Physica D 393 (2019) 9–23

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

A study of pattern forming systems with a fully nonlocal interaction kernel of possibly changing sign☆

Xiaofeng Ren[a,*,1], Jeremy Trageser[b]

[a] Department of Mathematics, George Washington University, Washington, DC 20052, United States
[b] Computer Science and Mathematics Division, Oak Ridge National Laboratory, Oak Ridge, TN 37831, United States

HIGHLIGHTS

- Pattern formation is driven by sign changing interaction kernels.
- A continuous interaction kernel allows discontinuous solutions.
- Presolutions lead to solutions.
- The more important solutions are the critical solutions.
- The minimizing solutions are the stable and the most relevant critical solutions.

ARTICLE INFO

Article history:
Received 19 August 2017
Received in revised form 1 November 2018
Accepted 29 December 2018
Available online 8 January 2019
Communicated by Y. Nishiura

Keywords:
Sign changing kernel

ABSTRACT

A nonlocal model with a possibly sign changing kernel is proposed to study pattern formation problems in physical and biological systems with self-organization properties. One defines pre-solutions of a related linear problem and gives sufficient conditions for pre-solutions to be jump discontinuous solutions of the nonlocal model. Among a multitude of solutions a selection criterion is used to single out more significant critical solutions. These solutions are further classified into minimizing solutions, maximizing solutions and saddle solutions. Minimizing solutions are the most relevant for patterned states and they exist if the kernel changes sign. A non-negative kernel on the other hand behaves in some ways like the gradient term in the standard local model.

1. Introduction

Patterned states arise in many physical and biological systems as orderly outcomes of self-organization principles. Examples include morphological phases in block copolymers, animal coats, and skin pigmentation. Common among these pattern-forming systems is that a deviation from homogeneity has a strong positive feedback on its further increase. On its own, this would lead to an unlimited increase and spreading. Consequently, pattern formation additionally requires a longer ranging confinement of the locally self-enhancing process.

In this paper we use a phase field model with a fully nonlocal self-interaction energy. For a phase field $u(x)$, $x \in D \subset \mathbb{R}^n$, the free energy of u is given by

$$I(u) = \frac{1}{4}\int_D\int_D J(x,y)(u(x)-u(y))^2\,dxdy + \int_D F(u)\,dx. \tag{1.1}$$

The first term in (1.1) is the self-interaction energy of the field u and the second term is the bulk energy of u. The function F is typically a double well potential, like $F(u) = (1/4)u^2(1-u)^2$ or a linear perturbation of this. The Euler–Lagrange equation of (1.1) is the integral equation

$$-J[u] + j(x)u + f(u) = 0 \ \text{ in } D \tag{1.2}$$

where

$$J[u](x) = \int_D J(x,y)u(y)\,dy, \quad j(x) = \int_D J(x,y)\,dy, \tag{1.3}$$

and f is the derivative of F. In (1.2) $J[\cdot]$ is an integral operator which makes the equation nonlocal. The functional (1.1) is defined in an

---

☆ Notice: This manuscript has been authored by UT-Battelle, LLC, under contract DE-AC05-00OR22725 with the US Department of Energy (DOE). The US government retains and the publisher, by accepting the article for publication, acknowledges that the US government retains a nonexclusive, paid-up, irrevocable, worldwide license to publish or reproduce the published form of this manuscript, or allow others to do so, for US government purposes. DOE will provide public access to these results of federally sponsored research in accordance with the DOE Public Access Plan (http://energy.gov/downloads/doe-public-access-plan).

* Corresponding author.
E-mail address: ren@gwu.edu (X. Ren).

1 Supported in part by National Science Foundation of United States of America (NSF) grants DMS-1311856 and DMS-1714371.

https://doi.org/10.1016/j.physd.2018.12.008
0167-2789/

L^p space, $1 \le p \le \infty$, and (1.2) holds almost everywhere with respect to the Lebesgue measure.

This problem was first studied by Bates, Fife, Ren, and Wang in [1] where the domain D is taken to be $\mathbb{R}$. Subsequent works on this problem include [2–6]. In all these papers the kernel J is assumed to be non-negative. A non-negative kernel penalizes spacial deviation of the phase field. This effect is analogous to the gradient term in the standard Allen–Cahn problem [7]:

$$I_{AC}(u) = \int_D \Big(\frac{1}{2}|\nabla u|^2 + F(u)\Big)\, dx. \tag{1.4}$$

This is a local problem whose Euler–Lagrange equation is the partial differential equation

$$-\Delta u + f(u) = 0,\ x \in D; \quad \frac{\partial u}{\partial \nu} = 0,\ x \in \partial D. \tag{1.5}$$

Like the gradient term in (1.4) a non-negative J in (1.1) captures the growth property in a self-organized system but misses the inhibition property. We argue that an inhibition property can be recovered if J changes sign.

In the Discussion section we demonstrate this viewpoint by showing that the FitzHugh–Nagumo system, an archetypical inhibitory system, leads to a functional like (1.1). The interaction kernel $J(x, y)$ should be positive if x is close to y so that short distance self-interaction is repulsive, but becomes negative if x is far from y so that long distance self-interaction can be attractive. We also explain a connection between our problem and the fractional Laplace operator that has been studied intensively in recent years.

While the paper [1] was devoted to traveling wave solutions of the gradient flow of (1.1), they also found critical points of (1.1) that are discontinuous. Whether or not (1.2) has discontinuous solutions depends on the nonlinearity f and the singularity of $J(x, y)$ when x and y are close. If the singularity is mild, then $J[u](x)$ is a continuous function of x. Then by (1.2), $j(x)u(x)+f(u(x))$ should also be continuous with respect to x. However if $j(x)u + f(u)$ is not monotone with respect to u, then $u(x)$ does not need to be continuous in order for $j(x)u(x) + f(u(x))$ to be continuous with respect to x.

