SIAM J. MATH. ANAL.
Vol. 50, No. 5, pp. 5072–5095

HIGHER DIMENSIONAL BUBBLE PROFILES IN A SHARP INTERFACE LIMIT OF THE FITZHUGH–NAGUMO SYSTEM*

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Abstract. The FitzHugh–Nagumo system gives rise to a nonlocal geometric variational problem defined on subsets of a domain. The energy of a subset contains three terms: its perimeter, its volume, and a long-range self-interaction term represented by the integral of the solution to a screened Poisson's equation. A bubble profile is a ball-shaped stationary set when the domain is the entire space. If the space dimension is three or higher, depending on the parameters of the problem, there can be zero, one, or two bubble profiles. This is in contrast to an earlier result for the two-dimensional space, from which one may have three bubble profiles. The stability of each bubble is determined from the eigenvalues of the linearized operator. Using a stable bubble profile, one constructs a stationary assembly of perturbed balls on a general bounded domain, when the parameters are properly chosen.

Key words. FitzHugh–Nagumo equations, singular limit, nonlocal geometric variational problem, bubble profile, stationary ball assembly

AMS subject classifications. 49J40, 33C10, 92C15, 35K57

DOI. 10.1137/17M1144933

1. Introduction.

In physical and biological systems, pattern formation results from orderly outcomes of self-organization principles. Examples include morphological phases in block copolymers, animal coats, and skin pigmentation in cell development. Common in these pattern-forming systems is that a deviation from homogeneity has a strong positive feedback on its further increase. In addition pattern formation provides a longer ranging confinement of the locally self-enhancing process.

The FitzHugh–Nagumo model was originally proposed for excitable systems such as neuron fields [17, 24] and is now of great interest to the scientific community as a breeding ground for patterns, like stripes and spots, and localized structures such as standing waves [5, 7, 8, 9, 10, 11, 27] and traveling waves [4, 6, 13]. It has been extensively studied as a paradigmatic activator-inhibitor system for patterns generated from homogeneous media destabilized by a spatial modulation. These patterns are robust in the sense that they are stable and exist for a wide range of parameters; see Turing [31].

When the parameters in the steady state FitzHugh–Nagumo equations are changed in a coordinate fashion within a specific range, in the limiting case we are led to studying a geometric variational problem on a domain $D \subset \mathbb{R}^N$; see [12] and references

*Received by the editors August 28, 2017; accepted for publication (in revised form) August 6, 2018; published electronically September 18, 2018.
http://www.siam.org/journals/sima/50-5/M114493.html
Funding: The work of the first author was supported by MOST grant 105-2115-M-007-009-MY3. The work of the fourth author was supported by NSF grants DMS-1311856 and DMS-1714371.
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therein. For a measurable subset Ω of D, the associated energy functional is

$$\mathcal{J}_D(\Omega) = \mathcal{P}_D(\Omega) - \alpha|\Omega| + \frac{\sigma}{2}\int_\Omega \mathcal{N}_D(\Omega)dx, \tag{1.1}$$

where α and σ are given positive constants.

In (1.1) Ω is a measurable subset of D, $|\Omega|$ is its Lebesgue measure, and $\mathcal{P}_D(\Omega)$ is the perimeter of Ω in D. In the case that Ω is of class C^1, $\mathcal{P}_D(\Omega)$ is the area of the part of the boundary of Ω that is inside D, namely, the area of $\partial\Omega \cap D$. One calls $\partial\Omega \cap D$ the interface of Ω because it separates Ω from $D\backslash\Omega$. For a general subset Ω of D,

$$\mathcal{P}_D(\Omega) = \sup\left\{\int_\Omega \operatorname{div} g(x)\,dx : \; g \in C_0^1(D, \mathbb{R}^N), \; |g(x)| \le 1 \; \forall x \in D\right\}. \tag{1.2}$$

In (1.2), $|g(x)|$ is the geometric norm of the vector $g(x)$. $\mathcal{J}_D(\Omega)$ is finite provided the admissible set of $\mathcal{J}_D$ is

$$\mathcal{A} = \big\{\Omega \subset D : \; \Omega \text{ is Lebesgue measurable}, \; |\Omega| < \infty, \; \chi_\Omega \in BV(D)\big\}, \tag{1.3}$$

where χ_Ω is the characteristic function associated with the set Ω.

The integral term in (1.1) is most novel and is where the nonlocality of the problem comes. In this term $\mathcal{N}_D$ is an operator that assigns each Ω the solution of the following screened Poisson equation (also known as the Helmholtz equation):

$$-\Delta\mathcal{N}_D(\Omega) + \mathcal{N}_D(\Omega) = \chi_\Omega \text{ in } D; \quad \partial_\nu \mathcal{N}_D(\Omega) = 0 \text{ on } \partial D. \tag{1.4}$$

Here ∂_ν is the outward normal deriviative. If D is bounded, then with some smoothness condition on D (say, $C^{2,\alpha}$ according to [18, section 6.7]), $\mathcal{N}_D$ is well defined. If $D = \mathbb{R}^N$, then

$$\mathcal{N}_{\mathbb{R}^N}(\Omega)(x) = \int_\Omega \frac{1}{(2\pi)^{N/2}} \frac{K_{\frac{N}{2}-1}(|x-y|)}{|x-y|^{\frac{N}{2}-1}} dy, \tag{1.5}$$

where $K_{\frac{N}{2}-1}$ is the $(\frac{N}{2}-1)$th order modified Bessel function of the second kind. A stationary set of (1.1) is a solution of the following equation:

$$(N-1)\mathcal{H}(\partial\Omega \cap D) - \alpha + \sigma\mathcal{N}_D(\Omega) = 0 \text{ on } \partial\Omega \cap D. \tag{1.6}$$

Here $\mathcal{H}(\partial\Omega)$ denotes the mean curvature of $\partial\Omega \cap D$ (the arithmetic mean of the principal curvatures), in the convention that a convex Ω has nonnegative mean curvature. In addition, if the interface, which is $\partial\Omega \cap D$, of a stationary set Ω meets the domain boundary ∂D, then the two surfaces intersect perpendicularly.

To motivate our study, we now explore the connection between (1.1) and the FitzHugh–Nagumo system. Write the FitzHugh–Nagumo system in the following form:

$$u_t = \epsilon^2 \Delta u - u\left(u - \frac{1}{2}\right)(u-1) + \epsilon\alpha - \epsilon\sigma v, \tag{1.7}$$

$$\gamma v_t = \Delta v - v + u \tag{1.8}$$

with the zero Neumann boundary condition for both u and v in case of a bounded domain. The assumptions in this paper are that $\alpha > 0$ and $\sigma > 0$ are fixed, and

$\epsilon > 0$ is small. They identify a parameter range leading to a limiting problem with sharp interface solutions. We recall that u is the activator and v is the inhibitor. Physically α measures the driving force toward a nontrivial state while σ represents the stabilizing inhibition mechanism. Their competition leads to interesting dynamics and the emergence of nontrivial patterns.

We investigate the stationary version of the system, where both u_t and v_t vanish. Solve (1.8) for v in terms of u so that $v = \mathcal{N}_D u$, where $\mathcal{N}_D u$ is the solution of (1.4) with χ_Ω replaced by u. Upon substitution (1.7) becomes

$$(1.9) \qquad -\epsilon^2 \Delta u + u\left(u - \frac{1}{2}\right)(u-1) - \epsilon\alpha + \epsilon\sigma \mathcal{N}_D u = 0 \text{ in } D; \quad \partial_\nu u = 0 \text{ on } \partial D.$$

With a variational structure, the solutions of (1.9) are the critical points of the functional

$$(1.10) \qquad \mathcal{I}_{D,\epsilon}(u) = \int_D \left(\frac{\epsilon^2}{2}|\nabla u|^2 + \frac{u^2(u-1)^2}{4} - \epsilon\alpha u + \frac{\epsilon\sigma u}{2}\mathcal{N}_D u \right) dx.$$

When D is bounded, this functional has a Γ-limit. More precisely, as $\epsilon \to 0$, $\epsilon^{-1}\mathcal{I}_{D,\epsilon}$ Γ-converges to the functional

$$(1.11) \qquad \mathcal{J}_D(\Omega) = \tau \mathcal{P}_D(\Omega) - \alpha|\Omega| + \frac{\sigma}{2}\int_\Omega \mathcal{N}_D(\Omega)\, dx.$$

The functional in (1.11) is the same as the one in (1.1) except for an immaterial constant τ, which is given by

$$(1.12) \qquad \tau = \int_0^1 \sqrt{\frac{u^2(u-1)^2}{2}}\, du = \frac{\sqrt{2}}{12}.$$

One can factor out τ in (1.11) by redefining α and σ. Hence $\mathcal{J}_D$ in (1.11) is equivalent to (1.1). Because of the Γ-convergence $\mathcal{J}_D$ is considered a singular limit of the FitzHugh–Nagumo system. Several properties follow once the Γ-convergence of $\epsilon^{-1}\mathcal{I}_{D,\epsilon}$ to $\mathcal{J}_D$ is established. A global minimizer of $\mathcal{I}_{D,\epsilon}$ converges to a global minimizer of $\mathcal{J}_D$ when $\epsilon \to 0$, at least along a subsequence. If $\mathcal{J}_D$ has a strict local minimizer, then nearby there is a local miminizer of $\mathcal{I}_{D,\epsilon}$ if ϵ is sufficiently small [15, 23, 21, 28, 3].

In an earlier paper [12] we studied bubble profiles in $\mathbb{R}^2$. By a bubble we refer to a ball-shaped stationary set of $\mathcal{J}_D$ when D is the entire space $\mathbb{R}^N$. In this paper we investigate the case $N \geq 3$. While it was proved in [12] that when $N = 2$, one may have zero, one, two, or even three bubble profiles, depending on the values of α and σ, here we show that when N is three or higher, there can be no more than two bubble profiles for any given pair of α and σ.

As will be seen in section 3, this difference arises because of the monotonicity property of the function

$$(1.13) \qquad b \to b^3\Big(I_{\nu-1}(b)K_{\nu-1}(b) - I_\nu(b)K_\nu(b)\Big), \ b > 0.$$

Using the notion of completely monotone functions and Bernstein's theorem, we prove that for all real value ν greater than or equal to 1, (1.13) is monotonically increasing if and only if $\nu \geq \frac{3}{2}$. Here $I_{\nu-1}$, $K_{\nu-1}$, I_ν and K_ν are the modified Bessel functions, and $\nu \geq \frac{3}{2}$ corresponds to $N = 2\nu \geq 3$. This result is of its own significance and we state it in Theorem 3.4. It is an optimal result because ν is allowed to be real.

The monotonicity property of (1.13) when $N \geq 3$ plays a key role in this paper. On the contrary when $\nu = 1$, i.e., $N = 2$, the quantity (1.13) is not monotone. An entirely different strategy was used in [12]. Moreover, the analysis in the higher dimensional case requires the use of spherical harmonics, Legendre polynomials, and Gegenbauer ultraspherical polynomials, which is more technically demanding than the $N = 2$ case.

In Theorem 3.6 we identify the ranges of (α, σ) that yield zero, one, or two bubble profiles. We find the eigenvalues of the linearized operators at the bubble profiles and, in Theorem 4.3, determine the stability of each bubble profile. Using bubble profiles and their stability properties we prove an existence theorem, Theorem 5.1: on a general bounded domain D, when α and σ are properly chosen, there exists a stationary set of $\mathcal{J}_D$ that is an assembly of perturbed balls.

