c}\mathbf{B}$$

(e.g. see [22, 39]), where c is the speed of light. This way, the equations of motion (5) of a charged particle under an electromagnetic field can be written in the standard Hamiltonian form with respect to $\omega_{\mathbf{B}}$:

$$\dot{x}^i = \left\{x^i, H\right\}_{\mathbf{B}}, \qquad \dot{p}_i = \{p_i, H\}_{\mathbf{B}}, \tag{1}$$

where the Poisson brackets $\{,\}_{\mathbf{B}}$ induced by $\omega_{\mathbf{B}}$ satisfy the following relations:

$$\left\{x^i, x^j\right\}_{\mathbf{B}} = 0, \qquad \left\{x^i, p_j\right\}_{\mathbf{B}} = \delta^i_j, \qquad \{p_i, p_j\}_{\mathbf{B}} = B_{ij}(x).$$

When $\mathbf{B}$ is constant, two important things happen:

1. the dynamics of the p_i decouples from the dynamics of the x^i and, correspondingly, $\{,\}_{\mathbf{B}}$ restricts to a Poisson structure on the space of momenta—this structure is called *magnetic brackets*;
2. the magnetic brackets are invariant by translations and so, for any lattice $L \simeq \mathbb{Z}^3 \subset \mathbb{R}^3$, they descend to the quotient $\mathbb{T}^3 \simeq \mathbb{R}^3/L$.

In the setting of the semiclassical approximation in solid state physics, a metal is modeled as a crystal lattice $L \simeq \mathbb{Z}^3$ of ions and the Hamiltonian $\varepsilon(p)$ is called *dispersion relation*, depends only on the quasimomenta p and is invariant under translations by vectors of L^*, the reciprocal lattice of L. In other words, $\varepsilon(p)$ is a well-defined smooth function on the Poisson manifold $(\mathbb{T}^3 \simeq (\mathbb{R}^3)^*/L^*, \{,\}_{\mathbf{B}})$. Formally, the corresponding equations of motion

$$\dot{p}_i = \left\{p_i, \varepsilon(p)\right\}_{\mathbf{B}} \tag{2}$$

admit the Casimir $b(p) = B^i p_i$ but notice that b, in $\mathbb{T}^3$, is only a *multi-valued* function, since so are the coordinates p_i. What is, instead, well-defined is the *differential* of b, namely the (constant) 1-form $db = B^i dp_i$, and the fact that b is a Casimir corresponds to the fact that the solutions of the Hamiltonian equations of motion above lie entirely on the leaves of the foliation $db = 0$. By analogy, in this setting we call db the *magnetic field* and we

denote it, with slight abuse of notation, again by $\mathbf{B} = B^i dp_i$. Note finally that, in the universal cover $\mathbb{R}^3$, the orbits of the solutions of the Hamiltonian equations are given by the intersection of planes perpendicular to $\mathbf{B}$ with the constant energy surfaces $\{\varepsilon(p) = c\}$. In other words, the orbits of momenta are level curves of a QP function in two variables with three quasiperiods.

The theory of Lifshitz, Azbel and Kaganov predicted that the magnetoresistance would depend qualitatively on the topology of the orbits of quasi-electrons' momenta that, therefore, would be *observable*. Many experiments followed and fully confirmed the correctness of this model (see Fig. 1 and the references in Sect. 2) but the theoretical efforts in this direction stopped, after about a decade, because no method was found to predict the topology of the orbits for a general dispersion relation.

In 1982 Novikov, in his celebrated paper extending Morse theory to multivalued functions [39], pointed out the equivalence of this problem with the study of the level sets of quasiperiodic functions (in two variables with three quasiperiods). Since then this problem was intensively studied analytically and numerically by Novikov [40, 42] and his pupils A.V. Zorich [48], S.P. Tsarev, I.A. Dynnikov [14, 17] and the first author [9, 14]. Lately contributions were also given by A. Skripchenko [47] jointly with Dynnikov [19] and A. Avila and P. Hubert [3, 4]. The following theorem collects the most important results on the case of 3 quasiperiods and shows the fundamental and unexpected fact that, for an open dense (possibly of full measure) set of directions, open orbits are finite deformations of a straight line, namely they lie inside a finite width strip parallel to a straight line and are cut by a generic line transversal to the strip in an odd number of points:

Theorem 1 (Zorich [48], Dynnikov [17]) *Let $\psi : \mathbb{R}^2 \to \mathbb{R}^3$ be an affine embedding and denote by $B_\psi \in (\mathbb{R}\mathrm{P}^2)^* \simeq \mathbb{R}\mathrm{P}^2$ the equivalence class of the 2-plane $\psi(\mathbb{R}^2)$. Assume that ψ is fully irrational. For any generic function $F \in C^\infty(\mathbb{T}^3)$ there exist two continuous functions $L_F, U_F : \mathbb{R}\mathrm{P}^2 \to \mathbb{R}$, with $L_F \le U_F$ pointwise, and a locally constant function*

$$\ell_F : \mathcal{D}_F = \{L_F < U_F\} \to H_2\big(\mathbb{T}^3, \mathbb{Z}\big)$$

such that:

1. *either $\mathcal{D}_F = \mathbb{R}\mathrm{P}^2$, and then ℓ_F is constant, or ℓ_F assumes infinitely many values, and then $\mathcal{E}_F = \mathbb{R}\mathrm{P}^2 \setminus \cup_{l \in \ell(\mathcal{D}_F)} \overline{\mathcal{D}_{l,F}}$, where $\mathcal{D}_{l,F} = \{\ell_F(B) = l\}$, is non-empty and uncountable*;
2. *all components of $F_{\psi_a} = c$ are compact iff $c \notin [L_F(B_\psi), U_F(B_\psi)]$*;
3. *if $B_\psi \in \mathcal{D}_F$ then*:
   (a) *$L_F(B_\psi) < U_F(B_\psi)$*;
   (b) *all F_{ψ_a} have non-singular open level sets*;
   (c) *all these open level sets are strongly asymptotic to $B_\psi \times \ell_F(B_\psi)$, namely to the straight line obtained as intersection of the plane $\psi(\mathbb{R}^2)$ with any lattice plane Π representing the integer 2-cycle $\ell_F(B_\psi)$*;
4. *if $B_\psi \in \mathcal{E}_F$ then*:
   (a) *$L_F(B_\psi) = U_F(B_\psi) = e$*;
   (b) *none of these non-singular open level sets of $F_{\psi_a} = e$ is strongly asymptotic to a straight line.*