Admitting discontinuous solutions makes the problem (1.1) very appealing in applications. Similar ideas have appeared in peridynamics which uses integral equations instead of partial differential equations to study damage in continuum mechanics [8–10]. This paper is concerned with solutions of (1.2) that are continuous except for a finite number of jump discontinuous points; see Section 2 for the precise definition of such solutions.

As an early work on discontinuous solutions of (1.2), we do not seek a theory for a general kernel. Instead we choose some special J's that allow us to carry out a thorough analysis of the problem. We solve this problem in one dimension, i.e. $D = (0, 1)$ and uses a piecewise linear function f as the nonlinearity.

Between Sections 2 and 5, J is the Green's function of the differential operator

$$-A\frac{d^2}{dx^2} + B,\ A > 0,\ B \in \mathbb{R} \tag{1.6}$$

with the zero Neumann boundary condition; namely for every $y \in (0, 1)$

$$\begin{aligned} &-A\frac{\partial^2 J(x,y)}{\partial x^2} + BJ(x,y) = \delta(x-y), \\ &\frac{\partial J(0,y)}{\partial x} = \frac{\partial J(1,y)}{\partial x} = 0. \end{aligned} \tag{1.7}$$

Here A and B must satisfy

$$\frac{B}{A} \ne -k^2\pi^2,\ k = 0, 1, 2, \ldots \tag{1.8}$$

More explicitly

$$J(x,y) = \begin{cases} \dfrac{\cosh\Big(\sqrt{\frac{B}{A}}(1-y)\Big)\cosh\Big(\sqrt{\frac{B}{A}}x\Big)}{\sqrt{AB}\sinh\sqrt{\frac{B}{A}}}, & x < y \\ \dfrac{\cosh\Big(\sqrt{\frac{B}{A}}(1-x)\Big)\cosh\Big(\sqrt{\frac{B}{A}}y\Big)}{\sqrt{AB}\sinh\sqrt{\frac{B}{A}}}, & x \ge y \end{cases}. \tag{1.9}$$

Note that B can be positive or negative. If B is negative then $\sqrt{\frac{B}{A}}$ is imaginary and (1.9) can be alternatively written as

$$J(x,y) = \begin{cases} -\dfrac{\cos\Big(\sqrt{|\frac{B}{A}|}(1-y)\Big)\cos\Big(\sqrt{|\frac{B}{A}|}x\Big)}{\sqrt{|AB|}\sin\sqrt{|\frac{B}{A}|}}, & x < y \\ -\dfrac{\cos\Big(\sqrt{|\frac{B}{A}|}(1-x)\Big)\cos\Big(\sqrt{|\frac{B}{A}|}y\Big)}{\sqrt{|AB|}\sin\sqrt{|\frac{B}{A}|}}, & x \ge y \end{cases}. \tag{1.10}$$

Fig. 1 shows three cases of J, one with $B > 0$ and two with $B < 0$. Integrating J we find

$$j(x) = B^{-1},\ x \in (0, 1). \tag{1.11}$$

The first notable property of this kernel is the mild singularity at $x = y$; J is continuous on $[0, 1] \times [0, 1]$ and differentiable except when $x = y$. Another property of J is its sign. When $B > 0$, J is everywhere positive; when $B < 0$, J may change sign. Fig. 1 shows graphs of J for several values of B.

In Section 6 we study another kernel:

$$J(x,y) = \begin{cases} \dfrac{3y^2 + 3x^2 - 6y + 2}{6A}, & x < y \\ \dfrac{3y^2 + 3x^2 - 6x + 2}{6A}, & x \ge y \end{cases}. \tag{1.12}$$

This kernel is the Green's function of the operator (1.6) when $B = 0$. Since (1.8) does not hold, this Green's function is the solution of a different equation:

$$\begin{aligned} &-A\frac{\partial^2 J(x,y)}{\partial x^2} = \delta(x-y) - 1, \quad \frac{\partial J(0,y)}{\partial x} = \frac{\partial J(1,y)}{\partial x} = 0, \\ &\int_0^1 J(x,y)\, dx = 0. \end{aligned} \tag{1.13}$$

The graph of this J is plotted in Fig. 1 as well. It has a simple shape and changes sign.

The starting point of our work is the notion of pre-solutions. A pre-solution satisfies an inhomogeneous linear differential equation; see Section 2. Every jump discontinuous solution of (1.2) is a pre-solution. However a pre-solution is not necessarily a jump discontinuous solution. In Section 3 we give sufficient conditions that yield jump discontinuous solutions from pre-solutions; see Theorems 3.1 and 3.2.