In the investigation of bubble solutions, van Heijster and Sandstede [32] dealt with the same type of activator-inhibitor models except that an additional equation for the second inhibitor is added in their model; in other words, by setting their coupling coefficient β to zero and taking out the third equation, their model is not different from ours. They treated the case of $\mathbb{R}^2$ and called the solutions planar radial spots instead of bubbles. Theorem 1.1 of [32], up to normalization when considering (1.7)–(1.8), asserts that when a simple root of

$$\frac{\sqrt{2}}{3r} + \alpha(2rI_1(r)K_0(r) - 1) = 0 \tag{1.14}$$

exists, say, $r = R_1$, then R_1 is the radius of the limiting bubble. For the stability, it is stated in the same Theorem 1.1 that the limiting profile is stable if $\hat{\lambda}(\ell) < 0$ for $\ell = 0, 2, 3, \ldots$, where

$$\hat{\lambda}(\ell) = 3\sqrt{2}\alpha R_1(I_1(R_1)K_1(R_1) - I_\ell(R_1)K_\ell(R_1)) + \frac{1}{R_1^2}(1 - \ell^2)\ . \tag{1.15}$$

Also, for (1.7)–(1.8), the results of [12] prevail over all the ratios for σ/α, while the case treated in [32] focuses on $\sigma/\alpha = 2$. It is interesting to note that if $\sigma/\alpha = 2$, stable bubbles can never exist in the higher dimensions no matter what α is, while they can for some α in $\mathbb{R}^2$. This validates the importance of the dimension of domains in such activator-inhibitor models. For the model with three species, the same methodology leads to equations parallel to (1.14)–(1.15). A complete understanding of the two-species case is a crucial first step. Our work in [12] and here completely answered the sovability question of (1.14) and the stability question of (1.15) in all dimensions.

Bubble profiles in the original FitzHugh–Nagumo system (1.7)–(1.8) were studied by Ohta, Mimura, and Kobayashi [25]. They considered a different parameter range: instead of $\epsilon\alpha$ and $\epsilon\sigma$ they put ϵ independent constants in these two places of (1.7). They also assumed that the nonlinearity is given by a piecewise linear function. Using matched asymptotics they derived an equation for the motion of an interface, found radially symmetric steady states in two and three dimensions, and determined their stability. Their formula [25, (6.1c)] is similar to our Lemma 4.1 with $N = 3$.

Throughout this paper, we assume that $N \geq 3$. In Lemma 3.3 and Theorem 3.4, N is a real number, not necessarily an integer.

2. Preliminary lemmas. In this paper I_ν and K_ν, $\nu \geq 0$, are the νth order modified Bessel functions of the first and second kind, respectively. These functions are positive on the positive real axis. In this work, we mainly focus on the cases when ν is an integer or a half integer, but the properties we introduce here hold for all $\nu \geq 0$.

LEMMA 2.1.

1. *Let $z > 0$ be fixed and $\nu \geq 0$. Then*

$$I_\nu(z)K_\nu(z) \text{ is decreasing in } \nu. \tag{2.1}$$

2. *Let $\nu \geq 0$ and define $g_\nu : [0,\infty) \to [0,\infty)$ to be $g_\nu(z) \equiv zI_{\nu+1}(z)K_\nu(z)$. Then g_ν is a strictly increasing function with $g_\nu'(z) = z(I_\nu\, K_\nu - I_{\nu+1}\, K_{\nu+1})$. Moreover $g_\nu(0) = 0$, $\lim_{z\to\infty} g_\nu(z) = 1/2$ and*

$$g_\nu(z) = \frac{1}{2}\left(1 - \frac{2\nu+1}{2z} + \frac{(2\nu-1)(2\nu+1)(2\nu+3)}{16z^3} + O\left(\frac{1}{z^4}\right)\right) \tag{2.2}$$

as $z \to \infty$.

Proof. Part 1 is proved in [2, Theorem 2]. For part 2 employing [1, (9.6.26)], namely,

$$I_{\nu+1}' = I_\nu - \frac{\nu+1}{z}I_{\nu+1},\ K_\nu' = -K_{\nu+1} + \frac{\nu}{z}K_\nu, \tag{2.3}$$

we obtain $g_\nu'(z) = z(I_\nu\, K_\nu - I_{\nu+1}\, K_{\nu+1}) > 0$. Thus g_ν is strictly increasing and its behavior as $z \to 0$ and $z \to \infty$ can be evaluated using formulas (9.6.7), (9.6.9), (9.7.1), and (9.7.2) in [1]. We compute a higher order asymptotic expansion for $I_{\nu+1}(z)K_\nu(z)$ for large z. Let $\mu = 4\nu^2$ and $\tilde{\mu} = 4(\nu+1)^2$. From the cited formulas,

$$\begin{aligned} &zI_{\nu+1}(z)K_\nu(z) \\ &\quad= \frac{1}{2}\left(1 - \frac{\tilde{\mu}-1}{8z} + \frac{(\tilde{\mu}-1)(\tilde{\mu}-9)}{2(8z)^2} - \frac{(\tilde{\mu}-1)(\tilde{\mu}-9)(\tilde{\mu}-25)}{3!(8z)^3} + O\left(\frac{1}{z^4}\right)\right) \\ &\quad\quad \left(1 + \frac{\mu-1}{8z} + \frac{(\mu-1)(\mu-9)}{2(8z)^2} + \frac{(\mu-1)(\mu-9)(\mu-25)}{3!(8z)^3} + O\left(\frac{1}{z^4}\right)\right) \\ &\quad= \frac{1}{2}\left(1 - \frac{2\nu+1}{2z} + \frac{(2\nu-1)(2\nu+1)(2\nu+3)}{16z^3} + O\left(\frac{1}{z^4}\right)\right) \end{aligned}$$

as $z \to \infty$. The $O(1/z^2)$ terms on the right-hand side happen to cancel. □

We give another lemma on the product of modified Bessel function I_ν and K_ν.

LEMMA 2.2. *Let $\nu \geq 1$ and $p_\nu = I_\nu K_\nu$ and $h_\nu = z^3(p_{\nu-1}(z) - p_\nu(z))$. Then*

$$\begin{aligned} p_{\nu-1}' &= 2I_\nu K_{\nu-1} - \frac{1}{z} + \frac{2(\nu-1)}{z}p_{\nu-1} \\ &= \frac{1}{2z^3}h_\nu' + \frac{2\nu-5}{2z}p_{\nu-1} - \frac{2\nu-3}{2z}p_\nu, \\ p_\nu' &= \frac{1}{z} - 2I_\nu K_{\nu-1} - \frac{2\nu}{z}p_\nu \\ &= -\frac{1}{2z^3}h_\nu' + \frac{2\nu+1}{2z}p_{\nu-1} - \frac{2\nu+3}{2z}p_\nu. \end{aligned}$$

Proof. Use [1, (9.6.26)]. □

Moreover, we have the following recurrence relation.

LEMMA 2.3. *Let $\nu \geq 1$ and $p_\nu = I_\nu K_\nu$. Then*

$$2\nu\, p_\nu'(z) = z(p_{\nu+1} - p_{\nu-1}).$$

Proof. The lemma immediately follows from the well-known recurrence relations [1, (9.6.26)]; see [26] for details. ∎

The polynomial P_n^N ($N = 2, 3, 4, \ldots$ and $n = 0, 1, 2, \ldots$) is called the Legendre polynomial of dimension N and degree n. For any fixed integer $N \geq 2$, the polynomials P_n^N, $n = 0, 1, \ldots$, are orthogonal with respect to the weight function $(1-t^2)^{\frac{N-3}{2}}$. Also P_n^N is an even function if n is even and an odd function if n is odd; it is normalized so that $P_n^N(1) = 1$. If $N = 2$ the resulting polynomials P_n^2 are the Chebyshev polynomials; if $N = 3$ the polynomials P_n^3 are the classical Legendre polynomials, usually denoted by P_n. We list the first few P_n^N's below:

$$\begin{aligned} P_0^N(t) &= 1, \quad P_1^N(t) = t, \quad P_2^N(t) = \frac{1}{N-1}\left(Nt^2 - 1\right), \\ P_3^N(t) &= \frac{1}{N-1} t\left((N+2)t^2 - 3\right). \end{aligned} \tag{2.4}$$

For more on the Legendre polynomials, see [20, section 3.3].

Recall that the volume of the unit ball in $\mathbb{R}^N$ is $\omega_N = \frac{\pi^{N/2}}{\Gamma(\frac{N}{2}+1)}$; the area of the unit sphere $\mathbb{S}^{N-1}$ is $N\omega_N$.

LEMMA 2.4. *Let $P_n^N(t)$ be the Legendre polynomials given above. The following identity holds for all $z > 0$:*

$$\begin{aligned} &\frac{z^{N-2}}{(2\pi)^{\frac{N}{2}}} \int_{-1}^{1} \left(z\sqrt{2-2t}\right)^{1-\frac{N}{2}} K_{\frac{N}{2}-1}\left(z\sqrt{2-2t}\right) P_n^N(t) \left(1-t^2\right)^{\frac{N-3}{2}} dt \\ &\quad = \frac{1}{(N-1)\omega_{N-1}} I_{n+\frac{N}{2}-1}(z) K_{n+\frac{N}{2}-1}(z). \end{aligned}$$

Proof. By setting $a = b = z$ in the last formula listed on [22, p. 90], we obtain

$$\begin{aligned} z^\nu \int_0^\pi (2-2\cos t)^{-\nu/2} K_\nu\left(z\sqrt{2-2\cos t}\right) C_n^\nu(\cos t) \sin^{2\nu} t\, dt \\ = \frac{2\pi\Gamma(n+2\nu)}{n!\, 2^\nu \Gamma(\nu)} I_{n+\nu}(z)\, K_{n+\nu}(z), \end{aligned} \tag{2.5}$$

where C_n^ν is the nth degree (Gegenbauer) ultraspherical polynomial. Choose $\nu = \frac{N}{2} - 1$. It is known [16, (4.39), p. 99] that C_n^ν is a constant multiple of P_n^N:

$$C_n^\nu = \binom{n+2\nu-1}{n} P_n^N = \frac{\Gamma(n+2\nu)}{\Gamma(2\nu)\,\Gamma(n+1)} P_n^N. \tag{2.6}$$

Putting this into (2.5) and using the duplication formula [1, (6.1.18)], we obtain

$$\begin{aligned} \frac{z^\nu}{(2\pi)^{\nu+1}} \int_0^\pi (2-2\cos t)^{-\nu/2} K_\nu(z\sqrt{2-2\cos t})\, P_n^N(\cos t) \sin^{2\nu} t\, dt \\ = \frac{\Gamma(\nu+\frac{1}{2})}{2\pi^{\nu+\frac{1}{2}}} I_{n+\nu}(z)\, K_{n+\nu}(z). \end{aligned}$$

This gives the lemma after employing the substitution $\tau = \cos t$ in the integral. ∎

3. Bubble profiles in $\mathbb{R}^N$. When $D = \mathbb{R}^N$ we simply write $\mathcal{J}$ instead of $\mathcal{J}_{\mathbb{R}^N}$ and similarly $\mathcal{N}$ to denote $\mathcal{N}_{\mathbb{R}^N}$.