Notice that, when ψ is not fully irrational (namely when its image $\psi(\mathbb{R}^2)$ contains reciprocal lattice vectors), open (periodic) orbits might arise for a larger closed connected interval of values of F_ψ but such orbits are *unstable*, namely they disappear for a generic perturbation of ψ.

Denote by $L \simeq \mathbb{Z}^3$ the lattice representing the metal ions, so that its reciprocal $L^* \simeq (\mathbb{Z}^3)^*$ is the set of all translations leaving F invariant. One of the main points of Theorem 1 is the discovery of a hidden topological first integral $\ell_F(B_\psi) \in H_2(\mathbb{R}^3/L^*, \mathbb{Z}) \simeq L$ that dictates the asymptotics of all (non-singular) open level sets when $B_\psi \in \mathcal{D}_F$. In the most interesting cases, the dependence of ℓ on B_ψ is of fractal nature (see Fig. 1 for some concrete example). It is not known yet whether $\mathcal{D}_F$ is a full measure subset of $\mathbb{R}\mathrm{P}^2$ or what is its Hausdorff dimension but it has been conjectured by Novikov [36] that, for a generic F, the measure of $\mathcal{E}_F$ is zero and its Hausdorff dimension is strictly between 1 and 2. This conjecture has been verified recently in the only concrete example, introduced by Dynnikov and the first author [14], where it was possible to find an explicit expression for ℓ_F: in [14] it was proved that the corresponding $\mathcal{E}_F$ has zero measure and later Avila, Hubert and Skripchenko in [4] showed that its Hausdorff measure is strictly smaller than 2.

The structure of the open orbits when $B_\psi \in \mathcal{E}_F$ is still not completely understood but significant advances have been made recently by Skripchenko and Dynnikov, that built examples of F such that each F_{ψ_a} has a unique open level set [47] and such that there are infinitely many [19], and by Avila, Hubert and Skripchenko [3] that evaluated bounds for the *diffusion rates* of such orbits.

2 Quasiperiodic Functions in Electron Transport Phenomena

To describe the applications of the Novikov problem in the transport phenomena in normal metals we have to start with a description of electron states in a crystal lattice, defined by bounded solutions of the Schrödinger equation

$$-\frac{\hbar^2}{2m}\Delta\psi + U(x, y, z)\psi = \varepsilon\psi \tag{3}$$

The potential $U(\mathbf{x}) = U(x, y, z)$ represents a periodic function in $\mathbb{R}^3$ with three different periods $\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3$:

$$U(\mathbf{x}+\mathbf{l}_1) \equiv U(\mathbf{x}+\mathbf{l}_2) \equiv U(\mathbf{x}+\mathbf{l}_3) \equiv U(\mathbf{x}),$$

which define the crystal lattice L of a metal.

The basis physical solutions of (3) can be chosen in the form of the Bloch functions $\psi_{\mathbf{p}}(\mathbf{x})$, satisfying the conditions

$$\psi_{\mathbf{p}}(\mathbf{x}+\mathbf{l}_1) \equiv e^{i(\mathbf{p},\mathbf{l}_1)/\hbar}\psi_{\mathbf{p}}(\mathbf{x}), \qquad \psi_{\mathbf{p}}(\mathbf{x}+\mathbf{l}_2) \equiv e^{i(\mathbf{p},\mathbf{l}_2)/\hbar}\psi_{\mathbf{p}}(\mathbf{x}),$$
$$\psi_{\mathbf{p}}(\mathbf{x}+\mathbf{l}_3) \equiv e^{i(\mathbf{p},\mathbf{l}_3)/\hbar}\psi_{\mathbf{p}}(\mathbf{x})$$

The real vector $\mathbf{p} = (p_1, p_2, p_3)$ represents the quasimomentum of an electron state and is defined in fact modulo the vectors

$$m_1\mathbf{a}_1 + m_2\mathbf{a}_2 + m_3\mathbf{a}_3, \quad m_1, m_2, m_3 \in \mathbb{Z}, \tag{4}$$

with the vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ defined by the relations

$$\mathbf{a}_1 = 2\pi\hbar\frac{\mathbf{l}_2 \times \mathbf{l}_3}{(\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3)}, \qquad \mathbf{a}_2 = 2\pi\hbar\frac{\mathbf{l}_3 \times \mathbf{l}_1}{(\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3)}, \qquad \mathbf{a}_3 = 2\pi\hbar\frac{\mathbf{l}_1 \times \mathbf{l}_2}{(\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3)},$$

where $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ denotes the mixed product of the three vectors.

The vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ give a basis of the reciprocal lattice L^* of a crystal, conjugate to the direct lattice L. In general, the full space of physical solutions of (3) consists of an infinite number of "energy bands" where the dependence of the parameter ε on the value of $\mathbf{p}$ is given by some three-periodic smooth functions $\varepsilon_s(\mathbf{p})$:

$$\varepsilon_s(\mathbf{p}+\mathbf{a}_1) \equiv \varepsilon_s(\mathbf{p}+\mathbf{a}_2) \equiv \varepsilon_s(\mathbf{p}+\mathbf{a}_3) \equiv \varepsilon_s(\mathbf{p})$$

Thus, the complete set of parameters specifying single-electron states in a crystal includes the number of the conduction band s, the quasimomentum value $\mathbf{p}$, and the spin variable σ. The last variable will in fact not be important in our considerations, so we will omit it in our further constructions.

