Stationary disc assemblies in a ternary system with long range interaction

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Abstract

The free energy of a ternary system, such as a triblock copolymer, is a sum of two parts: an interface energy determined by the size of the interfaces separating the micro-domains of the three constituents, and a long range interaction energy that serves to prevent unlimited micro-domain growth. In two dimensions a parameter range is identified where the system admits stable stationary disc assemblies. Such an assembly consists of perturbed discs made from either type-I constituent or type-II constituent. All the type-I discs have approximately the same radius and all the type-II discs also have approximately the same radius. The locations of the discs are determined by minimization of a function. Depending on the parameters, the discs of the two types can be mixed in an organized way, or mixed in a random way. They can also be fully separated. The first scenario offers a mathematical proof of the existence of a morphological phase for triblock copolymers conjectured by polymer scientists. The last scenario shows that the ternary system is capable of producing two levels of structure. The primary structure is at the microscopic level where discs form near perfect lattices. The secondary structure is at the macroscopic level forming two large regions, one filled with type-I discs and the other filled with type-II discs. A macroscopic, circular interface separates the two regions.

Key words. ternary system, triblock copolymer, cylindrical phase, disk assembly, primary structure, secondary structure

AMS Subject Classifications. 82B24 82D60 92C15

1 Introduction

A ternary physical or biological system, such as the iron-nickel-chromium ternary alloy, is comprised of three constituents. At low temperature, the constituents tend to separate and micro-domains rich in individual constituents emerge. Over time these micro-domains grow in size. This coarsening process reduces the total area of the interfaces separating the micro-domains.

However in many systems, coarsening tendencies are balanced by inhibitory forces which limit growth of micro-domains. Triblock copolymers are archetypal examples. In an ABC triblock copolymer a molecule is a subchain of type A monomers connected to a subchain of type B monomers which in turn is connected to a subchain of type C monomers. Because of the repulsion between the unlike monomers, the different type subchains tend to segregate. However since subchains are chemically bonded in molecules, segregation

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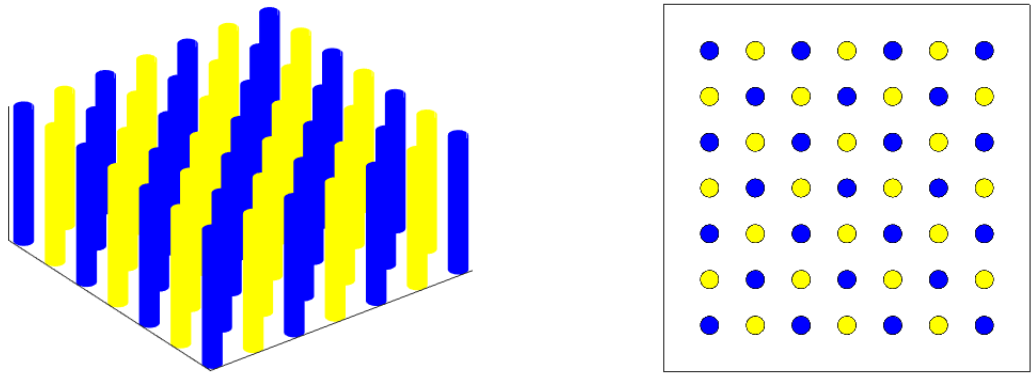

Figure 1: A triblock copolymer in the cylindrical phase and its cross section. Type A micro-domains are in blue and type B micro-domains are in yellow. The rest of the region is filled by type C monomers

cannot lead to a macroscopic phase separation; only micro-domains rich in individual type monomers emerge, forming morphological phases. Bonding of distinct monomer subchains provides an inhibition mechanism in block copolymers.

One of the morphological phases in triblock copolymers predicted by polymer scientists is the cylindrical phase [3, Figure 5(f)]. In this phase type A monomers form cylinder shaped micro-domains and type B monomers also form cylinder shaped micro-domains. Type C monomers fill the remaining region. A cross section of a triblock copolymer in the cylindrical phase shows an assembly of discs, some of type A monomers and some of type B monomers; see Figure 1. An application of the main result in this paper will mathematically confirm the existence of this phase as a stable stationary point of the free energy.

The ternary system considered here was originally derived by Ren and Wei in [22, 23] from Nakazawa and Ohta's density functional theory for triblock copolymers [15]. In two dimensions the system occupies a bounded domain D in $\mathbb{R}^2$ and has two sets of parameters. The first set consists of ω_1, ω_2, both in $(0,1)$. Moreover, with $\omega_3 = 1-(\omega_1+\omega_2)$, $\omega_3 \in (0,1)$ as well. The second set of parameters forms a symmetric two by two matrix γ,

$$\gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{12} & \gamma_{22} \end{bmatrix}. \tag{1.1}$$

For triblock copolymers, the matrix γ is necessarily positive definite [23]. In [5] Choksi and Ren studied diblock copolymer-homopolymer blends and arrived at the same model but with a positive semi-definite γ. For these blends γ has one positive eigenvalue and one zero eigenvalue; (ω_1, ω_2) is the eigenvector associated to the zero eigenvalue. In this paper γ is not necessarily positive definite. However, all the entries of γ are assumed to be positive in this work.

Let Ω_1, $\Omega_2 \subset D$ be two Lebesgue measurable subsets of D and be disjoint in the sense that $|\Omega_1 \cap \Omega_2| = 0$. Here $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^2$. The measures of Ω_i, $i = 1, 2$, are fixed at

$$|\Omega_i| = \omega_i |D|, \quad i = 1, 2. \tag{1.2}$$

Denote $\Omega_3 = D \backslash (\Omega_1 \cup \Omega_2)$, so $|\Omega_3| = \omega_3 |D|$. The free energy of the pair (Ω_1, Ω_2), or the triplet $(\Omega_1, \Omega_2, \Omega_3)$, is given by

$$\mathcal{J}(\Omega_1, \Omega_2) = \frac{1}{2} \sum_{i=1}^{3} \mathcal{P}_D(\Omega_i) + \sum_{i,j=1}^{2} \frac{\gamma_{ij}}{2} \int_D (-\Delta)^{-1/2} (\chi_{\Omega_i} - \omega_i) \, (-\Delta)^{-1/2} (\chi_{\Omega_j} - \omega_j) dx. \tag{1.3}$$

In functional (1.3), $\mathcal{P}_D(\Omega_i)$ denotes the perimeter of Ω_i in D. If Ω_i has piecewise C^1 boundary, then $\partial\Omega_i \cap D$, the part of the boundary of Ω_i that is inside D, is the interface that separates Ω_i from the other two

Ω_j's, $j \in \{1,2,3\}$, $j \neq i$, and $\mathcal{P}_D(\Omega_i)$ is simply the length of $\partial\Omega_i \cap D$. Note that $\partial\Omega_i$ denotes the boundary of Ω_i in $\mathbb{R}^2$. Part of $\partial\Omega_i$ may overlap with the domain boundary ∂D, but the interface $\partial\Omega_i \cap D$ does not include this part of $\partial\Omega_i$. For a general measurable Ω_i,

$$\mathcal{P}_D(\Omega_i) = \sup\Big\{ \int_{\Omega_i} \operatorname{div} g(x)\, dx : \ g \in C_0^1(D,\mathbb{R}^2),\ |g(x)| \le 1 \ \forall x \in D \Big\}, \tag{1.4}$$

and $\partial\Omega_i \cap D$ should be interpreted as the reduced boundary of Ω_i in D; see [34]. In the sum $\sum_{i=1}^3 \mathcal{P}_D(\Omega_i)$, each of the interfaces $\partial\Omega_i \cap D$, $i = 1,2,3$, is counted twice, so a half is placed in front of this sum to remove double counting. The first part of (1.3) is an interface energy equal to the total length of all interfaces.

The second part of (1.3) is a long range interaction term. For $f \in L^2(D)$ such that $\int_D f(x)dx = 0$, one solves the Poisson equation with the zero Neumann boundary condition and zero average:

$$-\Delta v = f \text{ in } D,\ \partial_\nu v = 0 \text{ on } \partial D,\ \int_D v(x)dx = 0. \tag{1.5}$$

If the domain D is sufficiently smooth, this equation has a unique solution in $H^2(D)$, and hence it defines an operator $(-\Delta)^{-1}$ from the space $\{f \in L^2(D) : \int_D f(x)dx = 0\}$ to the space $\{v \in H^2(D) : \partial_\nu v = 0 \text{ on } \partial D,\ \int_D v(x)dx = 0\}$; we write $v = (-\Delta)^{-1} f$. The operator $(-\Delta)^{-1}$ is positive definite and $(-\Delta)^{-1/2}$ is its positive square root. In (1.3) $(-\Delta)^{-1/2}$ acts on $\chi_{\Omega_i} - \omega_i$ where χ_{Ω_i} is the characteristic function of Ω_i ($\chi_{\Omega_i}(x) = 1$ if $x \in \Omega_i$ and 0 if $x \in D\backslash\Omega_i$).

A stationary point (Ω_1, Ω_2) of $\mathcal{J}$ consists of two disjoint subsets Ω_1 and Ω_2 of D, each bounded by piecewise smooth curves. It satisfies the following equations:

$$\begin{aligned}
\kappa_1 + \gamma_{11} I_{\Omega_1} + \gamma_{12} I_{\Omega_2} &= \lambda_1 \text{ on } \partial\Omega_1 \cap \partial\Omega_3 && (1.6)\\
\kappa_2 + \gamma_{12} I_{\Omega_1} + \gamma_{22} I_{\Omega_2} &= \lambda_2 \text{ on } \partial\Omega_2 \cap \partial\Omega_3 && (1.7)\\
\kappa_0 + (\gamma_{11} - \gamma_{12}) I_{\Omega_1} + (\gamma_{12} - \gamma_{22}) I_{\Omega_2} &= \lambda_1 - \lambda_2 \text{ on } \partial\Omega_1 \cap \partial\Omega_2 && (1.8)\\
T_1 + T_2 + T_0 &= \vec{0} \text{ at } \partial\Omega_1 \cap \partial\Omega_2 \cap \partial\Omega_3 && (1.9)\\
T_i &\perp \partial D \text{ at } \overline{\partial\Omega_i \cap \partial\Omega_3} \cap \partial D,\ i = 1,2 && (1.10)\\
T_0 &\perp \partial D \text{ at } \overline{\partial\Omega_1 \cap \partial\Omega_2} \cap \partial D. && (1.11)
\end{aligned}$$

The equation (1.6) holds on $\partial\Omega_1 \cap \partial\Omega_3$ which is the interface between Ω_1 and Ω_3. On the left side of (1.6) κ_1 is the curvature of this interface with respect to the normal vector that points inward into Ω_1. The I_{Ω_i}'s are shorthand notations:

$$I_{\Omega_i} = (-\Delta)^{-1}(\chi_{\Omega_i} - \omega_i),\ i = 1,2, \tag{1.12}$$

which we call inhibitors. The equations (1.7) holds on the interface between Ω_2 and Ω_3 and the curvature κ_2 is measured with respect to the normal vector pointing into Ω_2; the equation (1.8) holds on the interface between Ω_1 and Ω_2 and the curvature κ_0 is measured with respect to the normal vector pointing into Ω_1. On the right sides of these equations there are unknown constants λ_1 and λ_2. These are Lagrange multipliers associated with the constraints $|\Omega_i| = \omega_i |D|$, $i = 1,2$.

The three interfaces, $\partial\Omega_1 \cap \partial\Omega_3$, $\partial\Omega_2 \cap \partial\Omega_3$ and $\partial\Omega_1 \cap \partial\Omega_2$, may meet at a common point in D, which is termed a triple junction point. In (1.9) T_1, T_2 and T_0 are unit tangent vectors at triple junction points: T_1 is inward pointing and tangent to $\partial\Omega_1 \cap \partial\Omega_3$, T_2 is inward pointing and tangent to $\partial\Omega_2 \cap \partial\Omega_3$, and T_0 is inward pointing and tangent to $\partial\Omega_1 \cap \partial\Omega_2$. The equation (1.9) is equivalent to the condition that at any triple junction point the three interfaces meet at 120 degrees.

In the case that an interface meets the domain boundary ∂D, the equations (1.10) and (1.11) assert that it does so perpendicularly. Here T_1, T_2 and T_0 are again unit tangent vectors of $\partial\Omega_1 \cap \partial\Omega_3$, $\partial\Omega_2 \cap \partial\Omega_3$ and $\partial\Omega_1 \cap \partial\Omega_2$ respectively.

In this paper we are interested in stationary points (Ω_1, Ω_2) of $\mathcal{J}$ possessing the following properties.

1. Each of Ω_1 and Ω_2 is a union of multiple components.
2. Each component in Ω_1 or Ω_2 is close to a round disc.
3. Both Ω_1 and Ω_2 have positive distance from the domain boundary ∂D.
4. The components of Ω_1 and Ω_2 are well separated, in the sense that they are of positive distance from each other.

We call such a stationary point a stationary disc assembly. It solves the first two of the six equations in (1.6)-(1.11): (1.6) and (1.7). The rest are superfluous for disc assemblies.

We will find stationary disc assemblies when ω_1 and ω_2 are small and comparable. It is convenient to introduce a fixed $m \in (0,1)$ and a small ϵ so that

$$\omega_1|D| = \epsilon^2 m \text{ and } \omega_2|D| = \epsilon^2(1-m). \tag{1.13}$$

The area constraints (1.2) now take the form

$$|\Omega_1| = \epsilon^2 m \text{ and } |\Omega_2| = \epsilon^2(1-m). \tag{1.14}$$

Henceforth ϵ is the main parameter of our problem.

The premise in this paper is the range of γ_{11}, γ_{22}, and γ_{12} in terms of ϵ. They should all be positive and comparable, no less in order than $\frac{1}{\epsilon^3 \log \frac{1}{\epsilon}}$ and no greater in order than $\frac{1}{\epsilon^3}$. The precise conditions are in the following existence theorem.

Theorem 1.1. *Let D be a bounded and sufficiently smooth domain in $\mathbb{R}^2$. For $m \in (0,1)$, $K_1, K_2 \in \mathbb{N}$, $\eta > 0$, and $B > 0$, there exists $\epsilon_0 = \epsilon_0(D, m, K_1, K_2, \eta, B) > 0$ so that if*

1. *$0 < \epsilon < \epsilon_0$,*
2. *each entry $\gamma_{ij} > 0$ and each diagonal entry $\gamma_{ii} \in \Big(\frac{1+\eta}{\rho_i^3 \log \frac{1}{\rho_i}}, \frac{12-\eta}{\rho_i^3}\Big)$, where $\rho_1 = \epsilon(\frac{m}{K_1\pi})^{1/2}$ and $\rho_2 = \epsilon(\frac{1-m}{K_2\pi})^{1/2}$,*
3. *$\frac{\max\{\gamma_{ij}\}}{\min\{\gamma_{ij}\}} < B$,*

then $\mathcal{J}$ admits a stable stationary assembly of K_1 type-I perturbed discs and K_2 type-II perturbed discs, satisfying (1.6), (1.7), and (1.14).

Moreover, the radii of the type-I discs in the assembly are close to ρ_1 and the radii of the type-II discs are close to ρ_2. If

$$\frac{\gamma_{ij}}{|\gamma|} \to \Gamma_{ij} \text{ as } \epsilon \to 0, \tag{1.15}$$

the centers of the type-I discs are $\xi_1^{*,1}, ..., \xi_1^{*,K_1}$, and the centers of the type-II discs are $\xi_2^{*,1}, ..., \xi_2^{*,K_2}$, then $(\xi_1^{*,1}, ..., \xi_1^{*,K_1}, \xi_2^{*,1}, ..., \xi_2^{*,K_2})$ is close to a minimum of the function

$$\begin{aligned} F(\xi_1^1, ..., \xi_1^{K_1}, \xi_2^1, ..., \xi_2^{K_2}) \quad = \quad & \frac{\Gamma_{11} m^2}{K_1^2}\Big(\sum_{k=1}^{K_1} R(\xi_1^k, \xi_1^k) + \sum_{k=1}^{K_1}\sum_{l=1, l\neq k}^{K_1} G(\xi_1^k, \xi_1^l)\Big) \\ & + \frac{2\Gamma_{12} m(1-m)}{K_1 K_2}\sum_{k=1}^{K_1}\sum_{l=1}^{K_2} G(\xi_1^k, \xi_2^l) \\ & + \frac{\Gamma_{22}(1-m)^2}{K_2^2}\Big(\sum_{k=1}^{K_2} R(\xi_2^k, \xi_2^k) + \sum_{k=1}^{K_2}\sum_{l=1, l\neq k}^{K_2} G(\xi_2^k, \xi_2^l)\Big). \end{aligned} \tag{1.16}$$

For the precise meaning of the radius and the center of a perturbed disc, see (5.9). The smoothness condition on D in Theorem 1.1 is to ensure that $(-\Delta)^{-1}$ is well defined; any $C^{2,\alpha}$ domain meets the requirement [8, Section 6.7]. Note that ρ_1 and ρ_2 are the average radii of the type-I and type-II discs respectively; $|\gamma|$ is the operator norm of γ. The definition of the function F involves Green's function G of the $-\Delta$ operator on D with the Neumann boundary condition, and R which is the regular part of G.

A remark about condition 2 is in order. For $i = 1$ or 2, $\gamma_{ii} \in \left(\frac{1+\eta}{\rho_i^3 \log \frac{1}{\rho_i}}, \frac{12-\eta}{\rho_i^3}\right)$ is needed only if $K_i \geq 2$. If $K_i = 1$, this condition is relaxed to

$$\gamma_{ii} \in \left(0, \frac{12-\eta}{\rho_i^3}\right). \tag{1.17}$$

A motivation of this theorem is presented in Section 2. The theorem and its proof contain detailed information about stationary disc assemblies. Implications of the theorem include the determination of optimal numbers of discs in assemblies, locations of the two type discs, and overall self-organized patterns of discs. We will see how the parameters of our problem subtly affect the answers to these questions. When γ is positive definite, stationary disc assemblies found in Theorem 1.1 match the cylindrical morphological phase for triblock copolymers.

The proof of the theorem starts in Section 3. An assembly of K_1 exact type-I discs and K_2 exact type-II discs is used as an approximate solution. The radii and the centers of the discs are undetermined at this point. From Section 4 to Section 6, we prove that small and suitable perturbations of the discs will turn the assembly to almost a solution of (1.6) and (1.7). This assembly of perturbed discs becomes an exact solution if, in Section 7, the radii and centers of the perturbed discs are chosen properly. This approach is a Lyapunov-Schmidt reduction procedure tailored to fit our nonlocal geometric variational problem.

In Theorem 1.1 all entries of γ are assumed to be positive; by condition 3 all entries of Γ are also positive. It is natural to assume that γ_{11} and γ_{22} are positive, to limit the size of micro-domains. However γ_{12} may not be positive, even for triblock copolymers. We believe that the case of negative γ_{12} will yield patterns different from disc assemblies.

We end this introduction with a note on a binary counterpart of $\mathcal{J}$. It is a simpler analogy of a binary inhibitory system derived from the Ohta-Kawasaki theory [17] for diblock copolymers; see [16, 21]. The binary problem has been studied intensively in recent years. All stationary points in one dimension were known to be local minimizers [21]. Many stationary points in two and three dimensions were found that match morphological phases in diblock copolymers [18, 25, 24, 26, 27, 11, 12, 28, 29, 32]. Global minimizers were studied in [2, 31, 14, 4, 13, 9] for various parameter ranges. Applications of the second variation of the free energy functional and its connections to minimality and Gamma-convergence were found in [6, 1]. A binary analogy of F in (1.16), appeared in [24, 4, 9].

2 Motivation and implications

Before this work, only two types of stationary assemblies were known for inhibitory ternary systems in two dimensions: core-shell assemblies [20] and double bubble assemblies [30]; see Figure 2. In these assemblies, Ω_3 serves as the background while Ω_1 and Ω_2 form disconnected components. In a core-shell assembly each component consists of a core of type-I constituent surrounded by a shell of type-II constituent (also see [3, Figure 5 (b)]); in a double bubble assembly each component is a perturbed double bubble where one of the bubbles is made of type-I constituent and the other of type-II constituent.

A double bubble is enclosed by three circular arcs that meet in threes at two triple junction points forming 120 degree angles. It is the solution to the two area isoperimetric problem [7, 10]. For double bubble assemblies in ternary systems the following theorem was proved in [30].

Theorem 2.1. *Let $m \in (0,1)$, $K \in \mathbb{N}$, and $\iota \in (0,1]$. There exist positive numbers ϵ_0, $\widetilde{\sigma}$, and σ depending on the domain D, m, K, and ι only, such that if the following three conditions hold*

1. $0 < \epsilon < \epsilon_0$,

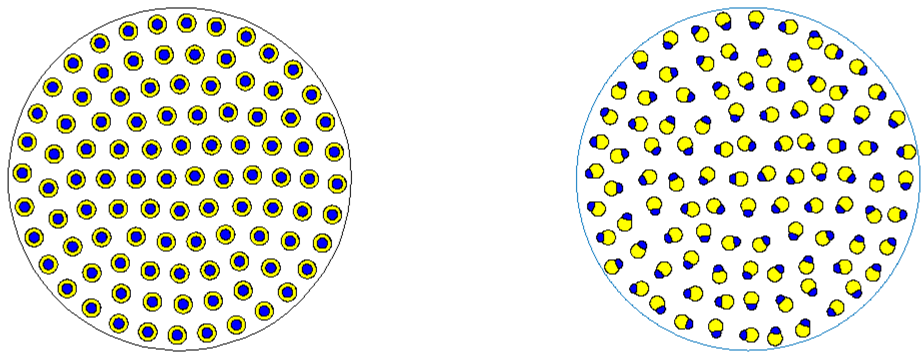

Figure 2: A core-shell assembly and a double bubble assembly.