LEMMA 3.1. *Let $B(0,r) \in \mathbb{R}^N$ be a ball centered at the origin with radius r. Then $\mathcal{N}(B(0,r))$ is a radially symmetric function on $\mathbb{R}^N$ and*

$$\mathcal{N}(B(0,r))(r) = rI_{\frac{N}{2}}(r)K_{\frac{N}{2}-1}(r). \tag{3.1}$$

Moreover,

$$\int_{B(0,r)} \mathcal{N}(B(0,r))(x)\,dx = \omega_N r^N - N\omega_N I_{\frac{N}{2}}(r)K_{\frac{N}{2}}(r)r^N. \tag{3.2}$$

Proof. Let $v = \mathcal{N}(B(0,r))$ and $t = |x|$. Then

$$v'' + \frac{N-1}{t}v' - v = \begin{cases} -1 & \text{if } 0 < t < r, \\ 0 & \text{if } r < t. \end{cases}$$

To match v and v' at the point r, we have

$$v(t) = \begin{cases} 1 + C_1 t^{-\frac{N}{2}+1} I_{\frac{N}{2}-1}(t) & \text{if } 0 < t < r, \\ C_2 t^{-\frac{N}{2}+1} K_{\frac{N}{2}-1}(t) & \text{if } r < t, \end{cases}$$

with

$$C_1 = -r^{\frac{N}{2}} K_{\frac{N}{2}}(r),\ C_2 = r^{\frac{N}{2}} I_{\frac{N}{2}}(r). \tag{3.3}$$

Further calculations yield (3.1). To derive (3.2), note

$$\begin{aligned}\int_{B(0,r)} \mathcal{N}(B(0,r))dx &= \int_{B(0,r)} v(x)dx = \int_{B(0,r)} (\Delta v + 1)dx \\ &= |B(0,r)| + \int_{\partial B(0,r)} \frac{\partial v}{\partial n} dS(x) = \omega_N r^N + v'(r)N\omega_N r^{N-1}.\end{aligned}$$

By (3.3), $v'(t) = -r^{\frac{N}{2}} I_{\frac{N}{2}}(r) t^{-\frac{N}{2}+1} K_{\frac{N}{2}}(t)$, and (3.2) follows. ∎

Let $j(r,\sigma,\alpha)$ denote $\mathcal{J}(B(0,r))$. From (1.1) and Lemma 3.1,

$$j(r,\sigma,\alpha) = N\omega_N r^{N-1} - \alpha\omega_N r^N + \frac{\sigma}{2}\left[\omega_N r^N - N\omega_N I_{\frac{N}{2}}(r)K_{\frac{N}{2}}(r)r^N\right]. \tag{3.4}$$

A ball $B(0,b)$ is a bubble profile if $\frac{\partial j}{\partial r}|_{r=b} = 0$, i.e., b solves the equation

$$\frac{N-1}{b} - \alpha + \sigma b I_{\frac{N}{2}}(b)K_{\frac{N}{2}-1}(b) = 0. \tag{3.5}$$

This follows either from (1.6) and (3.1) or from a differentiation of (3.4) helped by two more derivatives formulas:

$$(t^\nu I_\nu(t))' = t^\nu I_{\nu-1}(t),\ (t^\nu K_\nu(t))' = -t^\nu K_{\nu-1}(t) \tag{3.6}$$

[1, (9.6.28)]. Write σ as a function of b, $s_\alpha(b)$, after we fix the parameter α in (3.5),

$$\sigma = s_\alpha(b) \equiv \frac{\alpha b - N + 1}{b^2 I_{\frac{N}{2}}(b)K_{\frac{N}{2}-1}(b)}. \tag{3.7}$$

One calculates the derivative of $s_\alpha(b)$ and then employs (3.7) to eliminate α so that

$$s'_\alpha(b) = \frac{N-1-\sigma b^3\left[I_{\frac{N}{2}-1}(b)K_{\frac{N}{2}-1}(b) - I_{\frac{N}{2}}(b)K_{\frac{N}{2}}(b)\right]}{b^3 I_{\frac{N}{2}}(b)K_{\frac{N}{2}-1}(b)}. \tag{3.8}$$

Now observe $s'_\alpha(b) = 0$ on the curve $\mathcal{C}_0$ in the (σ, b)-quadrant, where $\mathcal{C}_0$ is defined by

$$\mathcal{C}_0 = \left\{(\sigma, b) \in \mathbb{R}^+ \times \mathbb{R}^+ : N-1-\sigma b^3\left[I_{\frac{N}{2}-1}(b)K_{\frac{N}{2}-1}(b) - I_{\frac{N}{2}}(b)K_{\frac{N}{2}}(b)\right] = 0\right\}. \tag{3.9}$$

This curve can be viewed as the graph of the function

$$\sigma = C_0(b) \equiv \frac{N-1}{b^3\left(I_{\frac{N}{2}-1}(b)K_{\frac{N}{2}-1}(b) - I_{\frac{N}{2}}(b)K_{\frac{N}{2}}(b)\right)}, \quad b \in (0, \infty). \tag{3.10}$$

This curve divides the (σ, b)-quadrant into two parts:

$$\mathcal{R}_i = \left\{(\sigma, b) \in \mathbb{R}^+ \times \mathbb{R}^+ : N-1-\sigma b^3\left[I_{\frac{N}{2}-1}(b)K_{\frac{N}{2}-1}(b) - I_{\frac{N}{2}}(b)K_{\frac{N}{2}}(b)\right] > 0\right\}; \tag{3.11}$$

$$\mathcal{R}_d = \left\{(\sigma, b) \in \mathbb{R}^+ \times \mathbb{R}^+ : N-1-\sigma b^3\left[I_{\frac{N}{2}-1}(b)K_{\frac{N}{2}-1}(b) - I_{\frac{N}{2}}(b)K_{\frac{N}{2}}(b)\right] < 0\right\}. \tag{3.12}$$

The part of curve $s_\alpha(b)$ that lies in $\mathcal{R}_i$ is increasing while the part of curve $s_\alpha(b)$ that lies in $\mathcal{R}_d$ is decreasing. We have the following properties for the shape of $s_\alpha(b)$.

LEMMA 3.2. *For each $\alpha > 0$, let $s_\alpha(b)$ be the curve given above.*

1. *Each curve s_α starts at the point $(0, \frac{N-1}{\alpha})$ with $s'_\alpha(\frac{N-1}{\alpha}) > 0$ and ends with a vertical asymptote $\sigma = 2\alpha$.*
2. *When s_α intersects $\mathcal{C}_0$, its slope is vertical, i.e., $s'_\alpha(b) = 0$ if $(\sigma, b) \in \mathcal{C}_0$.*

Proof. Since $\sigma = 0$ when $b = \frac{N-1}{\alpha}$ on the curve s_α, (3.8) implies

$$s'_\alpha\left(\frac{N-1}{\alpha}\right) = \frac{\alpha^3}{(N-1)^2 I_{\frac{N}{2}}\left(\frac{N-1}{\alpha}\right) K_{\frac{N}{2}-1}\left(\frac{N-1}{\alpha}\right)} > 0.$$

Moreover, by Lemma 2.1, part 2, one deduces that

$$\lim_{b\to+\infty} s_\alpha(b) = 2\alpha.$$

The second part of the lemma follows from the definition of $\mathcal{C}_0$. ◻

The shape of $\mathcal{C}_0$ is described in the following lemma. This is a crucial result. The definition of C_0 is meaningful even if N is a noninteger, real number, greater than or equal to 2.

LEMMA 3.3. *Let $N \geq 2$ be a real number, not necessarily an integer. The function C_0 has the following properties:*

1. $\lim_{b\to 0^+} C_0(b) = \infty$ *and* $\lim_{b\to\infty} C_0(b) = 4$.
2. $C'_0(b) < 0$ *for all $b > 0$ holds if and only if $N \geq 3$.*

Proof. Let $\nu = \frac{N}{2} \geq 1$ and define $h_\nu : (0, \infty) \to (0, \infty)$ to be

$$h_\nu(b) = b^3\Big(I_{\nu-1}(b)K_{\nu-1}(b) - I_\nu(b)K_\nu(b)\Big). \tag{3.13}$$

The monotonicity of C_0 is equivalent to the monotonicity of h_ν.

Using the asymptotic expansions of $I_\nu(b)$ and $K_\nu(b)$ for large b [1, (9.7.1)–(9.7.5)], we have that as $b \to \infty$,

$$I_\nu(b)K_\nu(b) = \frac{1}{2b}\left(1 - \frac{1}{2}\frac{4\nu^2-1}{(2b)^2} + \frac{3}{8}\frac{(4\nu^2-1)(4\nu^2-9)}{(2b)^4} + O\left(\frac{1}{b^6}\right)\right)$$

and a similar formula for $I_{\nu-1}(b)K_{\nu-1}(b)$. Thus

$$h_\nu(b) = \frac{2\nu-1}{4} - \frac{3(2\nu-3)(2\nu-1)(2\nu+1)}{32b^2} + O\left(\frac{1}{b^4}\right). \tag{3.14}$$

In particular

$$\lim_{b\to\infty} h_\nu(b) = \frac{2\nu-1}{4}. \tag{3.15}$$

For small b and $\nu > 1$, by [1, (9.6.7)–(9.6.9)],

$$I_{\nu-1}(b)K_{\nu-1}(b) \sim \frac{1}{2(\nu-1)},\ I_\nu(b)K_\nu(b) \sim \frac{1}{2\nu}.$$

They imply that

$$\lim_{b\to 0^+} h_\nu(b) = 0. \tag{3.16}$$

The same holds true if $\nu = 1$ by a similar argument. Part 1 of the lemma follows from (3.15) and (3.16).

By Lemmas 2.1 and 2.2, we find

$$h'_\nu(b) = b^2\Big((2\nu+1)I_{\nu-1}(b)K_{\nu-1}(b)+(2\nu-3)I_\nu(b)K_\nu(b)-2+4bI_\nu(b)K_{\nu-1}(b)\Big). \tag{3.17}$$

Consider the case $\nu > \frac{3}{2}$. We set

$$p_\nu(b) = I_\nu(b)K_\nu(b). \tag{3.18}$$

Define

$$H_\nu(b) = (2\nu+1)p_{\nu-1}(b) + (2\nu-3)p_\nu(b) - 2 + 4bI_\nu(b)K_{\nu-1}(b),$$

so that

$$h'_\nu(b) = b^2 H_\nu(b). \tag{3.19}$$

We know from (3.14) that $H_\nu(b) > 0$ for large b. If we can show that $H'_\nu(b) \le 0$ for all $b > 0$, then $H_\nu(b) > 0$ and $h'_\nu(b) > 0$ for all $b > 0$.

To this end, we use Lemmas 2.1 and 2.3 to derive

$$\begin{aligned} H'_\nu(b) &= (2\nu+1)\frac{b(p_\nu - p_{\nu-2})}{2(\nu-1)} + (2\nu-3)\frac{b(p_{\nu+1}-p_{\nu-1})}{2\nu} + 4b(p_{\nu-1}-p_\nu) \\ &= \frac{b}{2\nu(\nu-1)}\Big(-\nu(2\nu+1)p_{\nu-2} + 3(2\nu+1)(\nu-1)p_{\nu-1} \\ &\qquad - 3\nu(2\nu-3)p_\nu + (\nu-1)(2\nu-3)p_{\nu+1}\Big). \end{aligned} \tag{3.20}$$

By [2, statement (iv), p. 528], the function $q \to p_{\sqrt{q}}(b)$, $q > 0$, is completely monotone with respect to q for all $b > 0$. A theorem of Bernstein asserts that completely monotone functions are precisely the Laplace transforms of positive measures [33]. Therefore

$$p_\nu(b) = \int_{[0,\infty)} e^{-\nu^2 t} d\mu_b(t) \tag{3.21}$$

for some positive measure μ_b on $[0, \infty)$. This measure depends on b. Then by (3.20),

$$\begin{aligned}
&\frac{2\nu(\nu-1)}{b} H'_\nu(b) \\
&\quad = -\nu(2\nu+1)p_{\nu-2} + 3(2\nu+1)(\nu-1)p_{\nu-1} - 3\nu(2\nu-3)p_\nu + (\nu-1)(2\nu-3)p_{\nu+1} \\
&\quad = -\nu(2\nu+1)\int_{[0,\infty)} e^{-(\nu-2)^2 t} d\mu_b(t) + 3(2\nu+1)(\nu-1)\int_{[0,\infty)} e^{-(\nu-1)^2 t} d\mu_b(t) \\
&\qquad - 3\nu(2\nu-3)\int_{[0,\infty)} e^{-\nu t^2} d\mu_b(t) + (\nu-1)(2\nu-3)\int_{[0,\infty)} e^{-(\nu+1)^2 t} d\mu_b(t) \\
&\quad = \int_{[0,\infty)} f(t) e^{-\nu^2 t} d\mu_b(t),
\end{aligned}$$

where

$$f(t) = -\nu(2\nu+1)e^{(4\nu-4)t} + 3(2\nu+1)(\nu-1)e^{(2\nu-1)t} - 3\nu(2\nu-3) + (\nu-1)(2\nu-3)e^{-(2\nu+1)t}.$$