At this point a peculiar phenomenon arises. Often times there is a multitude of solutions to (1.2). One can almost arbitrarily prescribe the location of discontinuity points and construct solutions. A selection principle is needed so we can weed out unimportant solutions. This is accomplished in Section 4 where we introduce the notion of critical solutions. The idea is to use the points of discontinuity of a solution, say $\xi_1, \xi_2, \ldots, \xi_N$, as parameters and view the energy I as a function of these parameters. Namely $I = I(u(\cdot, \xi))$ where $\xi = (\xi_1, \xi_2, \ldots, \xi_N)$ and $u(\cdot, \xi)$ is the jump discontinuous solution whose discontinuous points are $\xi_1, \xi_2, \ldots, \xi_N$. A critical solution is characterized by parameters that yield a critical point of the energy. It turns out that the number of critical solutions is much smaller. It is proved in Theorem 4.4, that if $B > 0$ for

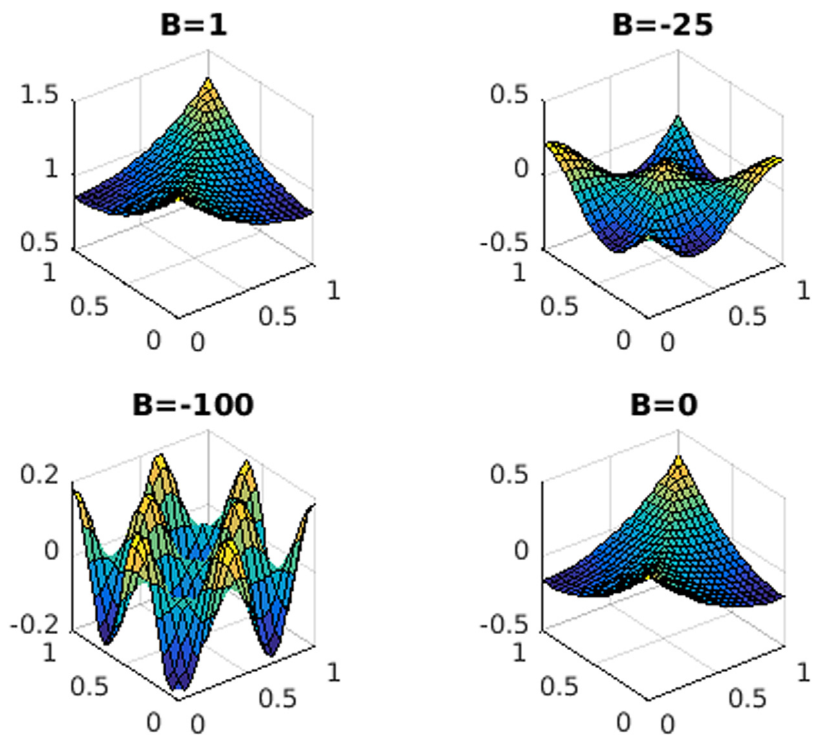

Fig. 1. Graphs of J for different values of B. $A = 1$ in all cases.

each $N \in \mathbb{N}$ there is only one critical solution with N jump discontinuity points, which jumps 'up' at the first discontinuous point. The discontinuous points $\xi_1, \xi_2, \ldots, \xi_N$ of this critical solution are equidistant:

$$2\xi_1 = \xi_2 - \xi_1 = \xi_3 - \xi_2 = \cdots = \xi_N - \xi_{N-1} = 2(1 - \xi_N). \quad (1.14)$$

There are no critical solutions if $-1 < B < 0$. And if $B < -1$ there may be other critical solutions in addition to the equidistant ones.

In Section 5 we proceed further to classify the critical solutions. We are particularly interested in the ones that locally minimize $I(u(\cdot, \xi))$, because these are stable and robust under perturbation. Theorem 5.3 shows that when $B > 0$, all critical solutions (which are precisely the equidistant ones) are locally maximizing; when $B < -1$ a criterion is found that determines the type of each critical solution. Minimizing solutions do exist in this case. It is known that the local problem (1.5) does not admit any stable non-constant solution [11]. In this aspect (1.1) with a non-negative kernel behaves like a local model.

Our journey from pre-solutions to jump discontinuous solutions, then to critical solutions, and finally to minimizing solutions is repeated in Section 6 for the kernel of (1.12). This J, because it changes sign, shares a number of properties with the kernel of (1.10) where $B < -1$. There are equidistant and non-equidistant critical solutions, some of which are minimizing.

2. Pre-solutions

We take f to be a piecewise linear function

$$f(u) = \begin{cases} u + 1 & \text{if } u < 0 \\ 0 & \text{if } u = 0 \\ u - 1 & \text{if } u > 0 \end{cases} \quad (2.1)$$

or

$$f(u) = u - \mathrm{sgn}(u) \quad (2.2)$$

where sgn is the sign function: $\mathrm{sgn}(u) = -1$ if $u < 0$, $\mathrm{sgn}(u) = 1$ if $u > 0$ and $\mathrm{sgn}(u) = 0$ if $u = 0$. Note that for this f,

$$F(u) = \frac{(u - \mathrm{sgn}(u))^2}{2} \quad (2.3)$$

Consequently, we transform (1.2) into

$$-J[u] + B^{-1}u + u - \mathrm{sgn}(u) = 0 \quad (2.4)$$

Define

$$v(x) = \int_0^1 J(x, y)u(y)dy, \quad (2.5)$$

which satisfies

$$-Av'' + Bv = u, \quad v'(0) = v'(1) = 0. \quad (2.6)$$

Then (2.4) can also be written as a system:

$$\begin{cases} -v + B^{-1}u + u - \mathrm{sgn}(u) = 0 \\ -Av'' + Bv = u \\ v'(0) = v'(1) = 0 \end{cases} \quad (2.7)$$

A solution u of (2.4) is termed jump discontinuous if there exist $\xi_1, \xi_2, \ldots, \xi_N$, $0 < \xi_1 < \xi_2 < \cdots < \xi_N < 1$, such that

1. u is continuous on each (ξ_{k-1}, ξ_k), $k = 1, 2, \ldots, N+1$, where $\xi_0 = 0$ and $\xi_{N+1} = 1$,
2. u on each (ξ_{k-1}, ξ_k) has a continuous extension to $[\xi_{k-1}, \xi_k]$,
3. $u(x) < 0$ if $x \in [\xi_{k-1}, \xi_k]$ and k is odd, and $u(x) > 0$ if $x \in [\xi_{k-1}, \xi_k]$ and k is even, where $u(\xi_{k-1}) = \lim_{x \to \xi_{k-1}^+} u(x)$ and $u(\xi_k) = \lim_{x \to \xi_k^-} u(x)$.

