SHRINKING TARGETS PROBLEMS FOR FLOWS ON HOMOGENEOUS SPACES

DUBI KELMER AND SHUCHENG YU

ABSTRACT. We study shrinking targets problems for discrete time flows on a homogenous space $\Gamma\backslash G$ with G a semisimple group and Γ an irreducible lattice. Our results apply to both diagonalizable and unipotent flows, and apply to very general families of shrinking targets. As a special case, we establish logarithm laws for cusp excursions of unipotent flows settling a problem raised by Athreya and Margulis.

1. INTRODUCTION

Consider an ergodic dynamical system given by the iteration of a measure preserving map $T : \mathcal{X} \to \mathcal{X}$ on a probability space $(\mathcal{X}, \mu)$. From ergodicity, it follows that generic orbits become dense, and shrinking target problems are a way to quantify the rate. That is, to determine how fast we can make a sequence of targets shrink so that a typical orbit will keep hitting the targets infinitely often.

A natural bound for this rate comes from the easy half of Borel-Cantelli, stating that for any sequence of sets, $\{B_m\}_{m\in\mathbb{N}}$, if $\sum_{m=1}^{\infty} \mu(B_m) < \infty$ then for a.e. $x \in \mathcal{X}$ from some point on $T^m x \notin B_m$. For chaotic dynamical systems, it could be expected that this bound is sharp, and much work has gone into proving this in various examples of fast mixing dynamical systems (under some regularity restrictions on the shrinking sets). In particular, this was done for shrinking cusp neighborhoods of homogenous spaces [Sul82, KM99, GS11, AM09, KM12, AM17, Yu17], and more generally for shrinking metric balls in a metric space [CK01, Dol04, Gal07, KZ17].

Recently, in [Kel17], the first author introduced a new method for attacking this problem for discrete time homogenous flows on (the frame bundle of) finite volume hyperbolic manifolds. This method works for any monotone family of shrinking targets in the hyperbolic manifold, and applies also for unipotent flows with arbitrarily slow polynomial mixing rate. In this paper we adapt this method to treat the general case of discrete time homogenous flows on a homogenous space $\mathcal{X} = \Gamma\backslash G$ with G a connected semisimple Lie group with finite center and no compact factors, and Γ an irreducible lattice.

1.1. **General setup, terminology, and notations.** Let G denote a connected semisimple Lie group with finite center and no compact factors, let $\Gamma \leq G$ be an irreducible lattice, and let μ denote the G-invariant probability measure on $\mathcal{X} = \Gamma\backslash G$, coming from the Haar measure of G. We fix once and for all a maximal compact subgroup $K \leq G$ and denote by $\mathcal{H} = G/K$ the corresponding symmetric space. We say that a subset $B \subseteq \mathcal{X}$ is spherical if it is invariant under the right action of K and we identify spherical sets as subsets of the locally symmetric space $\Gamma\backslash\mathcal{H}$.

One-parameter flows on $\mathcal{X} = \Gamma \backslash G$ are given by the right action of one-parameter subgroups of G. Explicitly, the one-parameter group generated by an element, X_0, in the Lie algebra, $\mathfrak{g} = \text{Lie}(G)$, is given by $\{h_t = \exp(tX_0) : t \in \mathbb{R}\}$. The corresponding discrete time flow is then given by the action of the discrete subgroup $H = \{h_m\}_{m \in \mathbb{Z}}$. We will always assume that the subgroup is unbounded and recall that, by Moore's ergodicity theorem, the action of any unbounded subgroup is ergodic and mixing.

We say that a family $\{B_t\}_{t>0}$ is a monotone family of shrinking targets if $B_t \subseteq B_s$ when $t \geq s$ and $\mu(B_t) \to 0$, and we say it is a family of spherical shrinking targets if all sets are spherical. Given an unbounded discrete time one-parameter flow $\{h_m\}_{m \in \mathbb{Z}}$, following [CK01], we say that a sequence of sets $\{B_m\}_{m \in \mathbb{N}}$ is *Borel-Cantelli* (BC) for $\{h_m\}_{m \in \mathbb{Z}}$, if for a.e. $x \in \mathcal{X}$ the set $\{m \in \mathbb{N} \mid xh_m \in B_m\}$ is unbounded, and we say it is *strongly Borel-Cantelli* (sBC) if for a.e. $x \in \mathcal{X}$

$$\lim_{m \to \infty} \frac{\#\{1 \leq j \leq m : xh_j \in B_j\}}{\sum_{1 \leq j \leq m} \mu(B_j)} = 1. \tag{1.1}$$

We say that a collection of sets is (strongly) Borel-Cantelli (resp. monotone sBC) if any sequence (resp. monotone sequence) from this collection with divergent measure is (strongly) Borel-Cantelli.

In what follows we adopt the notation $A(t) \ll B(t)$ or $A(t) = O(B(t))$ to indicate that there is a constant $c > 0$ such that $A(t) \leq cB(t)$, and we write $A(t) \asymp B(t)$ to indicate that $A(t) \ll B(t) \ll A(t)$. The implied constants may always depend on the group, G, the lattice, Γ, and the flow that we think of as fixed. We will use subscripts to indicate the dependance of the implied constants on any additional parameters.

1.2. Logarithm laws.

Our first result establishes a logarithm law for the first hitting time function (see [GK07] for the relation between such logarithm laws and Borel-Cantelli properties). Given an unbounded discrete time flow and a subset $B \subseteq \mathcal{X}$ the first hitting time function is defined for $x \in \mathcal{X}$ by

$$\tau_B(x) = \min\{m \in \mathbb{N} : xh_m \in B\}. \tag{1.2}$$

Theorem 1.1. *Assume that either G has property (T), or that G is of real rank one. Let $\{B_t\}_{t>0}$ denote a monotone family of spherical shrinking targets in $\mathcal{X} = \Gamma \backslash G$. Let $\{h_m\}_{m \in \mathbb{Z}}$ denote an unbounded discrete time flow on $\mathcal{X}$. Then for a.e. $x \in \mathcal{X}$*

$$\lim_{t \to \infty} \frac{\log(\tau_{B_t}(x))}{-\log(\mu(B_t))} = 1. \tag{1.3}$$

Remark 1. Since all simple groups without property (T) are of real rank one, Theorem 1.1 holds for all noncompact connected simple groups with finite center. In fact, the only groups of higher rank without property (T) are (almost) products $G = \prod G_i$ with at least one of the factors a simple rank one group without property (T). In this case, the same result still holds, unless the flow is a unipotent flow on one of those rank one factors. Even then, according to the congruence subgroup conjecture (see e.g. [Rap92, Rag04]), for higher rank groups the only irreducible lattices are congruence lattices. For congruence lattices, if we further assume the Selberg-Ramanujan Conjecture (see [Sha04, Sar05]), the resulting bounds on decay of matrix coefficients are sufficient to obtain the same result also in these cases (in fact, with the exception of G_i locally isomorphic to $\text{SL}_2(\mathbb{R})$ for some i the unconditional bounds

towards the Selberg-Ramanujan Conjecture obtained in [KS03, BB11, BC13, BC17] are already sufficient). Hence, conditional on these conjectures our result holds in full generality.

Since spherical sets in the homogenous space, $\Gamma\backslash G$, can be naturally identified with subsets of the locally symmetric space, $\Gamma\backslash\mathcal{H}$, our result holds for any monotone sequence of shrinking targets in $\Gamma\backslash\mathcal{H}$. A special case of such a sequence, that has received much attention, occurs when $\Gamma\backslash\mathcal{H}$ is not compact and the shrinking sets are cusp neighborhoods [Sul82, KM99, AM09, KM12, AM17, Yu17]. For these problems, logarithm laws can be expressed in terms of a distance function measuring how far out a point is in the cusp. Explicitly, given a continuous distance function on $\Gamma\backslash\mathcal{H}$ that we lift to a $K \times K$-invariant function $d(\cdot, \cdot)$ on $\Gamma\backslash G \times \Gamma\backslash G$, we can define spherical shrinking cusp neighborhoods of $\mathcal{X}$ by

$$B_t = \{x \in \mathcal{X} : d(x, x_0) > t\},$$

where $x_0 \in \mathcal{X}$ is some fixed base point. We assume that the measure of these cusp neighborhoods decay exponentially with rate $\varkappa > 0$ in the sense that

$$\lim_{t\to\infty} \frac{-\log\mu(B_t)}{t} = \varkappa. \tag{1.4}$$

In particular, this holds when the distance function[1] is determined by a right G-invariant, bi K-invariant Riemannian metric on G as in [KM99], or more generally by a norm-like pseudometric on G as in [AM17]. For any such distance function, a consequence of Theorem 1.1 applied to this family of shrinking targets is the following.

Corollary 1.2. *For G a connected semisimple Lie group with finite center and no compact factors, and $\Gamma \leq G$ an irreducible lattice, for any unbounded one-parameter flow $\{h_t\}_{t\in\mathbb{R}}$ on $\mathcal{X} = \Gamma\backslash G$, for a.e. $x \in \mathcal{X}$*

$$\limsup_{t\to\infty} \frac{d(xh_t, x_0)}{\log t} = \frac{1}{\varkappa}. \tag{1.5}$$

For diagonalizable flows, the logarithm law for cusp excursions (1.5) was established by Sullivan [Sul82] for $G = \mathrm{SO}_0(d, 1)$, and by Kleinbock and Margulis [KM99] in general. For unipotent flows, Athreya and Margulis [AM17] showed that there is some $c \in (0, 1]$ so that for a.e. $x \in \mathcal{X}$, $\limsup_{t\to\infty} \frac{d(xh_t, x_0)}{\log t} = \frac{c}{\varkappa}$, and raised the question if it is always the case that the constant $c = 1$. This question was previously answered affirmatively in some specific examples, such as unipotent flows on the space of lattices and on hyperbolic manifolds [AM09, KM12, Yu17]. Our result settles this problem completely and gives an affirmative answer to their question in general.

Remark 2. We emphasized this particular application for logarithm laws for cusp excursions because this problem has received considerable attention and was completely resolved by our results. We note, however, that our results apply in a much more general setting. For example, by taking the shrinking sets to be metric balls in $\Gamma\backslash\mathcal{H}$ we get similar logarithm laws, generalizing the results of [Mau06, KZ17, Kel17] on hyperbolic manifolds. Alternatively we can also take our shrinking sets to be shrinking neighborhoods of closed geodesics, or immersed submanifolds as done in [HP10] for hyperbolic manifolds, or, if one is so inclined, even take a sequence of cantor like sets (of positive measure) in $\Gamma\backslash\mathcal{H}$ shrinking to a point.

---

[1]While these distance function are defined on $\Gamma\backslash G$ and are not necessarily lifts from a distance function on $\Gamma\backslash\mathcal{H}$, the corresponding cusp neighborhoods can always be approximated by spherical cusp neighborhoods as shown in Corollary 5.3 below.

1.3. Summable decay of matrix coefficients.

A key ingredient in the proof of Theorem 1.1 is a uniform rate on decay of spherical matrix elements under one-parameter flows. We say $\varphi \in L^2(\Gamma \backslash G)$ is spherical if it is invariant under the action of K, and we identify the space of spherical functions with $L^2(\Gamma \backslash \mathcal{H}) = L^2(\Gamma \backslash G / K)$. Let π denote the regular representation of G on $L^2(\Gamma \backslash G)$ and let $L_0^2(\Gamma \backslash G)$ denote the space orthogonal to the constant functions. For any $\psi, \varphi \in L^2(\Gamma \backslash G)$ and $g \in G$ the corresponding matrix element is given by

$$\langle \pi(g)\varphi, \psi \rangle = \int_{\Gamma \backslash G} \varphi(xg)\overline{\psi(x)}d\mu(x).$$

By using the results of Oh [Oh98] on effective property (T), we show the following uniform bound on decay of matrix coefficients for one-parameter flows.

Theorem 1.3. *Assume that G has property (T) and let $\{h_t\}_{t \in \mathbb{R}}$ denote an unbounded one-parameter subgroup. For all sufficiently small $\epsilon > 0$, for any spherical $\varphi, \psi \in L_0^2(\Gamma \backslash G)$ and for all $|t| \geq 1$ we have*

$$|\langle \pi(h_t)\varphi, \psi \rangle| \ll_\epsilon \frac{\|\varphi\|_2 \|\psi\|_2}{|t|^{1-\epsilon}}.$$

It is remarkable that this uniform rate, which is exactly the rate needed for Theorem 1.1, can be obtained for all one-parameter flows for all semisimple groups with property (T). The rate of decay here is ϵ away from being summable, and, while this is good enough to establish a logarithm law, for other applications it just falls short. In many cases, however, we can establish a slightly better rate. In order to distinguish these cases we use the following terminology.

Definition 1.4. We say that a one-parameter flow on $\Gamma \backslash G$ has *Summable Decay (or SD)* if there is $\eta > 1$ such that for any spherical $\varphi, \psi \in L_0^2(\Gamma \backslash G)$, for all $|t| \geq 1$ we have

$$|\langle \pi(h_t)\varphi, \psi \rangle| \ll_\eta \frac{\|\varphi\|_2 \|\psi\|_2}{|t|^\eta}.$$

For SD flows we can show the stronger result implying that the bound coming from Borel-Cantelli is sharp.

Theorem 1.5. *For any discrete time one-parameter SD flow on $\Gamma \backslash G$, the collection of spherical subsets is strongly Borel-Cantelli.*

Remark 3. Although we state our results (Theorems 1.1 and 1.5) for the special case of homogeneous spaces, we note that our argument works in greater generality. Explicitly, given a dynamical system $(\mathcal{X}, \mu)$ with an ergodic action T and a family of functions $\mathcal{F} \subseteq L^2(\mathcal{X})$ such that for some $\eta > 1$ and for all $\psi, \phi \in \mathcal{F}$ and all $k \geq 1$

$$\langle \psi \circ T^k, \phi \rangle = \int_\mathcal{X} \psi(x)d\mu(x) \int_\mathcal{X} \phi(x)d\mu(x) + O_\eta\left(\|\psi\|\|\phi\|k^{-\eta}\right),$$

then following the same arguments, implies that the collection of sets with indicator functions in $\mathcal{F}$ is strongly Borel-Cantelli. Similarly, if we replace the condition $\eta > 1$ by $\eta = 1 - \epsilon$ for any $\epsilon > 0$, then the same argument shows that any monotone family of shrinking targets with indicator functions in $\mathcal{F}$ satisfies the logarithm law (1.3).

Following this result it would be useful to categorize precisely which one-parameter homogenous flows are SD. For diagonalizable flows, the exponential decay of matrix coefficients clearly imply summable decay. For unipotent flows, the decay is polynomial and we don't always have summable decay. When the group G is a simple Lie group of rank ≥ 2, the following result gives explicit conditions for when a one-parameter flow has summable decay, in terms of the restricted root system of G and the adjoint representation of the generator of the flow (see section 3.1 for more details).

Theorem 1.6. *Assume that G is simple with real rank ≥ 2. If the restricted root system is of type $B_n(n \geq 4), D_n(n \geq 4), E_6, E_7, E_8$ or F_4, then all unbounded one-parameter flows on $\Gamma \backslash G$ are SD. When the restricted root system is not of the above types, any unipotent one-parameter flow, $h_t = \exp(tX_0)$, with $\operatorname{ad}(X_0)^3 \neq 0$ is SD.*

Remark 4. For example, on $\Gamma \backslash \operatorname{SL}_3(\mathbb{R})$ the unipotent flow given by $h_t = \left(\begin{smallmatrix} 1 & 2t & 2t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{smallmatrix} \right)$ is SD, while the flow given by $h_t = \left(\begin{smallmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix} \right)$ is not SD.

Remark 5. For some applications it is useful to know the precise rate of decay for matrix coefficients along unipotent flows, and we remark that our method gives the following very explicit rate: When $h_t = \exp(tX_0)$ is unipotent, let l be the largest positive integer such that $\operatorname{ad}(X_0)^l \neq 0$. Then our method shows that for any spherical $\varphi, \psi \in L_0^2(\Gamma \backslash G)$, the matrix coefficients $\langle \pi(h_t)\varphi, \psi \rangle$ are bounded by $O_\epsilon \left(\frac{\|\varphi\|_2 \|\psi\|_2}{|t|^{l(1-\epsilon)}} \right)$ when the restricted root system is $B_n(n \geq 4), D_n(n \geq 4), E_6, E_7, E_8$ or F_4 and by $O_\epsilon \left(\frac{\|\varphi\|_2 \|\psi\|_2}{|t|^{l(1-\epsilon)/2}} \right)$ otherwise.

Remark 6. We can also characterize SD flows when G is semisimple with property (T). In such a case the flow will be SD unless the flow is essentially trivial except in one of the factors, with the restriction of the flow to this factor not SD.