Clearly $f(0) = 0$. Compute

$$\begin{aligned}
f'(t) &= -4(\nu-1)(2\nu+1)\nu e^{(4\nu-4)t} + 3(\nu-1)(2\nu+1)(2\nu-1)e^{(2\nu-1)t} \\
&\quad - (\nu-1)(2\nu+1)(2\nu-3)e^{-(2\nu+1)t} \\
&= (\nu-1)(2\nu+1)e^{(2\nu-1)t}\Big(3(2\nu-1) - 4\nu e^{(2\nu-3)t} - (2\nu-3)e^{-4\nu t}\Big).
\end{aligned}$$

By Young's inequality

$$AB \le \frac{A^p}{p} + \frac{B^q}{q},$$

with

$$p = \frac{3(2\nu-1)}{4\nu},\ q = \frac{3(2\nu-1)}{2\nu-3},\ A = e^{\frac{4\nu(2\nu-3)}{3(2\nu-1)}t},\ B = e^{-\frac{4\nu(2\nu-3)}{3(2\nu-1)}t},$$

we deduce

$$3(2\nu-1) < 4\nu e^{(2\nu-3)t} + (2\nu-3)e^{-4\nu t},$$

which shows that $f'(t) < 0$ for $t > 0$. Consequently $f(t) < 0$ for $t > 0$ and $H'_\nu(b) \le 0$ for all $b > 0$. This allows us to conclude that $H_\nu(b) > 0$ and $h'_\nu(b) > 0$ for all $b > 0$. Note that in the use of Young's inequality, $p > 1, q > 1$ because $\nu > \frac{3}{2}$

Now consider the case $\nu \in (1, \frac{3}{2})$. By (3.14), we see that $h_\nu(b)$ is decreasing when b is large. But by (3.17), for small b,

$$\begin{aligned}
h'_\nu(b) &= b^2\left(\frac{2\nu+1}{2(\nu-1)} + \frac{2\nu-3}{2\nu} - 2 + o(b)\right) \\
&= b^2\left(\frac{3}{2\nu(\nu-1)} + o(b)\right) > 0\ .
\end{aligned}$$

Then $h_\nu(b)$ is increasing when b is small. Hence h_ν, and C_0, cannot be monotone if $\nu \in (1, \frac{3}{2})$.

Finally for the borderline case $\nu = \frac{3}{2}$, since

$$\begin{aligned} I_{\frac{1}{2}}(b) &= \sqrt{\frac{2b}{\pi}}\ \frac{\sinh b}{b}, \\ K_{\frac{1}{2}}(b) &= \sqrt{\frac{\pi}{2b}}\ e^{-b}, \\ I_{\frac{3}{2}}(b) &= \sqrt{\frac{2b}{\pi}}\left(-\frac{\sinh b}{b^2} + \frac{\cosh b}{b}\right), \\ K_{\frac{3}{2}}(b) &= \sqrt{\frac{\pi}{2b}}\ e^{-b}\left(1 + \frac{1}{b}\right), \end{aligned}$$

by [1, (10.2.13) and (10.2.17)], we deduce

$$h_{\frac{3}{2}}(b) = b^3\left(I_{\frac{1}{2}}(b)K_{\frac{1}{2}}(b) - I_{\frac{3}{2}}(b)K_{\frac{3}{2}}(b)\right) = \frac{1}{2} - b^2 e^{-2b} - be^{-2b} - \frac{1}{2}e^{-2b}. \tag{3.22}$$

Hence

$$h'_{\frac{3}{2}}(b) = 2b^2 e^{-2b} > 0. \tag{3.23}$$

This completes the proof of the second part of the lemma. ∎

The monotonicity of h_ν should be a significant result in its own right, independent of the rest of this paper. We record it below for future reference.

Theorem 3.4. *Let $\nu \ge 1$ be a real number. Then $b^3(I_{\nu-1}(b)K_{\nu-1}(b) - I_\nu(b)K_\nu(b))$ is increasing with respect to $b \in (0,\infty)$ if and only if $\nu \ge \frac{3}{2}$.*

In [12] it was shown that when $\nu = 1$, there exists $\hat{b} \in (0,\infty)$ such that $b^3(I_0(b)K_0(b) - I_1(b)K_1(b))$ is increasing on $(0,\hat{b})$ and decreasing on $(\hat{b},\infty)$.

Knowing the shape of $\mathcal{C}_0$, one investigates how the curves s_α change as α varies.

Lemma 3.5. *Let $N \ge 3$ and $\mathcal{C}_0$ be as defined in* (3.9)*. There exist real analytic functions $\sigma_* : (2,\infty) \to (4,\infty)$ and $b_* : (2,\infty) \to (0,\infty)$ so that the following statements hold:*

1. *If $\alpha \in (0,2]$, then $\mathcal{C}_0$ always lies to the right of the curve $s_\alpha(b)$ in the σ-b quadrant.*
2. *If $\alpha \in (2,\infty)$, then $\mathcal{C}_0$ intersect transversally with $s_\alpha(b)$ only once at $(\sigma_*(\alpha), b_*(\alpha))$.*

Proof. If $0 < \alpha_1 < \alpha_2$, then the curve of s_{α_1} lies to the left of s_{α_2} in the σ-b quadrant. Each s_α starts at the point $(0, \frac{N-1}{\alpha})$, so when b is small, the curve is in the region $\mathcal{R}_i$. As b increases, the curve may touch C_0. When that happens at a point, say, (σ_1, b_1), since $C_0'(b_1) < 0$ by Lemma 3.3, part 2, and $s'_\alpha(b_1) = 0$ by Lemma 3.2, part 2, the curve s_α enters $\mathcal{R}_d$.

However, if s_α enters $\mathcal{R}_d$, it cannot exit $\mathcal{R}_d$. This is because if it exits $\mathcal{R}_d$ at a point, say, (σ_2, b_2), then $s'_\alpha(b_2) \le C_0'(b_2)$, which is inconsistent with $s'_\alpha(b_2) = 0$ and $C_0'(b_2) < 0$. In other words, each s_α can intersect $\mathcal{C}_0$ at most once.

Since $\lim_{b\to\infty} s_\alpha(b) = 2\alpha$ and $\lim_{b\to\infty} C_0(b) = 4$, for large b the curve s_α is in $\mathcal{R}_i$ if $\alpha < 2$ and s_α is in $\mathcal{R}_d$ if $\alpha > 2$. Hence when $\alpha < 2$, the entire curve s_α stays in $\mathcal{R}_i$; when $\alpha > 2$, the curve s_α enters $\mathcal{R}_d$ at one point and stays in $\mathcal{R}_d$ afterward.

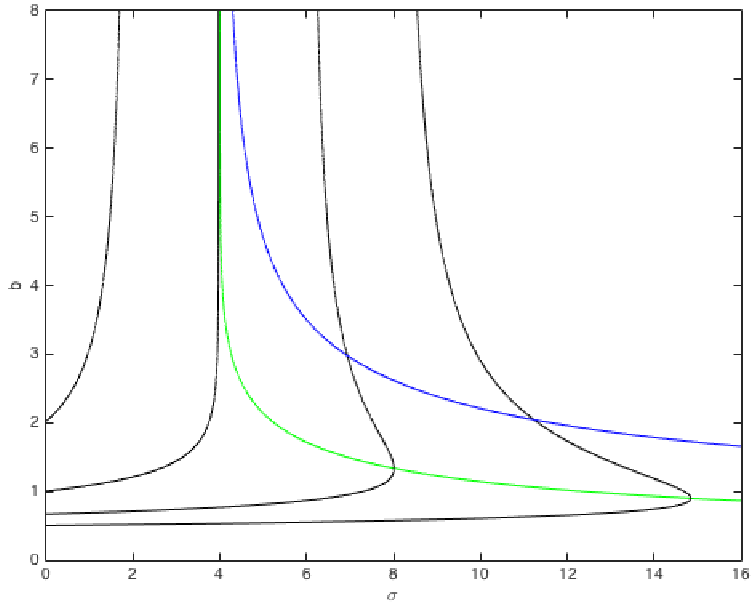

FIG. 1. $N = 3$. *The curve $\mathcal{C}_0$ (in green) and the curves s_α for various α. As α increases, the s_α curves shift to the right. Another curve $\mathcal{C}_2$ (in blue) is included here for the stability issue.*

Regarding the borderline case $\alpha = 2$, since $s_2(b) = \lim_{\alpha\to 2-} s_\alpha(b)$ for every $b > 0$, and for each $\alpha < 2$ and $b > 0$, $s_\alpha(b) < C_0(b)$, one deduces that $s_2(b) \le C_0(b)$ for all $b > 0$. The curve s_2 cannot touch $\mathcal{C}_0$ because, otherwise, s_2 would enter $\mathcal{R}_d$ and $s_2(b) > C_0(b)$ for some b. Therefore, like the $\alpha < 2$ case, $s_2(b) < C_0(b)$ for all $b > 0$ and the entire curve s_2 stays in $\mathcal{R}_i$.

For each $\alpha > 2$ denote the intersection point of s_α and $\mathcal{C}_0$ by $(\sigma_*(\alpha), b_*(\alpha))$. It is the unique solution of the following system for (σ, b):

$$\sigma = s_\alpha(b), \quad s_\alpha(b) = C_0(b). \tag{3.24}$$

Observe that $g(b,\alpha) \equiv s_\alpha(b) - C_0(b)$ is analytic. At the intersection point, $\frac{\partial g}{\partial b}(b_*(\alpha)) = -C_0'(b_*(\alpha)) > 0$. The implicit function theorem then requires b_* to be an analytic function of α. Similarly σ_* is analytic. ◻

Figure 1 shows the curve $\mathcal{C}_0$ and some s_α's in the case $N = 3$. The shapes of the s_α's differ depending on whether α is in $(0, 2]$ or $(2, \infty)$. The number of the bubble profiles can be read from this picture. The following theorem summarizes what we have discovered.

THEOREM 3.6. *Let $N \ge 3$. The numbers of bubble profiles are given as follows:*

1. *If $\alpha \in (0, 2]$, then*
   (a) *if $\sigma \in (0, 2\alpha)$, there is one bubble,*
   (b) *if $\sigma \in [2\alpha, \infty)$, there is no bubble.*
2. *If $\alpha \in (2, \infty)$, then*
   (a) *if $\sigma \in (0, 2\alpha]$, there is one bubble,*
   (b) *if $\sigma \in (2\alpha, \sigma_*(\alpha))$, there are two bubbles,*
   (c) *if $\sigma = \sigma_*(\alpha)$, there is one bubble,*
   (d) *if $\sigma \in (\sigma_*(\alpha), \infty)$, there is no bubble.*

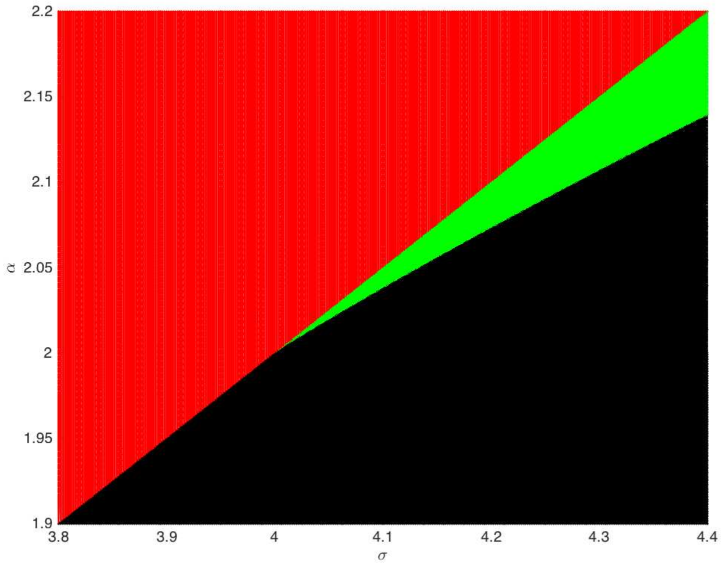

FIG. 2. $N = 3$. *Red denotes* 1 *radially unstable bubble; green is for* 2 *bubbles,* 1 *radially stable and the other radially unstable; black is for no bubble. The point surrounded by three colors is* $(\sigma, \alpha) = (4, 2)$. *The red region is separated from the others by the line* $\sigma = 2\alpha$. *The green region is separated from the black region by the curve* $\sigma = \sigma_*(\alpha)$.