In this paper, we are only interested in jump discontinuous solutions of (2.4). Note that $u(x) < 0$ on $(0, \xi_1)$ and $u(x) > 0$ on (ξ_1, ξ_2), so u jumps 'up' at the first discontinuous point ξ_1. One can certainly consider jump discontinuous solutions that jump 'down' at ξ_1. We do not consider these 'jump down' solutions in this paper. They may be treated in an equivalent fashion to the jump 'up' ones considered here.

Having a piecewise linear function f in our problem is not the primary reason for the existence of jump discontinuous solutions. If there is a jump discontinuous solution of (2.4), one can always modify the function $f(u)$ for u near 0 to turn it into a smooth function. As long as the discontinuous solution jumps past the region where the modification to f is made, the discontinuous solution remains to be a solution to the new problem with the modified, smooth nonlinearity.

If $B = -1$, then there is no jump discontinuous solution. This is because (2.4) implies $v = -\operatorname{sgn}(u)$ but v is continuous and $\operatorname{sgn}(u)$ has discontinuities at ξ_k if u is jump discontinuous. Thus we assume

$$B \neq -1 \tag{2.8}$$

throughout this paper.

We define the notion of pre-solutions at this point. Let $0 = \xi_1 < \xi_2 < \cdots < \xi_N < \xi_{N+1} = 1$ be a partition of $[0, 1]$ and $\xi = (\xi_1, \xi_2, \dots, \xi_N)$. A pre-solution u of the partition ξ is a solution of

$$-J[u] + B^{-1}u + u - \sum_{k=1}^{N+1} (-1)^k \chi_{(\xi_{k-1},\xi_k)} = 0 \tag{2.9}$$

Alternatively, if (2.8) holds, with $v = J[u]$, (2.9) implies

$$u = \frac{v + \sum_{k=1}^{N+1} (-1)^k \chi_{(\xi_{k-1},\xi_k)}}{B^{-1} + 1} \tag{2.10}$$

Then $-Av'' + Bv = u$ yields

$$-Av'' + \frac{B^2}{B+1} v = \frac{\sum_{k=1}^{N+1} (-1)^k \chi_{(\xi_{k-1},\xi_k)}}{B^{-1} + 1}, \quad v'(0) = v'(1) = 0. \tag{2.11}$$

Hence when (2.8) and (1.8) hold, we may equivalently call a solution v of (2.11) a pre-solution of ξ.

For the differential operator

$$-A\frac{d^2}{dx^2} + \frac{B^2}{B+1}, \tag{2.12}$$

with the zero Neumann boundary condition in (2.11), we impose the assumption

$$\frac{B^2}{A(B+1)} \neq -k^2\pi^2, \ k = 0, 1, 2, \dots \tag{2.13}$$

so that it is invertible. We set

$$\mu = \sqrt{\frac{B^2}{A(B+1)}} \tag{2.14}$$

with the convention that μ is positive when $\frac{B^2}{A(B+1)} > 0$ and μ is imaginary with a positive imaginary part if $\frac{B^2}{A(B+1)} < 0$. The Green's function of this operator, with the Neumann boundary condition, is

$$G(x, y) = \begin{cases} \dfrac{\cosh(\mu(1-y))\cosh(\mu x)}{A\mu \sinh \mu}, & x < y \\ \dfrac{\cosh(\mu(1-x))\cosh(\mu y)}{A\mu \sinh \mu}, & x \geq y \end{cases}. \tag{2.15}$$

When $B < -1$, μ is imaginary and the above can be written alternatively as

$$G(x, y) = \begin{cases} -\dfrac{\cos(|\mu|(1-y))\cos(|\mu| x)}{A|\mu| \sin |\mu|}, & x < y \\ -\dfrac{\cos(|\mu|(1-x))\cos(|\mu| y)}{A|\mu| \sin |\mu|}, & x \geq y \end{cases}. \tag{2.16}$$

If we have a jump discontinuous solution u to (2.4) with discontinuities at $\xi_1, \xi_2, \dots, \xi_N$, then u is also a pre-solution with the partition $\xi = (\xi_1, \xi_2, \dots, \xi_N)$. However a pre-solution is not necessarily a jump discontinuous solution. For a pre-solution u to be a jump discontinuous solution, it must satisfy the condition

$$\begin{cases} u(x) < 0 & \text{if } x \in [\xi_{k-1}, \xi_k], \ k = 1, 3, 5, \dots \\ u(x) > 0 & \text{if } x \in [\xi_{k-1}, \xi_k], \ k = 2, 4, 6, \dots \end{cases} \tag{2.17}$$

By (2.10) a pre-solution is continuous on each (ξ_{k-1}, ξ_k) and admits a continuous extension to $[\xi_{k-1}, \xi_k]$. In (2.17), we take $u(\xi_{k-1}) = \lim_{x\to\xi_{k-1}^+} u(x)$ and $u(\xi_k) = \lim_{x\to\xi_k^-} u(x)$ when considering $x \in [\xi_{k-1}, \xi_k]$.