1.4. **Groups of real rank one.** For a group of real rank one with property (T), every discrete time homogenous flow is SD, and hence the collection of spherical sets is sBC. For rank one groups without property (T), the question if all one-parameter flows on $\Gamma \backslash G$ are SD, depends on the spectral gap of Γ, that is, the size of the smallest non-trivial eigenvalue of the Laplacian on $L^2(\Gamma \backslash \mathcal{H})$. When the spectral gap is sufficiently large all flows are SD (see Corollary 3.8 below). However, there are also examples with a small spectral gap, for which unipotent flows may not be SD. Nevertheless, using a spectral decomposition it is still possible to prove that the collection of spherical sets is monotone Borell-Cantelli. This was done in [Kel17] for $G = \operatorname{SO}_0(d+1, 1)$ with $d \geq 2$ and the proof is similar for G locally isomorphic to $\operatorname{SU}(d, 1)$. Explicitly, for these groups we have the following.

Theorem 1.7. *Let G be locally isomorphic to $\operatorname{SO}(d+1, 1)$ or $\operatorname{SU}(d, 1)$ with $d \geq 2$. Then for any unbounded discrete time one-parameter flow on $\Gamma \backslash G$, the collection of spherical subsets is monotone Borel-Cantelli.*

Remark 7. In fact, our proof gives something stronger. For a sequence of spherical sets, $\{B_m\}_{m \in \mathbb{N}}$ (not necessarily monotone) with $\sum_m \mu(B_m) = \infty$, if we assume that $\{m\mu(B_m)\}_{m \in \mathbb{N}}$ is bounded then we can show this sequence is strongly Borel-Cantelli. If the sequence is monotone and $\{m\mu(B_m)\}_{m \in \mathbb{N}}$ is unbounded, we can show instead that there is a subsequence

$\{m_j\}$ such that for a.e. $x \in \mathcal{X}$

$$\lim_{j\to\infty} \frac{\#\{1 \le i \le m_j : xh_i \in B_{m_j}\}}{m_j \mu(B_{m_j})} = 1.$$

Remark 8. The case when G is locally isomorphic to $\mathrm{SO}(2,1)$ was considered in [Kel17]. In this case one needs some additional assumptions on the shrinking rate of family $\{B_m\}_{m\in\mathbb{N}}$ to show that it is BC for a unipotent flow.

1.5. **Orbits eventually always hitting.** Assume now that our flow is either SD or that G is of real rank one, so that any monotone sequence of spherical shrinking targets $\{B_m\}_{m\in\mathbb{N}}$ is BC. In such cases we wish to study the subtler point, of whether the finite orbits

$$xH_m^+ = \{xh_j : 1 \le j \le m\}, \tag{1.6}$$

eventually always hit or miss the targets B_m. Using the terminology introduced in [Kel17] we say that an orbit of a point $x \in \mathcal{X}$ is *eventually always hitting* if $xH_m^+ \cap B_m \ne \emptyset$ for all sufficiently large m, and *eventually always missing* if $xH_m^+ \cap B_m = \emptyset$ for all sufficiently large m. We denote by $\mathcal{A}_{\mathrm{ah}}$ and $\mathcal{A}_{\mathrm{am}}$ the set of points with such orbits respectively. For the eventually always missing set, we have the dichotomy given by the dynamical Borel-Cantelli Lemma: if $\sum_{m=1}^{\infty} \mu(B_m) < \infty$ then $\mathcal{A}_{\mathrm{am}}$ is of full measure and otherwise $\mathcal{A}_{\mathrm{am}}$ is a null set.

The eventually always hitting set, $\mathcal{A}_{\mathrm{ah}}$, is also either a null set or a set of full measure, but in this case we are not able to establish such an explicit dichotomy. Here we have the following partial result (extending the result of [Kel17] dealing with the case of $G = \mathrm{SO}_0(d+1,1)$).

Theorem 1.8. *Fix an unbounded discrete time one-parameter flow on $\Gamma \backslash G$ and assume that either the flow is SD, or that G is of real rank one, not locally isomorphic to $\mathrm{SO}(2,1)$. Let $\{B_m\}_{m\in\mathbb{N}}$ denote a monotone sequence of spherical shrinking targets. If*

$$\sum_{j=0}^{\infty} \frac{1}{2^j \mu(B_{2^j})} < \infty, \tag{1.7}$$

Then $\mathcal{A}_{\mathrm{ah}}$ is of full measure. Moreover, if we further assume that $\mu(B_{2m}) \asymp \mu(B_m)$, then for a.e. $x \in \mathcal{X}$, for all sufficiently large m

$$\#\{1 \le j \le m : xh_j \in B_m\} \asymp m\mu(B_m).$$

Remark 9. In the other direction, it was shown in [Kel17] that for any ergodic one-parameter flow, for any monotone sequence, $\{B_m\}_{m\in\mathbb{N}}$, of shrinking targets, if there is $c < 1$ such that the set $\{m : m\mu(B_m) \le c\}$ is unbounded then $\mathcal{A}_{\mathrm{ah}}$ is a null set. In particular, if we assume that $\mu(B_m)$ decays polynomially in the sense that $\mu(B_m) \asymp m^{-\eta}$ for some fixed η, then Theorem 1.8, implies that $\mathcal{A}_{\mathrm{ah}}$ is a set of full measure when $\eta < 1$, and a null set when $\eta > 1$. In this case, however, the same result already follows from Theorem 1.1 and hence holds also for flows that are not SD.

Remark 10. Finally we remark that in a recent work of Kleinbock and Wadleigh [KW17], they give examples of dynamical systems where they manage to get an explicit dichotomy for $\mathcal{A}_{\mathrm{ah}}$, and the dichotomy depends exactly on the convergence and divergence of the series of the type (1.7).

2. Shrinking target problems

We start by taking a closer look on shrinking target problems for a general ergodic $\mathbb{Z}$-action given by the action of a group, $H = \{h_m\}_{m \in \mathbb{Z}}$, on a probability space $(\mathcal{X}, \mu)$. Though we will later apply these results for the case of $\mathcal{X} = \Gamma \backslash G$, in this section we will not assume anything about the space $\mathcal{X}$ and the flow other than ergodicity.

2.1. **The hitting time problem.** Fix a positive integer i throughout this subsection. Instead of the first hitting time function given in (1.2), we consider the more general ith hitting time function: For any $x \in \mathcal{X}$ and any set $B \subset \mathcal{X}$ let

$$\tau_B^i(x) := \min\{m \in \mathbb{N} \mid |xH_m^+ \cap B| = i\} \tag{2.1}$$

measure the time needed for the orbit of x to enter the target B for the ith time (here xH_m^+ is as in (1.6)). In preparation for Theorem 1.1, our goal in this section is to give sufficient conditions on a monotone family of shrinking targets, $\{B_t\}_{t>0}$, implying that for a.e. $x \in \mathcal{X}$

$$\lim_{t \to \infty} \frac{\log \tau_{B_t}^i(x)}{-\log \mu(B_t)} = 1. \tag{2.2}$$

For any $0 < \delta < \frac{1}{2}$ we define the sets

$$\mathcal{L}_\delta^i := \{x \in \mathcal{X} \mid \liminf_{t \to \infty} \frac{\log \tau_{B_t}^i(x)}{-\log \mu(B_t)} \leq 1 - 2\delta\},$$

$$\mathcal{U}_\delta^i := \{x \in \mathcal{X} \mid \limsup_{t \to \infty} \frac{\log \tau_{B_t}^i(x)}{-\log \mu(B_t)} \geq 1 + 2\delta\},$$

and note that the condition $\mu(\mathcal{U}_\delta^i) = 0$ for all $0 < \delta < \frac{1}{2}$ implies that for a.e. $x \in \mathcal{X}$

$$\limsup_{t \to \infty} \frac{\log \tau_{B_t}^i(x)}{-\log \mu(B_t)} \leq 1 \tag{2.3}$$

and similarly the condition that $\mu(\mathcal{L}_\delta^i) = 0$ for all $0 < \delta < \frac{1}{2}$ implies that for a.e. $x \in \mathcal{X}$

$$\liminf_{t \to \infty} \frac{\log \tau_{B_t}^i(x)}{-\log \mu(B_t)} \geq 1. \tag{2.4}$$

Now, for any integer $m \geq 1$ and measurable subset $B \subset \mathcal{X}$, define the hitting set

$$\mathcal{H}_{m,B}^i := \{x \in \mathcal{X} \mid \tau_B^i(x) \leq m\} \tag{2.5}$$

and its complement

$$\mathcal{M}_{m,B}^i := \{x \in \mathcal{X} \mid \tau_B^i(x) > m\}. \tag{2.6}$$

Note that $x \in \mathcal{H}_{m,B}^i$ (resp. $x \in \mathcal{M}_{m,B}^i$) means the first m steps in the orbit of x hit the set B at least (resp. strictly less than) i times. For any $0 < \delta < \frac{1}{2}$ and any $t > 0$ let

$$m_\delta^\pm(t) = \lfloor \frac{1}{\mu(B_t)^{(1 \pm \delta)}} \rfloor.$$

If $x \in \mathcal{L}_\delta^i$, then there exists an unbounded sequence of t such that $\frac{\log \tau_{B_t}^i(x)}{-\log \mu(B_t)} < 1 - \delta$, or equivalently, $\tau_{B_t}^i(x) < \frac{1}{\mu(B_t)^{(1-\delta)}}$. Since $\tau_{B_t}^i(x)$ is integer-valued, this implies that $\tau_{B_t}^i(x) \leq \lfloor \frac{1}{\mu(B_t)^{(1-\delta)}} \rfloor$ for unbounded values of t. Hence, $x \in \mathcal{L}_\delta^i$ implies that $x \in \mathcal{H}_{m_\delta^-(t), B_t}^i$ for unbounded values

7

of t. Similarly, $x \in \mathcal{U}_\delta^i$ implies that $x \in \mathcal{M}_{m_\delta^+(t),B_t}^i$ for unbounded values of t. Let $\mathcal{N}$ denote the set of integers $\ell \geq 0$ such that $\{\mu(B_t) \mid t > 0\} \cap [\frac{1}{2^\ell}, \frac{1}{2^{\ell+1}})$ is nonempty. Note that $\mathcal{N}$ is unbounded since $\lim_{t \to \infty} \mu(B_t) = 0$. Thus

$$\mathcal{L}_\delta^i \subset \bigcap_{m=0}^\infty \bigcup_{\substack{\ell \geq m \\ \ell \in \mathcal{N}}} \bigcup_{\frac{1}{2^{\ell+1}} \leq \mu(B_t) < \frac{1}{2^\ell}} \mathcal{H}_{m_\delta^-(t),B_t}^i$$

and

$$\mathcal{U}_\delta^i \subset \bigcap_{m=0}^\infty \bigcup_{\substack{\ell \geq m \\ \ell \in \mathcal{N}}} \bigcup_{\frac{1}{2^{\ell+1}} \leq \mu(B_t) < \frac{1}{2^\ell}} \mathcal{M}_{m_\delta^+(t),B_t}^i.$$

For each $\ell \in \mathcal{N}$, let

$$\overline{B}_\ell := \bigcup_{\frac{1}{2^{\ell+1}} \leq \mu(B_t) < \frac{1}{2^\ell}} B_t \quad \text{and} \quad \underline{B}_\ell := \bigcap_{\frac{1}{2^{\ell+1}} \leq \mu(B_t) < \frac{1}{2^\ell}} B_t.$$

Since $\{B_t\}_{t>0}$ is monotone,

$$\frac{1}{2^{\ell+1}} \leq \mu\left(\underline{B}_\ell\right) \leq \mu\left(\overline{B}_\ell\right) \leq \frac{1}{2^\ell}.$$

Moreover, for any t such that $\mu(B_t) \in [\frac{1}{2^{\ell+1}}, \frac{1}{2^\ell})$,

$$\lfloor 2^{\ell(1\pm\delta)} \rfloor < m_\delta^\pm(t) \leq \lfloor 2^{(\ell+1)(1\pm\delta)} \rfloor.$$

By construction, for any $m \leq m'$ and $B \subset B'$, $\mathcal{H}_{m,B}^i \subset \mathcal{H}_{m',B'}^i$ and $\mathcal{M}_{m,B}^i \supset \mathcal{M}_{m',B'}^i$. Hence for any $\ell \in \mathcal{N}$, we have

$$\bigcup_{\frac{1}{2^{\ell+1}} \leq \mu(B_t) < \frac{1}{2^\ell}} \mathcal{H}_{m_\delta^-(t),B_t}^i \subset \mathcal{H}_{\lfloor 2^{(\ell+1)(1-\delta)} \rfloor, \overline{B}_\ell}^i \quad \text{and} \quad \bigcup_{\frac{1}{2^{\ell+1}} \leq \mu(B_t) < \frac{1}{2^\ell}} \mathcal{M}_{m_\delta^+(t),B_t}^i \subset \mathcal{M}_{\lfloor 2^{\ell(1+\delta)} \rfloor, \underline{B}_\ell}^i.$$

Combining the above arguments gives the following:

Lemma 2.1. *For a monotone family, $\{B_t\}_{t>0}$, of shrinking targets in $\mathcal{X}$. If for all sufficiently small $\delta > 0$*

$$\sum_{\ell \in \mathcal{N}} \mu\left(\mathcal{H}_{\lfloor 2^{(\ell+1)(1-\delta)} \rfloor, \overline{B}_\ell}^i\right) < \infty, \tag{2.7}$$

then the lower bound (2.4) holds for a.e. $x \in \mathcal{X}$. Similarly, if for all sufficiently small $\delta > 0$

$$\sum_{\ell \in \mathcal{N}} \mu\left(\mathcal{M}_{\lfloor 2^{\ell(1+\delta)} \rfloor, \underline{B}_\ell}^i\right) < \infty, \tag{2.8}$$

then the upper bound (2.3) holds for a.e. $x \in \mathcal{X}$.

In the following sections we shall show that (2.8) holds for one-parameter flows on homogenous spaces. The condition (2.7), on the other hand, holds in general without any extra assumption on the flow or the shrinking targets. Hence, the following lower bound holds in general.

8

Lemma 2.2. *Let $\{B_t\}_{t>0}$ be a monotone family of shrinking targets in $\mathcal{X}$. Then (2.7) holds for all $0 < \delta < \frac{1}{2}$. In particular, for a.e. $x \in \mathcal{X}$*

$$\liminf_{t\to\infty} \frac{\log \tau^i_{B_t}(x)}{-\log \mu(B_t)} \geq 1.$$

Proof. For any integer $m \geq 1$ and any measurable set $B \subset \mathcal{X}$, we first show the trivial estimate $\mu\left(\mathcal{H}^i_{m,B}\right) \leq m\mu(B)$. By definition $\mathcal{H}^i_{m,B} = \bigcup_{k=i}^m \{x \in \mathcal{X} \mid \tau^i_B(x) = k\}$ and by minimality $\{x \in \mathcal{X} \mid \tau^i_B(x) = k\} \subset \{x \in \mathcal{X} \mid xh_k \in B\} = Bh_{-k}$. Hence indeed $\mu\left(\mathcal{H}^i_{m,B}\right) \leq m\mu(B)$. For each $\ell \in \mathcal{N}$ applying this estimate to $\mathcal{H}^i_{\lfloor 2^{(\ell+1)(1-\delta)}\rfloor, \overline{B}_\ell}$ we get

$$\mu\left(\mathcal{H}^i_{\lfloor 2^{(\ell+1)(1-\delta)}\rfloor, \overline{B}_\ell}\right) \leq \lfloor 2^{(\ell+1)(1-\delta)}\rfloor\mu(\overline{B}_\ell) \ll 2^{-\delta\ell},$$

where for the last inequality we used $\mu(\overline{B}_\ell) \leq \frac{1}{2^\ell}$. Hence

$$\sum_{\ell \in \mathcal{N}} \mu\left(\mathcal{H}^i_{\lfloor 2^{(\ell+1)(1-\delta)}\rfloor, \overline{B}_\ell}\right) \ll \sum_{\ell=0}^{\infty} 2^{-\delta\ell} < \infty. \qquad \square$$

2.2. Orbits eventually always hitting.

Given a monotone sequence of shrinking targets $\{B_m\}_{m\in\mathbb{N}}$, we defined the eventually always hitting set to be

$$\mathcal{A}_{\mathrm{ah}} = \left\{x \in \mathcal{X} \mid xH^+_m \cap B_m \neq \emptyset \text{ for all sufficiently large } m\right\}.$$

In [Kel17, Proposition 12 and Lemmas 13,14] the first author gave sufficient conditions implying that $\mathcal{A}_{\mathrm{ah}}$ is a null or co-null set. For the readers convenience we summarize these results in the following:

Lemma 2.3. *Given a measure preserving ergodic $\mathbb{Z}$-action of a group H on a probability space $(\mathcal{X}, \mu)$, and a monotone sequence, $\{B_m\}_{m\in\mathbb{N}}$, of shrinking targets.*
  (1) If along some subsequence, we have that $m_j\mu(B_{m_j}) \leq c < 1$, then $\mu(\mathcal{A}_{\mathrm{ah}}) = 0$.
  (2) If $\sum_j \mu(\mathcal{M}^1_{2^{j-1}, B_{2^j}}) < \infty$ then $\mu(\mathcal{A}_{\mathrm{ah}}) = 1$. If in addition also $\mu(B_{2^j}) \asymp \mu(B_{2^{j+1}})$ and $\sum_j \mu(\mathcal{M}^1_{2^{j+1}, B_{2^j}}) < \infty$ then for a.e. $x \in \mathcal{X}$, for all sufficiently large m

$$\#(xH^+_m \cap B_m) \asymp m\mu(B_m).$$

The results of [Kel17] were given for a more general setting of $\mathbb{Z}^d$-actions. For $\mathbb{Z}$-actions we also have the following lemma stating that $\mathcal{A}_{\mathrm{ah}}$ is always either null or co-null.