Proof. The number of bubbles for given α and σ equals the number of intersection points between the curve s_α and the verticle σ-line in the σ-b quadrant.

When $\alpha \leq 2$, the curve s_α intersects any vertical σ-line with $\sigma < 2\alpha$ precisely once; hence there is one bubble. If a σ-line has $\sigma \geq 2\alpha$, it does not intersect s_α, hence no bubble.

When $\alpha > 2$, the curve s_α intersect a σ-line once if $\sigma \leq 2\alpha$. If the $\sigma \in (2\alpha, \sigma_*(\alpha))$, then s_α intersects the σ-line twice, hence two bubbles. If the σ is exactly $\sigma_*(\alpha)$, then s_α intersects the σ-line once, hence one bubble. If the $\sigma \in (\sigma_*, \infty)$, then the σ-line does not intersect s_α, hence no bubble. ∎

Figure 2 is obtained by numercially computing $\sigma_*(\alpha)$ for $N = 3$. It illustrates the main cases of the theorem in the σ-α quadrant. There is one bubble if (σ, α) in the red region, two bubbles in the green region, and no bubble in the black region.

4. Spectra of the bubbles. We now set up a framework to facilitate a discussion of the stability of the bubbles found in Theorem 3.6, i.e., we calculate the spectrum of the linearized operator at a bubble. Denote the inner product on $L^2(\mathbb{S}^{N-1})$ by

$$\langle \phi, \psi \rangle = \int_{\mathbb{S}^{N-1}} \phi(\theta)\psi(\theta) d\theta, \quad \phi, \psi \in L^2\left(\mathbb{S}^{N-1}\right).$$

Here $\theta = (\theta_1, \ldots, \theta_N)$ is a point on $\mathbb{S}^{N-1}$ and $d\theta$ refers to the surface area measure of $\mathbb{S}^{N-1}$.

Consider a bubble profile $B(0, b)$, i.e., b is a solution of (3.5). Let $\Omega = P_\phi$ be a perturbed ball given by

$$P_\phi = \left\{ r\theta : \ r \in \left[0, b(1 + N\phi(\theta))^{\frac{1}{N}}\right], \ \theta \in \mathbb{S}^{N-1} \right\}. \tag{4.1}$$

The energy $\mathcal{J}(P_\phi)$ of the perturbed ball P_ϕ is now treated as a functional of ϕ, i.e., $\mathcal{J}(P_\phi) = \mathcal{J}(\phi)$, and one can write it more explicitly as

$$\begin{aligned}\mathcal{J}(\phi) = b^{N-1} \int_{\mathbb{S}^{N-1}} (1+N\phi)^{\frac{N-2}{N}} \sqrt{(1+N\phi)^{\frac{2}{N}} + \frac{|\nabla_{\mathbb{S}^{N-1}}\phi|^2}{(1+N\phi)^{\frac{2N-2}{N}}}}\, d\theta \\ - \alpha b^N \int_{\mathbb{S}^{N-1}} \frac{1+N\phi}{N} d\theta \\ + \frac{\sigma}{2} \int_{P_\phi}\int_{P_\phi} \frac{1}{(2\pi)^{\frac{N}{2}}} \frac{K_{\frac{N}{2}-1}(|x-y|)}{|x-y|^{\frac{N}{2}-1}} dxdy.\end{aligned} \tag{4.2}$$

In the present setting, one can easily define a deformation of P_ϕ by deforming ϕ to

$$\phi + \epsilon\psi, \;\; \phi \in \mathrm{Dom}(\mathcal{J}) \text{ and } \psi \text{ sufficiently smooth.} \tag{4.3}$$

This gives rise to a deformation of P to $P_{\phi+\epsilon\psi}$. The first variation now is

$$\frac{d\mathcal{J}(\phi+\epsilon\psi)}{d\epsilon}\bigg|_{\epsilon=0} = \int_{\mathbb{S}^{N-1}} ((N-1)\mathcal{H}(\phi) - \alpha + \sigma\mathcal{N}(\phi))b^N \psi d\theta. \tag{4.4}$$

Now the mean curvature $\mathcal{H}$ and the nonlocal operator $\mathcal{N}$ are both regarded as acting on ϕ. Introduce a nonlinear operator $\mathcal{S} : \mathrm{Dom}(\mathcal{S}) \to L^2(\mathbb{S}^{N-1})$,

$$\mathcal{S}(\phi) = b^N\big((N-1)\mathcal{H}(\phi) - \alpha + \sigma\mathcal{N}(\phi)\big). \tag{4.5}$$

The Euler–Lagrange equation (1.6) now becomes the equation $\mathcal{S}(\phi) = 0$.

With the operator $\mathcal{S}$ one can write the first variation (4.4) more concisely as

$$\frac{d\mathcal{J}(\phi+\epsilon\psi)}{d\epsilon}\bigg|_{\epsilon=0} = \langle \mathcal{S}(\phi), \psi\rangle. \tag{4.6}$$

One can also find the second variation of $\mathcal{J}$:

$$\frac{d^2\mathcal{J}(\phi+\epsilon\psi)}{d\epsilon^2}\bigg|_{\epsilon=0} = \langle \mathcal{S}^{'}(\phi)\psi, \psi\rangle. \tag{4.7}$$

In (4.7), $\mathcal{S}^{'}$ is the Fréchet derivative of $\mathcal{S}$, so that $\mathcal{S}'(\phi)$ is a linear operator from $H^2(\mathbb{S}^{N-1})$ to $L^2(\mathbb{S}^{N-1})$.

Note that the spherical profile $B(0,b)$ corresponds to $\phi = 0$. Hence $\phi = 0$ is a critical point of $\mathcal{J}$, so $\mathcal{S}(0) = 0$. Furthermore, when $\phi = 0$, ψ gives a perturbation in the normal direction. Thus according to [14, (1.147), p. 41], one can show that

$$\begin{aligned}\mathcal{S}^{'}(0)\psi = b^{N-1}(-\Delta_{\mathbb{S}^{N-1}}\psi - (N-1)\psi) + \sigma b^{2N} \int_{\mathbb{S}^{N-1}} \frac{1}{(2\pi)^{\frac{N}{2}}} \frac{K_{\frac{N}{2}-1}(|b\theta - b\omega|)}{|b\theta - b\omega|^{\frac{N}{2}-1}} \psi(\omega) d\omega \\ + \sigma b^{N+1} \left(\int_{B(0,b)} \frac{1}{(2\pi)^{\frac{N}{2}}} \left(\frac{K_{\frac{N}{2}-1}(|b\theta - y|)}{|b\theta - y|^{\frac{N}{2}-1}} \right)^{'} \frac{b\theta - y}{|b\theta - y|} dy \cdot \theta \right) \psi(\theta),\end{aligned} \tag{4.8}$$

where $(\cdot)'$ denotes the derivative with respect to $|b\theta - y|$.

LEMMA 4.1. *Let $\mathcal{H}_n^N$, $n = 0, 1, 2, \ldots$, be the space of spherical harmonics of dimension N and degree n. Each of the $\mathcal{H}_n^N$ is an eigenspace of $\mathcal{S}'(0)$ with the corresponding eigenvalue*

$$\lambda_n = b^{N-1}(n+N-1)(n-1) + \sigma b^{N+2}\Big(I_{n+\frac{N}{2}-1}(b)K_{n+\frac{N}{2}-1}(b) - I_{\frac{N}{2}}(b)K_{\frac{N}{2}}(b)\Big). \tag{4.9}$$

Proof. According to [20, Theorem 3.2.11],

$$\Delta_{\mathbb{S}^{N-1}}\psi = -n(n+N-2)\psi, \tag{4.10}$$

where $\psi \in \mathcal{H}_n^N$. Then,

$$-\Delta_{\mathbb{S}^{N-1}}\psi - (N-1)\psi = (n-1)(n+N-1)\psi. \tag{4.11}$$

It remains to study the integral operator

$$\psi \to b^{2N}\int_{\mathbb{S}^{N-1}} \frac{1}{(2\pi)^{\frac{N}{2}}}\ \frac{K_{\frac{N}{2}-1}(|b\theta-b\omega|)}{|b\theta-b\omega|^{\frac{N}{2}-1}}\psi(\omega)d\omega \tag{4.12}$$

and the multiplication operator

$$\psi \to b^{N+1}\left(\int_{B(0,b)} \frac{1}{(2\pi)^{\frac{N}{2}}}\ \left(\frac{K_{\frac{N}{2}-1}(|b\theta-y|)}{|b\theta-y|^{\frac{N}{2}-1}}\right)' \frac{b\theta-y}{|b\theta-y|}dy\cdot\theta\right)\psi(\theta). \tag{4.13}$$

For the integral operator (4.12) take $\psi \in \mathcal{H}_n^N$ to deduce

$$\begin{aligned} &\int_{\mathbb{S}^{N-1}} \frac{1}{(2\pi)^{\frac{N}{2}}}\ \frac{K_{\frac{N}{2}-1}(|b\theta-b\omega|)}{|b\theta-b\omega|^{\frac{N}{2}-1}}\psi(\omega)d\omega \\ &\quad= \frac{(N-1)\omega_{N-1}}{(2\pi)^{\frac{N}{2}}}\left(\int_{-1}^{1}(b\sqrt{2-2t})^{1-N/2}K_{\frac{N}{2}-1}(b\sqrt{2-2t})P_n^N(t)(1-t^2)^{\frac{N-3}{2}}dt\right)\psi(\theta) \\ &\quad= b^{2-N}I_{n+\frac{N}{2}-1}(b)K_{n+\frac{N}{2}-1}(b)\psi(\theta). \end{aligned} \tag{4.14}$$

Here in the second to last step we have used the Funk–Hecke theorem [20, Theorem 3.4.1] or [16, Theorem 4.24], while in the last line, we have used Lemma 2.4. By (4.14) we conclude that the integral operator (4.12) acts on $\mathcal{H}_n^N$ like

$$\psi(\theta) \to b^{N+2}I_{n+\frac{N}{2}-1}(b)K_{n+\frac{N}{2}-1}(b)\psi(\theta),\ \psi \in \mathcal{H}_n^N. \tag{4.15}$$

For the multiplication operator (4.13), we use the divergence theorem to compute

$$\begin{aligned} &b^{N+1}\int_{B(0,b)} \frac{1}{(2\pi)^{\frac{N}{2}}}\ \left(\frac{K_{\frac{N}{2}-1}(|b\theta-y|)}{|b\theta-y|^{\frac{N}{2}-1}}\right)' \frac{b\theta-y}{|b\theta-y|}\ dy\cdot\theta \\ &\quad= -b^{N+1}\int_{B(0,b)} \frac{1}{(2\pi)^{\frac{N}{2}}}\ \nabla_y\cdot\left(\frac{K_{\frac{N}{2}-1}(|b\theta-y|)}{|b\theta-y|^{\frac{N}{2}-1}}\theta\right)\ dy \\ &\quad= -b^{2N}\int_{\partial B(0,1)} \frac{1}{(2\pi)^{\frac{N}{2}}}\ \frac{K_{\frac{N}{2}-1}(|b\theta-b\omega|)}{|b\theta-b\omega|^{\frac{N}{2}-1}}\theta\cdot\omega\ d\omega \\ &\quad= -b^{2N}\frac{(N-1)\omega_{N-1}}{(2\pi)^{\frac{N}{2}}}\ \int_{-1}^{1}\left(b\sqrt{2-2t}\right)^{1-\frac{N}{2}}K_{\frac{N}{2}-1}(b\sqrt{2-2t})P_1^N(t)\left(1-t^2\right)^{\frac{N-3}{2}}\ dt \\ &\quad= -b^{N+2}I_{\frac{N}{2}}(b)K_{\frac{N}{2}}(b). \end{aligned}$$