Lemma 2.1. *Suppose that ξ is a partition and (1.8), (2.8), and (2.13) hold. Then the pre-solution of ξ is given by*

$$v(x) = \frac{2}{B\sinh\mu}\Big(\cosh(\mu(1-x))\sum_{j=1}^{k-1}(-1)^j \sinh(\mu\xi_j) - \cosh(\mu x)\sum_{j=k}^{N}(-1)^j\sinh(\mu(1-\xi_j))\Big) + \frac{(-1)^k}{B}$$

for $x \in [\xi_{k-1}, \xi_k]$.

Proof. Since

$$v(x) = \int_0^1 G(x, y) \frac{\sum_{j=1}^{N+1} (-1)^j \chi_{(\xi_{j-1},\xi_j)}(y)}{B^{-1}+1}\, dy,$$

we deduce that, if $x \in (\xi_{k-1}, \xi_k)$,

$$\begin{aligned} v(x) = & \sum_{j=1}^{k-1} \int_{\xi_{j-1}}^{\xi_j} \frac{\cosh(\mu(1-x))\cosh(\mu y)}{A\mu\sinh\mu} \frac{(-1)^j}{B^{-1}+1}\, dy \\ & + \int_{\xi_{k-1}}^{x} \frac{\cosh(\mu(1-x))\cosh(\mu y)}{A\mu\sinh\mu} \frac{(-1)^k}{B^{-1}+1}\, dy \\ & + \int_{x}^{\xi_k} \frac{\cosh(\mu(1-y))\cosh(\mu x)}{A\mu\sinh\mu} \frac{(-1)^k}{B^{-1}+1}\, dy \\ & + \sum_{j=k+1}^{N+1} \int_{\xi_{j-1}}^{\xi_j} \frac{\cosh(\mu(1-y))\cosh(\mu x)}{A\mu\sinh\mu} \frac{(-1)^j}{B^{-1}+1}\, dy \\ = & \sum_{j=1}^{k-1} \frac{\cosh(\mu(1-x))(\sinh(\mu\xi_j) - \sinh(\mu\xi_{j-1}))(-1)^j}{A(B^{-1}+1)\mu^2\sinh\mu} \\ & + \frac{\cosh(\mu(1-x))(\sinh(\mu x) - \sinh(\mu\xi_{k-1}))(-1)^k}{A(B^{-1}+1)\mu^2\sinh\mu} \\ & + \frac{\cosh(\mu x)(-\sinh(\mu(1-\xi_k)) + \sinh(\mu(1-x)))(-1)^k}{A(B^{-1}+1)\mu^2\sinh\mu} \\ & + \sum_{j=k+1}^{N+1} \frac{\cosh(\mu x)(-\sinh(\mu(1-\xi_j)) + \sinh(\mu(1-\xi_{j-1})))(-1)^j}{A(B^{-1}+1)\mu^2\sinh\mu} \\ = & \sum_{j=1}^{k-1} \frac{2\cosh(\mu(1-x))\sinh(\mu\xi_j)(-1)^j}{B\sinh\mu} \\ & + \sum_{j=k+1}^{N+1} \frac{2\cosh(\mu x)\sinh(\mu(1-\xi_{j-1}))(-1)^j}{B\sinh\mu} \\ & + \frac{\cosh(\mu(1-x))\sinh(\mu x)(-1)^k}{B\sinh\mu} \\ & + \frac{\cosh(\mu x)\sinh(\mu(1-x))(-1)^k}{B\sinh\mu} \end{aligned}$$

$$= \sum_{j=1}^{k-1} \frac{2\cosh(\mu(1-x))\sinh(\mu\xi_j)(-1)^j}{B\sinh\mu} - \sum_{j=k}^{N} \frac{2\cosh(\mu x)\sinh(\mu(1-\xi_j))(-1)^j}{B\sinh\mu} + \frac{(-1)^k}{B}$$

proving the lemma. □

We call either v given in Lemma 2.1 or the corresponding u from (2.10) a pre-solution. The pre-solution u, or v, of a partition ξ is often written as $u(x,\xi)$, or $v(x,\xi)$, when its dependence on ξ needs to be emphasized.

Partitions of special interest are the equidistant ones, where $\xi_k = \frac{2k-1}{2N}$, $k = 1, 2, \dots, N$. The following follows from Lemma 2.1.

Corollary 2.2. *Suppose that ξ is an equidistant partition, $\xi_k = \frac{2k-1}{2N}$, $k = 1, 2, \dots, N$, and (1.8), (2.8), and (2.13) hold. Then the pre-solution is*

$$v(x) = \frac{(-1)^{k+1}}{B\cosh\left(\frac{\mu}{2N}\right)}\cosh\left(\mu\left(x - \frac{k-1}{N}\right)\right) + \frac{(-1)^k}{B},$$
$$\text{if } x \in [\xi_{k-1}, \xi_k]. \tag{2.18}$$

In particular,

$$v\left(\frac{2k-1}{2N}\right) = 0, \quad k = 1, \dots, N, \tag{2.19}$$

and the following symmetry properties hold: for all $x \in \left[0, \frac{1}{2N}\right]$

$$\begin{aligned} v(x) &= -v\left(\frac{1}{N} - x\right) = -v\left(\frac{1}{N} + x\right) \\ &= v\left(\frac{2}{N} - x\right) = v\left(\frac{2}{N} + x\right) \\ &= \cdots \\ &= (-1)^{N-1} v\left(\frac{N-1}{N} - x\right) = (-1)^{N-1} v\left(\frac{N-1}{N} + x\right) \\ &= (-1)^N v(1-x) \end{aligned} \tag{2.20}$$

3. Jump discontinuous solutions

Here we give sufficient conditions that yield jump discontinuous solutions from pre-solutions.