Lemma 2.4. *Let H be a measure preserving ergodic $\mathbb{Z}$-action on a probability space $(\mathcal{X}, \mu)$ and let $\{B_m\}_{m\in\mathbb{N}}$ denote a monotone sequence of shrinking targets. Then $\mathcal{A}_{\mathrm{ah}}$ has measure either zero or one.*

Proof. Suppose $\mu(\mathcal{A}_{\mathrm{ah}}) > 0$, we want to show that $\mu(\mathcal{A}_{\mathrm{ah}}) = 1$. Define the set

$$\mathcal{A}' := \{x \in \mathcal{A}_{\mathrm{ah}} \mid xh_k \in \mathcal{A}_{\mathrm{ah}} \text{ for all } k \geq 1\}.$$

It is clear that $\mathcal{A}'h_1 \subset \mathcal{A}'$. Hence by ergodicity it suffices to show that $\mu(\mathcal{A}') > 0$. To show this, we prove that $\mu(\mathcal{A}_{\mathrm{ah}}\backslash\mathcal{A}') = 0$. Note that

$$\mathcal{A}_{\mathrm{ah}}\backslash\mathcal{A}' = \{x \in \mathcal{A}_{\mathrm{ah}} \mid xh_k \notin \mathcal{A}_{\mathrm{ah}} \text{ for some } k \geq 1\} = \bigcup_{k=1}^{\infty} \{x \in \mathcal{A}_{\mathrm{ah}} \mid xh_k \notin \mathcal{A}_{\mathrm{ah}}\}.$$

9

Hence, it suffices to show that for any fixed k, the set $\{x \in \mathcal{A}_{\mathrm{ah}} | \, xh_k \notin \mathcal{A}_{\mathrm{ah}}\}$ has measure zero. We note that $x \in \mathcal{A}_{\mathrm{ah}}$ means that for all sufficiently large m, there exists some $1 \le j(m) \le m$ such that $xh_{j(m)} \in B_m$. If for all sufficiently large m we can take $j(m) > k$, then we have $xh_k h_{j(m)-k} \in B_m$ with $1 \le j(m) - k < m$, thus $xh_k \in \mathcal{A}_{\mathrm{ah}}$. Hence, if $x \in \mathcal{A}_{\mathrm{ah}}$ but $xh_k \notin \mathcal{A}_{\mathrm{ah}}$, then there exist infinitely many values of m for which there exists some $1 \le j(m) \le k$ such that $xh_{j(m)} \in B_m$ but $xh_i \notin B_m$ for all $k < i \le m$. In particular, for such x, there exists some $1 \le j \le k$ such that $xh_j \in B_m$ for infinitely many values of m. Since $\{B_m\}_{m \in \mathbb{N}}$ is monotone, this implies that $xh_j \in \bigcap_{m=1}^{\infty} B_m$. Hence the set

$$\{x \in \mathcal{A}_{\mathrm{ah}} \mid xh_k \notin \mathcal{A}_{\mathrm{ah}}\} \subset \bigcup_{j=1}^{k} \left(\bigcap_{m=1}^{\infty} B_m \right) h_{-j}$$

is of measure zero. $\qquad \square$

2.3. **Dynamical Borel-Cantelli and Quasi-independence.** The second part of the classical Borel-Cantelli lemma requires pairwise independence. The following argument, going back to Schmidt, shows that a weaker condition of quasi-independence is enough. Explicitly, let $\mathcal{F} = \{f_m\}_{m \in \mathbb{N}}$ denote a sequence of functions on the probability space $(\mathcal{X}, \mu)$ taking values in $[0, 1]$. For $m \in \mathbb{N}$ let $E_m^{\mathcal{F}} = \sum_{1 \le j \le m} \mu(f_j)$ and $S_m^{\mathcal{F}}(x) = \sum_{1 \le j \le m} f_j(x)$. We then have:

Lemma 2.5. [Spr79, Chapter I, Lemma 10] *Assuming that for some constant $C > 0$, for all $m, n \in \mathbb{N}$*

$$\int_{\mathcal{X}} \left(\sum_{i=m}^{n} f_i(x) - \sum_{i=m}^{n} \mu(f_i) \right)^2 d\mu(x) \le C \sum_{i=m}^{n} \mu(f_i), \tag{2.9}$$

then for any $\epsilon > 0$ for a.e. $x \in \mathcal{X}$

$$S_m^{\mathcal{F}}(x) = E_m^{\mathcal{F}} + O_\epsilon \left(\sqrt{E_m^{\mathcal{F}}} \log^{\frac{3}{2}+\epsilon}(E_m^{\mathcal{F}}) \right).$$

In particular, if $\left\{ E_m^{\mathcal{F}} \right\}_{m \in \mathbb{N}}$ is unbounded, then for a.e. $x \in \mathcal{X}$

$$\lim_{m \to \infty} \frac{S_m^{\mathcal{F}}(x)}{E_m^{\mathcal{F}}} = 1.$$

Given a $\mathbb{Z}$-action of a group $H = \{h_m\}_{m \in \mathbb{Z}}$ on $\mathcal{X}$, and a sequence of targets, $\{B_m\}_{m \in \mathbb{N}}$, let $f_j(x) = \chi_{B_j}(xh_j)$ so that $S_m^{\mathcal{F}}(x) = \#\{1 \le j \le m \mid xh_j \in B_j\}$. If the events, $xh_j \in B_j$, were pairwise independent, then the left hand side of (2.9) would be zero. This result shows that the weaker quasi-independence bound (2.9), is enough to show that the sequence $\{B_m\}_{m \in \mathbb{N}}$ is sBC for the flow.

3. Decay of matrix coefficients

We now collect the needed results on the decay of matrix coefficients for representations of semisimple Lie groups, and use them to prove Theorem 1.3, giving a uniform bound for decay of matrix coefficients for all one-parameter flows, as well as Theorem 1.6 classifying SD flows. We first give some background on semisimple Lie groups, their Lie algebras and restricted root systems. We then focus on simple groups and treat the cases of simple groups

of real rank one and simple groups of higher rank separately. Finally, we combine our results on simple groups to handle the general case of semisimple groups with property (T).

3.1. **Preliminaries on Lie groups.**

Let G be a connected semisimple Lie group with finite center and no compact factors, and let $\mathfrak{g}$ denote its Lie algebra. Fix a Cartan involution, θ, on $\mathfrak{g}$ and let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

denote the corresponding Cartan decomposition, where $\mathfrak{k}$ is the $+1$ eigenspace and $\mathfrak{p}$ is the -1 eigenspace of θ. Let $K \leq G$ denote the maximal compact subgroup with Lie algebra $\mathfrak{k}$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, and $\mathfrak{m}$ the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Denote by $\Phi = \Phi_{\mathbb{R}}(\mathfrak{a}, \mathfrak{g})$ the set of restricted roots with respect to the pair $(\mathfrak{a}, \mathfrak{g})$. Fix a set of simple roots $\Delta = \{\alpha_i \mid i \in I\}$ and let $\Phi^+ = \Phi_{\mathbb{R}}^+(\mathfrak{a}, \mathfrak{g})$ be the corresponding set of positive roots. Then $\mathfrak{g}$ has a root-space decomposition

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \bigoplus_{\lambda \in \Phi_{\mathbb{R}}(\mathfrak{a},\mathfrak{g})} \mathfrak{g}_\lambda$$

and an Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k},$$

where $\mathfrak{n} = \bigoplus_{\lambda \in \Phi_{\mathbb{R}}^+(\mathfrak{a},\mathfrak{g})} \mathfrak{g}_\lambda$. Let $G = NAK$ be the corresponding Iwasawa decomposition of G. Let $\mathfrak{a}^+ = \{X \in \mathfrak{a} \mid \alpha(X) \geq 0 \text{ for all } \alpha \in \Delta\}$ be the positive Weyl chamber determined by Δ, and A^+ the corresponding positive Weyl chamber in A. The choice of Δ determines a partial order on $\Phi_{\mathbb{R}}^+(\mathfrak{a}, \mathfrak{g})$ in the sense that $\lambda \geq \lambda'$ if and only if $\lambda(X) \geq \lambda'(X)$ for any $X \in \mathfrak{a}^+$, and we can fix a total order

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_L \tag{3.1}$$

on $\Phi_{\mathbb{R}}^+(\mathfrak{a}, \mathfrak{g})$ that is compatible with this partial order, where $L = |\Phi_{\mathbb{R}}^+(\mathfrak{a}, \mathfrak{g})|$ (see [Kna02, p. 155] for an example of such an order). Let $d_i = \dim(\mathfrak{g}_{\lambda_i})$ and $d_0 = \dim(\mathfrak{m} \oplus \mathfrak{a})$. The following lemma gives a nice matrix representation of the adjoint Lie algebra.

Lemma 3.1. (*cf.* [Kna02, Lemma 6.45]) *There exists a basis of $\mathfrak{g}$ compatible with the above root-space decomposition such that the matrices representing* $\mathrm{ad}(\mathfrak{g})$ *have the following properties:*

(1) The matrices of $\mathrm{ad}\,\mathfrak{k}$ *are skew symmetric,*
(2) the matrices of $\mathrm{ad}\,\mathfrak{n}$ *are upper triangular with* 0 *on the diagonal.*
(3) the matrices of $\mathrm{ad}\,\mathfrak{a}$ *are diagonal with real entries with*

$$ad(X) = \begin{pmatrix} \lambda_1(X)I_{d_1} & & & & & & \\ & \ddots & & & & & \\ & & \lambda_L(X)I_{d_L} & & & & \\ & & & 0I_{d_0} & & & \\ & & & & -\lambda_L(X)I_{d_L} & & \\ & & & & & \ddots & \\ & & & & & & -\lambda_1(X)I_{d_1} \end{pmatrix},$$

for $X \in \mathfrak{a}$ where I_{d_i} is the $d_i \times d_i$ identity matrix.

As an immediate consequence, we have the following description of $\mathrm{Ad}(G)$.

Corollary 3.2. *There exists a basis of $\mathfrak{g}$ such that the matrices representing $\mathrm{Ad}(G)$ have the following properties:*

(1) The matrices of $\mathrm{Ad}(K)$ are orthogonal,
(2) the matrices of $\mathrm{Ad}(N)$ are upper triangular with 1's on the diagonal,
(3) the matrices of $\mathrm{Ad}(A)$ are diagonal with real entries with

$$\mathrm{Ad}(\exp(X)) = \begin{pmatrix} e^{\lambda_1(X)}I_{d_1} & & & & & & \\ & \ddots & & & & & \\ & & e^{\lambda_L(X)}I_{d_L} & & & & \\ & & & I_{d_0} & & & \\ & & & & e^{-\lambda_L(X)}I_{d_L} & & \\ & & & & & \ddots & \\ & & & & & & e^{-\lambda_1(X)}I_{d_1} \end{pmatrix},$$

for all $X \in \mathfrak{a}$.

We will fix such a basis once and for all and use it to identify $\mathrm{ad}(\mathfrak{g})$ and $\mathrm{Ad}(G)$ with the corresponding matrix groups. For future reference, we prove the following short lemma.

Lemma 3.3. *For any nonzero $X \in \mathfrak{g}$ we have that $(\mathrm{ad}(X))^2 \neq 0$.*

Proof. It suffices to show that $\mathrm{ad}(X) \circ \mathrm{ad}(X)$ is nontrivial as an endomorphism of $\mathfrak{g}$. If $\mathrm{ad}(X)$ is not nilpotent there is nothing to show. If it is nilpotent, by [Hel78, Theorem 7.4, p. 432] there exist elements $H, Y \in \mathfrak{g}$ such that

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H.$$

Applying $\mathrm{ad}(X) \circ \mathrm{ad}(X)$ to Y we get

$$(\mathrm{ad}(X) \circ \mathrm{ad}(X))(Y) = [X, [X, Y]] = [X, H] = -2X \neq 0,$$

thus completing the proof. $\qquad\qquad\square$

3.2. Cartan decomposition of one-parameter subgroups. We now consider the case where G a simple, that is $\mathfrak{g}$ is simple as a real Lie algebra, and study the Cartan decomposition of a one-parameter subgroup $\{h_t = \exp(tX_0)\}$, with $X_0 \in \mathfrak{g}$.

Fix an order $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_L$ on the set of positive roots Φ^+ as in (3.1). Since G is simple, Φ is irreducible and the root $\lambda_1 \in \Phi^+$, is the highest root, characterized by the property that $\lambda_1(X) \geq \lambda(X)$ for any $\lambda \in \Phi^+$ and $X \in \mathfrak{a}^+$. Recalling the Cartan decomposition $G = KA^+K$, we can write

$$h_t = k_1(t) \exp(X(t)) k_2(t) \tag{3.2}$$

with $k_i(t) \in K$ and $X(t) \in \mathfrak{a}^+$, and note that $X(t) \in \mathfrak{a}^+$ is uniquely determined by (3.2). Now, by the complete additive Jordan decomposition, there exist three pairwise commuting elements $X_\mathfrak{n}$, $X_\mathfrak{a}$ and $X_\mathfrak{k}$ in $\mathfrak{g}$ such that $X_0 = X_\mathfrak{n} + X_\mathfrak{a} + X_\mathfrak{k}$ with $\mathrm{ad}(X_\mathfrak{n}), \mathrm{ad}(X_\mathfrak{a})$ and $\mathrm{ad}(X_\mathfrak{k})$ $\mathrm{Ad}(G)$-conjugate to elements in $\mathrm{ad}(\mathfrak{n}), \mathrm{ad}(\mathfrak{a})$ and $\mathrm{ad}(\mathfrak{k})$ respectively. We say a one-parameter subgroup is *quasi-diagonalizable* if $X_\mathfrak{a} \neq 0$, and that is *quasi-unipotent* if $X_\mathfrak{a} = 0$ and $X_\mathfrak{n} \neq 0$ (note that any unbounded subgroup is either quasi-diagonalizable or quasi-unipotent). We first prove the following:

Proposition 3.4. *Given an unbounded one-parameter subgroup* $h_t = k_1(t) \exp(X(t)) k_2(t)$, *if it is quasi-diagonalizable then there exists some constant* $c > 0$ *such that*

$$e^{\lambda_1(X(t))} \gg e^{ct},$$

and if it is quasi-unipotent then there exists some integer $l \geq 2$ *such that*

$$e^{\lambda_1(X(t))} \asymp |t|^l.$$

Proof. For any $g \in G$, let $\|g\|^2 := \mathrm{tr}\left(\mathrm{Ad}(g)^t \mathrm{Ad}(g)\right)$. That is, $\|g\|^2$ is the Hilbert-Schmidt norm of $\mathrm{Ad}(g)$, and equals the sum of squares of entries of $\mathrm{Ad}(g)$. By Cauchy-Schwartz, $\|g_1 g_2\| \leq \|g_1\|\|g_2\|$ for any $g_1, g_2 \in G$, so in particular, for any fixed $g_0 \in G$ and all $h \in G$ we have

$$\|g_0^{-1} h g_0\| \asymp_{g_0} \|h\|. \tag{3.3}$$

Now, on one hand, in view of (3.2) and Corollary 3.2,

$$\|h_t\|^2 = \mathrm{tr}\left(\mathrm{Ad}\left(\exp(X(t))\right)^2\right) = \sum_{i=1}^{L} d_i \left(e^{2\lambda_i(X(t))} + e^{-2\lambda_i(X(t))}\right) + d_0,$$

and since $X(t) \in \mathfrak{a}^+$ we have that $\lambda_1(X(t)) \geq \lambda_i(X(t)) \geq 0$ for all $\lambda_i \in \Phi^+$, hence,

$$\|h_t\|^2 \asymp e^{2\lambda_1(X(t))}.$$

On the other hand, since $X_{\mathfrak{n}}, X_{\mathfrak{a}}$ and $X_{\mathfrak{k}}$ are pairwise commuting,

$$\|h_t\| = \|\exp(tX_0)\| = \|\exp(tX_{\mathfrak{n}})\exp(tX_{\mathfrak{a}})\exp(tX_{\mathfrak{k}})\|.$$