Here we have used the fact that $\theta\cdot\omega$, as a function of ω, is a spherical harmonic of degree 1 according to [20, Lemma 3.2.3] and then applied [20, Theorem 3.4.1] and Lemma 2.4. Hence the multiplication operator (4.13) is simply

$$\psi \to -b^{N+2}I_{\frac{N}{2}}(b)K_{\frac{N}{2}}(b)\psi. \tag{4.16}$$

Then (4.9) follows from (4.11), (4.15), and (4.16). ∎

Note that when $n = 1$, $\lambda_1 = 0$. This is due to the translation invariance of the problem. A bubble profile $B(0, b)$ is stable if all the remaining eigenvalues are positive. For $n = 0, 2, 3, 4, \ldots$, let us define the curves

(4.17)
$$\mathcal{C}_n = \left\{(\sigma, b) : (n + N - 1)(n - 1) + \sigma b^3 \left(I_{n+\frac{N}{2}-1}(b)K_{n+\frac{N}{2}-1}(b) - I_{\frac{N}{2}}(b)K_{\frac{N}{2}}(b)\right) = 0\right\}$$

in the (σ, b) quadrant. Any (σ, b) point on the curve $\mathcal{C}_n$ corresponds to a bubble whose eigenvalue λ_n vanishes. Note that when $n = 0$, one has the same curve $\mathcal{C}_0$ previously defined in (3.9). One may regard $\mathcal{C}_n$ as the graph of the function

(4.18)
$$\sigma = C_n(b) \equiv \frac{(n + N - 1)(1 - n)}{b^3 \left(I_{n+\frac{N}{2}-1}(b)K_{n+\frac{N}{2}-1}(b) - I_{\frac{N}{2}}(b)K_{\frac{N}{2}}(b)\right)}, \ b > 0, \ n = 0, 2, 3, 4, \ldots.$$

Each $\mathcal{C}_n$ divides the (σ, b) quadrant into two regions:

(4.19)
$$\mathcal{R}_{n,s} = \left\{(n + N - 1)(n - 1) + \sigma b^3 \left(I_{n+\frac{N}{2}-1}(b)K_{n+\frac{N}{2}-1}(b) - I_{\frac{N}{2}}(b)K_{\frac{N}{2}}(b)\right) > 0\right\},$$

(4.20)
$$\mathcal{R}_{n,u} = \left\{(n + N - 1)(n - 1) + \sigma b^3 \left(I_{n+N/2-1}(b)K_{n+N/2-1}(b) - I_{\frac{N}{2}}(b)K_{\frac{N}{2}}(b)\right) < 0\right\}.$$

Note that
$$\mathcal{R}_{0,s} = \mathcal{R}_d, \quad \mathcal{R}_{0,u} = \mathcal{R}_i.$$

Note also that $\mathcal{R}_{0,s}$ lies to the right of the curve $\mathcal{C}_0$, while $\mathcal{R}_{n,s}$ is to the left of $\mathcal{C}_n$ for $n = 2, 3, \ldots$.

If (σ, b) is in $\mathcal{R}_{n,s}$, then the bubble $B(0, b)$ is stable with respect to the nth mode; if (σ, b) is in $\mathcal{R}_{n,u}$, then the bubble is unstable with respect to this mode. In the case $n = 0$, if $(\sigma, b) \in \mathcal{R}_{0,s}$ we say that the bubble is radially stable. A bubble is termed to be stable if it is stable with respect to all modes ($n = 0, 2, 3, 4, \ldots$). Hence, if $B(0, b)$ is stable, then (σ, b) must be in the intersection

$$\bigcap_{n=0,2,3,\ldots} \mathcal{R}_{n,s}. \tag{4.21}$$

The following lemma gives a simple description of this set.

Lemma 4.2. *For every $b > 0$,*
1. $C_{n+1}(b) > C_n(b)$ *when* $n = 2, 3, \ldots,$
2. $C_2(b) > C_0(b)$.

Proof. For part 1, since

$$b^3(C_{n+1}(b) - C_n(b)) = \frac{(2n + N - 1)p_{\frac{N}{2}} - (n + N)np_{n+\frac{N}{2}-1} + (n + N - 1)(n - 1)p_{n+\frac{N}{2}}}{(p_{\frac{N}{2}} - p_{n+\frac{N}{2}-1})(p_{\frac{N}{2}} - p_{n+\frac{N}{2}})},$$

it suffices to show the numerator on the right-hand side is positive. It is known [2, Theorem 1(4)] that the function $t \to p_{\sqrt{t}}(b)$ is log-convex for $t > 0$. Since a log-convex function is convex, for any $0 < \mu < 1$

$$p_{\sqrt{(1-\mu)\nu_1^2+\mu\nu_2^2}}(b) < (1 - \mu)p_{\nu_1}(b) + \mu p_{\nu_2}(b)\ .$$

Part 1 follows if $\nu_1 = \frac{N}{2}$, $\nu_2 = n + \frac{N}{2}$, and $\mu = \frac{(n+N-1)(n-1)}{(n+N)n}$. The proof of part 2 is similar. ∎

Lemma 4.2 implies that

$$\bigcap_{n=0,2,3,\ldots} \mathcal{R}_{n,s} = \mathcal{R}_d \cap \mathcal{R}_{2,s}. \tag{4.22}$$

Given σ and b, we define

$$T : (\sigma, b) \to (\sigma, \alpha) \equiv \left(\sigma, \frac{N-1}{b} + \sigma b I_{\frac{N}{2}}(b) K_{\frac{N}{2}-1}(b)\right). \tag{4.23}$$

When a given (σ, α) gives rise to two bubbles, only the one with the larger radius is stable with respect to radial perturbation. Consequently when T is restricted to the region $\mathcal{R}_d$, then the restriction, denoted as $T_{\mathcal{R}_d}$, is one-to-one. Denote the inverse of $T_{\mathcal{R}_d}$ by $T^{-1}_{\mathcal{R}_d}$, which maps from $T(\mathcal{R}_d)$ back to $\mathcal{R}_d$. Note that $T(\cap_{n=0,2,3,\ldots}\mathcal{R}_{n,s}) \subset T(\mathcal{R}_d)$ since $\cap_{n=0,2,3,\ldots}\mathcal{R}_{n,s} \subset \mathcal{R}_d$. With the help of T one can precisely identify regions for (σ, α) that give rise to radially stable bubbles and stable bubbles, respectively. The following theorem then follows from Lemmas 4.1 and 4.2 immediately.

THEOREM 4.3. *Let $N \geq 3$. If $(\sigma, \alpha) \in T(\mathcal{R}_d)$, then with $(\sigma, b) = T^{-1}_{\mathcal{R}_d}(\sigma, \alpha)$, $B(0, b)$ is a radially stable bubble. If $(\sigma, \alpha) \in T(\mathcal{R}_d \cap \mathcal{R}_{2,s})$, then with $(\sigma, b) = T^{-1}_{\mathcal{R}_d}(\sigma, \alpha)$, $B(0, b)$ is a stable bubble.*

The above theorem allows us to construct Figure 2, which will be convenient when we look for stable bubbles for a given (σ, α). In this picture any bubble from the red region is radially unstable. The larger of the two bubbles from the green region is radially stable but the other one is radially unstable. The radially stable one from the green region may or may not be stable. Figure 1 has already included the curve $\mathcal{C}_2$, in blue. The region $\bigcap_{n=0,2,3,\ldots} \mathcal{R}_{n,s} = \mathcal{R}_d \cap \mathcal{R}_{2,s}$ is bounded between the green curve and the blue curve. If (σ, b) is in this region and α is given accordingly by (4.23), then the bubble $B(0, b)$, corresponding to these σ and α, is stable.

5. Existence of bubble assemblies. In this section D is bounded and sufficiently smooth so a Green's function of (1.4) exists and is written as

$$G(x, y) = \frac{1}{(2\pi)^{\frac{N}{2}}} |x - y|^{1-N/2} K_{\frac{N}{2}-1}(|x - y|) + R(x, y), \tag{5.1}$$

where R is a smooth function of $(x, y) \in D \times D$. Define, for $\xi = (\xi^1, \ldots, \xi^K)$ where $\xi^1, \xi^2, \ldots, \xi^K$ are distinct points in D,

$$F(\xi) = \sum_{k=1}^{K} R\left(\xi^k, \xi^k\right) + \sum_{k=1}^{K} \sum_{l=1, l\neq k}^{K} G\left(\xi^k, \xi^l\right).$$

Since $G(x, y) \to \infty$ if $|x-y| \to 0$ and $R(x, x) \to \infty$ if $x \to \partial D$, F attains a minimum.

THEOREM 5.1. *Let D be a bounded and sufficiently smooth domain in $\mathbb{R}^N (N \geq 3), K \in \mathbb{N}$, and $\delta > 0$. There exists b_0 depending on D, K, and δ such that if*

1. $(\sigma, \alpha) \in T(\mathcal{R}_d)$,
2. $b < b_0$,
3. $\frac{(N-2)(N-1)N+\delta}{2b^3} < \sigma < \frac{N(N+1)(N+2)-\delta}{2b^3}$,

where $(\sigma, b) = T_{\mathcal{R}_d}^{-1}(\sigma, \alpha)$*, then* $\mathcal{J}_D$*, with* α *and* σ *being its parameters, admits a stationary set that is an assembly of* K *perturbed balls. Moreover,*

1. *the radii of all the perturbed balls are all approximately equal to* b*;*
2. *if the centers of the perturbed balls are* $\xi_*^1, \xi_*^2, \ldots, \xi_*^K$*, then as* $b \to 0$*,* $(\xi_*^1, \xi_*^2, \ldots, \xi_*^K)$ *converges to a minimum of* F*, possibly along a subsequence;*
3. *this stationary assembly is stable in some sense.*

Remark 1. The last statement about stability in the above theorem is vague; further discussion about what we mean is contained in the last paragraph of this section. We feel that the constructed assembly should be a local minimizer; however, we do not have a proof.

The proof of this theorem proceeds along the same line as the proof of [12, Theorem 5.1], so we only present a sketch here. The reader may consult [12] for missing details. Similar strategies were used in [29, 30] for a diblock copolymer problem. The diblock copolymer problem has a volume constraint and consequently there is always one and only one bubble for every parameter value.

We start with a construction of assemblies of exact balls whose radii are close to b. An important remark is in order. Condition 3 is derived from the bounds $C_0(b) < \sigma < C_2(b)$ for small b. When b is chosen from $T_{\mathcal{R}_d}^{-1}(\sigma, \alpha)$ and σ satisfies condition 3 of the theorem, the ball $B(0, b)$ is necessarily a stable bubble profile, studied in section 4. Henceforth, we construct a ball assembly from a stable bubble.