For a fixed energy band any two values of the quasimomentum that differ by any reciprocal lattice vector define the same physical electron state. As a result, we can actually claim that the space of electron states for a fixed energy band represents a three-dimensional torus $\mathbb{T}^3 = \mathbb{R}^3/L^*$, given by the factorization of the $\mathbf{p}$-space over the reciprocal lattice vectors. In the same way, every dispersion relation $\varepsilon_s(\mathbf{p})$ can be considered as a smooth function on $\mathbb{T}^3$ instead of the full Euclidean $\mathbf{p}$-space $\mathbb{R}^3$. Every function $\varepsilon_s(\mathbf{p})$ is naturally bounded by its minimal and maximal values $\varepsilon_s^{\min} \le \varepsilon_s(\mathbf{p}) \le \varepsilon_s^{\max}$, which define the boundaries of the corresponding energy band. Let us also note here that in the three-dimensional case the intervals $[\varepsilon_s^{\min}, \varepsilon_s^{\max}]$ can in general overlap with each other, so maybe it would be more rigorous to talk about different branches of the electron energy spectrum in a crystal.

Practically in any metal, the electron gas is highly degenerate and it can be assumed that all the electron states with energies below a certain value ε_F (the Fermi energy) are occupied, while states with energies greater than the Fermi energy are empty.[1] In the general case, we have here a certain finite number of completely filled energy bands, a finite number of partially filled bands (conduction bands), and an infinite number of empty energy bands. The full Fermi surface of a metal is given by the union of the surfaces $\varepsilon_s(\mathbf{p}) = \varepsilon_F$ for all partially filled energy bands and represents a 3-periodic smooth surface in the $\mathbf{p}$-space.

We would like to specially note here that we do not require that the Fermi surface consists of only one connected component. For us it is important, however, that the different connected components of the Fermi surface do not intersect each other. We note here that the latter property is satisfied, as a rule, also in the case when the Fermi surface is determined by several dispersion relations.

The form of the dispersion relation $\varepsilon(\mathbf{p})$ is very important for many quantum processes in crystals and, in particular, in normal metals. For us, the processes associated with transport phenomena in metals, for which the dynamics of quantum electron states in the presence of external electric and magnetic fields is important, will play a decisive role. It can immediately be noted that, since the magnitude of external electric and magnetic fields is much smaller than the magnitude of the intracrystalline fields, such dynamics is well described by the adiabatic approximation for the evolution of the functions $\psi_{\mathbf{p}}(\mathbf{r})$, which can be written in the form of a dynamical system determining the evolution of the values of the quasimomentum $\mathbf{p}$. Thus, in the presence of constant external electric and magnetic fields, the corresponding system can be written in the form (see e.g. [1])

$$\dot{\mathbf{p}} = \frac{e}{c}[\mathbf{v}_{gr} \times \mathbf{B}] + e\mathbf{E} \equiv \frac{e}{c}\big[\nabla\varepsilon(\mathbf{p}) \times \mathbf{B}\big] + e\mathbf{E} \tag{5}$$

[1] This "near to zero temperature" approximation holds in general for metals since for them $\varepsilon_F \simeq 10^4$–$10^5$ K, at least an order of magnitude above their melting point.

The electron transport properties are determined, at the leading order, by the properties of solutions of the kinetic equation for the one-particle distribution function $f(\mathbf{p}, t)$, which can be written in the general case in the form

$$f_t + \frac{e}{c}\sum_{k=1}^{3}\big[\nabla\varepsilon(\mathbf{p}) \times \mathbf{B}\big]^k \frac{\partial f}{\partial p^k} + e\sum_{k=1}^{3} E^k \frac{\partial f}{\partial p^k} = I[f](\mathbf{p}, t) \tag{6}$$

The functional $I[f](\mathbf{p}, t)$ is the collision integral, which in the general case determines the relaxation of the perturbations of the function $f(\mathbf{p}, t)$ to its temperature-equilibrium values

$$f_0(\mathbf{p}) = \frac{1}{e^{(\varepsilon(\mathbf{p})-\varepsilon_F)/T} + 1} \tag{7}$$

Quite often, all the necessary properties of the solutions of (6) can be obtained by introducing a certain typical relaxation time of the function $f(\mathbf{p}, t)$ to its equilibrium values (the mean free electron time) τ and replacing the collision integral by the value

$$-\big(f(\mathbf{p}, t) - f_0(\mathbf{p})\big)/\tau$$

When calculating the electronic transport properties in metals (such as electrical conductivity or electron thermal conductivity), the most interesting quantity is usually the response of the system to the application of, say, an electric field (or a temperature gradient) in the linear approximation in the value of $\mathbf{E}$. From this point of view, the electric field in (5) can be regarded as a small correction to the system

$$\dot{\mathbf{p}} = \frac{e}{c}\big[\nabla\varepsilon(\mathbf{p}) \times \mathbf{B}\big], \tag{8}$$

determining the evolution of electron states in the presence of a constant magnetic field. The geometry of the trajectories of the system (8) plays an important role for the corresponding electronic properties of a metal in the region $\omega_B\tau \gg 1$, where the cyclotron frequency ω_B is defined by the relation $\omega_B = eB/m^*c$. Let us note also here that the value of the effective electron mass m^* in a metal can, in general, differ noticeably from the free electron mass m.