Moreover, since $\mathrm{ad}(X_{\mathfrak{k}})$ is $\mathrm{Ad}(G)$-conjugate to some element in $\mathrm{ad}(\mathfrak{k})$, the one-parameter subgroup $\{\exp(tX_{\mathfrak{k}})\}_{t \in \mathbb{R}}$ is compact, and $\{\|\exp(tX_{\mathfrak{k}})\|\}_{t \in \mathbb{R}}$ is uniformly bounded from above (by constants depending only on the generator X_0). Hence by Cauchy-Schwartz

$$\|h_t\| \asymp \|\exp(tX_{\mathfrak{n}})\exp(tX_{\mathfrak{a}})\|.$$

If $X_{\mathfrak{a}}$ is nonzero, then there exists some $g \in G$ such that $\mathrm{Ad}(g)^{-1}\mathrm{ad}(X_{\mathfrak{a}})\mathrm{Ad}(g) = \mathrm{ad}(X'_{\mathfrak{a}})$ with $X'_{\mathfrak{a}} \in \mathfrak{a}$ nonzero. Thus by (3.3),

$$\|h_t\|^2 \asymp \|g^{-1}\exp(tX_{\mathfrak{n}})g\exp(tX'_{\mathfrak{a}})\|^2.$$

Since $\mathrm{ad}(X_{\mathfrak{n}})$ is nilpotent, the entries of $\mathrm{Ad}(g^{-1}\exp(tX_{\mathfrak{n}})g)$ are all polynomials in t and since it is invertible, each row has at least one nonzero entry. Since $X'_{\mathfrak{a}} \in \mathfrak{a}$, there exists some root $\lambda \in \Phi^+$ such that $|\lambda(X'_{\mathfrak{a}})| = c > 0$. In view of Corollary 3.2, the matrix $\mathrm{Ad}\left(g^{-1}\exp(tX_{\mathfrak{n}})g\exp(tX'_{\mathfrak{a}})\right)$ has at least one entry of the form $e^{ct}P(t)$, where $P(t)$ is some nontrivial polynomial in t. Hence in this case, for all sufficiently large t,

$$e^{2\lambda_1(X(t))} \asymp \|h_t\|^2 \geq e^{2ct}.$$

If $X_{\mathfrak{a}} = 0$, then $X_{\mathfrak{n}}$ is nonzero. Let l be the unique integer such that $(\mathrm{ad}(X_{\mathfrak{n}}))^l \neq 0$ and $(\mathrm{ad}(X_{\mathfrak{n}}))^{l+1} = 0$ and note that $l \geq 2$ by Lemma 3.3. In this case, the entries of $\mathrm{Ad}\left(\exp(tX_{\mathfrak{n}})\right)$ are all polynomials in t with degree less than or equal to l, and there exists some entry with degree exactly l. Hence in this case, for all sufficiently large t

$$e^{2\lambda_1(X(t))} \asymp \|h_t\|^2 \asymp \|\exp(tX_{\mathfrak{n}})\|^2 \asymp t^{2l}. \qquad \square$$

13

3.3. Simple groups of real rank one.

We recall that, up to local isomorphism, a rank one group G is in one of the following four families of groups: $\mathrm{SO}(d,1), \mathrm{SU}(d,1), \mathrm{Sp}(d,1)$, with $d \geq 2$ and F_4^{-20}. In these cases, the positive restricted root system $\Phi_{\mathbb{R}}^+(\mathfrak{a}, \mathfrak{g})$ consists of one or two elements. Let α be the unique element in $\Phi_{\mathbb{R}}^+(\mathfrak{a}, \mathfrak{g})$ such that $\frac{1}{2}\alpha \notin \Phi_{\mathbb{R}}^+(\mathfrak{a}, \mathfrak{g})$. Let p be the dimension of $\mathfrak{g}_\alpha$, q be the dimension of $\mathfrak{g}_{2\alpha}$ and $\rho = \frac{1}{2}(p + 2q)\alpha$ be half the sum of the positive roots with multiplicities. Explicitly we have that p, q and ρ are as follows:

	$\mathrm{SO}(d,1)$	$\mathrm{SU}(d,1)$	$\mathrm{Sp}(d,1)$	F_4^{-20}
p	$d-1$	$2(d-1)$	$4(d-1)$	8
q	0	1	3	7
ρ	$\frac{d-1}{2}\alpha$	$d\alpha$	$(2d+1)\alpha$	11α

Let $\mathfrak{a}_{\mathbb{C}}^*$ be the complexified dual of $\mathfrak{a}$ and fix X_1 to be the unique element in $\mathfrak{a}$ such that $\alpha(X_1) = 1$. We identify $\mathfrak{a}_{\mathbb{C}}^*$ with $\mathbb{C}$ via their values at X_1. Denote by $\widehat{G}$ the unitary dual of G and $\widehat{G}_K$ the spherical unitary dual. The spherical unitary dual can be parameterized by $\mathfrak{a}_{\mathbb{C}}^*/W$ where W is the Weyl group. Let $\rho_0 \in \mathfrak{a}_{\mathbb{C}}^*$ be defined by

$$\rho_0 = \begin{cases} \rho & \mathfrak{g} = \mathfrak{so}(d,1) \text{ or } \mathfrak{su}(d,1) \\ \rho - 2\alpha & \mathfrak{g} = \mathfrak{sp}(d,1) \\ \rho - 6\alpha & \mathfrak{g} = \mathfrak{f}_4^{-20}. \end{cases}$$

Then with the above identification between $\mathfrak{a}_{\mathbb{C}}^*$ and $\mathbb{C}$ we have the parametrization

$$\widehat{G}_K = \{\pi_s \mid s \in i\mathbb{R}_{\geq 0} \cup (0, \rho_0)\} \cup \{\pi_\rho\},$$

where the representations $\pi_s, s \in i\mathbb{R}_{\geq 0}$ are the (tempered) principal series representations, the representations $\pi_s, s \in (0, \rho_0)$ are the (non-tempered) complementary series (cf. [Kos69]), and π_ρ is the trivial representation. We note that in each representation $\pi_s \in \widehat{G}_K$ there is a unique (up to scaling) spherical vector.

For any lattice $\Gamma \leq G$ consider the right regular representation of G on $L^2(\Gamma \backslash G)$. We denote by $L^2_{\mathrm{temp}}(\Gamma \backslash G)$ the subspace that weakly contains only tempered representations. We then have a spectral decomposition for any spherical $f \in L^2(\Gamma \backslash G)$

$$f = \langle f, 1 \rangle + \sum_k \langle f, \varphi_k \rangle \varphi_k + f_{\mathrm{temp}}, \tag{3.4}$$

with $f_{\mathrm{temp}} \in L^2_{\mathrm{temp}}(\Gamma \backslash G)$ and $\varphi_k \in \pi_{s_k}$ with $s_k \in (0, \rho_0)$.

After identifying spherical functions in $L^2(\Gamma \backslash G)$ with functions in $L^2(\Gamma \backslash \mathcal{H})$, the vectors $\varphi_k \in \pi_{s_k}$ occurring in this decomposition are the exceptional Laplacian eigenfunction in $L^2(\Gamma \backslash \mathcal{H})$, with corresponding eigenvalues $\rho^2 - s_k^2$. In particular, the *spectral gap* for Γ, i.e., the gap between the trivial eigenvalue and the first non-trivial eigenvalue of the Laplacian on $L^2(\Gamma \backslash \mathcal{H})$, is $\tau(2\rho - \tau)$ where the parameter $\tau = \tau(\Gamma)$ is given by

$$\tau(\Gamma) := \min_k (\rho - s_k). \tag{3.5}$$

The (spherical) exceptional forms $\varphi_k \in \pi_{s_k}$ above are either cusp forms (vanishing at all cusps) or residual forms (obtained as residues of Eisenstein series). The exceptional cusp forms are uniformly bounded, but the residual forms can blow up at the cusps. To

control their growth we recall the following result from [Kel17, Lemma 7] on the L^p norms of exceptional forms.[2]

Proposition 3.5. *For any spherical exceptional form $\varphi_k \in \pi_{s_k}$ with $s_k \in (0, \rho_0)$ we have that $\varphi_k \in L^p(\Gamma \backslash G)$ for any $p < \frac{2\rho}{\rho - s_k}$.*

We now turn to estimate the decay of matrix coefficients for flows on $L^2(\Gamma \backslash G)$ for G a rank one group. For each $s \in i\mathbb{R}_{\geq 0} \cup (0, \rho_0)$ consider the spherical function defined by

$$\phi_s(g) = \langle \pi_s(g)v, v \rangle$$

where $v \in V_{\pi_s}$ is the unique unit spherical vector. For any $\epsilon > 0$, ϕ_s decays like

$$|\phi_s(\exp(tX_1))| \ll_\epsilon e^{-\rho(1-\epsilon)|t|} \tag{3.6}$$

if $s \in i\mathbb{R}_{\geq 0}$ and

$$|\phi_s(\exp(tX_1))| \ll_\epsilon e^{-(\rho-s)(1-\epsilon)|t|} \tag{3.7}$$

if $s \in (0, \rho_0)$. We note that (3.6) follows from the asymptotic behavior of the Harish-Chandra Ξ function (cf. [How82, section 7]), while for (3.7), we refer to [GV12, (5.1.18)]. Let $\lambda_1 \in \Phi_{\mathbb{R}}^+(\mathfrak{a}, \mathfrak{g})$ be the highest root as before, and define $\kappa = \kappa(G)$ to be the unique integer such that $\alpha = \frac{\kappa}{2}\lambda_1$. Explicitly, $\kappa = 2$ if G is locally isomorphic to $SO(d, 1)$ and $\kappa = 1$ otherwise. Applying (3.6) and (3.7) to the regular representation $(\pi, L_0^2(\Gamma \backslash G))$, we get the following:

Proposition 3.6. *Let G be a connected simple Lie group with finite center and real rank one, and let $\Gamma \leq G$ be a lattice in G. Let $\{h_t\}_{t \in \mathbb{R}}$ be an unbounded one-parameter subgroup of G. For any spherical tempered $\psi, \phi \in L^2_{\text{temp}}(\Gamma \backslash G)$, for all $|t| \geq 1$ we have*

$$|\langle \pi(h_t)\psi, \phi \rangle| \ll_\epsilon \frac{\|\psi\|_2 \|\phi\|_2}{|t|^{\kappa\rho(1-\epsilon)}}. \tag{3.8}$$

For any spherical non-tempered exceptional form $\varphi \in \pi_s$ with $s \in (0, \rho_0)$, for all $|t| \geq 1$ we have

$$|\langle \pi(h_t)\varphi, \varphi \rangle| \ll_\epsilon \frac{\|\varphi\|_2^2}{|t|^{\kappa(\rho-s)(1-\epsilon)}}. \tag{3.9}$$

Proof. Recall the Cartan decomposition $h_t = k_1(t)\exp(X(t))k_2(t)$ with $k_i(t) \in K$ and $X(t) \in \mathfrak{a}^+$. In view of (3.6) and (3.7), it suffices to show that $e^{\alpha(X(t))} \gg |t|^\kappa$. Recall that κ is defined precisely such that $\alpha = \frac{\kappa}{2}\lambda_1$ where λ_1 is the highest root. By Proposition 3.4, $e^{\lambda_1(X(t))} \gg t^2$, hence $e^{\alpha(X(t))} \gg (t^2)^{\frac{\kappa}{2}} = |t|^\kappa$. $\qquad\square$

Using Proposition 3.6 together with the spectral decomposition, we see that the rate of decay is controlled by the spectral gap parameter $\tau(\Gamma)$ as follows:

Theorem 3.7. *For G and Γ as in Proposition 3.6, for all sufficiently small $\epsilon > 0$, for any spherical $f_1, f_2 \in L_0^2(\Gamma \backslash G)$ and for all $|t| \geq 1$, we have*

$$|\langle \pi(h_t)f_1, f_2 \rangle| \ll_\epsilon \frac{\|f_1\|_2 \|f_2\|_2}{|t|^{\kappa\tau(\Gamma)(1-\epsilon)}}.$$

---

[2]The result in [Kel17] is stated for $G = SO_0(d, 1)$ but the proof is identical.

Proof. For any spherical $f_1, f_2 \in L_0^2(\Gamma \backslash G)$, applying the spectral decomposition (3.4) and Proposition 3.6 we get for any $|t| \geq 1$

$$|\langle \pi(h_t)f_1, f_2\rangle| \leq \sum_k \left| \langle f_1, \varphi_k\rangle \overline{\langle f_2, \varphi_k\rangle} \right| |\langle \pi(h_t)\varphi_k, \varphi_k\rangle| + \left| \langle \pi(h_t)f_{\text{temp}}^1, f_{\text{temp}}^2\rangle \right|$$

$$\ll_\epsilon \sum_k \frac{\left| \langle f_1, \varphi_k\rangle \overline{\langle f_2, \varphi_k\rangle} \right|}{|t|^{\kappa(\rho - s_k)(1-\epsilon)}} + \frac{\|f_{\text{temp}}^1\|_2 \|f_{\text{temp}}^2\|_2}{|t|^{\kappa\rho(1-\epsilon)}}$$

$$\leq \frac{\sum_k \left| \langle f_1, \varphi_k\rangle \overline{\langle f_2, \varphi_k\rangle} \right| + \|f_{\text{temp}}^1\|_2 \|f_{\text{temp}}^2\|_2}{|t|^{\kappa\tau(\Gamma)(1-\epsilon)}}$$

$$\leq \frac{\|f_1\|_2 \|f_2\|_2}{|t|^{\kappa\tau(\Gamma)(1-\epsilon)}},$$

where for the last inequality we used Cauchy-Schwartz. $\qquad\square$

As a direct corollary, we have the following:

Corollary 3.8. *For G a connected simple Lie group with finite center and real rank one, not locally isomorphic to $\mathrm{SO}(2,1)$ and $\Gamma \leq G$ a lattice in G. If $\tau(\Gamma) > \frac{1}{\kappa(G)}$, then any unbounded one-parameter flow is SD for $\Gamma \backslash G$. In particular, if G is locally isomorphic to $\mathrm{Sp}(d,1)$ with $d \geq 2$ or F_4^{-20}, then any unbounded one-parameter flow is SD for $\Gamma \backslash G$ for any lattice Γ.*

Remark 11. For G is locally isomorphic to $\mathrm{SO}(d+1, 1)$ and Γ a congruence lattice, the best known bounds towards the Selberg-Ramanujan conjecture [BS91, KS03, BB11, BC13, BC17] imply that $\tau(\Gamma) \geq \frac{25}{64}$ for $d = 1$, that $\tau(\Gamma) \geq \frac{25}{32}$ for $d = 2$ and that $\tau(\Gamma) \geq 1$ when $d \geq 3$. Similarly, when G is locally isomorphic to $\mathrm{SU}(d, 1)$, we have that $\tau(\Gamma) \geq \frac{6}{5}$ when $d = 2$ and that $\tau(\Gamma) \geq 2$ for $d \geq 3$. In particular, it follows that for any rank one group, G, not locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$, for any congruence lattice $\Gamma \leq G$, we have that $\kappa\tau(\Gamma) > 1$ and all unbounded one-parameter flows on $\Gamma \backslash G$ are SD.

3.4. **Simple groups of higher rank.** Next, we consider the case where G is a connected simple Lie group with finite center and real rank ≥ 2. Following [Oh98], two roots α and β are called strongly orthogonal if neither one of $\alpha \pm \beta$ is a root. Let $S(\Phi)$ denote the family of all subsets of Φ^+ whose elements are pairwise strongly orthogonal. We call an element $\mathcal{O}$ in $S(\Phi)$ a strongly orthogonal system. Let ϱ be the function on $S(\Phi)$ given by $\varrho(\mathcal{O}) = \sum_{\alpha \in \mathcal{O}} \alpha$.

Proposition 3.9. [Oh98, Proposition 2.3] *Let $\Delta = \{\alpha_i \mid i \in I\}$ be the set of simple roots as above. There exists a maximal strongly orthogonal system $\mathcal{Q}(\Phi)$ in $S(\Phi)$ in the sense that for any $\mathcal{O} \in S(\Phi)$, for each $i \in I$, the coefficient of α_i in $\varrho(\mathcal{Q}(\Phi))$ is greater than or equal to the coefficient of α_i in $\varrho(\mathcal{O})$.*

Let $\xi = \frac{1}{2}\varrho(\mathcal{Q}(\Phi)) = \frac{1}{2} \sum_{\alpha \in \mathcal{Q}(\Phi)} \alpha$, and consider the K-bi-invariant function, $F : G \to \mathbb{R}^+$, defined on A^+ via

$$F(\exp(X)) = e^{-\xi(X)}. \tag{3.10}$$

We then have the following result from [Oh98].