2. $\dfrac{\widetilde{\sigma}}{\epsilon^3 \log \frac{1}{\epsilon}} \leq \overline{\lambda}(\gamma) \leq \overline{\overline{\lambda}}(\gamma) < \dfrac{\sigma}{\epsilon^3}$,

3. $\iota\, \overline{\overline{\lambda}}(\gamma) \leq \overline{\lambda}(\gamma)$,

then there is a stable stationary assembly of K perturbed double bubbles, satisfying the constraints (1.14) and the equations (1.6)-(1.9).

Here $\overline{\lambda}(\gamma)$ and $\overline{\overline{\lambda}}(\gamma)$ are the two eigenvalues of γ, $\overline{\lambda}(\gamma) \leq \overline{\overline{\lambda}}(\gamma)$, so the theorem requires that γ be positive definite and the two eigenvalues remain comparable as $\epsilon \to 0$. It is interesting to see what happens if these conditions are stressed or violated. From a positive definite γ, one can increase $|\gamma_{12}|$ so the two eigenvalues become less comparable. Increasing $|\gamma_{12}|$ more gives one positive eigenvalue and one zero eigenvalue for γ. Increasing $|\gamma_{12}|$ still further yields one positive eigenvalue and one negative eigenvalue.

Wang, Ren and Zhao [33] performed numerical calculations with a gradient flow of a diffuse interface analogy of (1.3). If $|\gamma_{12}|$ is small compared to γ_{11} and γ_{22}, double bubble assemblies appear. If γ_{12} is positive and is tuned up to a larger value, double bubble assemblies lose to disc assemblies. Such a disc assembly consists of many discs, each made of type-I constituent or type-II constituent. The two type discs are well mixed in an organized way to form a good lattice structure. If γ_{12} is increased more, then the two type discs are not well mixed; they appear randomly in the assembly. Finally if γ_{12} is increased further, the two type discs no longer mix; they escape into two large separated regions. Theorem 1.1 is motivated by these numerical calculations. Many observed phenomena are captured in the theorem.

The numbers K_1 and K_2 are prescribed in the theorem. This implies that there are multiple stationary assemblies for given ϵ and γ. These assemblies differ in their numbers of discs. In the case that the γ_{ij}'s are of the order $\frac{1}{\epsilon^3 \log \frac{1}{\epsilon}}$ we can find the optimal numbers of discs. Suppose that

$$\gamma_{ij} = \frac{1}{\epsilon^3 \log \frac{1}{\epsilon}} \mathbf{\Gamma}_{ij}, \quad i, j = 1, 2, \tag{2.1}$$

where the $\mathbf{\Gamma}_{ij}$'s are positive and fixed. In relation to the Γ_{ij}'s in Theorem 1.1,

$$\Gamma_{ij} = \frac{\mathbf{\Gamma}_{ij}}{|\mathbf{\Gamma}|}. \tag{2.2}$$

To the leading order, the energy of the solution is

$$\sum_{k=1}^{K_1} \left(2\pi r_1^k + \frac{\gamma_{11}\pi}{4} (r_1^k)^4 \log \frac{1}{r_1^k} \right) + \sum_{k=1}^{K_2} \left(2\pi r_2^k + \frac{\gamma_{22}\pi}{4} (r_2^k)^4 \log \frac{1}{r_2^k} \right); \tag{2.3}$$

see Lemma 7.1. Let $r_i^k = \epsilon R_i^k$. Then the leading order of the above becomes

$$\epsilon\Big[\sum_{k=1}^{K_1}\Big(2\pi R_1^k + \frac{\mathbf{\Gamma}_{11}\pi}{4}(R_1^k)^4\Big) + \sum_{k=1}^{K_2}\Big(2\pi R_2^k + \frac{\mathbf{\Gamma}_{22}\pi}{4}(R_2^k)^4\Big)\Big] \tag{2.4}$$

The R_i^k's are subject to the constraints

$$\sum_{k=1}^{K_1}\pi(R_1^k)^2 = m, \quad \sum_{k=1}^{K_2}\pi(R_2^k)^2 = 1-m.$$

Since the theorem asserts that the perturbed discs of the same type have approximately the same radius, we take R_1^k and R_2^k to be R_1 and R_2 respectively where

$$R_1 = \frac{\rho_1}{\epsilon} = \sqrt{\frac{m}{K_1\pi}}, \ R_2 = \frac{\rho_2}{\epsilon} = \sqrt{\frac{1-m}{K_2\pi}}. \tag{2.5}$$

and simplify (2.4) to

$$\begin{aligned} & \epsilon K_1\left(2\pi\sqrt{\frac{m}{K_1\pi}} + \frac{\mathbf{\Gamma}_{11}\pi}{4}\Big(\frac{m}{K_1\pi}\Big)^2\right) + \epsilon K_2\left(2\pi\sqrt{\frac{1-m}{K_2\pi}} + \frac{\mathbf{\Gamma}_{22}\pi}{4}\Big(\frac{1-m}{K_2\pi}\Big)^2\right) \\ = \ & \epsilon\Big(2\sqrt{m\pi}K_1^{1/2} + \frac{\mathbf{\Gamma}_{11}m^2}{4\pi}K_1^{-1} + 2\sqrt{(1-m)\pi}K_2^{1/2} + \frac{\mathbf{\Gamma}_{22}(1-m)^2}{4\pi}K_2^{-1}\Big) \end{aligned}$$

With respect to K_1 and K_2 the above is minimized at

$$K_1 = \left(\frac{\mathbf{\Gamma}_{11}}{4}\right)^{2/3}\frac{m}{\pi}, \quad K_2 = \left(\frac{\mathbf{\Gamma}_{22}}{4}\right)^{2/3}\frac{1-m}{\pi}, \tag{2.6}$$

if one is willing to overlook the fact that K_1 and K_2 are integers. These are the optimal numbers of discs in the sense that the energy of the disc assembly is the smallest if K_1 and K_2 assume the values of (2.6). One should check that under (2.6),

$$\gamma_{11} = \frac{4}{\epsilon^3\log\frac{1}{\epsilon}}\Big(\frac{K_1\pi}{m}\Big)^{3/2} = \frac{4}{\rho_1^3\log\frac{1}{\rho_1}} + O\Big(\frac{1}{\rho_1^3\big(\log\frac{1}{\rho_1}\big)^2}\Big), \tag{2.7}$$

and

$$\gamma_{22} = \frac{4}{\epsilon^3\log\frac{1}{\epsilon}}\Big(\frac{K_2\pi}{1-m}\Big)^{3/2} = \frac{4}{\rho_2^3\log\frac{1}{\rho_2}} + O\Big(\frac{1}{\rho_2^3\big(\log\frac{1}{\rho_2}\big)^2}\Big), \tag{2.8}$$

so condition 2 of Theorem 1.1 is satisfied. Also note that with the optimal numbers of discs (2.6) the disc radii become

$$\rho_1 = \epsilon\Big(\frac{4}{\mathbf{\Gamma}_{11}}\Big)^{1/3}, \quad \rho_2 = \epsilon\Big(\frac{4}{\mathbf{\Gamma}_{22}}\Big)^{1/3}. \tag{2.9}$$

These values do not depend on m.

The locations of the discs in a stationary disc assembly can be found by numerical minimization of function F in (1.16). Recall that Green's function $G(x,y)$ as a function of x satisfies

$$-\Delta G(\cdot,y) = \delta(\cdot - y) - \frac{1}{|D|} \text{ in } D;\ \partial_\nu G(\cdot,y) = 0 \text{ on } \partial D;\ \int_D G(x,y)dx = 0 \tag{2.10}$$

for each $y \in D$. One can write G as a sum of two terms:

$$G(x,y) = \frac{1}{2\pi}\log\frac{1}{|x-y|} + R(x,y). \tag{2.11}$$

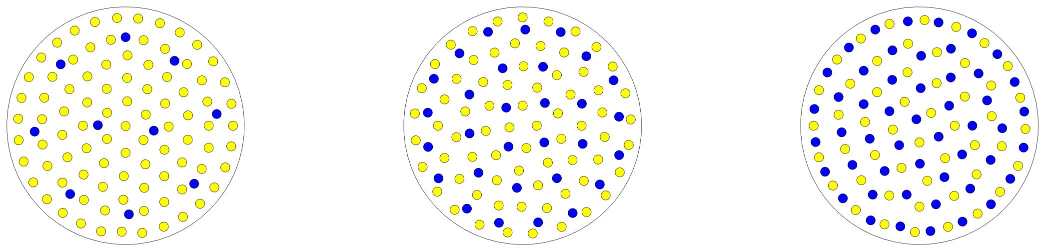

Figure 3: $Q_{11} = Q_{22} = 1$ and $Q_{12} = 0.2$. From left to right: $K_1 = 10$ and $K_2 = 90$, $K_1 = 30$ and $K_2 = 70$, $K_1 = K_2 = 50$.

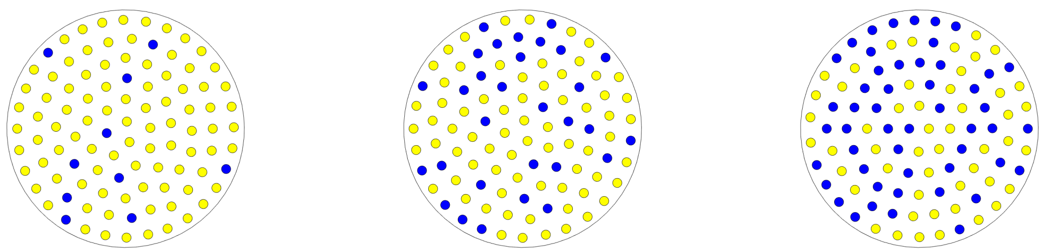

Figure 4: $Q_{11} = Q_{22} = Q_{12} = 1$. From left to right: $K_1 = 10$ and $K_2 = 90$, $K_1 = 30$ and $K_2 = 70$, $K_1 = K_2 = 50$.

The first term $\frac{1}{2\pi} \log \frac{1}{|x-y|}$ is the fundamental solution of the Laplace operator; the second term R is the regular part of Green's function, a smooth function of $(x, y) \in D \times D$. In the case that D is the unit disc,

$$G(x,y) = \frac{1}{2\pi} \log \frac{1}{|x-y|} + \frac{1}{2\pi} \Big[\frac{|x|^2}{2} + \frac{|y|^2}{2} + \log \frac{1}{|x\overline{y} - 1|} \Big] - \frac{3}{8\pi}, \tag{2.12}$$

where $\overline{y}$ is the complex conjugate of y and D is viewed as a subset of $\mathbb{C}$, so we have a closed formula for F. Let

$$Q_{11} = \frac{\Gamma_{11} m^2}{K_1^2}, \quad Q_{12} = \frac{\Gamma_{12} m(1-m)}{K_1 K_2}, \quad Q_{22} = \frac{\Gamma_{22}(1-m)^2}{K_2^2}, \quad Q = \left[\begin{array}{cc} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{array} \right], \tag{2.13}$$

where the Γ_{ij}'s are given in Theorem 1.1. Then Q_{ij} and K_i are the parameters that determine F.

We carry out numerical minimization of F, with D being the unit disc, for three main scenarios:

1. Q is positive definite,
2. Q has one positive eigenvalue and one zero eigenvalue,
3. Q has one positive and one negative eigenvalue.

Since the positivity of the matrix Q is equivalent to the positivity of the matrix Γ and Γ must be positive definite for triblock copolymers, the second and third scenarios do not apply to triblock copolymers.

In the first scenario we take $Q_{11} = Q_{22} = 1$ and $Q_{12} = 0.2$ and run numerical minimization for three cases of K_1 and K_2. Results are shown in Figure 3 where type-I discs are shown in blue and type-II discs in yellow. The two type discs are well mixed; they arrange themselves in a very organized, nearly periodic way, forming a lattice. We invite the reader to compare our Figure 3 to the triblock copolymer cylindrical morphological phase in [3, Figure 5 (f)].

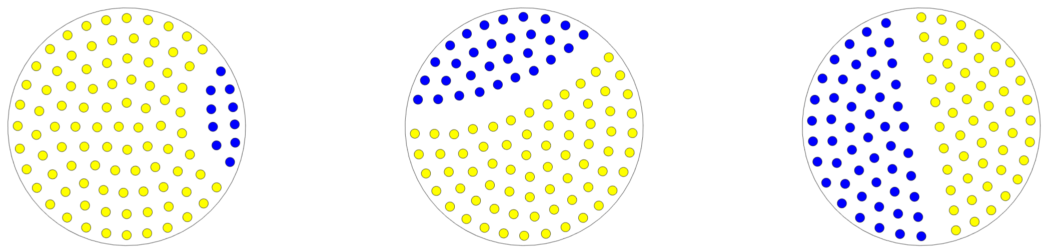

Figure 5: $Q_{11} = Q_{22} = 1$ and $Q_{12} = 1.2$. From left to right: $K_1 = 10$ and $K_2 = 90$, $K_1 = 30$ and $K_2 = 70$, $K_1 = K_2 = 50$.

In the second scenario, let $Q_{11} = Q_{22} = Q_{12} = 1$ so the two eigenvalues of Q are 2 and 0. For various cases of K_1 and K_2 we see in Figure 4 that the two type discs mix in a disorganized and random way.

In the last scenario we take $Q_{11} = Q_{22} = 1$ and $Q_{12} = 1.2$ so Q has one positive eigenvalue and one negative eigenvalue. For three cases of K_1 and K_2 we observe that the type-I and type-II discs segregate macroscopically. Two large regions appear where one holds the type-I discs and the other holds the type-II discs; see Figure 5.

When Q_{11} and Q_{22} are not equal, same conclusions hold. Those computational results are presented in Appendix.

Here we see that the functional $\mathcal{J}$ is capable of producing a primary structure and a secondary structure. A primary structure is visible at the microscopic level and for disc assemblies we see two type, near circular discs. A secondary structure appears at the macroscopic level. In the first scenario, because the two type discs are well mixed and well organized in a nearly periodic fashion, macroscopically one has a uniform structure. In the second scenario, because the two type discs are not mixed in an organized way, macroscopically one sees uneven concentration of the type-I and type-II constituents. In the third case macroscopically there appear two large domains, one with high concentration of type-I constituent and the other with high concentration of type-II constituent; the two regions are separated by a macroscopic interface of constant curvature.

3 Exact discs

In this section we construct an assembly of exact discs and compute its energy. Later it will be used as an approximate solution.

Let $\xi_1^1, ..., \xi_1^{K_1}, \xi_2^1, ..., \xi_2^{K_2}$ be $K_1 + K_2$ points in D and $r_1^1, \cdots, r_1^{K_1}, r_2^1, \cdots, r_2^{K_2}$ be $K_1 + K_2$ numbers. Denote by $B_i^k = \{x \in \mathbb{R}^2 : |x - \xi_i^k| < r_i^k\}$ an exact disc for $i = 1, 2$ and $k = 1, 2, ..., K_i$. Also introduce w_i^k, the scaled area of B_i^k, so that

$$\pi (r_i^k)^2 = \epsilon^2 w_i^k. \tag{3.1}$$

The w_i^k's form a point w in the set $\overline{W_h}$ which is the closure of

$$\begin{aligned} W_h \quad = \quad & \Big\{ w = \Big(w_1^1, ..., w_1^{K_1}, w_2^1, ..., w_2^{K_2} \Big) \in \mathbb{R}^{K_1 + K_2} : \Big| w_1^k - \frac{m}{K_1} \Big| < h, \ k = 1, 2, ..., K_1, \\ & \Big| w_2^k - \frac{1-m}{K_2} \Big| < h, \ k = 1, 2, ..., K_2, \sum_{k=1}^{K_1} w_1^k = m, \sum_{k=1}^{K_2} w_2^k = 1 - m \Big\}. \end{aligned} \tag{3.2}$$

For now assume that

$$0 < h < \min \left\{ \frac{m}{2K_1}, \frac{1-m}{2K_2} \right\}; \tag{3.3}$$

later h will be made sufficiently small. Initially w is fixed in $\overline{W_h}$. Later in Section 7 it will vary. Of course w_i^1,..., $w_i^{K_i}$ can vary only if $K_i > 1$. If $K_i = 1$, then w_i^1 is fixed at m if $i = 1$ or $1 - m$ if $i = 2$.

Recall the function F defined in (1.16). The domain of F is

$$\Xi = \{\xi = (\xi_1^1, \cdots, \xi_1^{K_1}, \xi_2^1, \cdots, \xi_2^{K_2}) : \ \xi_i^k \in D, \ i = 1, 2, \ k = 1, ..., K_i, \ \xi_i^k \neq \xi_j^l \ \text{ if } i \neq j \text{ or } k \neq l\}. \tag{3.4}$$

It is known that $R(x, x) \to \infty$ if $x \to \partial D$; see [19] for a more elaborate study of this property. Consequently $F(\xi) \to \infty$ as $\xi \to \partial \Xi$ where Ξ is viewed as a subset of $\mathbb{R}^{2(K_1+K_2)}$. One can find a small enough $\delta > 0$ such that

$$\min_{\xi \in \Xi} F(\xi) < \min_{\xi \in \Xi \setminus \Xi_\delta} F(\xi). \tag{3.5}$$

Here Ξ_δ is a subset of Ξ defined as

$$\Xi_\delta = \{\xi \in \Xi : d(\xi_i^k, \partial D) > \delta, \ i = 1, 2, \ k = 1, ..., K_i, \ d(\xi_i^k, \xi_j^l) > 2\delta \text{ if } i \neq j \text{ or } k \neq l\}. \tag{3.6}$$

In (3.6), with an abuse of notation, "d" stands for both the Euclidean distance in $\mathbb{R}^2$ between two points and the distance from a point to a set. Henceforth the number δ remains fixed.

The centers ξ_i^k of the discs B_i^k form a point ξ which is fixed in the closure of Ξ_δ:

$$\xi = (\xi_1^1, \cdots, \xi_1^{K_1}, \xi_2^1, \cdots, \xi_2^{K_2}) \in \overline{\Xi_\delta}. \tag{3.7}$$

Later in Section 7, ξ will vary. All quantitative estimates in this paper are uniform with respect to $w \in \overline{W_h}$ and $\xi \in \overline{\Xi_\delta}$.

Our initial requirement on ϵ_0 of Theorem 1.1, the bound for ϵ, is that ϵ_0 must be small enough so that

$$0 < r_i^k = \epsilon \sqrt{\frac{w_i^k}{\pi}} < \epsilon_0 \sqrt{\frac{w_i^k}{\pi}} < \frac{\delta}{2} \tag{3.8}$$

holds for all $w \in \overline{W_h}$. With this choice of ϵ_0 and with $\epsilon < \epsilon_0$, for $z_i^k \in B_i^k$ and $x \in \partial D$,

$$d(x, z_i^k) \geq d(x, \xi_i^k) - d(\xi_i^k, z_i^k) > \delta - r_i^k > \frac{\delta}{2}. \tag{3.9}$$

For $z_i^k \in B_i^k$ and $z_j^l \in B_j^l$, where $i \neq j$ or $k \neq l$,

$$d(z_i^k, z_j^l) \geq d(\xi_i^k, \xi_j^l) - d(\xi_i^k, z_i^k) - d(\xi_j^l, z_j^l) > 2\delta - r_i^k - r_j^l > \delta. \tag{3.10}$$

Hence the discs B_i^k are all inside D and well separated from each other. Moreover with $z_i^k \in B_i^k$, $z = (z_1^1, \cdots, z_1^{K_1}, z_2^1, \cdots, z_2^{K_2})$ is in $\Xi_{\frac{\delta}{2}}$, where the set $\Xi_{\frac{\delta}{2}}$ is defined as in (3.6).

Let $B_1 = \cup_{k=1}^{K_1} B_1^k$ and $B_2 = \cup_{k=1}^{K_2} B_2^k$. Then (B_1, B_2) is an assembly of exact discs. For now the centers ξ_i^k and the scaled areas w_i^k are taken arbitrarily from $\overline{\Xi_\delta}$ and $\overline{W_h}$ respectively. They will be determined in Lemma 7.3. Our first result gives $\mathcal{J}(B_1, B_2)$, the energy of the exact disc assembly.