In the semiclassical approximation one can also consider the motion of electron wave packets in the coordinate space, which is given by the relations

$$\dot{\mathbf{x}} = \mathbf{v}_{gr}(\mathbf{p}) \equiv \nabla\varepsilon(\mathbf{p}). \tag{9}$$

Writing (9) and (8) in coordinates shows that they are exactly the Hamiltonian equations (1) in case H only depends on $\mathbf{p}$, revealing the symplectic (rather than metric) nature of these equations despite the deceptive notation. Note moreover that, when $H = H(\mathbf{p})$, the Hamiltonian vector field ξ_H corresponding to H writes as

$$\xi_H = (\xi_H)^i \frac{\partial}{\partial x^i} + (\xi_H)_i \frac{\partial}{\partial p_i} = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial x^i} + B_{ij}\frac{\partial H}{\partial p_j}\frac{\partial}{\partial p_i},$$

so that $(\xi_H)^i(\xi_H)_i = 0$ identically (and invariantly) due to the antisymmetry of the coefficients $B_{ij} = \epsilon_{ijk}B^k$ of the magnetic field 2-form. Formally, this identity can be interpreted as the fact that, at every point, the $\mathbf{p}$ component of ξ_H is perpendicular to the projection of its $\mathbf{x}$ component on the plane perpendicular to $\mathbf{B}$. Due to the constancy of the symplectic form, one can also see that the projections of the electron trajectories in the $\mathbf{x}$-space onto a plane

orthogonal to **B** are obtained in this case from the trajectories of the system (8) in **p**-space with a rotation by an angle of $\pi/2$ in the same plane.

The latter circumstance clarifies the role of the shape of trajectories of the system (8) for electron transport phenomena and it can be also seen that the parameter $\omega_B\tau$ determines the average length of the electron motion along the trajectory between two acts of scattering by an impurity. The condition $\omega_B\tau \gg 1$ then leads to the manifestation of the features of the global geometry of the trajectories of system (8) for phenomena of this type.

Going back to system (8), namely to the Hamiltonian system in $(\mathbb{T}^3, \{,\}_{\mathbf{B}})$ given by (2), recall that Poisson brackets in odd dimension are necessarily degenerate and, in fact, $\{,\}_B$ has the (multivalued!) Casimir $b(\mathbf{p}) = B^i p_i$. On each of the Casimir's leaves, namely on each of the projections into $\mathbb{T}^3$ of the planes $\psi_a : \mathbb{R}^2 \to \mathbb{R}^3$ perpendicular to **B**, the system is non-degenerate and equivalent to the Hamiltonian system with QP Hamiltonian ε_{ψ_a}. Hence, since we are in dimension 2, the orbits of the solutions of (8) are just the (projection into $\mathbb{T}^3$ of) level sets of ε_{ψ_a}, whose structure has been summarized in Theorem 1.

The study of the question of the influence of the geometry of trajectories of the system (8) on the behavior of electron transport phenomena was started in the school of I.M. Lifshitz in the Fifties (see [24–29, 31]). Thus, in the work [31] it was first shown that the behavior of the electric conductivity tensor of a metal in strong magnetic fields is significantly different in the cases when the Fermi surface contains only closed trajectories and when open periodic trajectories appear on it. Let us always assume here that the coordinate system in the **x**-space is chosen in such a way that the z-axis coincides with the direction of the magnetic field. In addition, let us also assume that the direction of the x-axis in the second case coincides with the mean direction of the periodic open trajectories in the **p**-space (note here that the mean direction of the projection of the corresponding trajectories onto the plane orthogonal to **B** in the **x**-space coincides with the y-axis in this case). Then, according to [30], the analysis of (6) gives the following results for the asymptotic behavior of the conductivity tensor in the two cases above

$$\sigma^{ik} \simeq \frac{ne^2\tau}{m^*} \begin{pmatrix} (\omega_B\tau)^{-2} & (\omega_B\tau)^{-1} & (\omega_B\tau)^{-1} \\ (\omega_B\tau)^{-1} & (\omega_B\tau)^{-2} & (\omega_B\tau)^{-1} \\ (\omega_B\tau)^{-1} & (\omega_B\tau)^{-1} & * \end{pmatrix}, \quad \omega_B\tau \to \infty \tag{10}$$

(closed trajectories),

$$\sigma^{ik} \simeq \frac{ne^2\tau}{m^*} \begin{pmatrix} (\omega_B\tau)^{-2} & (\omega_B\tau)^{-1} & (\omega_B\tau)^{-1} \\ (\omega_B\tau)^{-1} & * & * \\ (\omega_B\tau)^{-1} & * & * \end{pmatrix}, \quad \omega_B\tau \to \infty \tag{11}$$

(open periodic trajectories).

We note here that the relations (10) and (11) should be understood only as asymptotic expressions and may contain additional dimensionless constants for each of the components σ^{ik}. We also use here the notation $*$ for arbitrary dimensionless constants of the order of unity.

It is easy to see that the main difference in the conductivity behavior in the cases considered above is the strong anisotropy of the conductivity in the plane, orthogonal to **B**, observed in the second case. This property is a direct consequence of the special form of the corresponding electron trajectories and makes it possible to measure the mean direction of the periodic open trajectories in the **p**-space.

In the works [27, 28], open trajectories of a more general type on Fermi surfaces of different shapes were considered. Let us say, that the trajectories, considered in [27, 28], are

not periodic in general, but also have a mean direction in the plane orthogonal to **B**. As a result, the conductivity behavior also exhibits strong anisotropic properties in strong magnetic fields in the presence of trajectories of this type on the Fermi surface. The works [24–26], as well as the book [31], provide a broad overview of the issues related to the electronic properties of metals, and in particular the issues related to transport phenomena in strong magnetic fields examined during that period. We would also like to give here a reference to the work [23] in which a return to this range of issues is made after a considerable time, and containing also aspects that arose in the later period.

As we have said above, the problem of the complete classification of possible types of trajectories of the system (8) was set by S.P. Novikov in the early Eighties and has now been studied with sufficient completeness, allowing to describe all essentially different types of open electron trajectories. In this chapter we will focus on the most significant physical results arising from the mathematical description of the trajectories of system (8), obtained in the recent decades.