Theorem 3.1. *Let (1.8) and (2.13) hold, ξ be a partition, and $u(\cdot,\xi)$ be the pre-solution of ξ.*

1. *If $B > 1$, then u is a jump discontinuous solution of (2.4).*
2. *If $B < -\frac{2N}{|\sin(|\mu|)|} - 1$, then u is a jump discontinuous solution of (2.4).*

Proof. Let $B > 1$. First consider the interval $[\xi_{k-1}, \xi_k]$ with an odd k. We show that $u(x) < 0$ if $x \in [\xi_{k-1}, \xi_k]$ where the value of u at ξ_{k-1} and ξ_k are taken to be the continuous extension of $u(x)$ for $x \in (\xi_{k-1}, \xi_k)$ in this proof. Note that, since

$$\sum_{j=1}^{k-1} (-1)^j \sinh(\mu\xi_j) < \sinh(\mu\xi_{k-1}),$$

$$-\sum_{j=k}^{N} (-1)^j \sinh(\mu(1-\xi_j)) < \sinh(\mu(1-\xi_k)),$$

by (2.10) and Lemma 2.1

$$\begin{aligned} (B^{-1}+1)u(x) &= v(x) - 1 \\ &< \frac{2}{B\sinh\mu}\Big(\cosh(\mu(1-x))\sinh(\mu\xi_{k-1}) \\ &\quad + \cosh(\mu x)\sinh(\mu(1-\xi_k))\Big) - \frac{1}{B} - 1 \end{aligned}$$

By the convexity of $\cosh(\mu(1-x))\sinh(\mu\xi_{k-1}) + \cosh(\mu x)\sinh(\mu(1-\xi_k))$ with respect to x, the maximum value of this quantity is attained either at $x = \xi_{k-1}$ or $x = \xi_k$. In the first case

$$\begin{aligned} &\cosh(\mu(1-\xi_{k-1}))\sinh(\mu\xi_{k-1}) + \cosh(\mu\xi_{k-1})\sinh(\mu(1-\xi_k)) \\ &\quad < \cosh(\mu(1-\xi_{k-1}))\sinh(\mu\xi_{k-1}) \\ &\qquad + \cosh(\mu\xi_{k-1})\sinh(\mu(1-\xi_{k-1})) \\ &\quad = \sinh\mu; \end{aligned}$$

in the second case

$$\begin{aligned} &\cosh(\mu(1-\xi_k))\sinh(\mu\xi_{k-1}) + \cosh(\mu\xi_k)\sinh(\mu(1-\xi_k)) \\ &\quad < \cosh(\mu(1-\xi_k))\sinh(\mu\xi_k) + \cosh(\mu\xi_k)\sinh(\mu(1-\xi_k)) \\ &\quad = \sinh\mu \end{aligned}$$

as well. Therefore, for $x \in [\xi_{k-1}, \xi_k]$ and k is odd,

$$u(x) < \frac{1}{B^{-1}+1}\left(\frac{2}{B\sinh\mu}\sinh\mu - \frac{1}{B} - 1\right) = \frac{1}{B^{-1}+1}\left(\frac{1}{B} - 1\right) < 0,$$

since $B > 1$.

Similarly one can show that, for $x \in [\xi_{k-1}, \xi_k]$ and k is even, $u(x) > 0$. This proves part 1.

Now let $B < -\frac{2N}{|\sin|\mu||} - 1$. Since μ is imaginary, we use $\sinh(\mathrm{i}\theta) = \mathrm{i}\sin\theta$ and $\cosh(\mathrm{i}\theta) = \cos\theta$ and Lemma 2.1 to derive that, for $x \in [\xi_{k-1}, \xi_k]$,

$$\begin{aligned} (B^{-1}+1)u(x) = \frac{2}{B\sin(|\mu|)}&\Bigg(\cos(|\mu|(1-x))\sum_{j=1}^{k-1}(-1)^j\sin(|\mu|\xi_j) \\ &+ -\cos(|\mu|x)\sum_{j=k}^{N}(-1)^j\sin(|\mu|(1-\xi_j))\Bigg) \\ &+ \frac{(-1)^k}{B} + (-1)^k \end{aligned}$$

Suppose that k is odd. Then, for $x \in [\xi_{k-1}, \xi_k]$,

$$\begin{aligned} (B^{-1}+1)u(x) &\le \frac{2}{|B\sin(|\mu|)|}\left(\sum_{j=1}^{k-1} 1 + \sum_{j=k}^{N} 1\right) - \frac{1}{B} - 1 \\ &= -\frac{2N}{B|\sin(|\mu|)|} - \frac{1}{B} - 1 \\ &< 0 \end{aligned}$$

since $B < -\frac{2N}{|\sin(|\mu|)|} - 1$. Consequently $u < 0$ on $[\xi_{k-1}, \xi_k]$ when k is odd, since $B^{-1} + 1 > 0$ here. Similarly, when k is even, $u > 0$ on $[\xi_{k-1}, \xi_k]$. Part 2 is proved. □

The bounds in Theorem 3.1 are as good as one can get for arbitrary partitions. For equidistant partitions the next theorem gives better sufficient conditions.