Theorem 3.10. *For G a connected simple group with finite center and real rank ≥ 2, for any $\epsilon > 0$ sufficiently small, for any nontrivial $\sigma \in \widehat{G}_K$ and K-invariant unit vector v_σ of σ*

$$|\langle \sigma(g)v_\sigma, v_\sigma\rangle| \ll_\epsilon F(g)^{1-\epsilon} \qquad \text{for any } g \in G. \tag{3.11}$$

Proof. When G is a linear group this is [Oh98, Theorem A]. In general, let $Z(G)$ be the center of G and V_σ be the Hilbert space of σ, we want to show $Z(G)$ acts trivially on V_σ. Since σ is irreducible and unitary (and hence admissible), $Z(G)$ acts on V_σ via some central character ω. That is, there is a character $\omega : Z(G) \to \mathbb{C}^\times$ such that $\sigma(z)v = \omega(z)v$ for any $z \in Z(G)$ and $v \in V_\sigma$. Since G has finite center, the maximal compact subgroup K contains the center ([Kna02, Theorem 6.31]). In particular, for any $z \in Z(G) \subset K$ we have $\sigma(z)v_\sigma = v_\sigma = \omega(z)v_\sigma$. Hence $\omega = 1$ and $Z(G)$ acts trivially on V_σ. Thus σ descends to an irreducible unitary representation of the adjoint group $\mathrm{Ad}(G)$. Identifying $\mathfrak{g}$ with $\mathrm{ad}(\mathfrak{g})$ and applying [Oh98, Theorem A] to the linear group $\mathrm{Ad}(G)$, we get that for any $g \in G$

$$|\langle \sigma(g)v_\sigma, v_\sigma \rangle| = |\langle \sigma\left(\mathrm{Ad}(g)\right)v_\sigma, v_\sigma \rangle| \ll_\epsilon F\left(\mathrm{Ad}(g)\right)^{1-\epsilon} = F(g)^{1-\epsilon}. \qquad \square$$

Applying these results to one-parameter subgroups we get:

Proposition 3.11. *Let G denote a connected simple Lie group with finite center and real rank ≥ 2 and $\{h_t\}_{t\in\mathbb{R}}$ an unbounded one-parameter subgroup. For any $\epsilon > 0$ sufficiently small, for any nontrivial $\sigma \in \widehat{G}_K$ and $|t| \geq 1$,*

$$|\langle \sigma(h_t)v_\sigma, v_\sigma \rangle| \ll_\epsilon |t|^{\epsilon-1}. \tag{3.12}$$

Proof. Since the singleton $\{\lambda_1\}$ constitutes a strongly orthogonal system, Proposition 3.9, implies that $\xi(X) \geq \frac{1}{2}\lambda_1(X)$ for any $X \in \mathfrak{a}^+$, leading to the bound $F(\exp(X)) \leq e^{-\lambda_1(X)/2}$. Now using the Cartan decomposition $h_t = k_1(t)\exp(X(t))k_2(t)$ and Theorem 3.10 we get that

$$|\langle \sigma(h_t)v_\sigma, v_\sigma \rangle| \ll_\epsilon \exp((\epsilon - 1)\xi(X(t))) \leq \exp(\tfrac{(\epsilon-1)}{2}\lambda_1(X(t))).$$

Now, by Proposition 3.4, we have that $e^{\lambda_1(X(t))} \gg t^2$, thus concluding the proof. $\qquad \square$

Remark 12. In [Oh02, p. 187-190] both ξ and the highest root λ_1 are explicitly given in terms of linear combinations of simple roots. By comparing ξ and λ_1 directly, we note that when $\Phi_\mathbb{R}(\mathfrak{a}, \mathfrak{g})$ is of type $B_n(n \geq 4)$, $D_n(n \geq 4)$, E_6, E_7, E_8 or F_4, we have that $\xi(X) \geq \lambda_1(X)$ for any $X \in \mathfrak{a}^+$. Hence in these cases we have the following summable decay:

$$|\langle \sigma(h_t)v_\sigma, v_\sigma \rangle| \ll_\epsilon |t|^{2(\epsilon-1)} \tag{3.13}$$

for any nontrivial $\sigma \in \widehat{G}_K$ and any $|t| \geq 1$. When $\Phi_\mathbb{R}(\mathfrak{a}, \mathfrak{g})$ is not of the above types, if the one-parameter subgroup is quasi-diagonalizable then the matrix coefficients decay exponentially, and if it is quasi-unipotent with $(\mathrm{ad}(X_\mathfrak{n}))^3 \neq 0$, then we can take $l = 3$ in Proposition 3.4 leading to the summable decay:

$$|\langle \sigma(h_t)v_\sigma, v_\sigma \rangle| \ll_\epsilon |t|^{\frac{3}{2}(\epsilon-1)} \tag{3.14}$$

for any nontrivial $\sigma \in \widehat{G}_K$ and any $|t| \geq 1$.

3.5. **Semisimple groups.** We now consider the general case where G is a connected semisimple Lie group with finite center and no compact factors. Let $\Gamma \leq G$ denote an irreducible lattice, and $\{h_t\}_{t\in\mathbb{R}}$ an unbounded one-parameter subgroup of G as before.

To deal with this case, first, note that there is a surjective homomorphism $\prod_{i=1}^m G_i \to G$ with finite kernel such that each G_i is a noncompact connected simple Lie group with finite center. Let $\tilde{\Gamma}$ be the preimage of Γ and $\{\tilde{h}_t\}$ be the identity component of the pre-image of $\{h_t\}$. By replacing $(\Gamma\backslash G, h_t)$ by $(\tilde{\Gamma}\backslash \prod_{i=1}^m G_i, \tilde{h}_t)$ we may assume, without loss of generality, that $G = \prod_{i=1}^m G_i$ and $h_t = (h_t^1, \ldots, h_t^m)$ is unbounded as a one-parameter subgroup of G.

Next, our maximal compact subgroup is of the form $K = \prod_{i=1}^m K_i$, with each K_i a maximal compact subgroup of G_i. By slightly abusing the terminology in [KM96][3], we say G_i is an *essential factor* of G if $\{h_t^i\}_{t\in\mathbb{R}}$ is unbounded in G_i. After reordering the factors, we can assume that $G_1, \cdots, G_k$ are all the essential factors (since $\{h_t\}$ is unbounded, we have $k \geq 1$). Let $\widehat{G}_\Gamma \subset \widehat{G}$ be the set of irreducible unitary representations that are weakly contained in $L_0^2(\Gamma\backslash G)$ and $\widehat{G}_{K,\Gamma} = \widehat{G}_\Gamma \cap \widehat{G}_K$. We first note that for any $\sigma \in \widehat{G}_{K,\Gamma}$, σ is of the form $\sigma = \otimes_{i=1}^m \sigma_i$ with each $\sigma_i \in \widehat{G}_{iK_i}$. Since $\Gamma \leq G$ is irreducible, each σ_i is nontrivial. Moreover, a K-invariant unit vector v_σ of σ is of the form $v_\sigma = \otimes_{i=1}^m v_{\sigma_i}$, where v_{σ_i} is the K_i-invariant unit vector of σ_i. Thus

$$|\langle\sigma(h_t)v_\sigma, v_\sigma\rangle| = \prod_{i=1}^m |\langle\sigma_i(h_t^i)v_{\sigma_i}, v_{\sigma_i}\rangle|.$$

First, we consider the case that G (and hence each of its factors) has property (T), that is, each G_i is either of real rank ≥ 2 or locally isomorphic to $\mathrm{Sp}(d,1)$ or F_4^{-20}. In this case, for $1 \leq i \leq k$ the factor $\{h_t^i\}$ is unbounded and by Proposition 3.11 and Corollary 3.8, for any $|t| \geq 1$ we have $|\langle\sigma_i(h_t^i)v_{\sigma_i}, v_{\sigma_i}\rangle| \ll_\epsilon \frac{1}{|t|^{(1-\epsilon)}}$. While for $k+1 \leq i \leq m$, we bound $|\langle\sigma_i(h_t^i)v_{\sigma_i}, v_{\sigma_i}\rangle| \leq 1$. We thus have for any $|t| \geq 1$

$$|\langle\sigma(h_t)v_\sigma, v_\sigma\rangle| \ll_\epsilon \frac{1}{|t|^{k(1-\epsilon)}}. \tag{3.15}$$

Proposition 3.12. *Let G be a connected semisimple Lie group with finite center and no compact factors, $\Gamma \leq G$ an irreducible lattice, and $\{h_t\}_{t\in\mathbb{R}}$ an unbounded one-parameter subgroup of G. If G has property (T), then for any spherical $\varphi, \psi \in L_0^2(\Gamma\backslash G)$ and for any $|t| \geq 1$, we have*

$$|\langle\pi(h_t)\varphi, \psi\rangle| \ll_\epsilon \frac{\|\varphi\|_2\|\psi\|_2}{|t|^{k(1-\epsilon)}}, \tag{3.16}$$

where $k \geq 1$ is the number of essential factors of the flow.

Proof. Using the direct integral decomposition of $L_0^2(\Gamma\backslash G)$, for any spherical $\varphi \in L_0^2(\Gamma\backslash G)$, φ can be written as

$$\varphi = \int_{\sigma\in\widehat{G}_{K,\Gamma}} \varphi_\sigma d\nu(\sigma),$$

where $\varphi_\sigma \in \sigma$ is spherical and ν is some Borel measure on $\widehat{G}_\Gamma$. Hence for any spherical $\varphi, \psi \in L_0^2(\Gamma\backslash G)$, we have

$$\langle\pi(h_t)\varphi, \psi\rangle = \int_{\sigma\in\widehat{G}_{K,\Gamma}} \langle\sigma(h_t)\varphi_\sigma, \psi_\sigma\rangle d\nu(\sigma).$$

---

[3]In [KM96], G_i is called an essential factor if $\{h_t^i\}_{t\in\mathbb{R}}$ is quasi-diagonalizable, while here we allow $\{h_t^i\}_{t\in\mathbb{R}}$ to be quasi-unipotent.

By (3.15) for any $|t| \geq 1$ we can bound

$$
\begin{aligned}
|\langle \pi(h_t)\varphi, \psi \rangle| &\leq \int_{\sigma \in \widehat{G}_{K,\Gamma}} |\langle \sigma(h_t)\varphi_\sigma, \psi_\sigma \rangle| \, d\nu(\sigma) \\
&\ll_\epsilon \frac{1}{|t|^{k(1-\epsilon)}} \int_{\sigma \in \widehat{G}_{K,\Gamma}} ||\varphi_\sigma||_2 ||\psi_\sigma||_2 d\nu(\sigma) \\
&\leq \frac{||\varphi||_2 ||\psi||_2}{|t|^{k(1-\epsilon)}},
\end{aligned}
$$

where for the last inequality we used Cauchy-Schwartz. $\qquad \square$

Proof of Theorem 1.3. The result follows from Proposition 3.12 (after noting that for an unbounded flow we have at least one essential factor so $k \geq 1$). We further note that the only possibility that the flow is not SD is that there is only one essential factor, say G_1, and $\{h_t^1\}_{t \in \mathbb{R}}$ (viewed as a one-parameter flow on $\Gamma \backslash G$) is not SD. $\qquad \square$

Proof of Theorem 1.6. The result follows from the above argument and Remark 12. $\qquad \square$

Remark 13. For a semisimple group G of real rank ≥ 2 without property (T) our result is only conditional. In this case, Margulis Arithmeticity Theorem states that any irreducible lattice $\Gamma \leq G$ is arithmetic in the sense that it is commensurable to a congruence lattice defined over some number field. Moreover, Serre's congruence subgroup conjecture (which is settled when $\Gamma \backslash G$ is not compact [Rap92]) states that in fact all irreducible lattices are congruence lattices. Now, when Γ is a congruence lattice the generalized Selberg-Ramanujan conjecture [Sha04, Sar05] gives very precise restrictions on which non-tempered representations may occur in the decomposition of $L^2(\Gamma \backslash G)$, and these representations all have fast decay of matrix coefficients. In particular, when Γ is a congruence lattice, the Selberg-Ramanujan conjecture implies that (3.16) holds for all flows on $\Gamma \backslash G$, and moreover, when G is not locally isomorphic to a product of copies of $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$, this already follows from the known bounds [KS03, BB11, BC13, BC17] towards the Selberg-Ramanujan conjecture.

4. Effective mean ergodic theorems and consequences

Let G be a connected semisimple Lie group with finite center and no compact factors, $\Gamma \leq G$ an irreducible lattice, $\mathcal{X} = \Gamma \backslash G$, and $H = \{h_m\}_{m \in \mathbb{Z}}$ an unbounded discrete time one-parameter flow on $\mathcal{X}$ generated by a one-parameter subgroup as before. For any $f \in L^2(\mathcal{X})$ and any integer $m \geq 1$, define the averaging operator

$$
\beta_m^+(f)(x) := \frac{1}{m} \sum_{j=1}^{m} f(xh_j).
$$

Since H acts ergodically on $\mathcal{X}$, the mean ergodic theorem states that

$$
\|\beta_m^+(f) - \mu(f)\|_2 \to 0,
$$

as $m \to \infty$ for any $f \in L^2(\mathcal{X})$, where $\mu(f) := \int_{\mathcal{X}} f d\mu$. In this section we adapt the method introduced in [GK17] and [Kel17], to prove two effective mean ergodic theorems using the explicit rate of decay of matrix coefficients obtained in the previous section. The arguments are slightly different for rank one groups and for groups with property (T), so we will treat them separately.

19

4.1. **Groups with Property (T).** When the group G has property (T), we can use the uniform result on decay of matrix coefficients for one-parameter flows to show the following.

Proposition 4.1. *Assume that G has property (T). Then for any unbounded discrete time one-parameter flow $H = \{h_m\}_{m \in \mathbb{Z}}$, for any $\epsilon > 0$ and for any spherical $f \in L^2(\mathcal{X})$ we have*

$$\|\beta_m^+(f) - \mu(f)\|_2 \ll_\epsilon \frac{\|f\|_2}{m^{\frac{1}{2}(1-\epsilon)}}. \tag{4.1}$$

If the flow is SD we have the slightly stronger bound

$$\|\beta_m^+(f) - \mu(f)\|_2 \ll \frac{\|f\|_2}{\sqrt{m}}. \tag{4.2}$$

Proof. Let $f_0 = f - \mu(f) \in L_0^2(\mathcal{X})$. Proposition 3.11 and Corollary 3.8 imply that for any small $\epsilon > 0$ for all $|k| \geq 1$

$$|\langle \pi(h_k)f_0, f_0 \rangle| \ll_\epsilon \frac{\|f_0\|_2^2}{|k|^{1-\epsilon}}.$$

Noting that $\beta_m^+(f_0) = \beta_m^+(f) - \mu(f)$ we have

$$\|\beta_m^+(f_0)\|_2^2 = \frac{1}{m^2} \left\langle \sum_{1 \leq i \leq m} \pi(h_i)f_0, \sum_{1 \leq j \leq m} \pi(h_j)f_0 \right\rangle = \frac{1}{m^2} \sum_{1 \leq i,j \leq m} \langle \pi(h_{i-j})f_0, f_0 \rangle$$

$$= \frac{1}{m^2} \sum_{|k| \leq m} \langle \pi(h_k)f_0, f_0 \rangle \, \#\{(i,j) \mid 1 \leq i,j \leq m, i-j = k\}$$

$$\leq \frac{1}{m} \sum_{|k| \leq m} |\langle \pi(h_k)f_0, f_0 \rangle| \ll_\epsilon \frac{1}{m}(1 + 2\sum_{k=1}^m \frac{1}{k^{1-\epsilon}})\|f_0\|_2^2 \ll \frac{\|f_0\|_2^2}{m^{1-\epsilon}} \leq \frac{\|f\|_2^2}{m^{1-\epsilon}}.$$

If the flow is SD the same argument with the stronger bound $|\langle \pi(h_k)f_0, f_0 \rangle| \ll_\eta \frac{\|f_0\|_2^2}{|k|^\eta}$, gives $\|\beta_m^+(f) - \mu(f)\|_2^2 \ll \frac{\|f\|_2^2}{m}$. $\qquad\square$

4.2. **Groups of real rank one.** For groups of real rank one without property (T) we have to take into account the contribution of the possible exceptional spectrum. The argument is similar to the one used in [Kel17, Theorem 15] for the orthogonal groups, and we include the details for the reader's convenience. Doing this leads to the following.

Proposition 4.2. *Let G be locally isomorphic to $\mathrm{SO}(d+1,1)$ or $\mathrm{SU}(d,1)$ with $d \geq 2$. Then for any unbounded discrete time one-parameter flow $H = \{h_m\}_{m \in \mathbb{Z}}$, for any sufficiently small $\epsilon > 0$ and for any spherical $f \in L^2(\mathcal{X})$ we have*

$$\|\beta_m^+(f) - \mu(f)\|_2 \ll_\epsilon \frac{\|f\|_2}{\sqrt{m}} + \sum_{s_k \in [\rho - \frac{1}{\kappa}, \rho)} \frac{|\langle f, \varphi_k \rangle|}{m^{\frac{\kappa}{2}(\rho - s_k)(1-\epsilon)}},$$

where φ_k are the finitely many spherical exceptional forms. When G is locally isomorphic to $\mathrm{SO}(2,1)$ we have the same result with the first term replaced by $\frac{\|f\|_2}{m^{\frac{1}{2}(1-\epsilon)}}$.