As we noted in the previous chapter, the most significant part in the classification of open trajectories of system (8) is the description of stable open trajectories obtained in the works of A.V. Zorich and I.A. Dynnikov. We shall try to describe here the most interesting physical consequences arising when such trajectories appear on the Fermi surface.

Since the orbits of the solutions of (8) are the level sets of all siblings ε_{ψ_a} of the quasiperiodic function in two variables with three quasiperiods ε_ψ, by Theorem 1 such trajectories always possess the following two remarkable properties:

1. Any stable open trajectory of system (8) in the **p**-space lies in a straight strip of finite width in a plane ψ_a;
2. For a fixed direction **B**, all stable open trajectories in the **p**-space have the same mean direction, given by $\mathbf{B} \times \ell_\varepsilon(\mathbf{B})$.

As pointed out in [43], the presence of stable open trajectories on the Fermi surface always entails a strong anisotropy of the conductivity tensor in the plane, orthogonal to **B**, in the limit $\omega_B \tau \to \infty$. Because of this, the topological quantum first integral $\ell_\varepsilon(\mathbf{B})$ is *observable* experimentally. The corresponding integer triples $\ell_\varepsilon(\mathbf{B}) = (M_1, M_2, M_3) \in H_2(\mathbb{T}^3, \mathbb{Z})$ were called in [36, 45] the *topological quantum numbers* observable in the conductivity of normal metals. Notice that $H_2(\mathbb{R}^3/L^*, \mathbb{Z})$ is the set of lattice planes of the reciprocal lattice L^* and therefore is naturally isomorphic to L, so that we can always think of $\ell_\varepsilon(\mathbf{B})$ as a direct lattice element in the **x** space. We call *Stability Zones* the sets $\mathcal{D}_{l,\varepsilon}$ defined by $\ell_\varepsilon(\mathbf{B}) = l$, so that for every $\mathbf{B} \in \mathcal{D}_{l,\varepsilon}$ all non-singular open orbits are strongly asymptotic to $\mathbf{B} \times l$.

Both the topological quantum first integral and the geometry of the Stability Zones contain important information about the electron spectrum in a crystal that is directly related to the determination of parameters of this spectrum in real materials. At the same time, both theoretical and experimental determination of the exact boundaries of the Stability Zones for a given dispersion relation represents a non-trivial problem that requires the use of rather special methods. As an example of a theoretical determination of the boundaries of the Stability Zones, we can cite the work [12], where such calculations were performed for a number of analytical dispersion relations that arise in real crystals. As can be seen from the work [12], an accurate calculation of the structure of the Stability Zones on the angular diagram requires the development of both rather serious topological and computational methods. We hope, on the other hand, that the methods used in [12] must be applicable to a large number of different examples of complex Fermi surfaces and will prove extremely useful in determination the parameters of the dispersion relations in real materials. It must also be said that the experimental determination of the structure of the Stability Zones in

real materials also presents a special problem because of a rather complicated analytical behavior of conductivity near their boundaries (see, e.g. [33]). In particular, the exact experimental determination of the mathematical boundaries of the Stability Zones also requires, in addition to direct study of conductivity, special experimental techniques ([34]).

Another very important achievement of mathematical research of the S.P. Novikov problem was the discovery of new, previously unknown, types of trajectories of system (8), which have very complicated (chaotic) behavior. The first trajectories of this type were constructed at the beginning of the Nineties by S.P. Tsarev[2] for "partially irrational" directions of **B** and have an obvious chaotic behavior on the Fermi surface. At the same time, the behavior of the Tsarev trajectories in planes orthogonal to **B** resembles the behavior of stable open trajectories, in particular, they all have asymptotic directions in these planes (although they do not lie in straight strips of finite width). As a result, the behavior of the conductivity tensor in the presence of the Tsarev trajectories on the Fermi surface is also very similar to its behavior in the presence of stable open trajectories, in particular, it has a strong anisotropy in this case. As already mentioned, trajectories of Tsarev type can appear only for directions of the magnetic field of irrationality 2 (the plane orthogonal to **B** contains a reciprocal lattice vector) and it can be shown (see [16]) that all chaotic trajectories arising for such directions of **B** have the properties described above.

The first examples of chaotic trajectories for directions of **B** of maximal irrationality were constructed by I.A. Dynnikov in the work [16]. Trajectories of this type have a strongly chaotic behavior both on the Fermi surface and in planes orthogonal to **B**. Its behavior in a plane orthogonal to **B** resembles in a certain sense the diffusion motion, which leads to the most complicated dependence of the conductivity on the value of B.

The most interesting moment in the behavior of conductivity in the presence of the Dynnikov trajectories is the blocking of the conductivity along the direction of **B** in strong magnetic fields ([32]), such that the entire Fermi surface area covered by the corresponding chaotic trajectories does not contribute to the conductivity along **B** in the limit $\omega_B \tau \to \infty$. As a result, for the corresponding directions on the angular diagram, rather sharp minima in conductivity along the direction of **B** should be observed in strong magnetic fields.

Another interesting feature of the conductivity behavior in the presence of the Dynnikov trajectories on the Fermi surface is the appearance of fractional powers of the parameter $\omega_B \tau$ in the dependence of the components of the conductivity tensor on the value of the magnetic field ([32]). It must be said that the analysis carried out for (6) in the presence of such trajectories actually used in this case an additional property (self-similarity) of trajectories constructed in [16], which, generally speaking, is not observed in the general case for Dynnikov chaotic trajectories. Quite recently, however, it was possible to show that the appearance of fractional powers of a parameter $\omega_B \tau$ in the conductivity behavior in the presence of trajectories of this type is actually a common fact and is connected with the existence of the so-called Zorich-Kontsevich-Forni indices [49] describing the behavior of such trajectories on a large scale [38].