Let $\xi = (\xi^1, \xi^2, \ldots, \xi^K)$, where $\xi^1, \xi^2, \ldots, \xi^K$ are distinct points in D, and make an approximate solution that is an assembly of small balls centered at the ξ^k's. The radii of the balls are $b\beta^k$, where the β^k's are not yet determined. Collectively write $\beta = (\beta^1, \beta^2, \ldots, \beta^K)$, where each β^k is close to 1 so $b\beta^k$ is close to b. Let P_0 be the union of balls P_0^k centered at ξ^k of radii $b\beta^k$:

$$P_0 = \cup_{k=1}^K P_0^k, \quad \text{where} \quad P_0^k = \left\{ x \in \mathbb{R}^N : \ \left| x - \xi^k \right| \le b\beta^k \right\}. \tag{5.2}$$

One estimates the energy of P_0 and finds

(5.3)

$$\begin{aligned} \mathcal{J}_D(P_0) = &\sum_{k=1}^K \Big(N\omega_N (b\beta^k)^{N-1} - \alpha \omega_N (b\beta^k)^N \Big) \\ &+ \sum_{k=1}^K \frac{\sigma}{2} \Big(\omega_N (b\beta^k)^N \big(1 - N I_{\frac{N}{2}}(b\beta^k) K_{\frac{N}{2}}(b\beta^k)\big) + \omega_N^2 (b\beta^k)^{2N} R(\xi^k, \xi^k) \Big) \\ &+ \sum_{k=1}^K \sum_{l=1, l\neq k}^K \frac{\sigma \omega_N^2 (b\beta^k)^N (b\beta^l)^N}{2} G(\xi^k, \xi^l) + O(\sigma b^{2N+2}). \end{aligned}$$

Next we proceed to perturb P_0. Let ϕ^k, $k = 1, 2, \ldots, K$, be functions defined on $\mathbb{S}^{N-1}$. Let

$$P_{\phi^k}^k = \left\{ \xi^k + r\theta : \ r \in \left[0, b\beta^k (1 + N\phi^k(\theta))^{\frac{1}{N}} \right], \ \theta \in \mathbb{S}^{N-1} \right\} \tag{5.4}$$

be a perturbed ball. Then let

$$P_\phi = \cup_{k=1}^K P_{\phi^k}^k \tag{5.5}$$

be an assembly of these perturbed balls. Here one writes $\phi = (\phi^1, \phi^2, \ldots, \phi^K)$ and P_ϕ to emphasize that this assembly depends on ϕ. In fact, P_ϕ also depends on

$\beta = (\beta^1, \beta^2, \ldots, \beta^K)$ and $\xi = (\xi^1, \xi^2, \ldots, \xi^K)$, but these dependencies will be explored later.

If we deform ϕ to

$$\phi + \varepsilon\psi, \tag{5.6}$$

then P_ϕ is deformed to $P_{\phi+\varepsilon\psi}$. The first variation is

$$\left.\frac{d\mathcal{J}_D(\phi+\varepsilon\psi)}{d\varepsilon}\right|_{\varepsilon=0} = \sum_{k=1}^{K}\int_0^{2\pi}\left((N-1)\mathcal{H}\left(\partial P^k_{\phi^k}\right) - \alpha + \sigma\mathcal{N}_D(P_\phi)\right)\left(b\beta^k\right)^N \psi^k\, d\theta. \tag{5.7}$$

Here we have treated $\mathcal{J}_D$ as a functional of ϕ and written $\mathcal{J}_D(\phi + \varepsilon\psi)$ for $\mathcal{J}_D(P_{\phi+\varepsilon\psi})$. We also introduce a nonlinear operator $\mathcal{S}_D$ whose kth component is

$$\mathcal{S}^k_D(\phi) = \left(b\beta^k\right)^N\left((N-1)\mathcal{H}\left(\partial P^k_{\phi^k}\right) - \alpha + \sigma\mathcal{N}_D(P_\phi)\right), \tag{5.8}$$

which is viewed as a function of θ. This operator is identified as the first variation of $\mathcal{J}$ since

$$\left.\frac{d\mathcal{J}_D(\phi+\varepsilon\psi)}{d\varepsilon}\right|_{\varepsilon=0} = \sum_{k=1}^{K}\int_{\mathbb{S}^{N-1}}\mathcal{S}^k_D(\phi)\psi^k\, d\theta. \tag{5.9}$$

An assembly P_ϕ of perturbed balls is a stationary set precisely when

$$\mathcal{S}_D(\phi) = 0. \tag{5.10}$$

To solve this equation, consider the second variation of $\mathcal{J}_D$. Let $\phi + \varepsilon_1\psi + \varepsilon_2\upsilon$ be a two-parameter deformation of ϕ. Then

$$\left.\frac{\partial^2\mathcal{J}_D(\phi+\varepsilon_1\psi+\varepsilon_2\upsilon)}{\partial\varepsilon_1\partial\varepsilon_2}\right|_{\varepsilon_1=\varepsilon_2=0} = \langle\mathcal{S}'_D(\phi)\psi, \upsilon\rangle. \tag{5.11}$$

Here $\mathcal{S}'_D(\phi)$ is the Fréchet derivative of $\mathcal{S}_D$ at ϕ and

$$\psi \to \mathcal{S}'_D(\phi)\psi \tag{5.12}$$

is a self-adjoint linear operator from $H^2(\mathbb{S}^{N-1}, \mathbb{R}^K) \subset L^2(\mathbb{S}^{N-1}, \mathbb{R}^K)$ to $L^2(\mathbb{S}^{N-1}, \mathbb{R}^K)$.

Equation (5.10) is solved in two steps. First one solves the equation up to the locations and radii of balls. More precisely introduce some subspaces

$$L^2_\flat\left(\mathbb{S}^{N-1}, \mathbb{R}^K\right) = \Big\{\phi \in L^2\left(\mathbb{S}^{N-1}, \mathbb{R}^K\right) :$$

$$\int_{\mathbb{S}^{N-1}}\phi^k(\theta)\, d\theta = \int_{\mathbb{S}^{N-1}}\phi^k(\theta)h(\theta)\, d\theta = 0 \ \forall h \in H^N_1,\ \forall k\Big\}, \tag{5.13}$$

$$H^2_\flat\left(\mathbb{S}^{N-1}, \mathbb{R}^K\right) = H^2\left(\mathbb{S}^{N-1}, \mathbb{R}^K\right) \cap L^2_\flat\left(\mathbb{S}^{N-1}, \mathbb{R}^K\right). \tag{5.14}$$

Recall that H^N_1 is the space of spherical harmonics of degree 1. Denote the orthogonal projection from $L^2(\mathbb{S}^{N-1}, \mathbb{R}^K)$ to $L^2_\flat(\mathbb{S}^{N-1}, \mathbb{R}^K)$ by Π. Geometrically one can interpret an element in $L^2_\flat(\mathbb{S}^{N-1}, \mathbb{R}^K)$ (or $H^2_\flat(\mathbb{S}^{N-1}, \mathbb{R}^K)$) as an assembly whose perturbed balls have well-defined centers ξ^k and well-defined radii $b\beta^k$. More specifically, the condition $\int_0^{2\pi}\phi^k(\theta)\, d\theta = 0$ implies that $b\beta^k$ can be interpreted as the radius of the perturbed ball $P^k_{\phi^k}$; the condition $\int_0^{2\pi} \phi^k(\theta)h(\theta)\,d\theta = 0$ for all $h \in H_1^N$ defines ξ^k as the center of $P^k_{\phi^k}$.

In the first step of solving (5.10) we find $\phi \in H^2_\flat(\mathbb{S}^{N-1}, \mathbb{R}^K)$, so that $\Pi\mathcal{S}_D(\phi) = 0$. This is done by a fixed point argument. One rewrites the equation as

$$-(\Pi\mathcal{S}'_D(0))^{-1}(\Pi\mathcal{S}_D(0) + \Pi\mathcal{R}(\phi)) = \phi, \tag{5.15}$$

where $\mathcal{R}(\phi)$ is the higher order part in the expansion of $\mathcal{S}(\phi)$, namely,

$$\mathcal{S}_D(\phi) = \mathcal{S}_D(0) + \mathcal{S}'_D(0)\phi + \mathcal{R}(\phi). \tag{5.16}$$

One shows that the operator $\Pi\mathcal{S}'_D(0)$ is positive definite on $H^2_\flat(\mathbb{S}^{N-1}, \mathbb{R}^K) \subset L^2_\flat(\mathbb{S}^{N-1}, \mathbb{R}^K)$. Here one needs the upper bound $\sigma < \frac{N(N+1)(N+2)-\delta}{2b^3}$ of condition 3. This is of no surprise as we consider the space $L^2_\flat(\mathbb{S}^{N-1}, \mathbb{R}^K)$ and $\lambda_n > 0$ in (4.9) for $n \geq 2$.

Once the fixed point is found, we denote it by $\phi_*(\cdot, \beta, \xi)$. As the above procedures are done with respect to the general β and ξ, it solves

$$\Pi\mathcal{S}_D(\phi_*(\cdot, \beta, \xi)) = 0. \tag{5.17}$$

In the second step, we choose appropriate β and ξ, denoted β_* and ξ_*, so that $\phi_*(\cdot, \beta_*, \xi_*)$ solves the equation $\mathcal{S}(\phi_*(\cdot, \beta_*, \xi_*)) = 0$. To this end, we use a variational argument. Let

$$J(\beta, \xi) = \mathcal{J}_{\mathcal{D}}(\phi_*(\cdot, \beta, \xi)). \tag{5.18}$$

One proves that if (β_*, ξ_*) is a minimum of J, then $\phi_*(\cdot, \beta_*, \xi_*)$ is the desired solution. It turns out that $J(\beta, \xi)$ can be approximated by the energy of P_0, the assembly of exact balls of radii $b\beta^k$ and centers ξ^k. Then

$$\begin{aligned} J(\beta, \xi) = &\sum_{k=1}^K \left(N\omega_N \left(b\beta^k\right)^{N-1} - \alpha\omega_N \left(b\beta^k\right)^N \right) \\ &+ \sum_{k=1}^K \frac{\sigma}{2}\left(\omega_N \left(b\beta^k\right)^N \left(1 - NI_{\frac{N}{2}}\left(b\beta^k\right) K_{\frac{N}{2}}\left(b\beta^k\right)\right) + \omega_N^2 (b\beta^k)^{2N} R\left(\xi^k, \xi^k\right) \right) \\ &+ \sum_{k=1}^K \sum_{l=1, l\neq k}^K \frac{\sigma\omega_N^2 (b\beta^k)^N \left(b\beta^l\right)^N}{2} G\left(\xi^k, \xi^l\right) + O\left(\sigma b^{2N+2}\right). \end{aligned} \tag{5.19}$$

We need to be a bit more precise about the domain of β and ξ. Let

$$\Xi = \left\{\xi = \left(\xi^1, \xi^2, \ldots, \xi^K\right) : \ \xi^k \in D,\ \xi^k \neq \xi^l \text{ if } k \neq l\right\}, \tag{5.20}$$

which is the domain of the function F given in (5). Then, for $\eta > 0$, define

$$\Xi_\eta = \left\{\xi = \left(\xi^1, \xi^2, \ldots, \xi^K\right) : \ \xi^k \in D,\ \operatorname{dist}\left(\xi^k, \partial D\right) > \eta,\ \operatorname{dist}\left(\xi^k, \xi^l\right) > 2\eta \text{ for } k \neq l\right\}. \tag{5.21}$$

By choosing η sufficiently small we ensure that the minimum of F on Ξ is achieved inside Ξ_η. Next let

$$B_\tau = \left\{\beta = \left(\beta^1, \beta^2, \ldots, \beta^K\right) : \ \beta^k \in (1-\tau, 1+\tau)\right\}. \tag{5.22}$$

The number τ will be specified and made small later. Now we take $\xi = (\xi^1, \xi^2, \ldots, \xi^K) \in \overline{\Xi_\eta}$ and $\beta = (\beta^1, \beta^2, \ldots, \beta^K) \in \overline{B_\tau}$.