Proof. Write $f = \langle f, 1 \rangle + \sum_k \langle f, \varphi_k \rangle \varphi_k + f_0$ with $f_0 \in L_{\mathrm{temp}}^2(\mathcal{X})$. Then

$$\|\beta_m^+(f) - \mu(f)\|_2 \leq \sum_k |\langle f, \varphi_k \rangle| \|\beta_m^+(\varphi_k)\|_2 + \|\beta_m^+(f_0)\|_2.$$

20

Using (3.8) and the same argument as in Proposition 4.1, we get

$$\|\beta_m^+(f_0)\|_2 \ll_\epsilon \begin{cases} \dfrac{\|f\|_2}{m^{\frac{1}{2}(1-\epsilon)}} & \mathfrak{g} = \mathfrak{so}(2,1) \\[4mm] \dfrac{\|f\|_2}{\sqrt{m}} & \text{otherwise.} \end{cases} \tag{4.3}$$

Similarly, for each of the spherical exceptional forms φ_k in π_{s_k} we can bound

$$\|\beta_m^+(\varphi_k)\|_2^2 \ll_\epsilon \frac{1}{m}\left(1 + \sum_{k=1}^m \frac{1}{k^{\kappa(\rho-s_k)(1-\epsilon)}}\right) \ll \begin{cases} \dfrac{1}{m} & s_k < \rho - \frac{1}{\kappa} \\[3mm] \dfrac{1}{m^{\kappa(\rho-s_k)(1-\epsilon)}} & s_k \geq \rho - \frac{1}{\kappa}. \end{cases}$$

Hence

$$\|\beta_m^+(\varphi_k)\|_2 \ll_\epsilon \begin{cases} \dfrac{1}{\sqrt{m}} & s_k < \rho - \frac{1}{\kappa} \\[3mm] \dfrac{1}{m^{\frac{\kappa}{2}(\rho-s_k)(1-\epsilon)}} & s_k \geq \rho - \frac{1}{\kappa}. \end{cases}$$

Combining this with (4.3) and using the bound $|\langle f, \varphi_k\rangle| \leq \|f\|_2$ for $s_k < \rho - \frac{1}{\kappa}$ concludes the proof. $\qquad\square$

4.3. **A variance estimate.** We now apply the above mean ergodic theorem for a variance estimate. Following [Kel17], for any integer $m \geq 1$ and $f \in L^2(\mathcal{X})$, we define the set

$$\mathcal{C}_{m,f} = \left\{ x \in \mathcal{X} \mid \left|\beta_m^+(f)(x) - \mu(f)\right| \geq \frac{\mu(f)}{2} \right\}.$$

Applying the mean ergodic theorem we obtain the following estimate for the measure of $\mathcal{C}_{m,f}$ when f is an indicator function of a spherical set.

Proposition 4.3. *For any spherical set $B \subset \mathcal{X}$ and $f = \chi_B$ its indicator function:*
   (1) If the flow is SD then

$$\mu(\mathcal{C}_{m,f}) \ll \frac{1}{m\mu(B)}. \tag{4.4}$$

   (2) If G has property (T), then for all sufficiently small $\epsilon > 0$

$$\mu(\mathcal{C}_{m,f}) \ll_\epsilon \frac{1}{m^{1-\epsilon}\mu(B)}. \tag{4.5}$$

   (3) If G is locally isomorphic to $\mathrm{SO}(d+1,1)$ or $\mathrm{SU}(d,1)$ with $d \geq 2$, then

$$\mu(\mathcal{C}_{m,f}) \ll \frac{1}{m\mu(B)} + \frac{1}{\left(m\mu(B)^{\frac{2}{3}}\right)^{2\tau(\Gamma)/3}}. \tag{4.6}$$

   (4) If G is locally isomorphic to $\mathrm{SO}(2,1)$, then for all sufficiently small $\epsilon > 0$

$$\mu(\mathcal{C}_{m,f}) \ll_\epsilon \frac{1}{m^{1-\epsilon}\mu(B)} + \frac{1}{\left(m^{1-\epsilon}\mu(B)^{1+\frac{\epsilon}{2}}\right)^{2\tau(\Gamma)}} \tag{4.7}$$

   where for the last two cases, $\tau(\Gamma)$ denotes the spectral gap parameter defined in (3.5).

Proof. Since $\mu(\mathcal{C}_{m,f}) \leq 1$ we may assume that $m\mu(B) \geq 1$ since otherwise the result holds trivially.

   Now, by definition if $x \in \mathcal{C}_{m,f}$, then $|\beta_m^+(f) - \mu(B)| \geq \frac{\mu(B)}{2}$ and hence

$$\left\|\beta_m^+(f) - \mu(B)\right\|_2^2 \geq \frac{1}{4}\int_{\mathcal{C}_{m,f}} \mu(B)^2 d\mu(x) = \frac{\mu(\mathcal{C}_{m,f})\mu(B)^2}{4}. \tag{4.8}$$

21

On the other hand, for SD flows by Proposition 4.1 we have

$$\|\beta_m^+(f) - \mu(B)\|_2^2 \ll \frac{\|f\|_2^2}{m} = \frac{\mu(B)}{m}. \tag{4.9}$$

Combining (4.8) and (4.9) we get (4.4).

If G has property (T), then again by Proposition 4.1 we have

$$\|\beta_m^+(f) - \mu(B)\|_2^2 \ll_\epsilon \frac{\|f\|_2^2}{m^{1-\epsilon}} = \frac{\mu(B)}{m^{1-\epsilon}}. \tag{4.10}$$

Combining (4.8) and (4.10), we get (4.5).

If G is locally isomorphically to $SO(d+1,1)$ or $SU(d,1)$ with $d \geq 2$, then by Proposition 4.2 we have

$$\|\beta_m^+(f) - \mu(B)\|_2^2 \ll_\epsilon \frac{\|f\|_2^2}{m} + \sum_{s_k \in [\rho - \frac{1}{\kappa}, \rho)} \frac{|\langle f, \varphi_k \rangle|^2}{m^{\kappa(\rho-s_k)(1-\epsilon)}}.$$

Recall that by Proposition 3.5, the exceptional forms $\varphi_k \in L^p(\mathcal{X})$ for any $1 < p < \frac{2\rho}{\rho-s_k}$. For $\epsilon > 0$ sufficiently small let $p_k = \frac{2\rho}{(1+\epsilon/2)(\rho-s_k)}$, $q_k = \frac{p_k-1}{p_k}$, and use Hölder inequality to bound

$$|\langle f, \varphi_k \rangle| \leq \|\varphi\|_{p_k} \|f\|_{q_k} \ll_\epsilon \|f\|_{q_k} = \mu(B)^{\frac{1}{q_k}}.$$

With this bound we get

$$\|\beta_m^+(f) - \mu(B)\|_2^2 \ll_\epsilon \frac{\mu(B)}{m} + \sum_{s_k \in [\rho - \frac{1}{\kappa}, \rho)} \frac{\mu(B)^{\frac{2}{q_k}}}{m^{\kappa(\rho-s_k)(1-\epsilon)}}. \tag{4.11}$$

Combining (4.8) and (4.11) we get

$$\mu(\mathcal{C}_{m,f}) \ll_\epsilon \frac{1}{m\mu(B)} + \sum_{s_k \in [\rho - \frac{1}{\kappa}, \rho)} \frac{1}{\left(m^{\kappa(1-\epsilon)} \mu(B)^{\frac{1+\frac{1}{2}\epsilon}{\rho}}\right)^{\rho-s_k}}. \tag{4.12}$$

Note that we always have that $\kappa\rho \geq 2$ so fixing $\epsilon < 1/3$ sufficiently small so that $\frac{1+\frac{1}{2}\epsilon}{\kappa\rho(1-\epsilon)} < \frac{2}{3}$ we get

$$\mu(\mathcal{C}_{m,f}) \ll \frac{1}{m\mu(B)} + \sum_{s_k \in [\rho - \frac{1}{\kappa}, \rho)} \frac{1}{\left(m^{\kappa(1-\epsilon)} \mu(B)^{\frac{1+\frac{1}{2}\epsilon}{\rho}}\right)^{\rho-s_k}}$$

$$= \frac{1}{m\mu(B)} + \sum_{s_k \in [\rho - \frac{1}{\kappa}, \rho)} \frac{1}{\left(m\mu(B)^{\frac{1+\frac{1}{2}\epsilon}{2\kappa\rho/3}}\right)^{\kappa(\rho-s_k)(1-\epsilon)}}$$

$$< \frac{1}{m\mu(B)} + \sum_{s_k \in [\rho - \frac{1}{\kappa}, \rho)} \frac{1}{\left(m\mu(B)^{\frac{2}{3}}\right)^{2\kappa(\rho-s_k)/3}}.$$

Finally, note that for all terms in the sum we have that $\tau(\Gamma) \leq \rho - s_k \leq \frac{1}{\kappa}$ so that

$$\mu(\mathcal{C}_{m,f}) \ll \frac{1}{m\mu(B)} + \frac{1}{\left(m\mu(B)^{\frac{2}{3}}\right)^{2\tau(\Gamma)/3}}.$$

22

For $\mathrm{SO}(2,1)$ we have that $\kappa = 2$ and $\rho = \frac{1}{2}$ and a similar argument gives (4.7) $\qquad\square$

5. Applications to shrinking target problems

We now combine all the ingredients and apply them to the shrinking target problems we described in the introduction. As before, throughout this section we let G denote a connected semisimple Lie group with finite center and no compact factors, $\Gamma \leq G$ an irreducible lattice $\{h_t\}_{t\in\mathbb{R}}$ an unbounded one-parameter subgroup, generating a discrete time flow given by the action of the discrete group $H = \{h_m\}_{m\in\mathbb{Z}}$ on the homogenous space $\mathcal{X} = \Gamma\backslash G$.

5.1. **The hitting time problem.** Let $\{B_t\}_{t>0}$ be a monotone family of spherical shrinking targets in $\mathcal{X}$. We recall that by Lemmas 2.1 and 2.2, in order to prove Theorem 1.1 it is enough to show that the estimate (2.8) holds. We show this in the following Lemma.

Lemma 5.1. *Assume that either G has property (T) or that it is simple of real rank one. Then (2.8) holds for any $0 < \delta < \frac{1}{2}$.*

Proof. Given a spherical measurable set, $B \subset \mathcal{X}$, let $f = \chi_B$ denote its indicator function, and for any $m \in \mathbb{N}$ let $\mathcal{M}^i_{m,B}$ be as defined in (2.6). In particular, for any $x \in \mathcal{M}^i_{m,B}$ we have that $\beta^+_m(f)(x) = \frac{k}{m}$ for some $0 \leq k < i$. Thus for any $m > \frac{2i}{\mu(B)}$, if $x \in \mathcal{M}^i_{m,B}$ then

$$|\beta^+_m(f)(x) - \mu(f)| = \mu(B) - \frac{k}{m} > \mu(B) - \frac{i}{m} > \frac{1}{2}\mu(B),$$

so $\mathcal{M}^i_{m,B} \subseteq \mathcal{C}_{m,f}$.

We recall that $\mathcal{N}$ is the set of integers $\ell \geq 0$ such that $\{\mu(B_t) \mid t > 0\} \cap [\frac{1}{2^\ell}, \frac{1}{2^{\ell+1}})$ is nonempty, and for each $\ell \in \mathcal{N}$, $\underline{B}_\ell = \bigcap_{\frac{1}{2^{\ell+1}} \leq \mu(B_t) < \frac{1}{2^\ell}} B_t$ with measure $\frac{1}{2^{\ell+1}} \leq \mu(\underline{B}_\ell) \leq \frac{1}{2^\ell}$. For each $\ell \in \mathcal{N}$, let $f_\ell = \chi_{\underline{B}_\ell}$. Fix $0 < \delta < \frac{1}{2}$ and let $j = \lfloor \frac{\log 4i}{\delta \log 2} \rfloor$. Then for any $\ell \in \mathcal{N} \cap [j, \infty)$, $\mathcal{M}^i_{\lfloor 2^{\ell(1+\delta)} \rfloor, \underline{B}_\ell} \subset \mathcal{C}_{\lfloor 2^{\ell(1+\delta)} \rfloor, f_\ell}$. Hence to prove (2.8) it is enough to show that

$$\sum_{\ell \in \mathcal{N} \cap [j,\infty)} \mu\left(\mathcal{C}_{\lfloor 2^{\ell(1+\delta)} \rfloor, f_\ell}\right) < \infty.$$

We now use Proposition 4.3 to estimate $\mu\left(\mathcal{C}_{\lfloor 2^{\ell(1+\delta)} \rfloor, f_\ell}\right)$. First, when G has property (T), Proposition 4.3 with $\epsilon = \frac{\delta}{2(1+\delta)}$ gives that

$$\mu\left(\mathcal{C}_{\lfloor 2^{\ell(1+\delta)} \rfloor, f_\ell}\right) \ll \frac{1}{\lfloor 2^{\ell(1+\delta)} \rfloor^{1-\epsilon} \mu(\underline{B}_\ell)} \ll 2^{-\delta\ell/2},$$

is summable. Next, for G locally isomorphic to $\mathrm{SO}(d+1,1)$ or $\mathrm{SU}(d,1)$ with $d \geq 2$, Proposition 4.3 implies that

$$\mu\left(\mathcal{C}_{\lfloor 2^{\ell(1+\delta)} \rfloor, f_\ell}\right) \ll_\epsilon \frac{1}{\lfloor 2^{\ell(1+\delta)} \rfloor \mu(\underline{B}_\ell)} + \frac{1}{\left(\lfloor 2^{\ell(1+\delta)} \rfloor \mu(\underline{B}_\ell)^{\frac{2}{3}}\right)^{2\tau(\Gamma)/3}}$$

$$\ll 2^{-\delta\ell} + 2^{-2\tau(\Gamma)\ell/9}$$

is also summable. Finally, when G is locally isomorphic to $\mathrm{SO}(2,1)$, Proposition 4.3 with $\epsilon = \frac{\delta}{3+2\delta}$ implies that

$$\mu\left(\mathcal{C}_{\lfloor 2^{\ell(1+\delta)} \rfloor, f_\ell}\right) \ll_\delta 2^{-\delta\ell/2} + 2^{-\delta\tau(\Gamma)\ell}$$

is summable. $\qquad\square$

5.2. Logarithm laws.

We now apply our results to the special case where $\Gamma\backslash G$ is not compact and the shrinking targets are cusp neighborhoods, to prove Corollary 1.2, establishing logarithm laws for one-parameter flows. Since $\Gamma\backslash G$ is not compact when G is of higher rank, we may assume that Γ is a congruence group. We also assume here that G is not locally isomorphic to a product of copies of $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$, and note that, in that case, Corollary 1.2 already follows from [KM12].

Now, fix a $K \times K$-invariant function $d(\cdot, \cdot)$ on $\Gamma\backslash G \times \Gamma\backslash G$ (coming from a distance function on $\Gamma\backslash\mathcal{H}$) satisfying (1.4) as in the introduction. For a fixed reference point $x_0 \in \mathcal{X}$ let

$$B_t = \{x \in \mathcal{X} \mid d(x, x_0) > t\}$$

be the corresponding spherical cusp neighborhood. We first note that the easy half of Borel-Cantelli lemma, together with a standard continuity argument, implies that the upper bound,

$$\limsup_{t\to\infty} \frac{d(xh_t, x_0)}{\log t} \leq \frac{1}{\varkappa},$$

holds for a.e. $x \in \mathcal{X}$. For the lower bound, note that by Theorem 1.1, when G has property (T) or is of real rank one the limit (1.3) holds for a.e. $x \in \mathcal{X}$. When G is of higher rank without property (T), since Γ is a congruence subgroup and G is not locally isomorphic to a product of copies of $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$, there is no factor of G locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$. Thus, using the known bounds towards the Selberg-Ramanujan conjecture (see Remarks 1 and 13) the limit (1.1) still holds for a.e. $x \in \mathcal{X}$. The proof of Corollary 1.2 now follows immediately from the following Lemma.