As mentioned in the previous section, the general properties of chaotic trajectories of Dynnikov type, as well as the properties of the set of directions of **B** at which such trajectories can be observed, are the subject of the most active research at the present time (e.g. see [3, 4, 9, 10, 12, 14, 17, 19, 47, 48]). Let us also note here that, in spite of the fact that Dynnikov chaotic trajectories are not trajectories of general position, they may nevertheless be typical for Fermi surfaces of a certain type (see [35]).

[2]Private communication.

Let us say, that, in addition to describing the new quantities and the new regimes observed in conductivity studies, a mathematical investigation of the Novikov problem actually made it possible to construct a complete classification of the possible types of conductivity behavior in strong magnetic fields, including all cases of both generic and non-generic position. Here we only point out that the most detailed mathematical consideration of the various situations possible for system (8) is presented in the work [17]. We also note that a detailed exposition of the physical consequences of the classification obtained can be found in the works [36, 37, 44].

3 Experimental and Numerical Study of Level Sets of Quasiperiodic Functions on $\mathbb{R}^2$ with 3 Quasiperiods

No algorithm is known able to obtain an analytical or approximate perturbative expression of the set $\mathcal{D}_F$ and the functions L_F, U_F relative to a general function $F \in C^\infty(\mathbb{T}^3)$. In fact, an analytical description for non-trivial $\mathcal{D}_F$ and ℓ_F has been found only in case of a very simple piecewise linear function [14]. Numerical methods are therefore necessary in order to get some intuition on the nature of such sets and maps and in order to predict theoretically from first principles the physical behavior of systems involving QP functions.

Using the *extrinsic* geometry results by Zorich [48] and Dynnikov [17] (see also [41]) about the foliations induced by bundles of parallel planes on a triply periodic surface, Dynnikov [15] was the first to study (semianalytically) a concrete case, namely the triply periodic function

$$\mathfrak{c}(x, y, z) = \cos(2\pi x) + \cos(2\pi y) + \cos(2\pi z).$$

The level sets $\mathfrak{c}_c$ have genus 3 and rank 3 for $c \in (-1, 1)$ and genus 0 and rank 0 otherwise. The function $\mathfrak{c}$ satisfies the property $T^*F = -F$, where T is the translation by $1/2$ in the three coordinate directions, so that when $\psi_a^*\mathfrak{c} = c$ has an open level set then also $\psi_a^*\mathfrak{c} = -\mathfrak{c}$ has, meaning ultimately that $L_\mathfrak{c} = -U_\mathfrak{c}$ and so that, in particular, in order to study $\mathcal{D}_\mathfrak{c}$ it is enough looking at the level $\mathfrak{c}_0$. Dynnikov was able to find the analytical expression for the 10 largest connected components of $\mathcal{D}_\mathfrak{c}$ and their corresponding values of $\ell_\mathfrak{c}$. The drawback of this method is that it does not look suitable to be implemented into a programming language.

In order to bypass this problem, the first author implemented the algorithm to evaluate $\ell_\mathfrak{c}(\mathbf{B})$ in the open source C++ library NTC [13]. NTC is built on top of the open source C++ library VTK, perhaps the most popular computational geometry library available online in the last two decades. VTK implements fundamental geometry operations such as generating, within some cuboid, the mesh for the level set of a given function or generating the mesh of the intersection between two such surfaces within some fixed cuboid.

While restricting an unbounded set to a bounded cuboid causes in general a big loss of information, it is not so for a periodic set since the whole information about it is contained inside a basic cell. Surprisingly enough, in the authors' knowledge, none of the general-purpose computational geometry libraries available online implement special algorithm for *periodic geometry*, although that is the only geometry where, quite remarkably, "*it is possible to keep infinity inside a bounded box*". In order to get an approximation for $\mathcal{D}_\mathfrak{c}$ and $\ell_\mathfrak{c}$ with NTC it is enough to fix a grid in $\mathbb{Q}\mathrm{P}^2$ and evaluate $\ell_\mathfrak{c}$ at all elements of the grid.

NTC currently supports functions with level surfaces of genus $g = 3$ and $g = 4$. The first is the simplest case with a non-trivial set $\mathcal{D}_\mathfrak{c}$, the second is the case of the Fermi surfaces of the noble metals Gold, Copper and Silver. In Fig. 1 (left) we show $\mathfrak{c}_0$, the whole SM $\mathcal{D}(\mathfrak{c})$ and a detail of it in the region $[0, 1]^2$ in the chart $B_z = 1$. A rough numerical evaluation of its