Lemma 3.1.

$$
\begin{aligned}
\mathcal{J}\left(B_1,B_2\right) &= \sum_{k=1}^{K_1} 2\pi r_1^k + \sum_{k=1}^{K_2} 2\pi r_2^k \\
&+\frac{\gamma_{11}\pi^2}{2}\left[\sum_{k=1}^{K_1}\left(\frac{\left(r_1^k\right)^4}{8\pi} - \frac{\left(r_1^k\right)^4\log r_1^k}{2\pi} + \left(r_1^k\right)^4 R(\xi_1^k,\xi_1^k)\right)\right. \\
&\left.+\sum_{k=1}^{K_1}\sum_{l=1,l\neq k}^{K_1}\left(r_1^k\right)^2\left(r_1^l\right)^2 G(\xi_1^k,\xi_1^l) + \sum_{k=1}^{K_1}\sum_{l=1}^{K_1}\left(\frac{\left(r_1^k\right)^2\left(r_1^l\right)^4}{8|D|} + \frac{\left(r_1^k\right)^4\left(r_1^l\right)^2}{8|D|}\right)\right] \\
&+\gamma_{12}\pi^2\sum_{k=1}^{K_1}\sum_{l=1}^{K_2}\left[\left(r_1^k\right)^2\left(r_2^l\right)^2 G(\xi_1^k,\xi_2^l) + \frac{\left(r_1^k\right)^2\left(r_2^l\right)^4}{8|D|} + \frac{\left(r_1^k\right)^4\left(r_2^l\right)^2}{8|D|}\right] \\
&+\frac{\gamma_{22}\pi^2}{2}\left[\sum_{k=1}^{K_2}\left(\frac{\left(r_2^k\right)^4}{8\pi} - \frac{\left(r_2^k\right)^4\log r_2^k}{2\pi} + \left(r_2^k\right)^4 R(\xi_2^k,\xi_2^k)\right)\right. \\
&\left.+\sum_{k=1}^{K_2}\sum_{l=1,l\neq k}^{K_2}\left(r_2^k\right)^2\left(r_2^l\right)^2 G(\xi_2^k,\xi_2^l) + \sum_{k=1}^{K_2}\sum_{l=1}^{K_2}\left(\frac{\left(r_2^k\right)^2\left(r_2^l\right)^4}{8|D|} + \frac{\left(r_2^k\right)^4\left(r_2^l\right)^2}{8|D|}\right)\right].
\end{aligned}
$$

Proof. The first part of $\mathcal{J}\left(B_1,B_2\right)$, denoted by $\mathcal{J}_s(B_1,B_2)$ for short range interaction, is just the total arc length:

$$
\mathcal{J}_s\left(B_1,B_2\right) = \sum_{k=1}^{K_1} 2\pi r_1^k + \sum_{k=1}^{K_2} 2\pi r_2^k. \tag{3.11}
$$

The second part of $\mathcal{J}\left(B_1,B_2\right)$, denoted by $\mathcal{J}_l(B_1,B_2)$ for long range interaction, is

$$
\begin{aligned}
\mathcal{J}_l\left(B_1,B_2\right) &= \frac{\gamma_{11}}{2}\int_D |\left(-\Delta\right)^{-1/2}\left(\chi_{B_1}-\omega_1\right)|^2\,dx \\
&+\gamma_{12}\int_D\left(\left(-\Delta\right)^{-1/2}\left(\chi_{B_1}-\omega_1\right)\right)\left(\left(-\Delta\right)^{-1/2}\left(\chi_{B_2}-\omega_2\right)\right)dx \\
&+\frac{\gamma_{22}}{2}\int_D |\left(-\Delta\right)^{-1/2}\left(\chi_{B_2}-\omega_2\right)|^2\,dx \\
&\equiv I + II + III,
\end{aligned} \tag{3.12}
$$

where

$$
I = \frac{\gamma_{11}}{2}\int_D |(-\Delta)^{-1/2}\left(\chi_{B_1}-\omega_1\right)|^2 dx = \frac{\gamma_{11}}{2}\int_{B_1} v_1\left(x\right)dx, \tag{3.13}
$$

$$
\begin{aligned}
II &= \gamma_{12}\int_D\left(\left(-\Delta\right)^{-1/2}\left(\chi_{B_1}-\omega_1\right)\right)\left(\left(-\Delta\right)^{-1/2}\left(\chi_{B_2}-\omega_2\right)\right)dx \\
&= \gamma_{12}\int_{B_2} v_1\left(x\right)dx,
\end{aligned} \tag{3.14}
$$

$$
III = \frac{\gamma_{22}}{2}\int_D |\left(-\Delta\right)^{-1/2}\left(\chi_{B_2}-\omega_2\right)|^2\,dx = \frac{\gamma_{22}}{2}\int_{B_2} v_2\left(x\right)dx. \tag{3.15}
$$

Here v_1 and v_2 are the solutions of

$$
-\Delta v_i = \chi_{B_i} - \omega_i \text{ in } D,\ \partial_\nu v_i = 0 \text{ on } \partial D,\ \int_D v_i = 0,\ i = 1,2.
$$

One can write

$$
v_1 = \sum_{k=1}^{K_1} v_1^k,\ v_2 = \sum_{k=1}^{K_2} v_2^k, \tag{3.16}
$$

where v_i^k solves

$$-\Delta v_i^k = \chi_{B_i^k} - \frac{\pi \left(r_i^k\right)^2}{|D|} \text{ in } D,\ \partial_\nu v_i^k = 0 \text{ on } \partial D,\ \int_D v_i^k = 0.$$

Define

$$P_i^k(x) = \begin{cases} -\frac{|x|^2}{4} + \frac{\left(r_i^k\right)^2}{4} - \frac{\left(r_i^k\right)^2}{2} \log r_i^k, & \text{if} |x| < r_i^k, \\ -\frac{\left(r_i^k\right)^2}{2} \log |x|, & \text{if} |x| \geq r_i^k. \end{cases} \tag{3.17}$$

Then $-\Delta P_i^k(x - \xi_i^k) = \chi_{B_i^k}$. Let $v_i^k = P_i^k(x - \xi_i^k) + Q(x, \xi_i^k)$, where Q solves

$$\begin{aligned} -\Delta Q(x, \xi_i^k) &= -\frac{\pi \left(r_i^k\right)^2}{|D|} \text{ in } D, \\ \partial_\nu Q(x, \xi_i^k) &= \partial_\nu \frac{\left(r_i^k\right)^2}{2} \log |x - \xi_i^k| \text{ on } \partial D, \\ \int_D Q(x, \xi_i^k) dx &= -\int_D P_i^k(x - \xi_i^k) dx. \end{aligned} \tag{3.18}$$

By (2.11) and (2.10),

$$\begin{aligned} -\Delta \left(\pi \left(r_i^k\right)^2 R(x, \xi_i^k)\right) &= -\frac{\pi \left(r_i^k\right)^2}{|D|} \text{ in } D, \\ \partial_\nu \left(\pi \left(r_i^k\right)^2 R(x, \xi_i^k)\right) &= \partial_\nu \frac{\left(r_i^k\right)^2}{2} \log |x - \xi_i^k| \text{ on } \partial D, \\ \int_D \left(\pi \left(r_i^k\right)^2 R(x, \xi_i^k)\right) dx &= \int_D \frac{\left(r_i^k\right)^2}{2} \log |x - \xi_i^k| \ dx. \end{aligned} \tag{3.19}$$

Comparing (3.18) to (3.19), we deduce

$$Q(x, \xi_i^k) = \pi \left(r_i^k\right)^2 R(x, \xi_i^k) + \frac{\pi (r_i^k)^4}{8|D|}. \tag{3.20}$$

Thus,

$$v_i^k(x) = P_i^k(x - \xi_i^k) + \pi \left(r_i^k\right)^2 R(x, \xi_i^k) + \frac{\pi (r_i^k)^4}{8|D|},\ i = 1, 2,\ k = 1, ..., K_i. \tag{3.21}$$

By (3.16),

$$\int_{B_1} v_1(x)\, dx = \sum_{l=1}^{K_1} \sum_{k=1}^{K_1} \int_{B_1^l} v_1^k(x)\, dx.$$

When $l = k$, by (3.17),

$$\int_{B_1^k} P_1^k(x - \xi_1^k) dx = \frac{\pi \left(r_1^k\right)^4}{8} - \frac{\pi \left(r_1^k\right)^4 \log r_1^k}{2}.$$

By (3.18), $Q(x, \xi_1^k) - \frac{\pi (r_1^k)^2}{4|D|} |x - \xi_1^k|^2$ is harmonic in x. The mean value property of harmonic functions and (3.20) imply

$$\int_{B_1^k} Q(x, \xi_1^k) dx = \pi^2 \left(r_1^k\right)^4 R\left(\xi_1^k, \xi_1^k\right) + \frac{\pi^2 (r_1^k)^6}{4|D|}. \tag{3.22}$$

When $l \neq k$, by (3.17) and the mean value property of harmonic functions,

$$\int_{B_1^l} P_1^k(x-\xi_1^k)dx = -\frac{\pi (r_1^l)^2 (r_1^k)^2}{2} \log|\xi_1^l - \xi_1^k|.$$

Similar to the derivation of (3.22), we have

$$\int_{B_1^l} Q(x,\xi_1^k)dx = \pi^2 \left(r_1^l\right)^2 \left(r_1^k\right)^2 R\left(\xi_1^l,\xi_1^k\right) + \frac{\left(r_1^l\right)^2\left(r_1^k\right)^4}{8|D|} + \frac{\left(r_1^k\right)^2\left(r_1^l\right)^4}{8|D|}.$$

Hence,

$$\begin{aligned}\int_{B_1} v_1(x)\,dx &= \pi^2 \Bigg[\sum_{k=1}^{K_1}\left(\frac{\left(r_1^k\right)^4}{8\pi} - \frac{\left(r_1^k\right)^4 \log r_1^k}{2\pi} + \left(r_1^k\right)^4 R\left(\xi_1^k,\xi_1^k\right)\right) \\ &\quad + \sum_{k=1}^{K_1}\sum_{l=1,l\neq k}^{K_1}\left(r_1^k\right)^2\left(r_1^l\right)^2 G\left(\xi_1^k,\xi_1^l\right) + \sum_{k=1}^{K_1}\sum_{l=1}^{K_1}\left(\frac{\left(r_1^k\right)^2\left(r_1^l\right)^4}{8|D|} + \frac{\left(r_1^k\right)^4\left(r_1^l\right)^2}{8|D|}\right)\Bigg].\end{aligned} \tag{3.23}$$

Similar calculations show that

$$\int_{B_2} v_1(x)\,dx = \pi^2 \sum_{k=1}^{K_1}\sum_{l=1}^{K_2}\left[\left(r_1^k\right)^2\left(r_2^l\right)^2 G\left(\xi_1^k,\xi_2^l\right) + \frac{\left(r_1^k\right)^2\left(r_2^l\right)^4}{8|D|} + \frac{\left(r_1^k\right)^4\left(r_2^l\right)^2}{8|D|}\right], \tag{3.24}$$

$$\begin{aligned}\int_{B_2} v_2(x)\,dx &= \pi^2 \Bigg[\sum_{k=1}^{K_2}\left(\frac{\left(r_2^k\right)^4}{8\pi} - \frac{\left(r_2^k\right)^4 \log r_2^k}{2\pi} + \left(r_2^k\right)^4 R\left(\xi_2^k,\xi_2^k\right)\right) \\ &\quad + \sum_{k=1}^{K_2}\sum_{l=1,l\neq k}^{K_2}\left(r_2^k\right)^2\left(r_2^l\right)^2 G\left(\xi_2^k,\xi_2^l\right) + \sum_{k=1}^{K_2}\sum_{l=1}^{K_2}\left(\frac{\left(r_2^k\right)^2\left(r_2^l\right)^4}{8|D|} + \frac{\left(r_2^k\right)^4\left(r_2^l\right)^2}{8|D|}\right)\Bigg].\end{aligned} \tag{3.25}$$

The lemma then follows from (3.23) - (3.25). □

4 Perturbed discs

We set up a framework to study assemblies of perturbed discs in this section. Let ϕ_i^k, $i=1,2$, $k=1,...,K_i$, be 2π periodic functions, collectively denoted as

$$\phi = (\phi_1^1, \cdots, \phi_1^{K_1}, \phi_2^1, \cdots, \phi_2^{K_2}). \tag{4.1}$$

Using ϕ, we define K_1+K_2 sets

$$\Omega_i^k = \left\{\xi_i^k + te^{\mathrm{i}\theta} : \theta\in[0,2\pi], t\in\left[0, \sqrt{(r_i^k)^2 + 2\phi_i^k(\theta)}\right]\right\},\ i=1,2,\ k=1,...,K_i. \tag{4.2}$$

Since our domain is in $\mathbb{R}^2$ we often use the complex notation for simplicity. In (4.2) $e^{\mathrm{i}\theta}$ is just $(\cos\theta,\sin\theta)$. The reader will see things like $e^{\mathrm{i}\theta}\cdot x$ which is the inner product of two vectors $e^{\mathrm{i}\theta}$ and x in $\mathbb{R}^2$.

Also in (4.2) $|\phi_i^k(\theta)|$ must be small compared to $(r_i^k)^2$, so Ω_i^k is a perturbation of B_i^k. The area of Ω_i^k is

$$|\Omega_i^k| = \int_0^{2\pi}\int_0^{\sqrt{(r_i^k)^2+2\phi_i^k(\theta)}} t\,dt\,d\theta = \pi (r_i^k)^2 + \int_0^{2\pi}\phi_i^k d\theta. \tag{4.3}$$

If the ϕ_i^k's satisfy

$$\sum_{k=1}^{K_i}\int_0^{2\pi}\phi_i^k d\theta = 0,\ i=1,2, \tag{4.4}$$

then $\Omega_1 = \cup_{k=1}^{K_1} \Omega_1^k$ and $\Omega_2 = \cup_{k=1}^{K_2} \Omega_2^k$ satisfy the area constraints $|\Omega_1| = \epsilon^2 m$ and $|\Omega_2| = \epsilon^2(1-m)$.

Let us define some Hilbert spaces. First

$$\mathcal{Z} = \Big\{ \phi = (\phi_1^1, \cdots, \phi_1^{K_1}, \phi_2^1, \cdots, \phi_2^{K_2}) : \ \phi_i^k \in L^2(S^1), \ i = 1,2, \ k = 1, ..., K_i, \sum_{k=1}^{K_i} \int_0^{2\pi} \phi_i^k d\theta = 0, \ i = 1,2 \Big\}. \tag{4.5}$$

Here S^1 is the unit circle identified with $[0, 2\pi]$. The inner product on $\mathcal{Z}$ is

$$\langle \phi, \psi \rangle = \sum_{i=1}^{2} \sum_{k=1}^{K_i} \int_0^{2\pi} \phi_i^k \psi_i^k d\theta. \tag{4.6}$$

In order to define the energy $\mathcal{J}$ on an assembly of such perturbed discs, one needs some smoothness on ϕ_i^k. Let

$$\mathcal{Y} = \big\{ \phi \in \mathcal{Z} : \ \phi_i^k \in H^1(S^1), \ i = 1,2, \ k = 1, ..., K_i \big\} \tag{4.7}$$

be a subspace of $\mathcal{Z}$. Here $H^1(S^1)$ is a usual Sobolev space on S^1. The norm of $\mathcal{Y}$ is given by

$$||\phi||_{\mathcal{Y}}^2 = \sum_{i=1}^{2} \sum_{k=1}^{K_i} \int_0^{2\pi} \left(((\phi_i^k)')^2 + (\phi_i^k)^2 \right) d\theta. \tag{4.8}$$

If ξ and w are held fixed, $\mathcal{J}$ is viewed as a functional of ϕ with the domain

$$Dom(\mathcal{J}) = \{ \phi \in \mathcal{Y} : ||\phi||_{\mathcal{Y}} < \beta \epsilon^2 \}. \tag{4.9}$$

Recall that for all $\epsilon < \epsilon_0$, $\xi \in \overline{\Xi_\delta}$ and $w \in \overline{W_h}$, the exactly disc assembly (B_1, B_2) determined by ϵ, ξ, and w has the property that $z = (z_1^1, \cdots, z_1^{K_1}, z_2^1, \cdots, z_2^{K_2}) \in \Xi_{\frac{\delta}{2}}$ if $z_i^k \in B_i^k$ for $i = 1,2$, and $k = 1, \cdots, K_i$. Choose β in (4.9) sufficiently small so that for all $\epsilon < \epsilon_0$, all $(\xi, w) \in \overline{\Xi_\delta} \times \overline{W_h}$, and all $\phi \in Dom(\mathcal{J})$, the perturbed disc assembly (Ω_1, Ω_2) specified by ϵ, ξ, w, and ϕ has the property that $z = (z_1^1, \cdots, z_1^{K_1}, z_2^1, \cdots, z_2^{K_2}) \in \Xi_{\frac{\delta}{4}}$ if $z_i^k \in \Omega_i^k$. Hence the perturbed discs Ω_i^k in (Ω_1, Ω_2) are well separated, and they all stay inside D, away from ∂D.

One writes $\mathcal{J}(\phi)$ for $\mathcal{J}(\Omega_1, \Omega_2)$. Then $\mathcal{J}(\phi) = \mathcal{J}_s(\phi) + \mathcal{J}_l(\phi)$, where $\mathcal{J}_s(\phi)$ and $\mathcal{J}_l(\phi)$ are given in terms of ϕ as

$$\mathcal{J}_s(\phi) = \sum_{i=1}^{2} \sum_{k=1}^{K_i} \int_0^{2\pi} \sqrt{ \left(r_i^k\right)^2 + 2\phi_i^k(\theta) + \frac{\left(\left(\phi_i^k\right)'(\theta) \right)^2}{\left(r_i^k\right)^2 + 2\phi_i^k(\theta)} } \, d\theta \tag{4.10}$$

$$\begin{aligned} \mathcal{J}_l(\phi) \ = \ & \frac{\gamma_{11}}{2} \sum_{k=1}^{K_1} \sum_{l=1}^{K_1} \int_0^{2\pi} \int_0^{\sqrt{\left(r_1^k\right)^2 + 2\phi_1^k(\theta)}} \int_0^{2\pi} \int_0^{\sqrt{\left(r_1^l\right)^2 + 2\phi_1^l(\eta)}} G(\xi_1^k + te^{\mathrm{i}\theta}, \xi_1^l + \tau e^{\mathrm{i}\eta}) t\tau \ d\tau d\eta dt d\theta \\ & + \gamma_{12} \sum_{k=1}^{K_1} \sum_{l=1}^{K_2} \int_0^{2\pi} \int_0^{\sqrt{\left(r_1^k\right)^2 + 2\phi_1^k(\theta)}} \int_0^{2\pi} \int_0^{\sqrt{\left(r_2^l\right)^2 + 2\phi_2^l(\eta)}} G(\xi_1^k + te^{\mathrm{i}\theta}, \xi_2^l + \tau e^{\mathrm{i}\eta}) t\tau \ d\tau d\eta dt d\theta \\ & + \frac{\gamma_{22}}{2} \sum_{k=1}^{K_2} \sum_{l=1}^{K_2} \int_0^{2\pi} \int_0^{\sqrt{\left(r_2^k\right)^2 + 2\phi_2^k(\theta)}} \int_0^{2\pi} \int_0^{\sqrt{\left(r_2^l\right)^2 + 2\phi_2^l(\eta)}} G(\xi_2^k + te^{\mathrm{i}\theta}, \xi_2^l + \tau e^{\mathrm{i}\eta}) t\tau \ d\tau d\eta dt d\theta \end{aligned} \tag{4.11}$$

Next consider the first variation of the functional $\mathcal{J}$ with respect to ϕ. For convenience introduce functions L_i^k:

$$L_i^k((\phi_i^k)', \phi_i^k) = \sqrt{\left(r_i^k\right)^2 + 2\phi_i^k(\theta) + \frac{\left((\phi_i^k)'(\theta)\right)^2}{\left(r_i^k\right)^2 + 2\phi_i^k(\theta)}}. \tag{4.12}$$

A deformation in $\mathcal{Y}$ is simply

$$\phi \to \phi + \varepsilon\psi \tag{4.13}$$

for $\phi \in Dom(\mathcal{J})$, $\psi \in \mathcal{Y}$ and $\varepsilon \in \mathbb{R}$. The first variation of $\mathcal{J}$ is the directional derivative

$$\left.\frac{d\mathcal{J}(\phi+\varepsilon\psi)}{d\varepsilon}\right|_{\varepsilon=0} = \left.\frac{d\mathcal{J}_s(\phi+\varepsilon\psi)}{d\varepsilon}\right|_{\varepsilon=0} + \left.\frac{d\mathcal{J}_l(\phi+\varepsilon\psi)}{d\varepsilon}\right|_{\varepsilon=0}, \tag{4.14}$$

where

$$\left.\frac{d\mathcal{J}_s(\phi+\varepsilon\psi)}{d\varepsilon}\right|_{\varepsilon=0} = \sum_{i=1}^{2}\sum_{k=1}^{K_i}\int_0^{2\pi}(D_1 L_i^k((\phi_i^k)', \phi_i^k)(\psi_i^k)' + D_2 L_i^k((\phi_i^k)', \phi_i^k)\psi_i^k)d\theta \tag{4.15}$$

(D_1 and D_2 denote differentiation with respect to the first and the second arguments of L_i^k respectively), and

$$\begin{aligned}\left.\frac{d\mathcal{J}_l(\phi+\varepsilon\psi)}{d\varepsilon}\right|_{\varepsilon=0} &= \gamma_{11}\sum_{k=1}^{K_1}\int_0^{2\pi} I_{\Omega_1}(\xi_1^k + \sqrt{\left(r_1^k\right)^2 + 2\phi_1^k(\theta)}e^{\mathrm{i}\theta})\psi_1^k(\theta)d\theta \\ &+\gamma_{12}\sum_{k=1}^{K_1}\int_0^{2\pi} I_{\Omega_2}(\xi_1^k + \sqrt{\left(r_1^k\right)^2 + 2\phi_1^k(\theta)}e^{\mathrm{i}\theta})\psi_1^k(\theta)d\theta \\ &+\gamma_{12}\sum_{k=1}^{K_2}\int_0^{2\pi} I_{\Omega_1}(\xi_2^k + \sqrt{\left(r_2^k\right)^2 + 2\phi_2^k(\theta)}e^{\mathrm{i}\theta})\psi_2^k(\theta)d\theta \\ &+\gamma_{22}\sum_{k=1}^{K_2}\int_0^{2\pi} I_{\Omega_2}(\xi_2^k + \sqrt{\left(r_2^k\right)^2 + 2\phi_2^k(\theta)}e^{\mathrm{i}\theta})\psi_2^k(\theta)d\theta.\end{aligned} \tag{4.16}$$