Lemma 5.2. *For any $x \in \mathcal{X}$ satisfying* (1.3) *we have that*

$$\limsup_{t\to\infty} \frac{d(xh_t, x_0)}{\log t} \geq \frac{1}{\varkappa}.$$

Proof. By (1.3) and (1.4), for any small $\epsilon > 0$, there exists $t_0 > 0$ such that for any $t \geq t_0$ we have that $\frac{\log \tau_{B_t}(x)}{-\log(\mu(B_t))} < 1 + \epsilon$ and $\frac{-\log \mu(B_t)}{t} < \varkappa + \epsilon$, hence, $\frac{\log \tau_{B_t}(x)}{t} < (\varkappa + \epsilon)(1 + \epsilon)$. Moreover, by the minimality of $\tau_{B_t}(x)$ we have that $xh_{\tau_{B_t}(x)} \in B_t$, or equivalently that $d(xh_{\tau_{B_t}(x)}, x_0) > t$. For any integer $\ell \in \mathbb{N}$ let $s_\ell = \tau_{B_\ell}(x)$. The condition that x satisfies (1.3), implies that $s_\ell \to \infty$ as $\ell \to \infty$, and for any $\ell \geq t_0$, we have

$$\frac{d(xh_{s_\ell}, x_0)}{\log s_\ell} = \frac{d(xh_{\tau_{B_\ell}(x)}, x_0)}{\log \tau_{B_\ell}(x)} > \frac{d(xh_{\tau_{B_\ell}(x)}, x_0)}{(\varkappa + \epsilon)(1 + \epsilon)\ell} > \frac{1}{(\varkappa + \epsilon)(1 + \epsilon)}.$$

To conclude, for any x satisfying (1.3) and for any $\epsilon > 0$ we can find an unbounded sequence $\{s_\ell\}$ such that $\frac{d(xh_{s_\ell}, x_0)}{\log s_\ell} > \frac{1}{(\varkappa+\epsilon)(1+\epsilon)}$, implying that $\limsup_{t\to\infty} \frac{d(xh_t, x_0)}{\log t} \geq \frac{1}{\varkappa}$. $\square$

In [KM99, AM17], the cusp neighborhoods are defined by a distance function $\tilde{d}(\cdot, \cdot)$ on $\Gamma\backslash G$ induced from a left G-invariant and bi-K-invariant metric (resp. norm like pseudo metric) on G, rather than a distance function on $\Gamma\backslash\mathcal{H}$ as in our case. More generally, [KM99, Section 1.6], considered cusp neighborhoods defined by a $\varkappa$-*distance like function*, that is, a function $\Delta : \mathcal{X} \to [0, \infty)$, that is uniformly continuous and satisfies that $\mu(\{x \in \mathcal{X} \mid \Delta(x) > t\}) \asymp e^{-\varkappa t}$ uniformly for all $t > 0$.

Remark 14. We refer to [KM99] for examples of $\varkappa$-distance like functions, and just point out here two important examples. First, as implied by the name, the function $x \mapsto \tilde{d}(x, x_0)$ for some fixed $x_0 \in \mathcal{X}$ is $\varkappa$-distance like for some $\varkappa > 0$ [KM99, Section 5]. Another important example of a d-distance like function is the shortest length function α_1 defined on the space of unimodular lattices $\mathcal{X}_d = \mathrm{SL}_d(\mathbb{Z}) \backslash \mathrm{SL}_d(\mathbb{R})$ by $\alpha_1(\Lambda) = \sup_{v \in \Lambda \backslash \{0\}} \log\left(\frac{1}{\|v\|_\infty}\right)$, where $\|\cdot\|_\infty$ denote the maximum norm on $\mathbb{R}^d$ [KM99, Section 7].

For a general κ-distance like functions Δ the shrinking targets $\tilde{B}_t = \{x \in \mathcal{X} \mid \Delta(x) > t\}$ may not be spherical and Corollary 1.2 does not apply directly. Nevertheless by replacing these non-spherical neighborhoods by spherical ones we get the following general consequence.

Corollary 5.3. *For $\mathcal{X} = \Gamma \backslash G$ as in Corollary 1.2 and Δ a κ-distance like function on $\mathcal{X}$ we have that for a.e. $x \in \mathcal{X}$*

$$\limsup_{t \to \infty} \frac{\Delta(x h_t)}{\log t} = \frac{1}{\varkappa}.$$

Proof. Replace the non-spherical function Δ by the spherical function $\Delta_K(x) := \inf_{k \in K} \Delta(xk)$, for which the corresponding cusp neighborhoods $B_t := \{x \in \mathcal{X} \mid \Delta_K(x) > t\}$ are clearly spherical. From uniform continuity of Δ, there exists some $\delta > 0$ such that $|\Delta(x) - \Delta(y)| < 1$ for any $x, y \in \mathcal{X}$ with $\tilde{d}(x, y) < \delta$ where $\tilde{d}(\cdot, \cdot)$ is the distance function on $\mathcal{X}$ induced from a left G-invariant and bi-K-invariant metric on G. In particular, since K is compact there is some $C > 0$ such that $|\Delta(x) - \Delta(xk)| < C$ for all $x \in \mathcal{X}$ and all $k \in K$. Thus the corresponding cusp neighborhoods satisfy $\tilde{B}_{t+C} \subset B_t \subset \tilde{B}_t$. Since by definition $\mu(\tilde{B}_t) \asymp e^{-\varkappa t}$, we also have $\mu(B_t) \asymp_C e^{-\varkappa t}$. Now applying Corollary 1.2 to $\{B_t\}_{t>0}$ and using the relation $\Delta(x) = \Delta_K(x) + O(1)$, we get that for a.e. $x \in \mathcal{X}$

$$\limsup_{t \to \infty} \frac{\Delta(x h_t)}{\log t} = \limsup_{t \to \infty} \frac{\Delta_K(x h_t)}{\log t} = \frac{1}{\varkappa}. \qquad \square$$

5.3. **Orbits eventually always hitting.** We now give the proof of Theorem 1.8. Given a monotone sequence of spherical shrinking targets $\{B_m\}_{m \in \mathbb{N}}$, let $f_m = \chi_{B_m}$. By Proposition 4.3, for SD flows

$$\mu\left(\mathcal{C}_{2^{j \pm 1}, f_{2^j}}\right) \ll \frac{1}{2^{j \pm 1} \mu(B_{2^j})} \asymp \frac{1}{2^j \mu(B_{2^j})}.$$

and if G is locally isomorphic to $\mathrm{SO}(d+1, 1)$ or $\mathrm{SU}(d, 1)$ with $d \geq 2$, then

$$\mu\left(\mathcal{C}_{2^{j \pm 1}, f_{2^j}}\right) \ll \frac{1}{2^j \mu(B_{2^j})} + \frac{1}{(2^j \mu(B_{2^j})^{\frac{2}{3}})^{2\tau(\Gamma)/3}}$$

$$< \frac{1}{2^j \mu(B_{2^j})} + \frac{2^{-2j/9}}{(2^j \mu(B_{2^j}))^{4\tau(\Gamma)/9}}.$$

Since $\sum_{j=0}^{\infty} \frac{1}{2^j \mu(B_{2^j})} < \infty$, there exists j_0 such that for any $j \geq j_0$, $2^j \mu(B_{2^j}) > 1$. Hence for all $j \geq j_0$, we have

$$\mu\left(\mathcal{C}_{2^{j \pm 1}, f_{2^j}}\right) \ll_\epsilon \frac{1}{2^j \mu(B_{2^j})} + 2^{-2j/9}.$$

25

Thus in both cases, (1.7) implies that $\sum_{j=0}^{\infty} \mu\left(\mathcal{C}_{2^{j\pm1}, f_{2^j}}\right) < \infty$, and since $\mathcal{M}_{m,B}^1 \subseteq \mathcal{C}_{m,f}$ with $f = \chi_B$, then $\sum_{j=0}^{\infty} \mu\left(\mathcal{M}_{2^{j\pm1}, B_{2^j}}^1\right) < \infty$ and Theorem 1.8 follows from Lemma 2.3.

5.4. **Dynamical Borel-Cantelli for SD flows.** We now give the proof of Theorem 1.5, by showing that rate of decay of matrix coefficients for SD flows is sufficient to show that for any sequence of spherical shrinking targets, $\{B_m\}_{m\in\mathbb{N}}$, the family of functions $f_m(x) = \chi_{B_m}(xh_m)$ satisfy condition (2.9) of Lemma 2.5.

Proposition 5.4. *Let $\{B_m\}_{m\in\mathbb{N}}$ denote any sequence of spherical sets in $\mathcal{X}$ and let $f_m(x) = \chi_{B_m}(xh_m)$. If the flow is SD then there exists some constant $C > 0$ such that for all $n > m \geq 1$*

$$\int_{\mathcal{X}} \left(\sum_{i=m}^{n} f_i(x) - \sum_{i=m}^{n} \mu(f_i)\right)^2 d\mu(x) \leq C \sum_{i=m}^{n} \mu(f_i).$$

Proof. We first note that by direct computation

$$\int_{\mathcal{X}} \left(\sum_{i=m}^{n} f_i(x) - \sum_{i=m}^{n} \mu(f_i)\right)^2 d\mu(x) = \sum_{m \leq i,j \leq n} \left(\int_{\mathcal{X}} f_i(x)f_j(x)d\mu(x) - \mu(f_i)\mu(f_j)\right).$$

To simplify notation, let $\mu_i = \mu(f_i) = \mu(B_i)$ and $\mu_{i,j} = \int_{\mathcal{X}} f_i(x)f_j(x)d\mu(x)$, and note that $\mu_{i,i} = \mu_i$ (since for $f_i(x) = \chi_{B_i}(xh_i)$, $f_i^2 = f_i$). With these notations we have

$$\int_{\mathcal{X}} \left(\sum_{i=m}^{n} f_i(x) - \sum_{i=m}^{n} \mu(f_i)\right)^2 d\mu(x) = \sum_{i=m}^{n} \mu_i + \sum_{m \leq i \neq j \leq n} \mu_{i,j} - \sum_{1 \leq i,j \leq n} \mu_i\mu_j$$

$$\leq \sum_{i=m}^{n} \mu_i + \sum_{m \leq i \neq j \leq n} (\mu_{i,j} - \mu_i\mu_j).$$

Thus it suffices to show that

$$\sum_{m \leq i \neq j \leq n} (\mu_{i,j} - \mu_i\mu_j) \ll \sum_{i=m}^{n} \mu_i.$$

Since we assume the flow is SD, there exists some constant $\eta > 1$ such that for any spherical $\varphi, \psi \in L_0^2(\Gamma \backslash G)$, for all $|t| \geq 1$

$$|\langle \pi(h_t)\varphi, \psi\rangle| \ll \frac{\|\varphi\|_2\|\psi\|_2}{|t|^\eta}.$$

In particular, for any $i \neq j$, taking $\varphi = \chi_{B_i} - \mu(B_i)$ and $\psi = \chi_{B_j} - \mu(B_j)$ we get that

$$|\mu_{i,j} - \mu_i\mu_j| = |\langle \pi(h_i)\varphi, \pi(h_j)\psi\rangle| = |\langle \pi(h_{i-j})\varphi, \psi\rangle|$$

$$\ll \frac{\|\varphi\|_2\|\psi\|_2}{|i-j|^\eta} \leq \frac{\sqrt{\mu_i\mu_j}}{|i-j|^\eta}.$$

It now suffices to show that

$$\sum_{m \leq i \neq j \leq n} \frac{\sqrt{\mu_i\mu_j}}{|j-i|^\eta} \ll \sum_{i=m}^{n} \mu_i.$$

26

We rewrite the sum on the left as

$$\sum_{m \le i \ne j \le n} \frac{\sqrt{\mu_i \mu_j}}{|j-i|^\eta} = \sum_{\substack{m \le i \ne j \le n \\ \mu_i \le \mu_j}} \frac{\sqrt{\mu_i \mu_j}}{|j-i|^\eta} + \sum_{\substack{m \le i \ne j \le n \\ \mu_i > \mu_j}} \frac{\sqrt{\mu_i \mu_j}}{|j-i|^\eta},$$

and using symmetry we can bound

$$\sum_{m \le i \ne j \le n} \frac{\sqrt{\mu_i \mu_j}}{|j-i|^\eta} \le 2 \sum_{\substack{m \le i \ne j \le n \\ \mu_i \le \mu_j}} \frac{\sqrt{\mu_i \mu_j}}{|j-i|^\eta} \le 2 \sum_{\substack{m \le i \ne j \le n \\ \mu_i \le \mu_j}} \frac{\mu_j}{|j-i|^\eta}$$

$$= 2 \sum_{m \le j \le n} \mu_j \Big(\sum_{\substack{m \le i \le n \\ \mu_i \le \mu_j, i \ne j}} \frac{1}{|j-i|^\eta} \Big).$$

We can bound the inner sum by the convergent series $2\sum_{i=1}^\infty \frac{1}{i^\eta}$ thus concluding the proof. $\square$

Proof of Theorem 1.5. Let $\{B_m\}_{m \in \mathbb{N}}$ denote any sequence of spherical sets in $\mathcal{X}$ satisfying that $\sum_{m=1}^\infty \mu(B_m) = \infty$. For $f_m(x) = \chi_{B_m}(xh_m)$ we have that

$$\sum_{1 \le j \le m} f_j(x) = \#\{1 \le j \le m : xh_j \in B_j\},$$

and hence, Proposition 5.4 and Lemma 2.5 imply that for a.e. $x \in \mathcal{X}$

$$\lim_{m \to \infty} \frac{\#\{1 \le j \le m : xh_j \in B_j\}}{\sum_{1 \le j \le m} \mu(B_j)} = 1. \qquad \square$$

5.5. **Dynamical Borel-Cantelli for rank one groups.** We now turn to the case where G is locally isomorphic to $\mathrm{SO}(d+1,1)$ or $\mathrm{SU}(d,1)$ with $d \ge 2$ (if G is of real rank one with property (T), then every unbounded flow is SD and we can apply Theorem 1.5). Combining bounds coming from the mean ergodic theorem, together with Lemma 2.5 we show that any monotone sequence of spherical shrinking targets, $\{B_m\}_{m \in \mathbb{N}}$, is BC. In this case, the argument is different when the sequence $\{m\mu(B_m)\}_{m \in \mathbb{N}}$ is bounded or unbounded, and we treat these cases separately. We first deal with the bounded case by showing the following.

Proposition 5.5. *For G locally isomorphic to $\mathrm{SO}(d+1,1)$ or $\mathrm{SU}(d,1)$ with $d \ge 2$, let $\{B_m\}_{m \in \mathbb{N}}$ denote a sequence of spherical sets satisfying that $\{m\mu(B_m)\}_{m \in \mathbb{N}}$ is uniformly bounded. Let $f_m(x) = \chi_{B_m}(xh_m)$, then there exists some constant $C > 0$ such that for all $n > m \ge 1$*

$$\int_{\mathcal{X}} \Big(\sum_{i=m}^n f_i(x) - \sum_{i=m}^n \mu(f_i) \Big)^2 d\mu(x) \le C \sum_{i=m}^n \mu(f_i).$$

Proof. As in the proof of Proposition 5.4, denote by $\mu_i = \mu(f_i) = \mu(B_i)$ and $\mu_{i,j} = \int_{\mathcal{X}} f_i(x) f_j(x) d\mu(x)$. Using the spectral decomposition we can write

$$\chi_{B_m} = \mu(B_m) + \sum_k \langle \chi_{B_m}, \varphi_k \rangle \varphi_k + f_m^0,$$

27

with $f_m^0 \in L^2_{\text{temp}}(\Gamma \backslash \mathcal{H})$. Hence for any $i \neq j$,

$$
\begin{aligned}
\mu_{i,j} &= \langle f_i, f_j \rangle = \langle \pi(h_i)\chi_{B_i}, \pi(h_j)\chi_{B_j} \rangle = \langle \pi(h_{i-j})\chi_{B_i}, \chi_{B_j} \rangle \\
&= \mu_i \mu_j + \sum_k \langle \chi_{B_i}, \varphi_k \rangle \overline{\langle \chi_{B_j}, \varphi_k \rangle} \langle \pi(h_{i-j})\varphi_k, \varphi_k \rangle + \langle \pi(h_{i-j})f_i^0, f_j^0 \rangle.
\end{aligned}
$$

Thus by Proposition 3.6 we have for any small $\epsilon > 0$

$$
|\mu_{i,j} - \mu_i \mu_j| \ll_\epsilon \sum_k \frac{|\langle \chi_{B_i}, \varphi_k \rangle \langle \chi_{B_j}, \varphi_k \rangle|}{|i-j|^{\kappa(\rho - s_k)(1-\epsilon)}} + \frac{\sqrt{\mu_i \mu_j}}{|i-j|^{\kappa\rho(1-\epsilon)}}.
$$

Let $\eta = \kappa\rho(1-\epsilon)$ and note that for all $0 < \epsilon < 1/4$ we have that $\eta > 3/2$ (since G is not locally isomorphic to $\mathrm{SO}(2,1)$). Hence, by the same arguments as in the proof of Proposition 5.4, for any $0 < \epsilon < \frac{1}{4}$ we can bound

$$
\sum_{m \leq i \neq j \leq n} \frac{\sqrt{\mu_i \mu_j}}{|j-i|^{\kappa\rho(1-\epsilon)}} \ll \sum_{i=m}^n \mu_i.
$$

Hence, it suffices to show that for each of the finitely many exceptional forms we have

$$
\sum_{m \leq i \neq j \leq n} \frac{|\langle \chi_{B_i}, \varphi_k \rangle \langle \chi_{B_j}, \varphi_k \rangle|}{|i-j|^{\kappa\rho(1-\epsilon)}} \ll_\epsilon \sum_{i=m}^n \mu_i.
$$