Theorem 3.2. *Let $\xi_k = \frac{(2k-1)}{2N}$, $k = 1, 2, \dots, N$, be an equidistant partition and (1.8) and (2.13) hold.*

1. *If $B > 0$, then the pre-solution of $\left\{\frac{(2k-1)}{2N}\right\}_{k=1}^N$ is a jump discontinuous solution of (2.4).*
2. *If $B \in (0, -1)$, then the pre-solution of $\left\{\frac{(2k-1)}{2N}\right\}_{k=1}^N$ is not a solution of (2.4).*
3. *If $B < -1$ and*
$$B < \frac{\cos(|\mu|x)}{\cos\left(\frac{|\mu|}{2N}\right)} - 1, \quad \text{for all } x \in \left[0, \frac{1}{2N}\right],$$
then the pre-solution of $\left\{\frac{(2k-1)}{2N}\right\}_{k=1}^N$ is a jump discontinuous solution of (2.4).

Proof. Let u be the pre-solution of $\left\{\frac{(2k-1)}{2N}\right\}_{k=1}^{N}$. By the symmetry of the pre-solution in Corollary 2.2, u is a jump discontinuous solution of (2.4) provided that u is negative on the first interval $[0, \xi_1] = [0, \frac{1}{2N}]$. For $x \in [0, \frac{1}{2N}]$,

$$(B^{-1}+1)u(x) = v(x) - 1 = \frac{\cosh(\mu x)}{B\cosh\left(\frac{\mu}{2N}\right)} - \frac{1}{B} - 1$$

If $B > 0$, then μ is positive and the right side of the above equation is negative since $0 < \frac{\cosh(\mu x)}{\cosh(\frac{\mu}{2N})} \le 1$. Hence $u(x) < 0$ and part 1 follows.

If $B \in (-1, 0)$, μ is positive and on $[0, \frac{1}{2N}]$,

$$(B^{-1}+1)u(x) = v(x) - 1 = \frac{\cosh(\mu x)}{B\cosh(\mu \xi)} - \frac{1}{B} - 1$$

Then

$$u(\xi_1-) = \lim_{x \to \frac{1}{2N}-} \frac{\cosh(\mu x)}{(B+1)\cosh(\mu\xi)} - 1 = \frac{1}{B+1} - 1 > 0$$

Hence the pre-solution $u(x)$ is positive when $x \in [0, \xi_1]$ is close to ξ_1. If u were a solution of (2.4), then (2.4) and (2.9) imply

$$\operatorname{sgn}(u) = \sum_{k=1}^{N+1} (-1)^k \chi_{(\xi_{k-1},\xi_k)}$$

which does not hold if $x \in (0, \xi_1)$ and x is close to ξ_1. This proves part 2.

If $B < -1$, then μ is imaginary and for $x \in [0, \frac{1}{2N}]$,

$$(B^{-1}+1)u(x) = \frac{\cos(|\mu|x)}{B\cos\left(\frac{|\mu|}{2N}\right)} - \frac{1}{B} - 1$$

Then the right side is negative if $B < \frac{\cos(|\mu|x)}{\cos\left(\frac{|\mu|}{2N}\right)} - 1$ for all $x \in [0, \frac{1}{2N}]$, from which part 3 follows. □

4. Critical solutions

In this section we denote the pre-solution of partition ξ by $u(\cdot, \xi)$ where $\xi = (\xi_1, \xi_2, \ldots, \xi_N)$. We view ξ as a parameter and $u(\cdot, \xi)$ as a family of pre-solutions parametrized by ξ. The corresponding $v = J[u]$ is denoted $v(\cdot, \xi)$ as well. The energy of $u(\cdot, \xi)$ is $I(u(\cdot, \xi))$, which is now viewed as a function of ξ. For simplicity we write $I(\xi)$ for $I(u(\cdot, \xi))$.

If the pre-solution $u(\cdot, \zeta)$ is a jump discontinuous solution of (2.4) where $\{\zeta_k\}_{k=1}^N$ is a particular partition, then for any partition $\{\xi_k\}_{k=1}^N$ that is sufficiently close to $\{\zeta_k\}_{k=1}^N$ (namely, each ξ_k is sufficiently close to ζ_k), the pre-solution $u(\cdot, \xi)$ of partition ξ is also a jump discontinuous solution of (2.4). Hence the set of $\xi \in \mathbb{R}^N$ where $u(\cdot, \xi)$ is a jump discontinuous solution of (2.4) is open.

We introduce the notion of critical solutions. Let $u(\cdot, \xi)$ be a pre-solution family parametrized by partition $\{\xi_k\}_{k=1}^N$ and $\{\zeta_k\}_{k=1}^N$ be a particular partition. Then the pre-solution $u(\cdot, \zeta)$ is called a critical solution if $u(\cdot, \zeta)$ is a jump discontinuous solution of (2.4) and ζ is a critical point of the function $\xi \to I(u(\cdot, \xi))$. Depending on the type of this critical point, $u(\cdot, \zeta)$ is called a minimizing solution (or maximizing solution, saddle solution, resp.) if ζ is a minimum (or maximum, saddle, resp.) of $\xi \to I(u(\cdot, \xi))$.

Lemma 4.1. *Let (1.8), (2.8), and (2.13) hold. If $v(\cdot, \xi)$ is a pre-solution parametrized by ξ, then*

$$\frac{\partial v(x,\xi)}{\partial \xi_j} = \frac{2(-1)^j}{B^{-1}+1} G(x, \xi_j), \quad j = 1, 2, \ldots, N$$

Proof. Since

$$v(x,\xi) = \int_0^1 G(x,y) \sum_{k=1}^{N+1} \frac{(-1)^k \chi_{(\xi_{k-1},\xi_k)}(y)}{B^{-1}+1}\, dy$$

differentiating with respect to ξ_j yields the lemma. □

Lemma 4.2. *Let (1.8), (2.8), and (2.13) hold. Suppose that the pre-solution $u(\cdot, \xi)$ is a jump discontinuous solution of (2.4) at ξ. Then*

$$\frac{\partial I}{\partial \xi_j} = \frac{2(-1)^{j+1}}{B^{-1}+1} v(\xi_j, \xi), \quad j = 1, 2, \ldots, N$$

In particular, $\frac{\partial I(\xi)}{\partial \xi_j} = 0$ if and only if $v(\xi_j, \xi) = 0$.