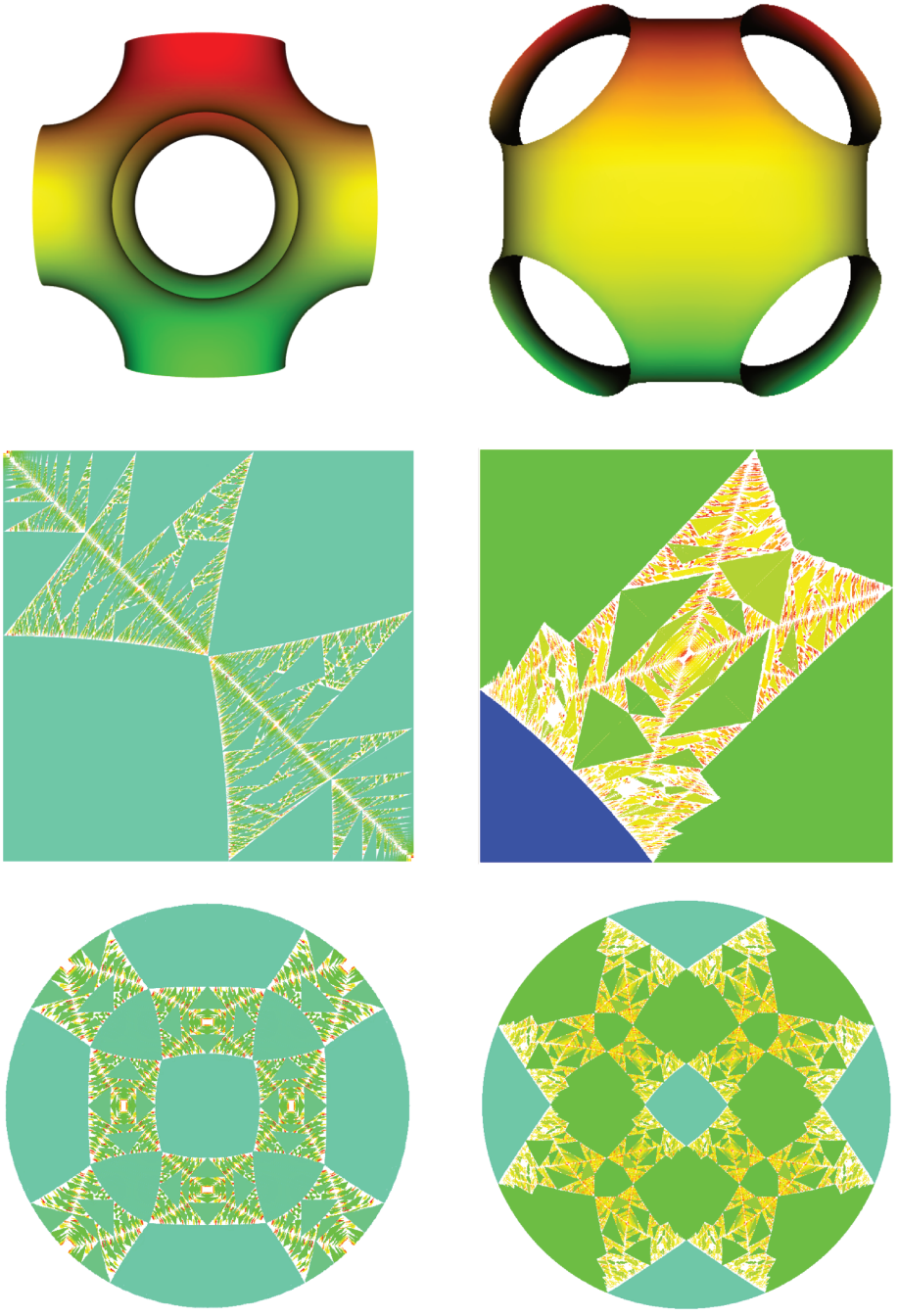

Fig. 1 From top to bottom: the surfaces $\mathfrak{c}_0$ (left) and $\mathfrak{d}_0$ (right) in the corresponding first Brillouin zone; the sets $\mathcal{D}_{\mathfrak{c}} \subset \mathbb{R}\mathrm{P}^2$ (left) and $\mathcal{D}_{\mathfrak{d}} \subset \mathbb{R}\mathrm{P}^2$ (right) in the square $[0, 1]^2$ of the projective chart $B_z = 1$; the same sets in the whole $\mathbb{R}\mathrm{P}^2$, represented as a disc with opposite boundary points identified. The colors of the stability zones of $\mathcal{D}_{\mathfrak{c}}$ and $\mathcal{D}_{\mathfrak{d}}$ correspond to the norms of the topological first integral associated to them, passing from green to red as the norm goes to infinity (Color figure online)

box dimension gives an estimate of about 1.83, in agreement with Novikov's Conjecture 1. In Fig. 1 (right) we show the set $\mathcal{D}_{\mathfrak{d}}$ for the map

$$\mathfrak{d}(x, y, z) = \cos(2\pi x)\cos(2\pi y) + \cos(2\pi y)\cos(2\pi z) + \cos(2\pi z)\cos(2\pi x),$$

whose regular level sets $\mathfrak{d}_c$ are either spheres (for $c < -1$ and $c > 0$) or genus-4 surfaces (for $-1 < c < 0$). Each of the genus-4 level sets has topological rank 4. Note also that $\mathfrak{d}$, besides being invariant by integer translations along the coordinate axes, is invariant with respect to translations by $1/2$ along the cube diagonals, namely it has a bcc invariance. A rough numerical evaluation of its box dimension of about 1.69, again in agreement with Novikov's Conjecture 1. A striking confirmation of the correctness of these numerical data is shown in [14]. In that article it is discussed the case of a simple piecewise linear function F where the first author and Dynnikov were able to find an analytical expression for ℓ_F; the numerical data for that case agrees at 100% level with the analytical ones.

We switch now to the experimental data. As mentioned in the previous sections, according to the semiclassical approximation the topology of the level sets of the QP function ε_ψ given by the restriction of the Fermi energy function to some plane ψ perpendicular to $\mathbf{B}$ dictates the asymptotic behavior of the magnetoresistance for $\|\mathbf{B}\| \to \infty$ and so it can be detected experimentally. Starting from the end of the Fifties, *stereographic maps* were experimentally obtained for many metals, mostly by Pippard, Alekseevskii and Gaidukov (e.g. see [2, 21, 46]). The maps for Gold (left) and Silver (right) are shown in the middle column of Fig. 2. In these stereographic maps, $\mathbb{R}\mathrm{P}^2$ is represented as a disc and regions are shaded for those directions of the magnetic field open orbits are detected and left blank otherwise. Mathematically, this corresponds to the fact that we look only at a single level set $\varepsilon_\psi = c$, the Fermi energy level, for every sibling of ψ and, correspondingly, we define a *reduced* map $\ell_{\varepsilon,c}(\mathbf{B})$ that, for any $\mathbf{B} \in \mathcal{D}_\varepsilon$, is equal to $\ell_\varepsilon(\mathbf{B})$ if $c \in [L_\varepsilon, U_\varepsilon]$ and to $(0, 0, 0)$ (meaning absence of open orbits) otherwise. We denote by $\mathcal{D}_{\varepsilon,c}$ the subset of $\mathcal{D}_\varepsilon$ where $\ell_{\varepsilon,c}(\mathbf{B}) \neq (0, 0, 0)$.