Now let $0 < \epsilon_0 < \frac{1}{4}$ be sufficiently small such that $\frac{2\rho}{(\rho+s_k)(1-\epsilon_0)} < 2$ for all s_k. For $0 < \epsilon \leq \epsilon_0$ (to be determined later) let $q_k = \frac{2\rho}{(\rho+s_k)(1-\epsilon)}$ and let $\eta_k = \kappa(\rho - s_k)(1-\epsilon)$. Note that $\frac{1}{2} < \frac{1}{q_k} < 1$. Recalling that the exceptional form $\varphi_k \in \pi_{s_k}$ is contained in $L^p(\mathcal{X})$ for any $1 < p < \frac{2\rho}{\rho - s_k}$ we can bound

$$
|\langle \chi_{B_i}, \varphi_k \rangle \langle \chi_{B_j}, \varphi_k \rangle| \ll_\epsilon \mu_i^{\frac{1}{q_k}} \mu_j^{\frac{1}{q_k}},
$$

and hence together with symmetry we can bound

$$
\sum_{m \leq i \neq j \leq n} \frac{|\langle \chi_{B_i}, \varphi_k \rangle \langle \chi_{B_j}, \varphi_k \rangle|}{|i-j|^{\kappa(\rho - s_k)(1-\epsilon)}} \ll \sum_{m \leq i \neq j \leq n} \frac{\mu_i^{\frac{1}{q_k}} \mu_j^{\frac{1}{q_k}}}{|j-i|^{\eta_k}} \ll \sum_{\substack{m \leq i \neq j \leq n \\ \mu_j \leq \mu_i}} \frac{\mu_i^{\frac{1}{q_k}} \mu_j^{\frac{1}{q_k}}}{|j-i|^{\eta_k}}.
$$

Since $\frac{1}{q_k} < 1$, for $\mu_j \leq \mu_i$ we can bound $\mu_i^{\frac{1}{q_k}} \mu_j^{\frac{1}{q_k}} \leq \mu_i \mu_j^{\frac{2}{q_k}-1}$ to get that

$$
\sum_{\substack{m \leq i \neq j \leq n \\ \mu_j \leq \mu_i}} \frac{\mu_i^{\frac{1}{q_k}} \mu_j^{\frac{1}{q_k}}}{|j-i|^{\eta_k}} \leq \sum_{\substack{m \leq i \neq j \leq n \\ \mu_j \leq \mu_i}} \frac{\mu_i \mu_j^{\frac{2}{q_k}-1}}{|j-i|^{\eta_k}}.
$$

Now using the assumption that $\{\ell\mu_\ell\}_{\ell \in \mathbb{N}}$ is uniformly bounded, (and noting that $\frac{2}{q_k} > 1$) we can bound $\mu_j^{\frac{2}{q_k}-1} \ll \frac{1}{j^{\frac{2}{q_k}-1}}$ so that

$$
\sum_{\substack{m \leq i \neq j \leq n \\ \mu_j \leq \mu_i}} \frac{\mu_i \mu_j^{\frac{2}{q_k}-1}}{|j-i|^{\eta_k}} \ll \sum_{m \leq i \leq n} \mu_i \left(\sum_{\substack{m \leq j \leq n \\ \mu_j \leq \mu_i, j \neq i}} \frac{1}{j^{\frac{2}{q_k}-1}|j-i|^{\eta_k}} \right).
$$

28

For the cases we consider we have that $\kappa - \frac{1}{\rho} \geq 1/2$ and we can estimate the exponent

$$\frac{2}{q_k} - 1 + \eta_k = \frac{(\rho + s_k)(1 - \epsilon)}{\rho} - 1 + \kappa(\rho - s_k)(1 - \epsilon) \geq (1 - \epsilon)\left(2 + \tfrac{\tau}{2}\right) - 1,$$

where $\tau = \tau(\Gamma)$ is the spectral gap parameter for Γ. Taking $\epsilon = \min\{\frac{1}{2}\epsilon_0, \frac{\tau}{2\tau+8}\}$ we get that $\frac{2}{q_k} - 1 + \eta_k \geq 1 + \frac{1}{4}\tau$. Hence for any $m \leq i \leq n$, the sum

$$\sum_{\substack{m \leq j \leq n \\ \mu_j \leq \mu_i, j \neq i}} \frac{1}{j^{\frac{2}{q_k} - 1}|j - i|^{\eta_k}} \leq \sum_{\substack{j > 0 \\ j \neq i}} \left(\frac{1}{j^{\frac{2}{q_k} - 1 + \eta_k}} + \frac{1}{|i - j|^{\frac{2}{q_k} - 1 + \eta_k}} \right)$$

$$\ll \sum_{\ell \neq 0} \frac{1}{|\ell|^{\frac{2}{q_k} - 1 + \eta_k}} \leq \sum_{\ell \neq 0} \frac{1}{|\ell|^{1 + \frac{1}{4}\tau}} < \infty,$$

is uniformly bounded and thus concluding the proof. $\qquad\square$

Next, we consider the case where $\{m\mu(B_m)\}_{m \in \mathbb{N}}$ is unbounded. For this case we use results of the effective mean ergodic theorem to show the following.

Proposition 5.6. *For G locally isomorphic to $\mathrm{SO}(d + 1, 1)$ or $\mathrm{SU}(d, 1)$ with $d \geq 2$, let $\{B_m\}_{m \in \mathbb{N}}$ be a monotone family of spherical shrinking targets in $\mathcal{X} = \Gamma\backslash G$ satisfying that $\sum_{m=1}^{\infty} \mu(B_m) = \infty$ and that $\{m\mu(B_m)\}_{m \in \mathbb{N}}$ is unbounded. Then there is a subsequence m_j with $m_j\mu(B_{m_j}) \to \infty$ satisfying that for a.e. $x \in \mathcal{X}$*

$$\lim_{j \to \infty} \frac{\#(xH_{m_j}^+ \cap B_{m_j})}{m_j\mu(B_{m_j})} = 1.$$

Proof. For any $m \in \mathbb{N}$ let $f_m = \frac{\chi_{B_m}}{\mu(B_m)}$ and note that $\beta_m^+(f_m)(x) = \frac{\#(xH_m^+ \cap B_m)}{m\mu(B_m)}$. Hence it suffices to show that there is some subsequence $\{m_j\}$ such that $\beta_{m_j}(f_{m_j})(x) \to 1$ as $j \to \infty$ for a.e. $x \in \mathcal{X}$. By Proposition 4.2 we have

$$\|\beta_m^+(f_m) - 1\|_2 \ll_\epsilon \frac{1}{\sqrt{m\mu(B_m)}} + \sum_{s_k \in [\rho - \frac{1}{\kappa}, \rho)} \frac{|\langle f_m, \varphi_k \rangle|}{m^{\frac{\kappa}{2}(\rho - s_k)(1 - \epsilon)}}.$$

Recall that we can bound $|\langle f_m, \varphi_k \rangle| \ll_q \|f_m\|_q$ for any $q > \frac{2\rho}{\rho + s_k}$. Now for $0 < \epsilon < 1 - \frac{1}{\kappa\rho}$, for any $s_k \in [\rho - \frac{1}{\kappa}, \rho)$ we have $q_k = \frac{1}{1 - \frac{\kappa}{2}(\rho - s_k)(1 - \epsilon)} > \frac{2\rho}{\rho + s_k}$ and hence $|\langle f_m, \varphi_k \rangle| \ll_\epsilon \|f_m\|_{q_k} = \frac{1}{\mu(B_m)^{1 - \frac{1}{q_k}}} = \frac{1}{\mu(B_m)^{\frac{\kappa}{2}(\rho - s_k)(1 - \epsilon)}}$. Thus for any $0 < \epsilon < 1 - \frac{1}{\kappa\rho}$ we have

$$\|\beta_m^+(f_m) - 1\|_2 \ll_\epsilon \frac{1}{\sqrt{m\mu(B_m)}} + \sum_{s_k \in [\rho - \frac{1}{\kappa}, \rho)} \frac{1}{(m\mu(B_m))^{\frac{\kappa}{2}(\rho - s_k)(1 - \epsilon)}}.$$

Since $\{m\mu(B_m)\}_{m \in \mathbb{N}}$ is unbounded, there is a subsequence satisfying that $m_j\mu(B_{m_j}) \to \infty$, for which $\|\beta_{m_j}(f_{m_j}) - 1\|_2 \to 0$ as $j \to \infty$. Passing to another subsequence, if necessary, we get that $\beta_{m_j}(f_{m_j})(x) \to 1$ for a.e. $x \in \mathcal{X}$. $\qquad\square$

We now combine the two cases to complete the proof.

Proof of Theorem 1.7. Let G be locally isomorphic to $\mathrm{SO}(d+1,1)$ or $\mathrm{SU}(d,1)$ with $d \geq 2$, and let $\{B_m\}_{m \in \mathbb{N}}$ denote a monotone family of spherical shrinking targets with $\sum_m \mu(B_m) = \infty$. Now, if the sequence $\{m\mu(B_m)\}_{m \in \mathbb{N}}$ is bounded then by Proposition 5.5 we have that for a.e. $x \in \mathcal{X}$,

$$\lim_{m \to \infty} \frac{\#\{1 \leq j \leq m : xh_j \in B_j\}}{\sum_{1 \leq j \leq m} \mu(B_j)} = 1.$$

If the sequence $\{m\mu(B_m)\}_{m \in \mathbb{N}}$ is unbounded then by Proposition 5.6 there is a subsequence m_j such that for a.e. $x \in \mathcal{X}$

$$\lim_{j \to \infty} \frac{\#\{1 \leq i \leq m_j : xh_i \in B_{m_j}\}}{m_j \mu(B_{m_j})} = 1.$$

In both cases, for a.e. $x \in \mathcal{X}$ the set $\{m \in \mathbb{N} : xh_m \in B_m\}$ is unbounded, so $\{B_m\}_{m \in \mathbb{N}}$ is BC for this flow. $\qquad\square$

REFERENCES

[AM09] J. S. Athreya and G. A. Margulis. Logarithm laws for unipotent flows. I. *J. Mod. Dyn.*, 3(3):359–378, 2009. 1, 3

[AM17] J. S. Athreya and G. A. Margulis. Logarithm laws for unipotent flows, II. *J. Mod. Dyn.*, 11:1–16, 2017. 1, 3, 24

[BB11] V. Blomer and F. Brumley. On the Ramanujan conjecture over number fields. *Ann. of Math. (2)*, 174(1):581–605, 2011. 3, 16, 19

[BC13] N. Bergeron and L. Clozel. Quelques conséquences des travaux d'Arthur pour le spectre et la topologie des variétés hyperboliques. *Invent. Math.*, 192(3):505–532, 2013. 3, 16, 19

[BC17] N. Bergeron and L. Clozel. Sur le spectre et la topologie des variétés hyperboliques de congruence: les cas complexe et quaternionien. *Mathematische Annalen*, 368(3):1333–1358, Aug 2017. 3, 16, 19

[BS91] M. Burger and P. Sarnak. Ramanujan duals II. *Invent. Math*, 106:1–11, 1991. 16

[CK01] N. Chernov and D. Kleinbock. Dynamical Borel-Cantelli lemmas for Gibbs measures. *Israel J. Math.*, 122:1–27, 2001. 1, 2

[Dol04] D. Dolgopyat. Limit theorems for partially hyperbolic systems. *Trans. Amer. Math. Soc.*, 356(4):1637–1689 (electronic), 2004. 1

[Gal07] S. Galatolo. Dimension and hitting time in rapidly mixing systems. *Math. Res. Lett.*, 14(5):797–805, 2007. 1

[GK07] S. Galatolo and D. H. Kim. The dynamical Borel-Cantelli lemma and the waiting time problems. *Indag. Math. (N.S.)*, 18(3):421–434, 2007. 2

[GK17] A. Ghosh and D. Kelmer. Shrinking targets for semisimple groups. *Bull. Lond. Math. Soc.*, 49(2):235–245, 2017. 19

[GS11] A. Gorodnik and N. A. Shah. Khinchin's theorem for approximation by integral points on quadratic varieties. *Math. Ann.*, 350(2):357–380, 2011. 1

[GV12] R. Gangolli and V. S. Varadarajan. *Harmonic analysis of spherical functions on real reductive groups*, volume 101. Springer Science & Business Media, 2012. 15

[Hel78] S. Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 80 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1978. 12

[How82] R. Howe. On a notion of rank for unitary representations of the classical groups. In *Harmonic analysis and group representations*, pages 223–331. Liguori, Naples, 1982. 15

[HP10] Sa'ar Hersonsky and Frédéric Paulin. On the almost sure spiraling of geodesics in negatively curved manifolds. *J. Differential Geom.*, 85(2):271–314, 2010. 3

[Kel17] Dubi Kelmer. Shrinking targets for discrete time flows on hyperbolic manifolds. *Geom. Funct. Anal.*, 27(5):1257–1287, 2017. 1, 3, 5, 6, 9, 15, 19, 20, 21

[KM96]   D. Y. Kleinbock and G. A. Margulis. Bounded orbits of nonquasiunipotent flows on homogeneous spaces. *Amer. Math. Soc. Transl.*, 171:141–172, 1996. 18

[KM99]   D. Y. Kleinbock and G. A. Margulis. Logarithm laws for flows on homogeneous spaces. *Invent. Math.*, 138(3):451–494, 1999. 1, 3, 24, 25

[KM12]   D. Kelmer and A. Mohammadi. Logarithm laws for one parameter unipotent flows. *Geom. Funct. Anal.*, 22(3):756–784, 2012. 1, 3, 24

[Kna02]   A. W. Knapp. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 2002. 11, 17

[Kos69]   B. Kostant. On the existence and irreducibility of certain series of representations. *Bull. Amer. Math. Soc.*, 75:627–642, 1969. 14

[KS03]   H. H. Kim and P. Sarnak. Refined estimates towards the Ramanujan and Selberg conjectures, appendix to H. H. Kim, Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2, with appendix 1 by Dinakar Ramakrishnan and appendix 2 by Kim and Peter Sarnak. *J. Amer. Math. Soc.*, 16(1), 2003. 3, 16, 19

[KW17]   D. Kleinbock and N. Wadleigh. An inhomogeneous dirichlet theorem via shrinking targets. *arXiv preprint arXiv:1709.04082*, 2017. 6

[KZ17]   D. Kleinbock and X. Zhao. An application of lattice points counting to shrinking target problems. *Discrete Contin. Dyn. Syst.*, 38, 155–168, 2018. 1, 3

[Mau06]   François Maucourant. Dynamical Borel-Cantelli lemma for hyperbolic spaces. *Israel J. Math.*, 152:143–155, 2006. 3

[Oh98]   H. Oh. Tempered subgroups and representations with minimal decay of matrix coefficients. *Bull. Soc. Math. France*, 126(3):355–380, 1998. 4, 16, 17

[Oh02]   H. Oh. Uniform pointwise bounds for matrix coefficients of unitary representations and applications to Kazhdan constants. *Duke Math. J.*, 113(1):133–192, 2002. 17

[Rag04]   M. S. Raghunathan. The congruence subgroup problem. *Proc. Indian Acad. Sci. Math. Sci.*, 114(4):299–308, 2004. 2

[Rap92]   A. S. Rapinchuk. Congruence subgroup problem for algebraic groups: old and new. *Astérisque*, (209):11, 73–84, 1992. Journées Arithmétiques, 1991 (Geneva). 2, 19

[Sar05]   P. Sarnak. Notes on the generalized Ramanujan conjectures. In *Harmonic analysis, the trace formula, and Shimura varieties*, volume 4 of *Clay Math. Proc.*, pages 659–685. Amer. Math. Soc., Providence, RI, 2005. 2, 19

[Sha04]   F. Shahidi. On the Ramanujan conjecture for quasisplit groups. *Asian J. Math.*, 8(4):813–835, 2004. 2, 19

[Spr79]   V. G. Sprindžuk. *Metric theory of Diophantine approximations*. V. H. Winston & Sons, Washington, D.C.; A Halsted Press Book, John Wiley & Sons, New York-Toronto, Ont.-London, 1979. Translated from the Russian and edited by Richard A. Silverman, With a foreword by Donald J. Newman, Scripta Series in Mathematics. 10

[Sul82]   D. Sullivan. Disjoint spheres, approximation by imaginary quadratic numbers, and the logarithm law for geodesics. *Acta Math.*, 149(3-4):215–237, 1982. 1, 3

[Yu17]   S. Yu. Logarithm laws for unipotent flows on hyperbolic manifolds. *J. Mod. Dyn.*, 11(03):447–476, 2017. 1, 3

DEPARTMENT OF MATHEMATICS, BOSTON COLLEGE, CHESTNUT HILL MA 02467-3806, USA
Email address: kelmer@bc.edu

DEPARTMENT OF MATHEMATICS, BOSTON COLLEGE, CHESTNUT HILL MA 02467-3806, USA
Email address: shucheng.yu@bc.edu