We would like to have operators $\mathcal{S}_s$ and $\mathcal{S}_l$ so that

$$\left.\frac{d\mathcal{J}_s(\phi+\varepsilon\psi)}{d\varepsilon}\right|_{\varepsilon=0} = \langle \mathcal{S}_s(\phi), \psi\rangle, \ \left.\frac{d\mathcal{J}_l(\phi+\varepsilon\psi)}{d\varepsilon}\right|_{\varepsilon=0} = \langle \mathcal{S}_l(\phi), \psi\rangle. \tag{4.17}$$

This is always possible for $\mathcal{J}_l$, but for $\mathcal{J}_s$, one must restrict ϕ to a smaller, more smooth space. Define

$$\mathcal{X} = \left\{\phi \in \mathcal{Z} : \ \phi_i^k \in H^2(S^1), \ i = 1,2, \ k = 1, ..., K_i\right\} \tag{4.18}$$

with the norm

$$||\phi||_{\mathcal{X}}^2 = \sum_{i=1}^{2}\sum_{k=1}^{K_i}\int_0^{2\pi}\left(((\phi_i^k)'')^2 + ((\phi_i^k)')^2 + (\phi_i^k)^2\right)d\theta. \tag{4.19}$$

Clearly $\mathcal{X} \subset \mathcal{Y} \subset \mathcal{Z}$. One can define $\mathcal{S} = \mathcal{S}_s + \mathcal{S}_l$ on

$$Dom(\mathcal{S}) = \{\phi \in \mathcal{X} : ||\phi||_{\mathcal{X}} < \beta\epsilon^2\} \tag{4.20}$$

where β is the same as the number in (4.9). Therefore $Dom(\mathcal{S}) \subset Dom(\mathcal{J})$. The nonlinear operators $\mathcal{S}_s$ and $\mathcal{S}_l$ map from $Dom(\mathcal{S})$ to $\mathcal{Z}$ as follows. The component $\mathcal{S}_{s,i}^k(\phi)$ is given as

$$\mathcal{S}_{s,i}^k(\phi) = \mathcal{H}_i^k(\phi_i^k) - \lambda_{s,i}(\phi_i) \tag{4.21}$$

where

$$\mathcal{H}_i^k\left(\phi_i^k\right)(\theta) = \frac{\left(r_i^k\right)^2 + 2\phi_i^k + \frac{3\left(\left(\phi_i^k\right)'\right)^2}{\left(r_i^k\right)^2+2\phi_i^k} - \left(\phi_i^k\right)''}{\left(\left(r_i^k\right)^2 + 2\phi_i^k + \frac{\left(\left(\phi_i^k\right)'\right)^2}{\left(r_i^k\right)^2+2\phi_i^k}\right)^{\frac{3}{2}}} \tag{4.22}$$

is the curvature operator, and $\lambda_{s,i}(\phi_i)$ is a number, depending on $\phi_i = (\phi_i^1, \cdots, \phi_i^{K_i})$, so chosen that

$$\sum_{k=1}^{K_i} \int_0^{2\pi} \mathcal{S}_{s,i}^k(\phi)\, d\theta = 0,\ i = 1, 2. \tag{4.23}$$

Note that $\lambda_{s,i}(\phi_i)$ is the same for all $k = 1, 2, \cdots, K_i$. The components of $\mathcal{S}_l(\phi)$ are

$$\begin{aligned}
\mathcal{S}_{l,1}^k(\phi) &= \gamma_{11} I_{\Omega_1}(\xi_1^k + \sqrt{\left(r_1^k\right)^2 + 2\phi_1^k(\theta)}e^{\mathrm{i}\theta}) + \gamma_{12} I_{\Omega_2}(\xi_1^k + \sqrt{\left(r_1^k\right)^2 + 2\phi_1^k(\theta)}e^{\mathrm{i}\theta}) - \lambda_{l,1}(\phi) && (4.24)\\
\mathcal{S}_{l,2}^k(\phi) &= \gamma_{12} I_{\Omega_1}(\xi_2^k + \sqrt{\left(r_2^k\right)^2 + 2\phi_2^k(\theta)}e^{\mathrm{i}\theta}) + \gamma_{22} I_{\Omega_2}(\xi_2^k + \sqrt{\left(r_2^k\right)^2 + 2\phi_2^k(\theta)}e^{\mathrm{i}\theta}) - \lambda_{l,2}(\phi) && (4.25)
\end{aligned}$$

where $\lambda_{l,i}(\phi)$ are numbers that ensure

$$\sum_{k=1}^{K_i} \int_0^{2\pi} \mathcal{S}_{l,i}^k(\phi)\, d\theta = 0,\ i = 1, 2. \tag{4.26}$$

We need to write $\mathcal{S}_{l,i}^k(\phi)$ more explicitly. Let

$$\begin{aligned}
\mathcal{A}_1^k\left(\phi_1^k\right) &= -\frac{\gamma_{11}}{2\pi} \int_0^{2\pi} \int_0^{\sqrt{\left(r_1^k\right)^2+2\phi_1^k(\eta)}} \log\left|\sqrt{\left(r_1^k\right)^2 + 2\phi_1^k(\theta)}e^{\mathrm{i}\theta} - te^{\mathrm{i}\eta}\right| tdtd\eta \\
\mathcal{B}_1^k\left(\phi_1^k\right) &= \gamma_{11} \int_0^{2\pi} \int_0^{\sqrt{\left(r_1^k\right)^2+2\phi_1^k(\eta)}} R(\xi_1^k + \sqrt{\left(r_1^k\right)^2 + 2\phi_1^k(\theta)}e^{\mathrm{i}\theta}, \xi_1^k + te^{\mathrm{i}\eta}) tdtd\eta.
\end{aligned}$$

When $1 \le l \le K_1$ and $l \ne k$,

$$\mathcal{C}_1^{kl}\left(\phi_1^k, \phi_1^l\right) = \gamma_{11} \int_0^{2\pi} \int_0^{\sqrt{\left(r_1^l\right)^2+2\phi_1^l(\eta)}} G(\xi_1^k + \sqrt{\left(r_1^k\right)^2 + 2\phi_1^k(\theta)}e^{\mathrm{i}\theta}, \xi_1^l + te^{\mathrm{i}\eta}) tdtd\eta.$$

When $1 \le l \le K_2$,

$$\mathcal{D}_1^{kl}\left(\phi_1^k, \phi_2^l\right) = \gamma_{12} \int_0^{2\pi} \int_0^{\sqrt{\left(r_2^l\right)^2+2\phi_2^l(\eta)}} G(\xi_1^k + \sqrt{\left(r_1^k\right)^2 + 2\phi_1^k(\theta)}e^{\mathrm{i}\theta}, \xi_2^l + te^{\mathrm{i}\eta}) tdtd\eta.$$

Similarly,

$$\begin{aligned}
\mathcal{A}_2^k\left(\phi_2^k\right) &= -\frac{\gamma_{22}}{2\pi} \int_0^{2\pi} \int_0^{\sqrt{\left(r_2^k\right)^2+2\phi_2^k(\eta)}} \log\left|\sqrt{\left(r_2^k\right)^2 + 2\phi_2^k(\theta)}e^{\mathrm{i}\theta} - te^{\mathrm{i}\eta}\right| tdtd\eta \\
\mathcal{B}_2^k\left(\phi_2^k\right) &= \gamma_{22} \int_0^{2\pi} \int_0^{\sqrt{\left(r_2^k\right)^2+2\phi_2^k(\eta)}} R(\xi_2^k + \sqrt{\left(r_2^k\right)^2 + 2\phi_2^k(\theta)}e^{\mathrm{i}\theta}, \xi_2^k + te^{\mathrm{i}\eta}) tdtd\eta.
\end{aligned}$$

When $1 \le l \le K_2$ and $l \ne k$,

$$\mathcal{C}_2^{kl}\left(\phi_2^k, \phi_2^l\right) = \gamma_{22} \int_0^{2\pi} \int_0^{\sqrt{\left(r_2^l\right)^2 + 2\phi_2^l(\eta)}} G(\xi_2^k + \sqrt{\left(r_2^k\right)^2 + 2\phi_2^k(\theta)} e^{\mathrm{i}\theta}, \xi_2^l + te^{\mathrm{i}\eta}) t dt d\eta.$$

When $1 \le l \le K_1$,

$$\mathcal{D}_2^{kl}\left(\phi_2^k, \phi_1^l\right) = \gamma_{12} \int_0^{2\pi} \int_0^{\sqrt{\left(r_1^l\right)^2 + 2\phi_1^l(\eta)}} G(\xi_2^k + \sqrt{\left(r_2^k\right)^2 + 2\phi_2^k(\theta)} e^{\mathrm{i}\theta}, \xi_1^l + te^{\mathrm{i}\eta}) t dt d\eta.$$

Then we obtain

$$\mathcal{S}_1^k(\phi) = \mathcal{H}_1^k\left(\phi_1^k\right) + \mathcal{A}_1^k\left(\phi_1^k\right) + \mathcal{B}_1^k\left(\phi_1^k\right) + \sum_{l=1, l\neq k}^{K_1} \mathcal{C}_1^{kl}\left(\phi_1^k, \phi_1^l\right) + \sum_{l=1}^{K_2} \mathcal{D}_1^{kl}\left(\phi_1^k, \phi_2^l\right) - \lambda_1(\phi), \quad (4.27)$$

$$\mathcal{S}_2^k(\phi) = \mathcal{H}_2^k\left(\phi_2^k\right) + \mathcal{A}_2^k\left(\phi_2^k\right) + \mathcal{B}_2^k\left(\phi_2^k\right) + \sum_{l=1, l\neq k}^{K_2} \mathcal{C}_2^{kl}\left(\phi_2^k, \phi_2^l\right) + \sum_{l=1}^{K_1} \mathcal{D}_2^{kl}\left(\phi_2^k, \phi_1^l\right) - \lambda_2(\phi). \quad (4.28)$$

Here $\lambda_1(\phi)$ and $\lambda_2(\phi)$ are two numbers, independent of k, so chosen that

$$\sum_{k=1}^{K_i} \int_0^{2\pi} \mathcal{S}_i^k(\phi)\, d\theta = 0,\ i = 1, 2. \quad (4.29)$$

The equations (1.6) and (1.7) now become

$$\mathcal{S}(\phi) = 0. \quad (4.30)$$

When ξ and w are fixed, the exact disc assembly (B_1, B_2) is represented by $\phi = 0$ in $\mathcal{X}$. We insert this assembly into the left sides of (1.6) and (1.7) and see how much it deviates from a solution. In the present notation the left sides of (1.6) and (1.7) are $\mathcal{S}_i^k(0) + \lambda_i(0)$. In the following we write ∇R or ∇G to denote the gradient of R or G with respect to the first set of variables; namely $\nabla R(x, y) = \left(\frac{\partial R(x,y)}{\partial x_1}, \frac{\partial R(x,y)}{\partial x_2}\right)$.

Lemma 4.1.

$$\begin{aligned}
\mathcal{S}_1^k(0) + \lambda_1(0) &= \frac{1}{r_1^k} + \gamma_{11}\Bigg[-\frac{(r_1^k)^2}{2}\log r_1^k + \pi(r_1^k)^2\left(R(\xi_1^k, \xi_1^k) + \nabla R(\xi_1^k, \xi_1^k)\cdot r_1^k e^{\mathrm{i}\theta}\right) \\
&\quad + \sum_{l=1, l\neq k}^{K_1} \pi(r_1^l)^2\left(G(\xi_1^k, \xi_1^l) + \nabla G(\xi_1^k, \xi_1^l)\cdot r_1^k e^{\mathrm{i}\theta}\right)\Bigg] \\
&\quad + \gamma_{12}\left[\sum_{l=1}^{K_2} \pi(r_2^l)^2\left(G(\xi_1^k, \xi_2^l) + \nabla G(\xi_1^k, \xi_2^l)\cdot r_1^k e^{\mathrm{i}\theta}\right)\right] + O(|\gamma|\epsilon^4) \\
\mathcal{S}_2^k(0) + \lambda_2(0) &= \frac{1}{r_2^k} + \gamma_{22}\Bigg[-\frac{(r_2^k)^2}{2}\log r_2^k + \pi(r_2^k)^2\left(R(\xi_2^k, \xi_2^k) + \nabla R(\xi_2^k, \xi_2^k)\cdot r_2^k e^{\mathrm{i}\theta}\right) \\
&\quad + \sum_{l=1, l\neq k}^{K_2} \pi(r_2^l)^2\left(G(\xi_2^k, \xi_2^l) + \nabla G(\xi_2^k, \xi_2^l)\cdot r_2^k e^{\mathrm{i}\theta}\right)\Bigg] \\
&\quad + \gamma_{12}\left[\sum_{l=1}^{K_1} \pi(r_1^l)^2\left(G(\xi_2^k, \xi_1^l) + \nabla G(\xi_2^k, \xi_1^l)\cdot r_2^k e^{\mathrm{i}\theta}\right)\right] + O(|\gamma|\epsilon^4).
\end{aligned}$$

Proof. The interfaces in the exact disc assembly are circles, so

$$\mathcal{H}_i^k(0) = \frac{1}{r_i^k}. \tag{4.31}$$

Recall v_i^l from (3.21), and $v_1 = \sum_{l=1}^{K_1} v_1^l, v_2 = \sum_{l=1}^{K_2} v_2^l$. Then

$$\begin{aligned} I_{B_1}(\xi_1^k + r_1^k e^{\mathrm{i}\theta}) &= v_1(\xi_1^k + r_1^k e^{\mathrm{i}\theta}) = \sum_{l=1}^{K_1} v_1^l(\xi_1^k + r_1^k e^{\mathrm{i}\theta}) \\ &= -\frac{(r_1^k)^2}{2}\log r_1^k + \pi(r_1^k)^2\left(R(\xi_1^k,\xi_1^k) + \nabla R(\xi_1^k,\xi_1^k)\cdot r_1^k e^{\mathrm{i}\theta}\right) \\ &\quad + \sum_{l=1,l\neq k}^{K_1} \pi(r_1^l)^2\left(G(\xi_1^k,\xi_1^l) + \nabla G(\xi_1^k,\xi_1^l)\cdot r_1^k e^{\mathrm{i}\theta}\right) + O(\epsilon^4), \end{aligned} \tag{4.32}$$

$$\begin{aligned} I_{B_2}(\xi_1^k + r_1^k e^{\mathrm{i}\theta}) &= v_2(\xi_1^k + r_1^k e^{\mathrm{i}\theta}) = \sum_{l=1}^{K_2} v_2^l(\xi_1^k + r_1^k e^{\mathrm{i}\theta}) \\ &= \sum_{l=1}^{K_2} \pi(r_2^l)^2\left(G(\xi_1^k,\xi_2^l) + \nabla G(\xi_1^k,\xi_2^l)\cdot r_1^k e^{\mathrm{i}\theta}\right) + O(\epsilon^4), \end{aligned} \tag{4.33}$$

$$\begin{aligned} I_{B_1}(\xi_2^k + r_2^k e^{\mathrm{i}\theta}) &= v_1(\xi_2^k + r_2^k e^{\mathrm{i}\theta}) = \sum_{l=1}^{K_1} v_1^l(\xi_2^k + r_2^k e^{\mathrm{i}\theta}) \\ &= \sum_{l=1}^{K_1} \pi(r_1^l)^2\left(G(\xi_2^k,\xi_1^l) + \nabla G(\xi_2^k,\xi_1^l)\cdot r_2^k e^{\mathrm{i}\theta}\right) + O(\epsilon^4), \end{aligned} \tag{4.34}$$

$$\begin{aligned} I_{B_2}(\xi_2^k + r_2^k e^{\mathrm{i}\theta}) &= v_2(\xi_2^k + r_2^k e^{\mathrm{i}\theta}) = \sum_{l=1}^{K_2} v_2^l(\xi_2^k + r_2^k e^{\mathrm{i}\theta}) \\ &= -\frac{(r_2^k)^2}{2}\log r_2^k + \pi(r_2^k)^2\left(R(\xi_2^k,\xi_2^k) + \nabla R(\xi_2^k,\xi_2^k)\cdot r_2^k e^{\mathrm{i}\theta}\right) \\ &\quad + \sum_{l=1,l\neq k}^{K_2} \pi(r_2^l)^2\left(G(\xi_2^k,\xi_2^l) + \nabla G(\xi_2^k,\xi_2^l)\cdot r_2^k e^{\mathrm{i}\theta}\right) + O(\epsilon^4). \end{aligned} \tag{4.35}$$

The lemma follows from (4.31) - (4.35) . □

5 Linear analysis

The Fréchet derivative of $\mathcal{S}$ at ϕ is denoted by $\mathcal{S}'(\phi)$. It can also be interpreted as the second variation of $\mathcal{J}$ because

$$\left.\frac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_2}\right|_{\varepsilon_1=\varepsilon_2=0} \mathcal{J}(\phi + \varepsilon_1 u + \varepsilon_2 v) = \langle \mathcal{S}'(\phi)(u), v\rangle. \tag{5.1}$$

In this section we study $\mathcal{S}'(0)$, i.e. the linearized operator at the exact disc assembly (B_1, B_2). By (4.27) and (4.28), consider the Fréchet derivative of each of the terms in $\mathcal{S}$. Calculations show that

$$\left(\mathcal{H}_i^k\right)'(0)(u_i^k) = -\frac{1}{(r_i^k)^3}((u_i^k)'' + u_i^k)$$

$$\left(\mathcal{A}_1^k\right)'(0)\left(u_1^k\right)(\theta) = -\frac{\gamma_{11}}{2\pi}\int_0^{2\pi} u_1^k(\eta)\log|r_1^k e^{\mathrm{i}\theta} - r_1^k e^{\mathrm{i}\eta}|d\eta - \frac{\gamma_{11}}{2}u_1^k(\theta)$$

$$\begin{aligned}\left(\mathcal{B}_1^k\right)'(0)\left(u_1^k\right)(\theta) &= \gamma_{11}\int_0^{2\pi} u_1^k(\eta)\, R(\xi_1^k + r_1^k e^{\mathrm{i}\theta}, \xi_1^k + r_1^k e^{\mathrm{i}\eta})d\eta \\ &\quad +\gamma_{11}\frac{u_1^k(\theta)}{r_1^k}\int_{B_1^k} \nabla R\left(\xi_1^k + r_1^k e^{\mathrm{i}\theta}, y\right)\cdot e^{\mathrm{i}\theta}dy\end{aligned}$$

$$\begin{aligned}\left(\mathcal{C}_1^{kl}\right)'(0)\left(u_1^k, u_1^l\right)(\theta) &= \gamma_{11}\int_0^{2\pi} u_1^l(\eta)\, G(\xi_1^k + r_1^k e^{\mathrm{i}\theta}, \xi_1^l + r_1^l e^{\mathrm{i}\eta})d\eta \\ &\quad +\gamma_{11}\frac{u_1^k(\theta)}{r_1^k}\int_{B_1^l} \nabla G\left(\xi_1^k + r_1^k e^{\mathrm{i}\theta}, y\right)\cdot e^{\mathrm{i}\theta}dy\end{aligned}$$

$$\begin{aligned}\left(\mathcal{D}_1^{kl}\right)'(0)\left(u_1^k, u_2^l\right)(\theta) &= \gamma_{12}\int_0^{2\pi} u_2^l(\eta)\, G(\xi_1^k + r_1^k e^{\mathrm{i}\theta}, \xi_2^l + r_2^l e^{\mathrm{i}\eta})d\eta \\ &\quad +\gamma_{12}\frac{u_1^k(\theta)}{r_1^k}\int_{B_2^l} \nabla G\left(\xi_1^k + r_1^k e^{\mathrm{i}\theta}, y\right)\cdot e^{\mathrm{i}\theta}dy\end{aligned}$$

$$\left(\mathcal{A}_2^k\right)'(0)\left(u_2^k\right)(\theta) = -\frac{\gamma_{22}}{2\pi}\int_0^{2\pi} u_2^k(\eta)\log|r_2^k e^{\mathrm{i}\theta} - r_2^k e^{\mathrm{i}\eta}|d\eta - \frac{\gamma_{22}}{2}u_2^k(\theta)$$

$$\begin{aligned}\left(\mathcal{B}_2^k\right)'(0)\left(u_2^k\right)(\theta) &= \gamma_{22}\int_0^{2\pi} u_2^k(\eta)\, R(\xi_2^k + r_2^k e^{\mathrm{i}\theta}, \xi_2^k + r_2^k e^{\mathrm{i}\eta})d\eta \\ &\quad +\gamma_{22}\frac{u_2^k(\theta)}{r_2^k}\int_{B_2^k} \nabla R\left(\xi_2^k + r_2^k e^{\mathrm{i}\theta}, y\right)\cdot e^{\mathrm{i}\theta}dy\end{aligned}$$

$$\begin{aligned}\left(\mathcal{C}_2^{kl}\right)'(0)\left(u_2^k, u_2^l\right)(\theta) &= \gamma_{22}\int_0^{2\pi} u_2^l(\eta)\, G(\xi_2^k + r_2^k e^{\mathrm{i}\theta}, \xi_2^l + r_2^l e^{\mathrm{i}\eta})d\eta \\ &\quad +\gamma_{22}\frac{u_2^k(\theta)}{r_2^k}\int_{B_2^l} \nabla G\left(\xi_2^k + r_2^k e^{\mathrm{i}\theta}, y\right)\cdot e^{\mathrm{i}\theta}dy\end{aligned}$$

$$\begin{aligned}\left(\mathcal{D}_2^{kl}\right)'(0)\left(u_2^k, u_1^l\right)(\theta) &= \gamma_{12}\int_0^{2\pi} u_1^l(\eta)\, G(\xi_2^k + r_2^k e^{\mathrm{i}\theta}, \xi_1^l + r_1^l e^{\mathrm{i}\eta})d\eta \\ &\quad +\gamma_{12}\frac{u_2^k(\theta)}{r_2^k}\int_{B_1^l} \nabla G\left(\xi_2^k + r_2^k e^{\mathrm{i}\theta}, y\right)\cdot e^{\mathrm{i}\theta}dy\end{aligned}$$

Separate $\mathcal{S}'(0)$ into a dominant part $\mathcal{E}$ and a minor part $\mathcal{F}$. The components of $\mathcal{E}$ are

$$\mathcal{E}_i^k(u)(\theta) = -\frac{1}{(r_i^k)^3}\left((u_i^k)''(\theta) + u_i^k(\theta)\right) - \frac{\gamma_{ii}}{2\pi}\int_0^{2\pi} u_i^k(\eta)\log|r_i^k e^{\mathrm{i}\theta} - r_i^k e^{\mathrm{i}\eta}|d\eta - \frac{\gamma_{ii}}{2}u_i^k(\theta) - e_i(u).$$

The real valued linear operator e_i is independent of k. It is so chosen that $\mathcal{E}$ maps from $\mathcal{X}$ to $\mathcal{Z}$. The rest of $\mathcal{S}'(0)$ is denoted by $\mathcal{F}$. Note that $\mathcal{E}$ is determined by $\mathcal{H}$ and $\mathcal{A}$, and $\mathcal{F}$ is determined by $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$.