No comparison of these experimental data with theoretical prediction was possible for about half a century because of the lack of knowledge about the levels of QP functions. In the bottom row of Fig. 2 we show the numerical approximations of the sets $\mathcal{D}_{\varepsilon,c}$ relative to approximated expressions of the Fermi surfaces retrieved from the physics literature. Note that a strong magnetic field (of the order of 10 Tesla) is needed in order for this phenomenon to become visible and these old experimental data was taken right at the threshold (with magnetic fields of about 1 Tesla). Similarly, the trigonometric approximations we used for the Fermi energy functions is far from being the best approximation available to date (but it was the simplest and quickest to implement in the NTC library). Yet, the match between experimental data and theoretical prediction is remarkably high.

We point out that the reason for using such old experimental data is that, after about a decade of great excitement that saw a large number of theoretical and experimental articles dedicated to the subject, the interest of the solid state community in the topic decreased a lot, possible exactly because no way was found to reproduce the experimental data from first principles, and so in our knowledge no new stereographic maps were produced since the Sixties. Recently, though, some new experimental result, in particular on the role of dislocations in the deformation of the map $\ell_{\varepsilon,c}$ for Copper, has been published by M. Niewczas and his student Q. Bian [5, 6], giving some hope for the appearance of accurate stereographic maps in a near future. Now that we have the possibility, it would be indeed extremely interesting to have some more reliable experimental data to compare to.

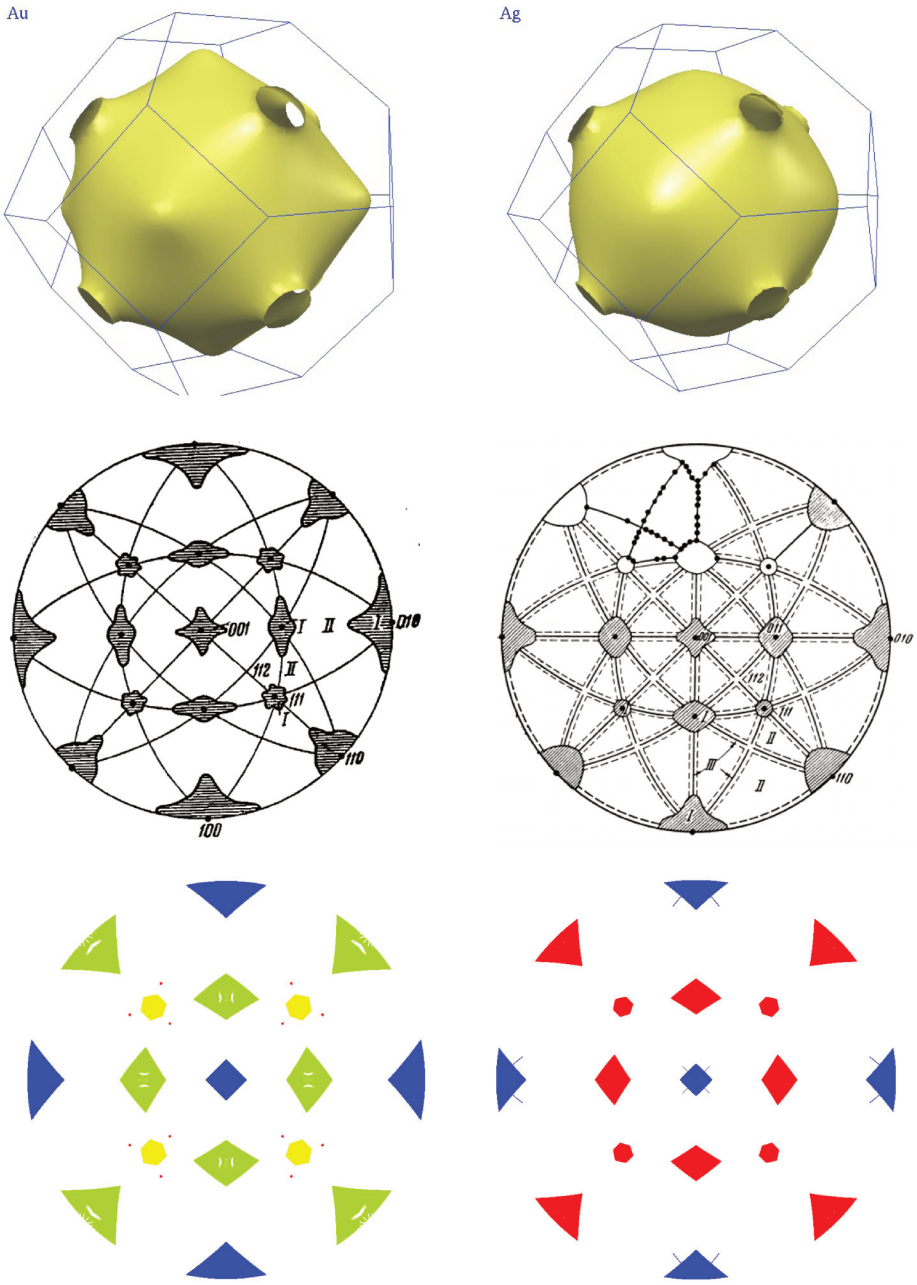

Fig. 2 From top to bottom: Fermi surfaces of Gold and Silver in the corresponding first Brillouin zone; experimental data on the topology of quasi-electrons trajectories in Gold [21] and Silver [2]—the shaded regions correspond to directions of the magnetic field for which open trajectories arise (detectable through measurements of magnetoresistance); numerical results on the topology of planar sections of the Fermi surfaces of Gold and Silver [11, 12]

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