Let (β_*, ξ_*) be a minimum of J on $\overline{B_\tau} \times \overline{\Xi_\eta}$. We need to show that if b is small, then (β_*, ξ_*) is in $B_\tau \times \Xi_\eta$, the interior of $\overline{B_\tau} \times \overline{\Xi_\eta}$, so that (β_*, ξ_*) is a critical point of J. Then the equation $\mathcal{S}(\phi_*(\cdot, \beta_*, \xi_*)) = 0$ follows. Let $b \to 0$ and (β_*, ξ_*) converge, possibly along a subsequence, to $(\beta_\circ, \xi_\circ) \in \overline{B_\tau} \times \overline{\Xi_\eta}$.

We need good estimates for $I_\nu(z)$ and $K_\nu(z)$ for small z. For I_ν we have

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{z^2}{4}\right)^k}{k!\Gamma(\nu + k + 1)} \tag{5.23}$$

(see [1, (9.6.10)] or [19, (8.445)]), from which we derive

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \left(\frac{1}{\Gamma(\nu+1)} + \frac{z^2}{4\Gamma(\nu+2)} + O\left(z^4\right)\right). \tag{5.24}$$

For K_ν, when ν is a positive integer

$$\begin{aligned} K_\nu(z) = {} & \frac{1}{2}\left(\frac{z}{2}\right)^{-\nu} \sum_{k=0}^{\nu-1} \frac{(\nu-k-1)!}{k!}\left(-\frac{z^2}{4}\right)^k \\ & + (-1)^{\nu+1} \log\left(\frac{z}{2}\right) I_\nu(z) \\ & + (-1)^\nu \frac{1}{2}\left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \{\psi(k+1) + \psi(\nu+k+1)\} \frac{\left(\frac{z^2}{4}\right)^k}{k!(\nu+k)!} \end{aligned} \tag{5.25}$$

(see [1, (9.6.11)] or [19, (8.446)]; ψ here is the psi function); when ν is a nonnegative integer plus $\frac{1}{2}$, with $\nu = n + \frac{1}{2}$,

$$K_{n+\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!(2z)^k} \tag{5.26}$$

(see [1, (10.2.15)] or [19, (8.468)]). One derives that if $\nu \geq 3/2$ is an integer or an integer plus $\frac{1}{2}$ and $\nu \neq 2$,

$$K_\nu(z) = \left(\frac{z}{2}\right)^{-\nu} \left(\frac{\Gamma(\nu)}{2} - \frac{\Gamma(\nu-1)}{8} z^2 + O\left(z^4\right)\right). \tag{5.27}$$

If $\nu = 2$, then the $O(z^4)$ term above in (5.27) is replaced by $O(z^4 \log z)$. Following (5.24) and (5.27), one finds

$$I_\nu(z) K_\nu(z) = \frac{1}{2\nu} - \frac{z^2}{4(\nu-1)\nu(\nu+1)} + O\left(z^4\right) \tag{5.28}$$

if $\nu \neq 2$. And if $\nu = 2$, the $O(z^4)$ term becomes $O(z^4 \log z)$.

Now we first claim that $\beta_\circ = (1, 1, \ldots, 1)$, which is in B_τ. One needs the estimate

$$N - 1 - \alpha b + \sigma b^2 \left(\frac{b}{2}\right)^{\frac{N}{2}} \left(\frac{1}{\Gamma(\frac{N}{2}+1)} + o(1)\right) \left(\frac{b}{2}\right)^{1-\frac{N}{2}} \left(\frac{\Gamma(\frac{N}{2}-1)}{2} + o(1)\right) = 0 \tag{5.29}$$

for α. Here (5.29) follows from (3.5) and [1, (9.6.7) and (9.6.9)].

Assume that $\sigma b^3 \to \gamma$. Because of condition 3 in Theorem 5.1,

$$\gamma \in \left[\frac{(N-2)(N-1)N+\delta}{2}, \frac{N(N+1)(N+2)-\delta}{2}\right]. \tag{5.30}$$

To find a uniform limit of $\frac{1}{\sigma b^{N+2}} J(\beta,\xi)$ as $b \to 0$, we appeal to (5.19) and derive the following limits:

$$\frac{1}{\sigma b^{N+2}}\left(N\omega_N (b\beta^k)^{N-1}\right) \to \frac{N\omega_N}{\gamma}(\beta^k)^{N-1}, \tag{5.31}$$

$$\frac{1}{\sigma b^{N+2}}\left(-\alpha\omega_N (b\beta^k)^{N}\right) \to -\omega_N\left(\frac{N-1}{\gamma}+\frac{1}{(N-2)N}\right)\left(\beta^k\right)^N \tag{5.32}$$

by (5.29),

$$\frac{1}{\sigma b^{N+2}}\left(\frac{\sigma\omega_N\left(b\beta^k\right)^N}{2}\left(1-NI_{\frac{N}{2}}\left(b\beta^k\right)K_{\frac{N}{2}}\left(b\beta^k\right)\right)\right) \to \frac{\omega_N}{(N-2)(N+2)}\left(\beta^k\right)^{N+2} \tag{5.33}$$

by (5.28), and

$$\begin{aligned}\frac{1}{\sigma b^{N+2}}&\left(\sum_{k=1}^{K}\frac{\sigma\omega_N^2\left(b\beta^k\right)^{2N}}{2}R\left(\xi^k,\xi^k\right)\right.\\ &\left.+\sum_{k=1}^{K}\sum_{l=1,l\neq k}^{K}\frac{\sigma\omega_N^2\left(b\beta^k\right)^N\left(b\beta^l\right)^N}{2}G\left(\xi^k,\xi^l\right)+O\left(\sigma b^{2N+2}\right)\right) \to 0.\end{aligned} \tag{5.34}$$

Then (5.19) implies that

$$\begin{aligned}\frac{J(\beta,\xi)}{\sigma b^{N+2}} \to \sum_{k=1}^{K}&\left(\frac{N\omega_N}{\gamma}\left(\beta^k\right)^{N-1}-\omega_N\left(\frac{N-1}{\gamma}+\frac{1}{(N-2)N}\right)\left(\beta^k\right)^N\right.\\ &\left.+\frac{\omega_N}{(N-2)(N+2)}\left(\beta^k\right)^{N+2}\right)\end{aligned} \tag{5.35}$$

uniformly on $\overline{B_\tau} \times \overline{\Xi_\eta}$ as $b \to 0$. Consequently

$$\begin{aligned}&\frac{1}{\sigma b^{N+2}}\left(J(\beta_*,\xi_*)-J((1,1,\dots,1),\xi_*)\right)\\ &\to \sum_{k=1}^{K}\left(\frac{N\omega_N}{\gamma}\left(\beta_\circ^k\right)^{N-1}-\omega_N\left(\frac{N-1}{\gamma}+\frac{1}{(N-2)N}\right)\left(\beta_\circ^k\right)^N\right.\\ &\qquad\left.+\frac{\omega_N}{(N-2)(N+2)}\left(\beta_\circ^k\right)^{N+2}\right)\\ &-\sum_{k=1}^{K}\left(\frac{N\omega_N}{\gamma}1^{N-1}-\omega_N\left(\frac{N-1}{\gamma}+\frac{1}{(N-2)N}\right)1^N+\frac{\omega_N}{(N-2)(N+2)}1^{N+2}\right).\end{aligned} \tag{5.36}$$

The function

$$f(t)=\frac{N\omega_N}{\gamma}\ t^{N-1}-\omega_N\left(\frac{N-1}{\gamma}+\frac{1}{(N-2)N}\right)\ t^N+\frac{\omega_N}{(N-2)(N+2)}\ t^{N+2} \tag{5.37}$$

has $t = 1$ as a strict local minimum, since $f''(1) > 0$ precisely when $\gamma > \frac{(N-2)(N-1)N}{2}$. Here we have used the lower bound $\frac{(N-2)(N-1)N+\delta}{2b^3} < \sigma$ of condition 3. Fix τ in (5.22) to be small enough so that in the interval $(1-2\tau, 1+2\tau)$, $w = 1$ is the only critical point of the function f. If $\beta_\circ$ were not $(1, 1, \ldots, 1)$, then the right side of (5.36) is positive since each $\beta_\circ^k$ is in $[1-\tau, 1+\tau]$. Consequently when b is sufficiently small $J(\beta_*, \xi_*) > J((1, 1, \ldots, 1), \xi_*)$, a contradiction to the choice of (β_*, ξ_*).

Next we show that $\xi_\circ$ is a minimum of F on Ξ. Let ξ_m be a minimum of F. Then $\xi_m \in \Xi_\eta$. By Lemma 5.19 and the just proved fact that $\beta_* \to (1, 1, \ldots, 1)$,

(5.38)

$$\begin{aligned}
&\frac{1}{\sigma b^{2N}}\big(J(\beta_*, \xi_*) - J(\beta_*, \xi_m)\big) \\
&\quad = \sum_{k=1}^{K} \frac{\omega_N^2 (\beta_*^k)^{2N}}{2} R\left(\xi_*^k, \xi_*^k\right) + \sum_{k=1}^{K} \sum_{l=1, l\neq k}^{K} \frac{\omega_N^2 (\beta_*^k)^N (\beta_*^l)^N}{2} G\left(\xi_*^k, \xi_*^l\right) \\
&\quad\quad - \sum_{k=1}^{K} \frac{\omega_N^2 (\beta_*^k)^{2N}}{2} R\left(\xi_m^k, \xi_m^k\right) - \sum_{k=1}^{K} \sum_{l=1, l\neq k}^{K} \frac{\omega_N^2 (\beta_*^k)^N \left(\beta_*^l\right)^N}{2} G\left(\xi_m^k, \xi_m^l\right) + O\left(b^2\right) \\
&\quad \to \frac{\omega_N^2}{2}\big(F(\xi_\circ) - F(\xi_m)\big).
\end{aligned}$$

If $\xi_\circ$ were not a minimum of F, then the last line of (5.38) would be positive, a contradiction to the choice of (β_*, ξ_*). Now that $\xi_\circ$ is a minimum of F, $\xi_\circ$ is necessarily in Ξ_η. Therefore $\xi_* \in \Xi_\eta$ when b is sufficiently small.

We have shown that (β_*, ξ_*) is a critical point of J. It follows that $P_{\phi_*(\cdot, \beta_*, \xi_*)}$ is a stationary assembly of $\mathcal{J}_D$. The first additional assertion of the theorem that all the perturbed discs in $P_{\phi_*(\cdot, \beta_*, \xi_*)}$ have approximately the same radius comes from the fact that $\beta_* \to (1, 1, \ldots, 1)$; the second assertion follows from the facts that $\xi_* \to \xi_\circ$ and $\xi_\circ$ is a minimum of F.

Our assertion that $P_{\phi_*(\cdot, \beta_*, \xi_*)}$ is stable comes from its local minimization property. Recall that $P_{\phi_*(\cdot, \beta_*, \xi_*)}$ is found in two steps. First for each $(\beta, \xi) \in \overline{B_\tau} \times \overline{\Xi_\eta}$, a fixed point $P_{\phi_*(\cdot, \beta, \xi)}$ is constructed in a class of assemblies with well-defined centers and radii. The fixed point is actually locally minimizing $\mathcal{J}_D$ within this class of assemblies by the positivity of the operator $\Pi \mathcal{S}_D'(0)$. In the second step $\mathcal{J}_D$ is minimized among the $P_{\phi_*(\cdot, \beta, \xi)}$'s, where (β, ξ) ranges in $\overline{B_\tau} \times \overline{\Xi_\eta}$, and $P_{\phi_*(\cdot, \beta_*, \xi_*)}$ emerges as a minimum. As a minimum of locally minimizing assemblies within classes of well-defined centers and radii, $P_{\phi_*(\cdot, \beta_*, \xi_*)}$ is locally energy minimizing with respect to deformations both within and without the class of assemblies with well-defined centers and radii; hence, in this sense we claim that $P_{\phi_*(\cdot, \beta_*, \xi_*)}$ is stable.

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