Decompose

$$\mathcal{Z} = \overset{\infty}{\underset{n=0}{\oplus}} \mathcal{Z}(n), \tag{5.2}$$

where, when $n \neq 0$,

$$\begin{aligned} \mathcal{Z}(n) \quad &= \quad \Big\{ \mathbf{a} \cos n\theta + \mathbf{b} \sin n\theta : \mathbf{a} = \left(a_1^1, \cdots, a_1^{K_1}, a_2^1, \cdots, a_2^{K_2}\right) \in \mathbb{R}^{K_1+K_2}, \\ &\qquad \mathbf{b} = \left(b_1^1, \cdots, b_1^{K_1}, b_2^1, \cdots, b_2^{K_2}\right) \in \mathbb{R}^{K_1+K_2} \Big\}, \end{aligned} \tag{5.3}$$

and

$$\mathcal{Z}(0) \quad = \quad \Big\{ \mathbf{c} = \left(c_1^1, \cdots, c_1^{K_1}, c_2^1, \cdots, c_2^{K_2}\right) \in \mathbb{R}^{K_1+K_2} : \sum_{k=1}^{K_i} c_i^k = 0, \ i = 1, 2 \Big\}. \tag{5.4}$$

Below we will see that each $\mathcal{Z}(n)$ is invariant under $\mathcal{E}$; the eigenvectors of $\mathcal{E}$ are to be found in these $\mathcal{Z}(n)$'s.

Let $\mathbf{1}_i^k = (0, \cdots, 0, 1, 0, \cdots, 0)$ be a vector in $\mathbb{R}^{K_1+K_2}$. If $i = 1$, the k-th component of the vector is 1 and all the other components are 0. If $i = 2$, the $(K_1 + k)$-th component of the vector is 1 and all the other components are 0.

When $n \neq 0$, by the formula

$$\log |1 - e^{\mathrm{i}\theta}| = -\sum_{\tilde{n}=1}^{\infty} \frac{\cos \tilde{n}\theta}{\tilde{n}},$$

we deduce

$$\mathcal{E}_i^k(\mathbf{1}_j^p \cos(n\theta)) \quad = \quad \begin{cases} \left(\frac{n^2-1}{(r_i^k)^3} + \frac{\gamma_{ii}}{2n} - \frac{\gamma_{ii}}{2}\right) \cos(n\theta), & \text{if } j = i \text{ and } p = k, \\ 0 & \text{otherwise.} \end{cases}$$

The same holds if $\mathbf{1}_j^p \cos n\theta$ is replaced by $\mathbf{1}_j^p \sin n\theta$. Thus,

$$\mathcal{E}(\mathbf{1}_i^k \cos(n\theta)) \quad = \quad \left(\frac{n^2-1}{(r_i^k)^3} + \frac{\gamma_{ii}}{2n} - \frac{\gamma_{ii}}{2}\right) \mathbf{1}_i^k \cos(n\theta), \tag{5.5}$$

$$\mathcal{E}(\mathbf{1}_i^k \sin(n\theta)) \quad = \quad \left(\frac{n^2-1}{(r_i^k)^3} + \frac{\gamma_{ii}}{2n} - \frac{\gamma_{ii}}{2}\right) \mathbf{1}_i^k \sin(n\theta). \tag{5.6}$$

This means that, when $n \neq 0$, in $\mathcal{Z}(n)$ there are $K_1 + K_2$ eigenvalues of $\mathcal{E}$ which are

$$\mu_{i,n}^k := \frac{n^2-1}{(r_i^k)^3} + \frac{\gamma_{ii}}{2n} - \frac{\gamma_{ii}}{2}, \ i = 1, 2, \ k = 1, ..., K_i, \tag{5.7}$$

whose multiplicity is 2. The corresponding eigenvectors are $\mathbf{1}_i^k \cos(n\theta)$ and $\mathbf{1}_i^k \sin(n\theta)$.

When $n = 0$, for every $\mathbf{c} \in \mathcal{Z}(0)$,

$$\mathcal{E}_i^k(\mathbf{c}) = \left(-\frac{1}{(r_i^k)^3} + \gamma_{ii} \log \frac{1}{r_i^k} - \frac{\gamma_{ii}}{2}\right) c_i^k - \frac{1}{K_i} \sum_{l=1}^{K_i} \left(-\frac{1}{(r_i^l)^3} + \gamma_{ii} \log \frac{1}{r_i^l} - \frac{\gamma_{ii}}{2}\right) c_i^l, \tag{5.8}$$

This shows that $\mathcal{E}(\mathbf{c}) \in \mathcal{Z}(0)$ when $\mathbf{c} \in \mathcal{Z}(0)$. There are $K_1 + K_2 - 2$ eigenvalues in $\mathcal{Z}(0)$ counting multiplicity.

Although an exact disc B_i^k has the well defined center ξ_i^k and the radius r_i^k, after perturbation to Ω_i^k by ϕ_i^k, the notions of center and radius are no longer meaningful. Nevertheless, there is a special class of perturbations that preserve centers and radii, including the area of each perturbed disc. Let Π be the orthogonal projection operator from $\mathcal{Z}$ to a subspace $\mathcal{Z}_\flat$, where

$$\mathcal{Z}_\flat = \left\{ \phi \in \mathcal{Z} : \int_0^{2\pi} \phi_i^k \, d\theta = \int_0^{2\pi} \phi_i^k \cos\theta \, d\theta = \int_0^{2\pi} \phi_i^k \sin\theta \, d\theta = 0, \; i = 1, 2, \; k = 1, ..., K_i \right\}. \tag{5.9}$$

Note that

$$\mathcal{Z}_\flat = \overset{\infty}{\underset{n=2}{\oplus}} \mathcal{Z}(n). \tag{5.10}$$

Also define

$$\mathcal{Y}_\flat = \mathcal{Y} \cap \mathcal{Z}_\flat, \quad \mathcal{X}_\flat = \mathcal{X} \cap \mathcal{Z}_\flat. \tag{5.11}$$

When $\phi \in \mathcal{Z}_\flat$, the perturbed disc described by ϕ_i^k is considered to be centered at ξ_i^k of radius r_i^k. If $\phi \in \mathcal{Z} \backslash \mathcal{Z}_\flat$, then ξ_i^k or r_i^k cannot be interpreted as a center or a radius.

We are more interested in $\Pi \mathcal{S}'(0)$ and $\Pi \mathcal{E}$ restricted to $\mathcal{X}_\flat$ instead of $\mathcal{S}'(0)$ and $\mathcal{E}$ on $\mathcal{X}$. Since $\mathcal{E}$ maps $\mathcal{X}_\flat$ into $\mathcal{Z}_\flat$, $\Pi \mathcal{E} = \mathcal{E}$ on $\mathcal{X}_\flat$.

Lemma 5.1. *There exist $c_2 > 0$ and $h > 0$ such that if $\xi \in \overline{\Xi_\delta}$ and $w \in \overline{W_h}$, then*

1. $$\langle \Pi \mathcal{E}(u), u \rangle \; \geq \; 2c_2 \epsilon^{-3} ||u||_{\mathcal{Y}}^2$$

   and

2. $$||\Pi \mathcal{E}(u)||_{\mathcal{Z}} \; \geq \; 2c_2 \epsilon^{-3} ||u||_{\mathcal{X}}$$

for all $u \in \mathcal{X}_\flat$.

Proof. By (5.10) we consider

$$\mu_{i,n}^k = \frac{(n-1)}{2n(r_i^k)^3} \Big(2n(n+1) - \gamma_{ii} (r_i^k)^3 \Big)$$

of (5.7) for $n \geq 2$. According to condition 2 of Theorem 1.1,

$$\gamma_{ii} (\rho_i)^3 < 12 - \eta. \tag{5.12}$$

By (3.1) and the definition of ρ_i in Theorem 1.1,

$$(r_i^k)^2 = \frac{\epsilon^2}{\pi} w_i^k, \; (\rho_1)^2 = \frac{\epsilon^2}{\pi} \frac{m}{K_1}, \; (\rho_2)^2 = \frac{\epsilon^2}{\pi} \frac{1-m}{K_2}. \tag{5.13}$$

Choose h sufficiently small, so that, in addition to (3.3), one has the property

$$\gamma_{ii} (r_i^k)^3 < 12 - \frac{\eta}{2} \tag{5.14}$$

for all $w \in \overline{W_h}$. Thus, when $n \geq 2$,

$$\mu_{i,n}^k > \frac{(n-1)}{2n(r_i^k)^3} \Big(2n(n+1) - 12 + \frac{\eta}{2} \Big). \tag{5.15}$$

It follows that there exists $c > 0$, independent of n and ϵ, such that

$$\mu_{i,n}^k \geq c \, \epsilon^{-3} n^2. \tag{5.16}$$

Both parts of the lemma follow from (5.16). ☐

Henceforth h in the definition (3.2) of W_h is chosen in accordance with Lemma 5.1. We use c_2 with a lower case c and a subscript 2 to indicate that the constant is used for a lower bound of the second variation of $\mathcal{J}$ at 0. The second part $\mathcal{F}$ of $\mathcal{S}'(0)$ is minor.

Lemma 5.2. *There exists $C_2 > 0$ independent of ϵ and γ such that*

$$\|\mathcal{F}(u)\|_{\mathcal{Z}} \leq C_2|\gamma|\epsilon\,\|u\|_{\mathcal{Z}}$$

for all $u \in \mathcal{X}_\flat$.

Proof. The operator $\mathcal{F}$ is given by

$$\begin{aligned}
\mathcal{F}_1^k(u)(\theta) &= \gamma_{11}\int_0^{2\pi} u_1^k(\eta)\, R(\xi_1^k + r_1^k e^{\mathrm{i}\theta}, \xi_1^k + r_1^k e^{\mathrm{i}\eta})d\eta \\
&\quad +\gamma_{11}\frac{u_1^k(\theta)}{r_1^k}\int_{B_1^k} \nabla R\left(\xi_1^k + r_1^k e^{\mathrm{i}\theta}, y\right)\cdot e^{\mathrm{i}\theta}dy \\
&\quad +\sum_{l=1,l\neq k}^{K_1}\gamma_{11}\int_0^{2\pi} u_1^l(\eta)\, G(\xi_1^k + r_1^k e^{\mathrm{i}\theta}, \xi_1^l + r_1^l e^{\mathrm{i}\eta})d\eta \\
&\quad +\sum_{l=1,l\neq k}^{K_1}\gamma_{11}\frac{u_1^k(\theta)}{r_1^k}\int_{B_1^l} \nabla G\left(\xi_1^k + r_1^k e^{\mathrm{i}\theta}, y\right)\cdot e^{\mathrm{i}\theta}dy \\
&\quad +\sum_{l=1}^{K_2}\gamma_{12}\int_0^{2\pi} u_2^l(\eta)\, G(\xi_1^k + r_1^k e^{\mathrm{i}\theta}, \xi_2^l + r_2^l e^{\mathrm{i}\eta})d\eta \\
&\quad +\sum_{l=1}^{K_2}\gamma_{12}\frac{u_1^k(\theta)}{r_1^k}\int_{B_2^l} \nabla G\left(\xi_1^k + r_1^k e^{\mathrm{i}\theta}, y\right)\cdot e^{\mathrm{i}\theta}dy - f_1(u),
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_2^k(u)(\theta) &= \gamma_{22}\int_0^{2\pi} u_2^k(\eta)\, R(\xi_2^k + r_2^k e^{\mathrm{i}\theta}, \xi_2^k + r_2^k e^{\mathrm{i}\eta})d\eta \\
&\quad +\gamma_{22}\frac{u_2^k(\theta)}{r_2^k}\int_{B_2^k} \nabla R\left(\xi_2^k + r_2^k e^{\mathrm{i}\theta}, y\right)\cdot e^{\mathrm{i}\theta}dy \\
&\quad +\sum_{l=1,l\neq k}^{K_2}\gamma_{22}\int_0^{2\pi} u_2^l(\eta)\, G(\xi_2^k + r_2^k e^{\mathrm{i}\theta}, \xi_2^l + r_2^l e^{\mathrm{i}\eta})d\eta \\
&\quad +\sum_{l=1,l\neq k}^{K_2}\gamma_{22}\frac{u_2^k(\theta)}{r_2^k}\int_{B_2^l} \nabla G\left(\xi_2^k + r_2^k e^{\mathrm{i}\theta}, y\right)\cdot e^{\mathrm{i}\theta}dy \\
&\quad +\sum_{l=1}^{K_1}\gamma_{12}\int_0^{2\pi} u_1^l(\eta)\, G(\xi_2^k + r_2^k e^{\mathrm{i}\theta}, \xi_1^l + r_1^l e^{\mathrm{i}\eta})d\eta \\
&\quad +\sum_{l=1}^{K_1}\gamma_{12}\frac{u_2^k(\theta)}{r_2^k}\int_{B_1^l} \nabla G\left(\xi_2^k + r_2^k e^{\mathrm{i}\theta}, y\right)\cdot e^{\mathrm{i}\theta}dy - f_2(u)
\end{aligned}$$

where $f_1(u)$ and $f_2(u)$ are real valued and independent of k. They ensure that $\mathcal{F}(u)$ is in $\mathcal{Z}$.

Recall that ∇R (and ∇G) denotes the gradient of R with respect to the first set of variables. Let $\widetilde{\nabla} R$ (and $\widetilde{\nabla} G$) denote the gradient of R with respect to the second set of variables, i.e. $\widetilde{\nabla} R(x, y) = (\frac{\partial R(x,y)}{\partial y_1}, \frac{\partial R(x,y)}{\partial y_2})$. Because

$$\begin{aligned}
R(\xi_i^k + r_i^k e^{\mathrm{i}\theta}, \xi_i^k + r_i^k e^{\mathrm{i}\eta}) &= R(\xi_i^k, \xi_i^k) + \nabla R(\xi_i^k, \xi_i^k)\cdot r_i^k e^{\mathrm{i}\theta} + \widetilde{\nabla} R(\xi_i^k, \xi_i^k)\cdot r_i^k e^{\mathrm{i}\eta} + O(\epsilon^2), \\
G(\xi_i^k + r_i^k e^{\mathrm{i}\theta}, \xi_j^l + r_j^l e^{\mathrm{i}\eta}) &= G(\xi_i^k, \xi_j^l) + \nabla G(\xi_i^k, \xi_j^l)\cdot r_i^k e^{\mathrm{i}\theta} + \widetilde{\nabla} G(\xi_i^k, \xi_j^l)\cdot r_j^l e^{\mathrm{i}\eta} + O(\epsilon^2),
\end{aligned}$$

and $\int_0^{2\pi} u_i^k(\eta)\,d\eta = \int_0^{2\pi} u_i^k(\eta)\cos\eta\,d\eta = \int_0^{2\pi} u_i^k \sin\eta\,d\eta = 0$, we obtain that

$$\begin{aligned}\left\| \int_0^{2\pi} u_i^k(\eta) R(\xi_i^k + r_i^k e^{\mathrm{i}\theta}, \xi_i^k + r_i^k e^{\mathrm{i}\eta})\,d\eta \right\|_{L^2(S^1)} &\leq C\epsilon^2 \left\|u_i^k\right\|_{L^2(S^1)}, \\ \left\| \int_0^{2\pi} u_j^l(\eta) G(\xi_i^k + r_i^k e^{\mathrm{i}\theta}, \xi_j^l + r_j^l e^{\mathrm{i}\eta})\,d\eta \right\|_{L^2(S^1)} &\leq C\epsilon^2 \left\|u_j^l\right\|_{L^2(S^1)}.\end{aligned}$$

Since the area of an exact disc is of order $O(\epsilon^2)$,

$$\begin{aligned}\left\| \frac{u_i^k(\theta)}{r_i^k} \int_{B_i^k} \nabla R(\xi_i^k + r_i^k e^{\mathrm{i}\theta}, y)\cdot e^{\mathrm{i}\theta}\,dy \right\|_{L^2(S^1)} &\leq C\epsilon \left\|u_i^k\right\|_{L^2(S^1)}, \\ \left\| \frac{u_i^k(\theta)}{r_i^k} \int_{B_j^l} \nabla G(\xi_i^k + r_i^k e^{\mathrm{i}\theta}, y)\cdot e^{\mathrm{i}\theta}\,dy \right\|_{L^2(S^1)} &\leq C\epsilon \left\|u_i^k\right\|_{L^2(S^1)}.\end{aligned}$$

The condition

$$\sum_{k=1}^{K_i} \int_0^{2\pi} \mathcal{F}_i^k(u)(\theta)d\theta = 0,\ i = 1, 2,$$

implies that

$$|f_i(u)| \leq C|\gamma|\epsilon \left\|u\right\|_{\mathcal{Z}},\ i = 1, 2.$$

The lemma then follows. □

Lemma 5.3. *There exist $c_2 > 0$ and $\epsilon_0 > 0$ such that when $\epsilon < \epsilon_0$,*

1. $$\langle \Pi\mathcal{S}'(0)(u), u\rangle \geq c_2\epsilon^{-3}\|u\|_{\mathcal{Y}}^2,\ \textit{for all } u \in \mathcal{X}_\flat,$$

2. $$\|\Pi\mathcal{S}'(0)(u)\|_{\mathcal{Z}} \geq c_2\epsilon^{-3}\|u\|_{\mathcal{X}},\ \textit{for all } u \in \mathcal{X}_\flat,$$

3. *the operator $\Pi\mathcal{S}'(0)$ is one-to-one and onto from $\mathcal{X}_\flat$ to $\mathcal{Z}_\flat$.*

Here c_2 is the same as the c_2 in Lemma 5.1.

Proof. By conditions 2 and 3 of Theorem 1.1, there exists $\hat{C} > 0$ such that

$$\epsilon^3|\gamma| \leq \hat{C}. \tag{5.17}$$

According to Lemma 5.2,

$$\|\mathcal{F}(u)\|_{\mathcal{Z}} \leq C_2|\gamma|\epsilon \left\|u\right\|_{\mathcal{Z}} \leq C_2\hat{C}\epsilon^{-2} \left\|u\right\|_{\mathcal{Z}} \leq c_2\epsilon^{-3}\|u\|_{\mathcal{Z}} \tag{5.18}$$

for all $u \in \mathcal{X}_\flat$, if ϵ is sufficiently small. By Lemma 5.1.1 and (5.18)

$$\begin{aligned}\langle \Pi\mathcal{S}'(0)(u), u\rangle &= \langle \Pi\mathcal{E}(u), u\rangle + \langle \Pi\mathcal{F}(u), u\rangle \\ &\geq 2c_2\epsilon^{-3}\|u\|_{\mathcal{Y}}^2 - c_2\epsilon^{-3}\|u\|_{\mathcal{Z}}^2 \\ &\geq c_2\epsilon^{-3}\|u\|_{\mathcal{Y}}^2,\end{aligned}$$

for all $u \in \mathcal{X}_\flat$. This proves part 1.

By Lemma 5.1.2 and (5.18)

$$\begin{aligned}\|\Pi\mathcal{S}'(0)(u)\|_{\mathcal{Z}} &\geq \|\Pi\mathcal{E}(u)\|_{\mathcal{Z}} - \|\Pi\mathcal{F}(u)\|_{\mathcal{Z}} \\ &\geq 2c_2\epsilon^{-3}\|u\|_{\mathcal{X}} - c_2\epsilon^{-3}\|u\|_{\mathcal{Z}} \\ &\geq c_2\epsilon^{-3}\|u\|_{\mathcal{X}},\end{aligned}$$

for all $u \in \mathcal{X}_\flat$. This proves part 2.

For the third part it suffices to show that $\Pi\mathcal{S}'(0)$ is from $\mathcal{X}_\flat$ onto $\mathcal{Z}_\flat$. Note that $\Pi\mathcal{S}'(0)$ is an unbounded self-adjoint operator on $\mathcal{Z}_\flat$ with the domain $\mathcal{X}_\flat \subset \mathcal{Z}_\flat$. If $v \in \mathcal{Z}_\flat$ is perpendicular to the range of $\Pi\mathcal{S}'(0)$ i.e. $\langle \Pi\mathcal{S}'(0)(u), v\rangle = 0$ for all $u \in \mathcal{X}_\flat$, then the self-adjointness of $\Pi\mathcal{S}'(0)$ implies that $u \in \mathcal{X}_\flat$ and $\Pi\mathcal{S}'(0)(v) = 0$. By part 2, $v = 0$. Hence, the range of $\Pi\mathcal{S}'(0)$ is dense in $\mathcal{Z}_\flat$. Part 2 also implies that the range of $\Pi\mathcal{S}'(0)$ is a closed subspace of $\mathcal{Z}_\flat$. Therefore $\Pi\mathcal{S}'(0)$ is onto. □

Finally one needs an estimate on the second Fréchet derivative of $\mathcal{S}$, i.e. the third variation of $\mathcal{J}$.

Lemma 5.4. *There exists $C_3 > 0$ such that for all $\phi \in Dom(\mathcal{S})$, the following estimates hold for all $u \in \mathcal{X}$ and $v \in \mathcal{X}$,*

1. $$|\langle \mathcal{S}''(\phi)(u,v), v\rangle| \le C_3 \left(\epsilon^{-5} + |\gamma|\epsilon^{-2}\right) \|u\|_{\mathcal{X}} \|v\|^2_{\mathcal{Y}},$$

2. $$||\mathcal{S}''(\phi)(u,v)||_{\mathcal{Z}} \le C_3 \left(\epsilon^{-5} + |\gamma|\epsilon^{-2}\right) \|u\|_{\mathcal{X}} \|v\|_{\mathcal{X}}.$$

Note the use of C_3 in these upper bounds for the third variation of $\mathcal{J}$.

Proof. One proves this lemma by showing that, for all $u = (u_1^1, \cdots, u_1^{K_1}, u_2^1, \cdots, u_2^{K_2})$ and $v = (v_1^1, \cdots, v_1^{K_1}, v_2^1, \cdots, v_2^{K_2})$ in $\mathcal{X}$,

$$\begin{aligned}
\left\|\left(\mathcal{H}_i^k\right)''(\phi_i^k)(u_i^k, v_i^k)\right\|_{L^2(S^1)} &\le C_3\epsilon^{-5}\left\|u_i^k\right\|_{H^2(S^1)}\left\|v_i^k\right\|_{H^2(S^1)} \\
\left\|\left(\mathcal{A}_i^k\right)''(\phi_i^k)(u_i^k, v_i^k)\right\|_{L^2(S^1)} &\le C_3|\gamma|\epsilon^{-2}\left\|u_i^k\right\|_{H^1(S^1)}\left\|v_i^k\right\|_{H^1(S^1)} \\
\left\|\left(\mathcal{B}_i^k\right)''(\phi_i^k)(u_i^k, v_i^k)\right\|_{L^2(S^1)} &\le C_3|\gamma|\epsilon^{-1}\left\|u_i^k\right\|_{H^1(S^1)}\left\|v_i^k\right\|_{H^1(S^1)} \\
\left\|\left(\mathcal{C}_i^{kl}\right)''(\phi_i^k, \phi_i^l)(u_i^k, u_i^l)(v_i^k, v_i^l)\right\|_{L^2(S^1)} &\le C_3|\gamma|\epsilon^{-1}\left(\left\|u_i^k\right\|_{H^1(S^1)} + \left\|u_i^l\right\|_{H^1(S^1)}\right)\left(\left\|v_i^k\right\|_{H^1(S^1)} + \left\|v_i^l\right\|_{H^1(S^1)}\right) \\
\left\|\left(\mathcal{D}_i^{kl}\right)''(\phi_i^k, \phi_j^l)(u_i^k, u_j^l)(v_i^k, v_j^l)\right\|_{L^2(S^1)} &\le C_3|\gamma|\epsilon^{-1}\left(\left\|u_i^k\right\|_{H^1(S^1)} + \left\|u_j^l\right\|_{H^1(S^1)}\right)\left(\left\|v_i^k\right\|_{H^1(S^1)} + \left\|v_j^l\right\|_{H^1(S^1)}\right) \\
|\lambda_i''(\phi)(u,v)| &\le C_3\left(\epsilon^{-5} + |\gamma|\epsilon^{-2}\right)\|u\|_{\mathcal{X}}\|v\|_{\mathcal{X}}.
\end{aligned}$$

We will not include the derivations of these estimates here. See the proofs of [25, Lemma 3.2] and [24, Lemma 6.1] for similar results. □

6 Reduction

In this section it will be proved that, for each $(\xi, w) \in \overline{\Xi_\delta} \times \overline{W_h}$, there exists $\phi^*(\cdot, \xi, w)$ such that

$$\Pi\mathcal{S}(\phi^*(\cdot, \xi, w)) = 0. \tag{6.1}$$

In other words, there exist $A_i^k, B_i^k, C_i^k \in \mathbb{R}$ such that

$$\mathcal{S}_i^k(\phi^*)(\theta) = A_i^k + B_i^k\cos\theta + C_i^k\sin\theta,\ i = 1, 2,\ k = 1, 2, ..., K_i. \tag{6.2}$$

Note that ϕ^* is found in $\mathcal{X}_\flat$, so the discs in the assembly $\phi^*(\cdot, \xi, w)$ are centered at ξ_i^k of radii r_i^k. In the next section we will find a particular (ξ, w), denoted by (ξ^*, w^*), at which $A_i^k = B_i^k = C_i^k = 0$, i.e. $\mathcal{S}(\phi^*(\cdot, \xi^*, w^*)) = 0$. This means that by finding $\phi^*(\cdot, \xi^*, w^*)$ the original problem (1.6) and (1.7) is reduced to a problem of finding a (ξ^*, w^*) in the set $\overline{\Xi_\delta} \times \overline{W_h}$.

Lemma 6.1. *When ϵ is small enough, there exists $\phi^* = \phi^*(\theta, \xi, w) \in \mathcal{X}_\flat$ for every $(\xi, w) \in \overline{\Xi_\delta} \times \overline{W_h}$ such that $\phi^*(\cdot, \xi, w)$ solves (6.1) and $\|\phi^*(\cdot, \xi, w)\|_{\mathcal{X}} \le \frac{2C_1}{c_2}|\gamma|\epsilon^7$.*

Proof. Write $\mathcal{S}(\phi)$ as

$$\mathcal{S}(\phi) = \mathcal{S}(0) + \mathcal{S}'(0)(\phi) + \mathcal{R}(\phi) \tag{6.3}$$

where $\mathcal{R}(\phi)$, defined by (6.3), is a higher order term. Turn (6.1) to a fixed point form:

$$\phi = \mathcal{T}(\phi) \tag{6.4}$$

where

$$\mathcal{T}(\phi) = -(\Pi\mathcal{S}'(0))^{-1}(\Pi\mathcal{S}(0) + \Pi\mathcal{R}(\phi)). \tag{6.5}$$

is an operator defined on $\mathcal{W} = \{\phi \in \mathcal{X}_\flat : \|\phi\|_{\mathcal{X}} \le b\epsilon^2\} \subset Dom(\mathcal{S})$. Here $b > 0$ is to be determined.

By Lemma 4.1, $\mathcal{S}(0)$ is a sum of a θ independent part, a $e^{i\theta}$ part, and a quantity of order $O(|\gamma|\epsilon^4)$. After one applies the projection operator Π to $\mathcal{S}(0)$, the θ independent part and $e^{i\theta}$ part vanish by the definition of $\mathcal{Z}_\flat$, so there exists $C_1 > 0$ so that

$$\|\Pi\mathcal{S}(0)\|_{\mathcal{Z}} \le C_1|\gamma|\epsilon^4. \tag{6.6}$$

We use C_1 to remind the reader that it is used in an upper bound estimate of the first variation of $\mathcal{J}$. By Lemma 5.3.2,

$$\|(\Pi\mathcal{S}'(0))^{-1}\Pi\mathcal{S}(0)\|_{\mathcal{X}} \le \frac{C_1}{c_2}|\gamma|\epsilon^7. \tag{6.7}$$

Lemma 5.4.2 implies that

$$\|\mathcal{R}(\phi)\|_{\mathcal{Z}} \le C_3\left(\epsilon^{-5} + |\gamma|\epsilon^{-2}\right)\|\phi\|^2_{\mathcal{X}}. \tag{6.8}$$

By Lemma 5.3.2,

$$\|(\Pi\mathcal{S}'(0))^{-1}\Pi\mathcal{R}(\phi)\|_{\mathcal{X}} \quad \le \quad \frac{C_3}{c_2}\left(\epsilon^{-2} + |\gamma|\epsilon\right)\|\phi\|^2_{\mathcal{X}}. \tag{6.9}$$

For $\phi \in \mathcal{W}$, by (5.17), (6.5), (6.7), and (6.9) we obtain

$$\|\mathcal{T}(\phi)\|_{\mathcal{X}} \le \frac{C_1}{c_2}|\gamma|\epsilon^7 + \frac{C_3}{c_2}\left(\epsilon^2 + |\gamma|\epsilon^5\right)b^2 \le \Big(\frac{C_1}{c_2}\hat{C}\epsilon^2 + \frac{C_3}{c_2}b^2 + \frac{C_3}{c_2}\hat{C}b^2\Big)\epsilon^2.$$

Take

$$b = \min\left\{\frac{c_2}{4C_3(1+\hat{C})}, \frac{\beta}{2}\right\} \tag{6.10}$$

where β comes from (4.20), the domain of $\mathcal{S}$. Let ϵ be small enough such that $\frac{C_1}{c_2}\hat{C}\epsilon^2 < \frac{b}{2}$ and Lemma 5.3 holds. Then

$$\|\mathcal{T}(\phi)\|_{\mathcal{X}} \le b\epsilon^2.$$

Therefore $\mathcal{T}$ maps $\mathcal{W}$ into itself.

Next show that $\mathcal{T}$ is a contraction. Let $\phi_1, \phi_2 \in \mathcal{W}$. First note that

$$\mathcal{T}(\phi_1) - \mathcal{T}(\phi_2) = -(\Pi\mathcal{S}'(0))^{-1}(\Pi)\left(\mathcal{R}(\phi_1) - \mathcal{R}(\phi_2)\right). \tag{6.11}$$

Because

$$\mathcal{R}(\phi_1) - \mathcal{R}(\phi_2) = \mathcal{S}(\phi_1) - \mathcal{S}(\phi_2) - \mathcal{S}'(0)(\phi_1 - \phi_2), \tag{6.12}$$

we deduce, with the help of Lemma 5.4.2, that

$$\begin{aligned}
& \|\mathcal{R}(\phi_1) - \mathcal{R}(\phi_2)\|_{\mathcal{Z}} \\
\le\; & \|\mathcal{S}'(\phi_2)(\phi_1 - \phi_2) - \mathcal{S}'(0)(\phi_1 - \phi_2)\|_{\mathcal{Z}} + \frac{C_3}{2}\left(\epsilon^{-5} + |\gamma|\epsilon^{-2}\right)\|\phi_1 - \phi_2\|^2_{\mathcal{X}} \\
\le\; & C_3\left(\epsilon^{-5} + |\gamma|\epsilon^{-2}\right)\|\phi_2\|_{\mathcal{X}}\|\phi_1 - \phi_2\|_{\mathcal{X}} + \frac{C_3}{2}\left(\epsilon^{-5} + |\gamma|\epsilon^{-2}\right)\|\phi_1 - \phi_2\|^2_{\mathcal{X}} \\
\le\; & C_3\left(\epsilon^{-5} + |\gamma|\epsilon^{-2}\right)(b+b)\,\epsilon^2\|\phi_1 - \phi_2\|_{\mathcal{X}} \\
\le\; & 2b\,C_3\left(\epsilon^{-3} + |\gamma|\right)\|\phi_1 - \phi_2\|_{\mathcal{X}}.
\end{aligned}$$

Then Lemma 5.3.2, (5.17) and (6.10) imply that

$$\|\mathcal{T}(\phi_1) - \mathcal{T}(\phi_2)\|_{\mathcal{X}} \le \frac{2b\, C_3}{c_2}\left(1+\hat{C}\right)\|\phi_1-\phi_2\|_{\mathcal{X}} \le \frac{1}{2}\|\phi_1-\phi_2\|_{\mathcal{X}}. \tag{6.13}$$

Hence $\mathcal{T}$ is a contraction mapping, and a unique fixed point ϕ^* exists in $\mathcal{W}$.

By the definition of $\mathcal{W}$, $\|\phi^*\|_{\mathcal{X}} = O(\epsilon^2)$. However, this can be improved, if we revisit the equation $\phi^* = \mathcal{T}(\phi^*)$ and derive from (6.5), (6.7) and (6.9) that

$$\|\phi^*\|_{\mathcal{X}} \le \|(\Pi\mathcal{S}'(0))^{-1}\Pi\mathcal{S}(0)\|_{\mathcal{X}} + \|(\Pi\mathcal{S}'(0))^{-1}\Pi\mathcal{R}(\phi)\|_{\mathcal{X}} \le \frac{C_1}{c_2}|\gamma|\epsilon^7 + \frac{C_3}{c_2}\left(\epsilon^{-2}+|\gamma|\epsilon\right)\|\phi^*\|^2_{\mathcal{X}}.$$

Rewrite the above as

$$\left(1 - \frac{C_3}{c_2}\left(\epsilon^{-2}+|\gamma|\epsilon\right)\|\phi^*\|_{\mathcal{X}}\right)\|\phi^*\|_{\mathcal{X}} \le \frac{C_1}{c_2}|\gamma|\epsilon^7. \tag{6.14}$$

In (6.14) estimate

$$\frac{C_3}{c_2}\left(\epsilon^{-2}+|\gamma|\epsilon\right)\|\phi^*\|_{\mathcal{X}} \le \frac{C_3}{c_2}\left(1+|\gamma|\epsilon^3\right)b \le \frac{C_3}{c_2}\left(1+\hat{C}\right)b \le \frac{1}{4} \tag{6.15}$$

by (5.17) and (6.10). The estimate of ϕ^* follows from (6.14) and (6.15). □

We state a result regarding $\mathcal{S}'$ at $\phi^*(\cdot,\xi,w)$. It implies that ϕ^* is stable with respect to deformations within $\mathcal{X}_\flat$.

Lemma 6.2. *When ϵ is sufficiently small, for all $u \in \mathcal{X}_\flat$*

1.
$$\langle \Pi\mathcal{S}'(\phi^*)(u), u\rangle \ge \frac{c_2}{2}\epsilon^{-3}\|u\|^2_{\mathcal{Y}}$$

2.
$$\|\Pi\mathcal{S}'(\phi^*)(u)\|_{\mathcal{Z}} \ge \frac{c_2}{2}\epsilon^{-3}\|u\|_{\mathcal{X}}.$$

Here c_2 is the same constant as the one in Lemma 5.3.

Proof. Lemma 5.3.1, Lemma 5.4.1 and Lemma 6.1 imply that

$$\begin{aligned}\langle \Pi\mathcal{S}'(\phi^*)(u), u\rangle &= \langle \Pi\mathcal{S}'(0)(u), u\rangle + \langle \Pi(\mathcal{S}'(\phi^*) - \mathcal{S}'(0))u, u\rangle \\ &\ge c_2\epsilon^{-3}\|u\|^2_{\mathcal{Y}} - C_3\left(\epsilon^{-5}+|\gamma|\epsilon^{-2}\right)\|\phi^*\|_{\mathcal{X}}\|u\|^2_{\mathcal{Y}} \\ &\ge \left(c_2 - \frac{2C_1C_3\hat{C}}{c_2}(1+\hat{C})\epsilon^2\right)\epsilon^{-3}\|u\|^2_{\mathcal{Y}}.\end{aligned}$$

The first part of the lemma follows if ϵ is sufficiently small so that $\frac{2C_1C_3\hat{C}}{c_2}(1+\hat{C})\epsilon^2 \le \frac{c_2}{2}$.

By Lemma 5.3.2, Lemma 5.4.2 and Lemma 6.1,

$$\begin{aligned}\|\Pi\mathcal{S}'(\phi^*)(u)\|_{\mathcal{Z}} &\ge \|\Pi\mathcal{S}'(0)(u)\|_{\mathcal{Z}} - \|\Pi(\mathcal{S}'(\phi^*) - \mathcal{S}'(0))(u)\|_{\mathcal{Z}} \\ &\ge c_2\epsilon^{-3}\|u\|_{\mathcal{X}} - C_3\left(\epsilon^{-5}+|\gamma|\epsilon^{-2}\right)\|\phi^*\|_{\mathcal{X}}\|u\|_{\mathcal{X}} \\ &\ge \left(c_2 - \frac{2C_1C_3\hat{C}}{c_2}(1+\hat{C})\epsilon^2\right)\epsilon^{-3}\|u\|_{\mathcal{X}}.\end{aligned}$$

As before, the second part follows if ϵ is small so that $\frac{2C_1C_3\hat{C}}{c_2}(1+\hat{C})\epsilon^2 \le \frac{c_2}{2}$. □

7 Existence

It was found, in Lemma 6.1, that for each $(\xi, w) \in \overline{\Xi_\delta} \times \overline{W_h}$, there exists $\phi^*(\cdot, \xi, w) \in \mathcal{X}_\flat$ such that $\Pi\mathcal{S}(\phi^*(\cdot, \xi, w)) = 0$. In this section, we will find a particular (ξ, w), denoted by (ξ^*, w^*), such that $\mathcal{S}(\phi^*(\cdot, \xi^*, w^*)) = 0$. The starting point is a good estimate of the energy of $\phi^*(\cdot, \xi, w)$.

Lemma 7.1. *It holds uniformly for all $(\xi, w) \in \overline{\Xi_\delta} \times \overline{W_h}$ and sufficiently small ϵ,*

$$\begin{aligned}
\mathcal{J}(\phi^*(\cdot, \xi, w)) \quad = \quad & \sum_{i=1}^{2} \sum_{k=1}^{K_i} \left(2\pi r_i^k + \frac{\gamma_{ii}\pi^2}{2} \left(\frac{(r_i^k)^4}{8\pi} - \frac{\left(r_i^k\right)^4 \log r_i^k}{2\pi} \right) \right) \\
& + \frac{\gamma_{11}\pi^2}{2} \left(\sum_{k=1}^{K_1} \left(r_1^k\right)^4 R\left(\xi_1^k, \xi_1^k\right) + \sum_{k=1}^{K_1} \sum_{l=1, l\neq k}^{K_1} \left(r_1^k\right)^2 \left(r_1^l\right)^2 G\left(\xi_1^k, \xi_1^l\right) \right) \\
& + \gamma_{12}\pi^2 \sum_{k=1}^{K_1} \sum_{l=1}^{K_2} \left(r_1^k\right)^2 \left(r_2^l\right)^2 G\left(\xi_1^k, \xi_2^l\right) \\
& + \frac{\gamma_{22}\pi^2}{2} \left(\sum_{k=1}^{K_2} \left(r_2^k\right)^4 R\left(\xi_2^k, \xi_2^k\right) + \sum_{k=1}^{K_2} \sum_{l=1, l\neq k}^{K_2} \left(r_2^k\right)^2 \left(r_2^l\right)^2 G\left(\xi_2^k, \xi_2^l\right) \right) \\
& + O(|\gamma|\epsilon^6).
\end{aligned}$$

Proof. Expanding $\mathcal{J}(\phi^*)$ yields

$$\mathcal{J}(\phi^*) \quad = \quad \mathcal{J}(0) + \langle \mathcal{S}(0), \phi^* \rangle + \frac{1}{2} \langle \mathcal{S}'(0)(\phi^*), \phi^* \rangle + \frac{1}{6} \langle \mathcal{S}''(\tau\phi^*)(\phi^*, \phi^*), \phi^* \rangle \tag{7.1}$$

for some $\tau \in (0, 1)$. On the other hand expanding $\mathcal{S}(\phi^*)$, and then applying Π on both sides give

$$\|\Pi\mathcal{S}(\phi^*) - \Pi\mathcal{S}(0) - \Pi\mathcal{S}'(0)(\phi^*)\|_{\mathcal{Z}} \leq \sup_{\tau \in (0,1)} \frac{1}{2} \|\Pi\mathcal{S}''(\tau\phi^*)(\phi^*, \phi^*)\|_{\mathcal{Z}}. \tag{7.2}$$

Since $\Pi\mathcal{S}(\phi^*) = 0$, (7.2) shows that

$$\|\Pi\mathcal{S}(0) + \Pi\mathcal{S}'(0)(\phi^*)\|_{\mathcal{Z}} \leq \sup_{\tau \in (0,1)} \frac{1}{2} \|\Pi\mathcal{S}''(\tau\phi^*)(\phi^*, \phi^*)\|_{\mathcal{Z}},$$

which implies that

$$\|\langle \Pi\mathcal{S}(0), \phi^* \rangle + \langle \Pi\mathcal{S}'(0)(\phi^*), \phi^* \rangle\|_{\mathcal{Z}} \leq \left(\sup_{\tau \in (0,1)} \frac{1}{2} \|\Pi\mathcal{S}''(\tau\phi^*)(\phi^*, \phi^*)\|_{\mathcal{Z}} \right) \|\phi^*\|_{\mathcal{X}}. \tag{7.3}$$

Since $\phi^* \in \mathcal{X}_\flat$,

$$\langle \Pi\mathcal{S}(0), \phi^* \rangle = \langle \mathcal{S}(0), \phi^* \rangle, \quad \langle \Pi\mathcal{S}'(0)(\phi^*), \phi^* \rangle = \langle \mathcal{S}'(0)(\phi^*), \phi^* \rangle. \tag{7.4}$$

Then (7.3) shows that

$$\|\langle \mathcal{S}(0), \phi^* \rangle + \langle \mathcal{S}'(0)(\phi^*), \phi^* \rangle\|_{\mathcal{Z}} \leq \left(\sup_{\tau \in (0,1)} \frac{1}{2} \|\Pi\mathcal{S}''(\tau(\phi^*))(\phi^*, \phi^*)\|_{\mathcal{Z}} \right) \|\phi^*\|_{\mathcal{X}}. \tag{7.5}$$

By (7.5), (7.1) yields that

$$|\mathcal{J}(\phi^*) - \mathcal{J}(0) - \frac{1}{2} \langle \mathcal{S}(0), \phi^* \rangle| \leq \frac{5}{12} \left(\sup_{\tau \in (0,1)} \|\Pi\mathcal{S}''(\tau(\phi^*))(\phi^*, \phi^*)\|_{\mathcal{Z}} \right) \|\phi^*\|_{\mathcal{X}}.$$

Therefore (6.6), (7.4), Lemma 5.4.2 and Lemma 6.1 imply that

$$\begin{aligned}|\mathcal{J}(\phi^*)-\mathcal{J}(0)| &\leq \frac{1}{2}|\langle\mathcal{S}(0),\phi^*\rangle|+\frac{5}{12}\left(\sup_{\tau\in(0,1)}\|\Pi\mathcal{S}''(\tau(\phi^*))(\phi^*,\phi^*)\|_{\mathcal{Z}}\right)\|\phi^*\|_{\mathcal{X}}\\ &\leq \frac{1}{2}(C_1|\gamma|\epsilon^4)\frac{2C_1}{c_2}|\gamma|\epsilon^7+\frac{5}{12}C_3\left(\epsilon^{-5}+|\gamma|\epsilon^{-2}\right)\left(\frac{2C_1}{c_2}|\gamma|\epsilon^7\right)^3\\ &= |\gamma|^2\epsilon^{11}\left(\frac{C_1^2}{c_2}+\frac{10C_3C_1^3}{3c_2^3}(1+|\gamma|\epsilon^3)|\gamma|\epsilon^5\right).\end{aligned}$$

Finally one uses Lemma 3.1 and (5.17) to complete the proof. □

As (ξ,w) varies in $\overline{\Xi_\delta}\times\overline{W_h}$, the energy of $\phi^*(\cdot,\xi,w)$, $\mathcal{J}(\phi^*(\cdot,\xi,w))$, is now viewed as a function of (ξ,w) on $\overline{\Xi_\delta}\times\overline{W_h}$. We treat both $\overline{\Xi_\delta}$ and $\overline{W_h}$ as manifolds with boundary. The product of the two, $\overline{\Xi_\delta}\times\overline{W_h}$, is also a manifold with boundary. It turns out that every critical point of $\mathcal{J}(\phi^*(\cdot,\xi,w))$ in $\Xi_\delta\times W_h$, the interior of $\overline{\Xi_\delta}\times\overline{W_h}$, corresponds to a stationary point of $\mathcal{J}$.

Lemma 7.2. *If (ξ_c,w_c) is an (interior) critical point of the function $(\xi,w)\to\mathcal{J}(\phi^*(\cdot,\xi,w))$, then $\phi^*(\cdot,\xi_c,w_c)$ solves $\mathcal{S}(\phi^*(\cdot,\xi_c,w_c))=0$, i.e. the disc assembly represented by $\phi^*(\cdot,\xi_c,w_c)$ is a stationary point of $\mathcal{J}$.*

Proof. Let $\Omega=(\Omega_1,\Omega_2)$ be the perturbed disc assembly characterized by $\phi^*(\cdot,\xi,w)$. Suppose that the boundary of Ω_i^k, the k-th component of Ω_i, $i=1,2$, $k=1,...,K_i$, is parametrized by $\mathbf{R}_i^k$; namely

$$\mathbf{R}_i^k(\theta,\xi,w)=\xi_i^k+\sqrt{\frac{\epsilon^2}{\pi}w_i^k+2\phi_i^{*,k}(\theta,\xi,w)}\,e^{\mathrm{i}\theta},\quad k=1,\cdots,K_i,\ i=1,2. \tag{7.6}$$

Here we write $\frac{\epsilon^2}{\pi}w_i^k$ in place of $(r_i^k)^2$ to emphasize the dependence on w. The unit tangent and normal vectors of $\mathbf{R}_i^k$ are

$$\mathbf{T}_i^k(\theta,\xi,w)=\frac{\frac{\partial\mathbf{R}_i^k(\theta,\xi,w)}{\partial\theta}}{\left|\frac{\partial\mathbf{R}_i^k(\theta,\xi,w)}{\partial\theta}\right|},\quad \mathbf{N}_i^k(\theta,\xi,w)=\mathrm{i}\mathbf{T}_i^k(\theta,\xi,w), \tag{7.7}$$

respectively. Recall that we identify vectors in plane with complex numbers; hence the use of complex multiplication in (7.7). Note that $\mathbf{N}_i^k(\theta,\xi,w)$ is inward pointing with respect to Ω_i^k.

Let one of $w_1^1,\cdots,w_1^{K_1}$, $w_2^1,\cdots,w_2^{K_2}$, say w_i^k, vary, and keep the others fixed; all of $\xi_1^1,\cdots,\xi_1^{K_1}$, $\xi_2^1,\cdots,\xi_2^{K_2}$ are also fixed. One treats w_i^k as a deformation parameter and $\mathbf{R}_j^l$, $j=1,2$ and $l=1,...,K_j$, as a deformation. Unlike the deformation (4.13), this deformation does not stay within the class of assemblies with the fixed centers and radii. Nevertheless, one can still obtain a first variation formula of this deformation:

$$\frac{\partial\mathcal{J}(\phi^*(\cdot,\xi,w))}{\partial w_i^k}=-\sum_{j=1}^2\sum_{l=1}^{K_j}\int_{\partial\Omega_j^l}\kappa_j\mathbf{N}_j^l\cdot\mathbf{X}_j^l\,ds-\sum_{j=1}^2\sum_{l=1}^{K_j}\int_{\partial\Omega_j^l}(\gamma_{j1}I_{\Omega_1}+\gamma_{j2}I_{\Omega_2})\mathbf{N}_j^l\cdot\mathbf{X}_j^l\,ds \tag{7.8}$$

see [30, Lemma 2.4] for a similar formula. Here $\mathbf{N}_j^l$ is the normal vector defined in (7.7) and $\mathbf{X}_j^l$ is the infinitesimal element of the deformation:

$$\mathbf{X}_j^l(\theta,\xi,w)=\frac{\partial\mathbf{R}_j^l(\theta,\xi,w)}{\partial w_i^k}. \tag{7.9}$$

Following the setup in Section 4, especially (4.27) and (4.28), one rewrites (7.8) as

$$\frac{\partial\mathcal{J}(\phi^*(\cdot,\xi,w))}{\partial w_i^k}=-\sum_{j=1}^2\sum_{l=1}^{K_j}\int_{\partial\Omega_j^l}(\mathcal{S}_j^l(\phi^*)+\lambda_j(\phi^*))\mathbf{N}_j^l\cdot\mathbf{X}_j^l ds. \tag{7.10}$$

Since $\Pi\mathcal{S}(\phi^*)=0$, there exist $A_j^l(\xi,w), B_j^l(\xi,w), C_j^l(\xi,w)\in\mathbb{R}$, $j=1,2$, $l=1,...,K_j$, such that

$$\mathcal{S}_j^l(\phi^*(\cdot,\xi,w)) = A_j^l(\xi,w) + B_j^l(\xi,w)\cos\theta + C_j^l(\xi,w)\sin\theta,\ \ j=1,2. \tag{7.11}$$

Also, because $\sum_{l=1}^{K_j}\int_0^{2\pi}\mathcal{S}_j^l(\phi^*)\,d\theta = 0$ by (4.29),

$$\sum_{l=1}^{K_j} A_j^l(\xi,w)=0,\ \ j=1,2. \tag{7.12}$$

Moreover,

$$\begin{aligned}
\mathbf{N}_j^l\cdot\mathbf{X}_j^l\frac{ds}{d\theta} &= \mathrm{i}\frac{\partial \mathbf{R}_j^l(\theta,\xi,w)}{\partial\theta}\cdot\mathbf{X}_j^l(\theta,\xi,w)\\
&= \left(\frac{\frac{\partial\phi_j^{*,l}}{\partial\theta}}{\sqrt{\frac{\epsilon^2}{\pi}w_j^l+2\phi_j^{*,l}}}e^{\mathrm{i}\theta}\mathrm{i} - \sqrt{\frac{\epsilon^2}{\pi}w_j^l+2\phi_j^{*,l}}\,e^{\mathrm{i}\theta}\right)\cdot\begin{cases}\dfrac{\frac{\epsilon^2}{2\pi}+\frac{\partial\phi_i^{*,k}}{\partial w_i^k}}{\sqrt{\frac{\epsilon^2}{\pi}w_i^k+2\phi_i^{*,k}}}e^{\mathrm{i}\theta} & \text{if } j=i \text{ and } l=k\\ \dfrac{\frac{\partial\phi_j^{*,l}}{\partial w_i^k}}{\sqrt{\frac{\epsilon^2}{\pi}w_j^l+2\phi_j^{*,l}}}e^{\mathrm{i}\theta} & \text{otherwise}\end{cases}\\
&= \begin{cases} -\frac{\epsilon^2}{2\pi}-\frac{\partial\phi_i^{*,k}}{\partial w_i^k} & \text{if } j=i \text{ and } l=k\\ -\frac{\partial\phi_j^{*,l}}{\partial w_i^k} & \text{otherwise}\end{cases}.
\end{aligned} \tag{7.13}$$

Using the facts that $\int_0^{2\pi}\phi_j^{*,l}d\theta=\int_0^{2\pi}\phi_j^{*,l}\cos\theta d\theta=\int_0^{2\pi}\phi_j^{*,l}\sin\theta d\theta=0$, one obtains

$$\int_0^{2\pi}\frac{\partial\phi_j^{*,l}}{\partial w_i^k}\,d\theta=\int_0^{2\pi}\frac{\partial\phi_j^{*,l}}{\partial w_i^k}\cos\theta\,d\theta=\int_0^{2\pi}\frac{\partial\phi_j^{*,l}}{\partial w_i^k}\sin\theta\,d\theta=0. \tag{7.14}$$

It follows from (7.10), (7.11), (7.13), and (7.14) that

$$\frac{\partial\mathcal{J}(\phi^*(\cdot,\xi,w))}{\partial w_i^k}=\epsilon^2(A_i^k(\xi,w)+\lambda_i). \tag{7.15}$$

At the critical point (ξ_c,w_c) of $\mathcal{J}$,

$$\left.\frac{\partial\mathcal{J}(\phi^*(\cdot,\xi,w))}{\partial w_i^k}\right|_{(\xi,w)=(\xi_c,w_c)}=\mu_i,\quad i=1,2,\ k=1,...,K_i. \tag{7.16}$$

Here μ_1 and μ_2 are the Lagrange multipliers associated with the constraints $\sum_{k=1}^{K_1}w_1^k=m$ and $\sum_{k=1}^{K_2}w_2^k=1-m$. Then (7.15) and (7.16) imply

$$A_i^k(\xi_c,w_c)=\epsilon^{-2}\mu_i-\lambda_i,\quad i=1,2,\ k=1,...,K_i. \tag{7.17}$$

This shows that $A_i^k(\xi_c,w_c)$ is independent of k. By (7.12) we conclude that

$$A_i^k(\xi_c,w_c)=0,\quad i=1,2,\ k=1,...,K_i. \tag{7.18}$$

Next let the first component of ξ_i^k, denoted $\xi_i^{k,1}$, vary and keep the other components of ξ and all components of w fixed. Again treat $\xi_i^{k,1}$ as a deformation parameter and $\mathbf{R}$ as a deformation. This deformation also goes beyond the class of assemblies with fixed centers and radii. Now let $\mathbf{X}$ be the infinitesimal element of this deformation:

$$\mathbf{X}_j^l(\theta,\xi,w)=\frac{\partial\mathbf{R}_j^l(\theta,\xi,w)}{\partial\xi_i^{k,1}}. \tag{7.19}$$

Again compute

$$
\begin{aligned}
\mathbf{N}_j^l \cdot \mathbf{X}_j^l \frac{ds}{d\theta} &= \mathrm{i}\frac{\partial \mathbf{R}_j^l(\theta,\xi,w)}{\partial\theta} \cdot \mathbf{X}_j^l(\theta,\xi,w) \\
&= \left(\frac{\frac{\partial \phi_j^{*,l}}{\partial\theta}}{\sqrt{\frac{\epsilon^2}{\pi} w_j^l + 2\phi_j^{*,l}}} e^{\mathrm{i}\theta}\mathrm{i} - \sqrt{\frac{\epsilon^2}{\pi} w_j^l + 2\phi_j^{*,l}} e^{\mathrm{i}\theta} \right) \cdot \begin{cases} 1 + \frac{\frac{\partial \phi_i^{*,k}}{\partial \xi_i^{k,1}}}{\sqrt{\frac{\epsilon^2}{\pi} w_i^k + 2\phi_i^{*,k}}} e^{\mathrm{i}\theta} & \text{if } j = i \text{ and } l = k \\ \frac{\frac{\partial \phi_j^{*,l}}{\partial \xi_i^{k,1}}}{\sqrt{\frac{\epsilon^2}{\pi} w_j^l + 2\phi_j^{*,l}}} e^{\mathrm{i}\theta} & \text{otherwise} \end{cases} \\
&= \begin{cases} -\frac{\frac{\partial \phi_i^{*,k}}{\partial\theta}}{\sqrt{\frac{\epsilon^2}{\pi} w_i^k + 2\phi_i^{*,k}}} \sin\theta - \sqrt{\frac{\epsilon^2}{\pi} w_i^k + 2\phi_i^{*,k}} \cos\theta - \frac{\partial \phi_i^{*,k}}{\partial \xi_i^{k,1}} & \text{if } j = i \text{ and } l = k \\ -\frac{\partial \phi_j^{*,l}}{\partial \xi_i^{k,1}} & \text{otherwise} \end{cases} .
\end{aligned} \tag{7.20}
$$

Similar to (7.14), one has

$$
\int_0^{2\pi} \frac{\partial \phi_j^{*,l}}{\partial \xi_i^{k,1}} \, d\theta = \int_0^{2\pi} \frac{\partial \phi_j^{*,l}}{\partial \xi_i^{k,1}} \cos\theta \, d\theta = \int_0^{2\pi} \frac{\partial \phi_j^{*,l}}{\partial \xi_i^{k,1}} \sin\theta \, d\theta = 0. \tag{7.21}
$$

Consequently

$$
\begin{aligned}
\frac{\partial \mathcal{J}(\phi^*(\cdot,\xi,w))}{\partial \xi_i^{k,1}} &= -\int_0^{2\pi} \left(A_i^k(\xi,w) + B_i^k(\xi,w)\cos\theta + C_i^k(\xi,w)\sin\theta + \lambda_i \right) \\
&\quad \left(-\frac{\frac{\partial \phi_i^{*,k}}{\partial\theta}}{\sqrt{\frac{\epsilon^2}{\pi} w_i^k + 2\phi_i^{*,k}}} \sin\theta - \sqrt{\frac{\epsilon^2}{\pi} w_i^k + 2\phi_i^{*,k}} \cos\theta \right) d\theta \\
&= (A_i^k(\xi,w) + \lambda_i) \int_0^{2\pi} \left(\frac{\frac{\partial \phi_i^{*,k}}{\partial\theta}}{\sqrt{\frac{\epsilon^2}{\pi} w_i^k + 2\phi_i^{*,k}}} \sin\theta + \sqrt{\frac{\epsilon^2}{\pi} w_i^k + 2\phi_i^{*,k}} \cos\theta \right) d\theta \\
&\quad + B_i^k(\xi,w) \int_0^{2\pi} \left(\frac{\frac{\partial \phi_i^{*,k}}{\partial\theta}}{\sqrt{\frac{\epsilon^2}{\pi} w_i^k + 2\phi_i^{*,k}}} \sin\theta + \sqrt{\frac{\epsilon^2}{\pi} w_i^k + 2\phi_i^{*,k}} \cos\theta \right) \cos\theta \; d\theta \\
&\quad + C_i^k(\xi,w) \int_0^{2\pi} \left(\frac{\frac{\partial \phi_i^{*,k}}{\partial\theta}}{\sqrt{\frac{\epsilon^2}{\pi} w_i^k + 2\phi_i^{*,k}}} \sin\theta + \sqrt{\frac{\epsilon^2}{\pi} w_i^k + 2\phi_i^{*,k}} \cos\theta \right) \sin\theta \; d\theta.
\end{aligned} \tag{7.22}
$$

As in [30, Lemma 2.4], the variation of the area of Ω_i^k under this deformation is given by

$$
\begin{aligned}
\frac{\partial}{\partial \xi_i^{k,1}} \int_{\Omega_i^k} dx &= -\int_{\partial\Omega_i^k} \mathbf{N}_i^k \cdot \mathbf{X}_i^k ds \\
&= \int_0^{2\pi} \left(\frac{\frac{\partial \phi_i^{*,k}}{\partial\theta}}{\sqrt{\frac{\epsilon^2}{\pi} w_i^k + 2\phi_i^{*,k}}} \sin\theta + \sqrt{\frac{\epsilon^2}{\pi} w_i^k + 2\phi_i^{*,k}} \cos\theta \right) d\theta.
\end{aligned} \tag{7.23}
$$

On the other hand, the area of Ω_i^k is fixed at $\epsilon^2 w_i^k$ under this deformation, so the integral in (7.23) vanishes, and (7.22) is simplified to

$$\begin{aligned}\frac{\partial \mathcal{J}(\phi^*(\cdot,\xi,w))}{\partial \xi_i^{k,1}} &= B_i^k(\xi,w)\int_0^{2\pi}\left(\frac{\frac{\partial \phi_i^{*,k}}{\partial\theta}}{\sqrt{\frac{\epsilon^2}{\pi}w_i^k+2\phi_i^{*,k}}}\sin\theta+\sqrt{\frac{\epsilon^2}{\pi}w_i^k+2\phi_i^{*,k}}\cos\theta\right)\cos\theta\,d\theta \\ &\quad +C_i^k(\xi,w)\int_0^{2\pi}\left(\frac{\frac{\partial \phi_i^{*,k}}{\partial\theta}}{\sqrt{\frac{\epsilon^2}{\pi}w_i^k+2\phi_i^{*,k}}}\sin\theta+\sqrt{\frac{\epsilon^2}{\pi}w_i^k+2\phi_i^{*,k}}\cos\theta\right)\sin\theta\,d\theta. \quad (7.24)\end{aligned}$$

Since $\|\phi^*\|_{\mathcal{X}} = O(|\gamma|\epsilon^7)$ according to Lemma 6.1, the two integrals in (7.24) are estimated as follows:

$$\int_0^{2\pi}\left(\frac{\frac{\partial \phi_i^{*,k}}{\partial\theta}}{\sqrt{\frac{\epsilon^2}{\pi}w_i^k+2\phi_i^{*,k}}}\sin\theta+\sqrt{\frac{\epsilon^2}{\pi}w_i^k+2\phi_i^{*,k}}\cos\theta\right)\cos\theta\,d\theta = \epsilon\sqrt{w_i^k\pi}+O(|\gamma|\epsilon^6), \quad (7.25)$$

$$\int_0^{2\pi}\left(\frac{\frac{\partial \phi_i^{*,k}}{\partial\theta}}{\sqrt{\frac{\epsilon^2}{\pi}w_i^k+2\phi_i^{*,k}}}\sin\theta+\sqrt{\frac{\epsilon^2}{\pi}w_i^k+2\phi_i^{*,k}}\cos\theta\right)\sin\theta\,d\theta = O(|\gamma|\epsilon^6). \quad (7.26)$$

At (ξ_c, w_c),

$$\left.\frac{\partial \mathcal{J}(\phi^*(\cdot,\xi,w))}{\partial \xi_i^{k,1}}\right|_{(\xi,w)=(\xi_c,w_c)} = 0. \quad (7.27)$$

Then (7.24), (7.25), (7.26), and (7.27) show that

$$B_i^k(\xi_c,w_c)\left(\epsilon\sqrt{w_i^k\pi}+O(|\gamma|\epsilon^6)\right)+C_i^k(\xi_c,w_c)\left(0+O(|\gamma|\epsilon^6)\right)=0. \quad (7.28)$$

Finally if $\xi_i^{k,2}$, the second component of ξ_i^k, is taken as a deformation parameter, then in an analogous way one deduces

$$B_i^k(\xi_c,w_c)\left(0+O(|\gamma|\epsilon^6)\right)+C_i^k(\xi_c,w_c)\left(\epsilon\sqrt{w_i^k\pi}+O(|\gamma|\epsilon^6)\right)=0. \quad (7.29)$$

The equations (7.28) and (7.29) form a linear homogeneous system for $B_i^k(\xi_c, w_c)$ and $C_i^k(\xi_c, w_c)$. This system is non-singular if ϵ is small. Hence

$$B_i^k(\xi_c,w_c) = C_i^k(\xi_c,w_c) = 0, \quad i=1,2, \ k=1,...,K_i. \quad (7.30)$$

Combining (7.30) with (7.18) one deduces that $\mathcal{S}(\phi^*(\cdot,\xi_c,w_c)) = 0$. □

Since $\overline{\Xi_\delta}\times\overline{W_h}$ is compact, there exists $(\xi^*, w^*) \in \overline{\Xi_\delta}\times\overline{W_h}$, that minimizes $(\xi, w) \to \mathcal{J}(\phi^*(\cdot,\xi,w))$. The next lemma asserts that (ξ^*, w^*) is attained in the interior of $\overline{\Xi_\delta}\times\overline{W_h}$, and hence is a critical point of $\mathcal{J}(\phi^*(\cdot,\xi,w))$.

Lemma 7.3. *Let $(\xi^*, w^*) \in \overline{\Xi_\delta}\times\overline{W_h}$ be a minimum of the function $(\xi, w) \to \mathcal{J}(\phi^*(\cdot,\xi,w))$. When h and ϵ are sufficiently small, (ξ^*, w^*) must be in $\Xi_\delta \times W_h$, the interior of $\overline{\Xi_\delta}\times\overline{W_h}$.*

Proof. Suppose $(\xi^*, w^*) \to (\xi^\circ, w^\circ)$ as $\epsilon \to 0$, possibly along a subsequence. It suffices to prove that (ξ°, w°) is in $\Xi_\delta \times W_h$.

First show that $w^\circ = \widetilde{w}$ where

$$\widetilde{w} = \left(\frac{m}{K_1},\cdots,\frac{m}{K_1},\frac{1-m}{K_2},\cdots,\frac{1-m}{K_2}\right). \quad (7.31)$$

Since $w_1^k = \epsilon^{-2}\pi(r_1^k)^2 = \frac{m}{K_1}\frac{(r_1^k)^2}{\rho_1^2}$ and $w_2^k = \epsilon^{-2}\pi(r_2^k)^2 = \frac{1-m}{K_2}\frac{(r_2^k)^2}{\rho_2^2}$, Lemma 7.1 implies that

$$\begin{aligned}
\mathcal{J}\left(\phi^*(\cdot,\xi,w)\right) &= \sum_{i=1}^{2}\sum_{k=1}^{K_i}\left(2\pi r_i^k - \frac{\gamma_{ii}\pi^2}{2}\frac{\left(r_i^k\right)^4\log r_i^k}{2\pi}\right) + O(|\gamma|\epsilon^4) \\
&= \frac{\gamma_{11}\pi}{4}\left(\frac{K_1}{m}\right)^2\rho_1^4\log\frac{1}{\rho_1}\sum_{k=1}^{K_1}\left(\frac{8}{\gamma_{11}\rho_1^3\log\frac{1}{\rho_1}}\left(\frac{K_1}{m}\right)^{-\frac{3}{2}}\sqrt{w_1^k} + \left(w_1^k\right)^2\right) \\
&\quad + \frac{\gamma_{22}\pi}{4}\left(\frac{K_2}{1-m}\right)^2\rho_2^4\log\frac{1}{\rho_2}\sum_{k=1}^{K_2}\left(\frac{8}{\gamma_{22}\rho_2^3\log\frac{1}{\rho_2}}\left(\frac{K_2}{1-m}\right)^{-\frac{3}{2}}\sqrt{w_2^k} + \left(w_2^k\right)^2\right) \\
&\quad + O(|\gamma|\epsilon^4),
\end{aligned} \tag{7.32}$$

Let

$$g_1(x) = \frac{8}{\gamma_{11}\rho_1^3\log\frac{1}{\rho_1}}\left(\frac{K_1}{m}\right)^{-\frac{3}{2}}\sqrt{x} + x^2 \quad \text{and} \quad g_2(x) = \frac{8}{\gamma_{22}\rho_2^3\log\frac{1}{\rho_2}}\left(\frac{K_2}{1-m}\right)^{-\frac{3}{2}}\sqrt{x} + x^2$$

be two functions. By condition 2 of Theorem 1.1,

$$\frac{8}{\gamma_{ii}\rho_i^3\log\frac{1}{\rho_i}} \leq \frac{8}{1+\eta}, \quad i = 1,2. \tag{7.33}$$

Then g_1 is convex on on $((\frac{1}{1+\eta})^{\frac{2}{3}}\frac{m}{K_1},\infty)$ and $g_2(x)$ is convex on $((\frac{1}{1+\eta})^{\frac{2}{3}}\frac{1-m}{K_2},\infty)$. Decrease h, last specified in Lemma 5.1, of W_h if necessarily so that when $w \in \overline{W_h}$,

$$w_1^k > \left(\frac{1}{1+\eta}\right)^{\frac{2}{3}}\frac{m}{K_1},\ k = 1,...,K_1, \quad w_2^k > \left(\frac{1}{1+\eta}\right)^{\frac{2}{3}}\frac{1-m}{K_2},\ k = 1,...,K_2. \tag{7.34}$$

Because $\sum_{k=1}^{K_1} w_1^k = m$ and $\sum_{k=1}^{K_2} w_2^k = 1-m$, by the convexity of g_1 and g_2,

$$\begin{aligned}
&\frac{\gamma_{11}\pi}{4}\left(\frac{K_1}{m}\right)^2\rho_1^4\log\frac{1}{\rho_1}\sum_{k=1}^{K_1}\left(\frac{8}{\gamma_{11}\rho_1^3\log\frac{1}{\rho_1}}\left(\frac{K_1}{m}\right)^{-\frac{3}{2}}\sqrt{w_1^k} + \left(w_1^k\right)^2\right) \\
&+\frac{\gamma_{22}\pi}{4}\left(\frac{K_2}{1-m}\right)^2\rho_2^4\log\frac{1}{\rho_2}\sum_{k=1}^{K_2}\left(\frac{8}{\gamma_{22}\rho_2^3\log\frac{1}{\rho_2}}\left(\frac{K_2}{1-m}\right)^{-\frac{3}{2}}\sqrt{w_2^k} + \left(w_2^k\right)^2\right)
\end{aligned} \tag{7.35}$$

is minimized at $w = \widetilde{w}$. If w° were not $\widetilde{w}$, then $\mathcal{J}(\phi^*(\cdot,\xi^*,w^*)) > \mathcal{J}(\phi^*(\cdot,\xi^*,\widetilde{w}))$ when ϵ is sufficiently small, a contradiction to the assumption that (ξ^*,w^*) is a minimum of $\mathcal{J}(\phi^*(\cdot,\xi,w))$.

Next we show that

$$F(\xi^\circ) = \min_{\xi\in\Xi} F(\xi). \tag{7.36}$$

Take $w = w^*$ and denote the corresponding r by r^*. By Lemma 7.1, we deduce that

$$\begin{aligned}
F_\epsilon(\xi) :&= \frac{2}{|\gamma|\epsilon^4}\left\{\mathcal{J}(\phi^*(\cdot,\xi,w^*)) - \sum_{i=1}^{2}\sum_{k=1}^{K_i}\left[2\pi r_i^{*,k} + \frac{\gamma_{ii}\pi^2}{2}\left(\frac{(r_i^{*,k})^4}{8\pi} - \frac{\left(r_i^{*,k}\right)^4\log r_i^{*,k}}{2\pi}\right)\right]\right\} \\
&= \frac{2}{|\gamma|}\left\{\frac{\gamma_{11}}{2}\left(\sum_{k=1}^{K_1}\left(w_1^{*,k}\right)^2 R\left(\xi_1^k,\xi_1^k\right) + \sum_{k=1}^{K_1}\sum_{l=1,l\neq k}^{K_1} w_1^{*,k}w_1^{*,l}G\left(\xi_1^k,\xi_1^l\right)\right)\right. \\
&\quad + \gamma_{12}\sum_{k=1}^{K_1}\sum_{l=1}^{K_2} w_1^{*,k}w_2^{*,l}G\left(\xi_1^k,\xi_2^l\right) \\
&\quad \left. + \frac{\gamma_{22}}{2}\left(\sum_{k=1}^{K_2}\left(w_2^{*,k}\right)^2 R\left(\xi_2^k,\xi_2^k\right) + \sum_{k=1}^{K_2}\sum_{l=1,l\neq k}^{K_2} w_2^{*,k}w_2^{*,l}G\left(\xi_2^k,\xi_2^l\right)\right)\right\} + O\left(\epsilon^2\right).
\end{aligned}$$

According to (1.15), (1.16) and the fact that $w^* \to \widetilde{w}$, one obtains that as $\epsilon \to 0$,

$$\begin{aligned} F_\epsilon(\xi) \quad \to \quad & \frac{\Gamma_{11} m^2}{K_1^2}\Big(\sum_{k=1}^{K_1} R(\xi_1^k,\xi_1^k) + \sum_{k=1}^{K_1}\sum_{l=1,l\neq k}^{K_1} G(\xi_1^k,\xi_1^l)\Big) \\ & + \frac{2\Gamma_{12} m(1-m)}{K_1K_2}\sum_{k=1}^{K_1}\sum_{l=1}^{K_2} G(\xi_1^k,\xi_2^l) \\ & + \frac{\Gamma_{22}(1-m)^2}{K_2^2}\Big(\sum_{k=1}^{K_2} R(\xi_2^k,\xi_2^k) + \sum_{k=1}^{K_2}\sum_{l=1,l\neq k}^{K_2} G(\xi_2^k,\xi_2^l)\Big) \\ & = F\left(\xi\right), \quad \text{uniformly with respect to } \xi \in \overline{\Xi_\delta}. \end{aligned}$$

If ξ° were not a minimum of F, then let $\tilde{\xi}$ be a minimum of F so that $F(\tilde{\xi}) < F(\xi^\circ)$. Then $F_\epsilon(\tilde{\xi}) < F_\epsilon(\xi^*)$ when ϵ is sufficiently small. Consequently, by the definition of F_ϵ, $\mathcal{J}(\phi^*(\cdot,\tilde{\xi},w^*)) < \mathcal{J}(\phi^*(\cdot,\xi^*,w^*))$ when ϵ is sufficiently small, a contradiction to the fact that (ξ^*,w^*) is a minimum of $\mathcal{J}(\phi^*(\cdot,\xi,w))$.

Since $w^\circ = \widetilde{w}$ is in W_h and any minimum of F is attained in Ξ_δ by (3.5), (ξ°,w°) is in $\Xi_\delta \times W_h$. $\square$

Note that the lower bound

$$\gamma_{ii} > \frac{1+\eta}{\rho_i^3 \log\frac{1}{\rho_i}} \tag{7.37}$$

in condition 2 of Theorem 1.1 is only used in (7.33). In the case $K_i = 1$ where $i=1$ or 2, there is only one w_i^k, which is w_i^1, and it is fixed at m if $i=1$ or $1-m$ if $i=2$. There is no need for minimization with respect to this w_i^k. Then one can relax the bound on γ_{ii} in this case to (1.17).

Proof of Theorem 1.1. By Lemma 7.3, the minimum (ξ^*,w^*) of the function $(\xi,w) \to \mathcal{J}(\phi^*(\cdot,\xi,w))$ is attained in the interior of $\overline{\Xi_\delta} \times \overline{W_h}$, so it is a critical point. Lemma 7.2 then asserts that $\phi^*(\cdot,\xi^*,w^*)$ is a stationary point of $\mathcal{J}$.

Our assertion that $\phi^*(\cdot,\xi^*,w^*)$ is a stable assembly is based on the fact that this stationary point is obtained in successive (local) minimization steps. In Section 6 for each (ξ,w) in $\overline{\Xi_\delta} \times \overline{W_h}$, $\phi^*(\cdot,\xi,w)$ was found as a fixed point. Because of Lemma 6.2.1, $\phi^*(\cdot,\xi,w)$ is locally minimizing in $\mathcal{X}_\flat$, that is in the class of assemblies whose discs are centered at ξ_i^k and of area $\epsilon^2 w_i^k$. Then in Lemma 7.3, (ξ^*,w^*) is taken to be the minimum in $\overline{\Xi_\delta} \times \overline{W_h}$. $\square$

Appendix: More minimization of F

We present more numerical minimization tests for F. Again D is the unit disc and three scenarios are considered: Q has two positive eigenvalues, Q has one positive and one zero eigenvalues, and Q has one positive and one negative eigenvalues. However Q_{11} is no longer equal to Q_{22} in this appendix. Type-I discs are plotted in blue and type-II discs in yellow.

For scenario one, we tested $Q_{11} = 0.5$, $Q_{22} = 1.5$, $Q_{12} = 0.2$, and various values for K_1 and K_2. Results are shown in Figure 6. Again, like in Figure 3, the two type discs are well mixed in an organized way; macroscopically the distinct constituents are distributed uniformly.

Regarding scenario two, we tested $Q_{11} = 0.5$, $Q_{22} = 4.5$, and $Q_{12} = 1.5$, so the two eigenvalues are 0 and 5; see Figure 7. As in Figure 4, the two type discs are not well mixed.

For scenario three, we tested $Q_{11} = 0.5$, $Q_{22} = 1.5$, and $Q_{12} = 1$; see Figure 8. These outcomes are similar to the ones in Figure 5. The main difference is that regions of type-I discs are more crowded than regions of type-II discs. For example when $K_1 = K_2 = 50$, the blue disc region is smaller than the yellow disc region. This phenomenon is attributed to the fact $Q_{11} < Q_{22}$.

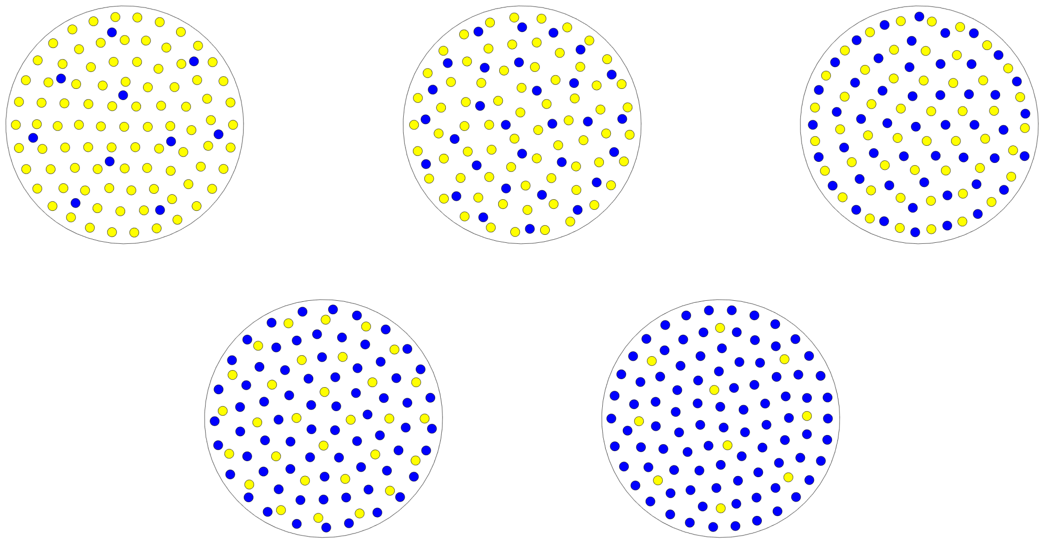

Figure 6: $Q_{11} = 0.5$, $Q_{22} = 1.5$ and $Q_{12} = 0.2$. First row from left to right: $K_1 = 10$ and $K_2 = 90$, $K_1 = 30$ and $K_2 = 70$, $K_1 = K_2 = 50$. Second row from left to right: $K_1 = 70$ and $K_2 = 30$, $K_1 = 90$ and $K_2 = 10$.

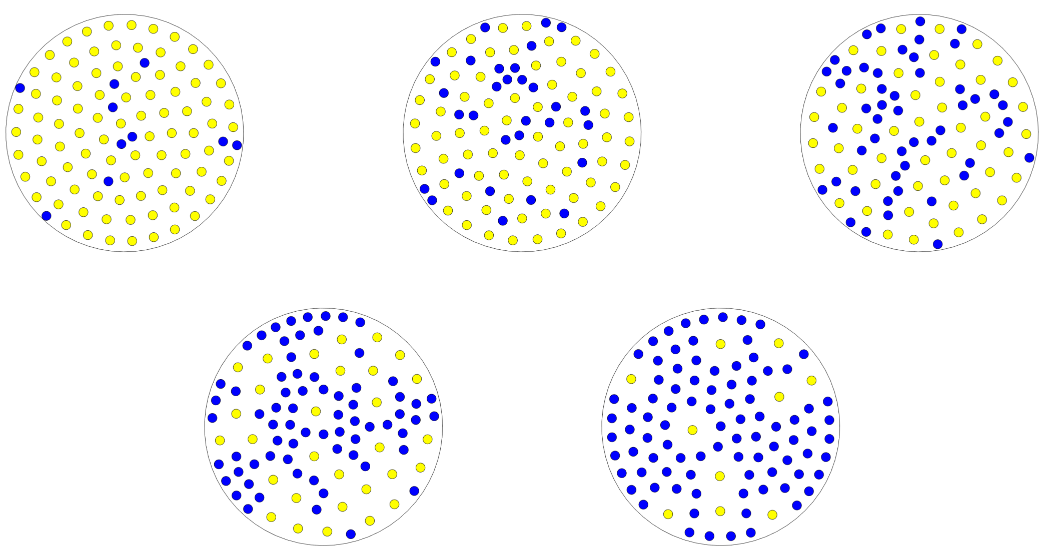

Figure 7: $Q_{11} = 0.5$, $Q_{22} = 4.5$ and $Q_{12} = 1.5$. First row from left to right: $K_1 = 10$ and $K_2 = 90$, $K_1 = 30$ and $K_2 = 70$, $K_1 = K_2 = 50$. Second row from left to right: $K_1 = 70$ and $K_2 = 30$, $K_1 = 90$ and $K_2 = 10$.

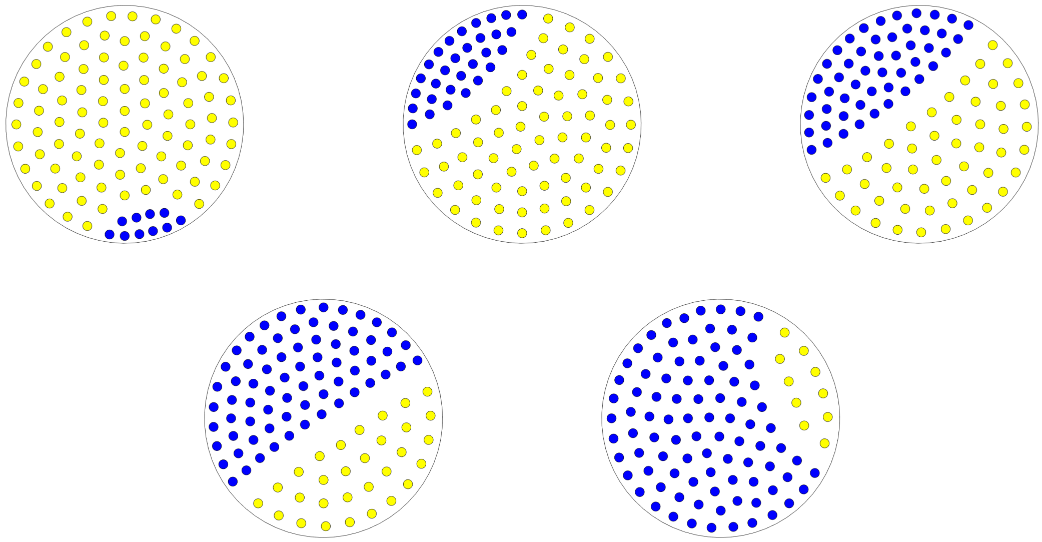

Figure 8: $Q_{11} = 0.5$, $Q_{22} = 1.5$ and $Q_{12} = 1$. First row from left to right: $K_1 = 10$ and $K_2 = 90$, $K_1 = 30$ and $K_2 = 70$, $K_1 = K_2 = 50$. Second row from left to right: $K_1 = 70$ and $K_2 = 30$, $K_1 = 90$ and $K_2 = 10$.

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