Proof. Let ξ be in the region where $u(\cdot, \xi)$ is a jump discontinuous solution of (2.4). By (2.4) and with $v = J[u]$,

$$\begin{aligned}
I(\xi) = I(u(\cdot,\xi)) &= \frac{1}{4}\int_0^1\int_0^1 J(x,y)(u(x,\xi)-u(y,\xi))^2\, dxdy \\
&\quad + \int_0^1 F(u(x,\xi))\, dx \\
&= \frac{1}{2}\int_0^1 \left(-J[u]u + B^{-1}u^2 + (u - \operatorname{sgn}(u))^2\right) dx \\
&= \frac{1}{2}\int_0^1 \Big(-(B^{-1}u + u - \operatorname{sgn}(u))u + B^{-1}u^2 \\
&\quad + (u-\operatorname{sgn}(u))^2\Big)\, dx \\
&= \frac{1}{2}\int_0^1 (1 - \operatorname{sgn}(u)u)\, dx \\
&= \frac{1}{2}\int_0^1 \left(1 - \operatorname{sgn}(u)\frac{v + \operatorname{sgn}(u)}{B^{-1}+1}\right) dx \\
&= \frac{1}{2}\int_0^1 \left(1 - \frac{1}{B^{-1}+1} - \frac{\operatorname{sgn}(u)v}{B^{-1}+1}\right) dx \\
&= \frac{B^{-1}}{2(B^{-1}+1)} - \frac{1}{2(B^{-1}+1)}\int_0^1 \left(v\sum_{k=1}^{N+1}(-1)^k \chi_{(\xi_{k-1},\xi_k)}\right) dx
\end{aligned}$$

so we arrive at the formula

$$\begin{aligned}
I(\xi) &= \frac{B^{-1}}{2(B^{-1}+1)} - \frac{1}{2(B^{-1}+1)} \\
&\quad \times \int_0^1 \left(v(x,\xi)\sum_{k=1}^{N+1}(-1)^k \chi_{(\xi_{k-1},\xi_k)}(x)\right) dx
\end{aligned} \tag{4.1}$$

Differentiation of (4.1) with respect to ξ_j and Lemma 4.1 imply

$$\begin{aligned}
\frac{\partial I}{\partial \xi_j} &= -\frac{1}{2(B^{-1}+1)}\int_0^1 \left(\frac{\partial v(x,\xi)}{\partial \xi_j}\sum_{k=1}^{N+1}(-1)^k\chi_{(\xi_{k-1},\xi_k)}(x)\right) dx \\
&\quad - \frac{(-1)^j v(\xi_j,\xi)}{B^{-1}+1} \\
&= \frac{(-1)^{j+1}}{B^{-1}+1}\int_0^1 \left(\frac{G(x,\xi_j)}{B^{-1}+1}\sum_{k=1}^{N+1}(-1)^k\chi_{(\xi_{k-1},\xi_k)}(x)\right) dx \\
&\quad - \frac{(-1)^j v(\xi_j,\xi)}{B^{-1}+1} \\
&= \frac{(-1)^{j+1}v(\xi_j,\xi)}{B^{-1}+1} - \frac{(-1)^j v(\xi_j,\xi)}{B^{-1}+1} \\
&= \frac{2(-1)^{j+1}v(\xi_j,\xi)}{B^{-1}+1}
\end{aligned}$$

proving the lemma. □

Remark 4.3. By Corollary 2.2 and Lemma 4.2, if the pre-solution of an equidistant partition is a jump discontinuous solution, then it is a critical solution.

Before stating the main theorem in this section we introduce the notation $a \equiv b \mod (\frac{\pi}{|\mu|})\mathbb{Z}$, which means that there is an integer n such that $a = b + \frac{n\pi}{|\mu|}$.

Theorem 4.4. *Let (1.8) and (2.13) hold, and $u(\cdot,\xi)$ be the pre-solution of a partition $\xi = (\xi_1, \xi_2, \ldots, \xi_N)$.*

1. *If $B > 0$, then $u(\cdot,\xi)$ is a critical solution if and only if $\xi_k = \frac{2k-1}{2N}$, i.e. ξ is an equidistant partition.*
2. *If $B \in (-1, 0)$, then no pre-solution is a critical solution.*
3. *If $B < -1$ and the pre-solution $u(\cdot,\xi)$ is a jump discontinuous solution of (2.4), then $u(\cdot,\xi)$ is a critical solution if and only if*

$$\xi_1 \equiv \frac{\xi_2-\xi_1}{2} \equiv \frac{\xi_3-\xi_2}{2} \equiv \cdots \equiv \frac{\xi_N-\xi_{N-1}}{2} \equiv 1-\xi_N \mod \left(\frac{\pi}{|\mu|}\right)\mathbb{Z} \tag{4.2}$$

Proof. Part 1. Let u be the pre-solution of an equidistant partition. Theorem 3.2.1 shows that u is necessarily a jump discontinuous solution of (2.4). Corollary 2.2 asserts that the corresponding v satisfies $v(\xi_k) = 0$, $k = 1, 2, \ldots, N$, and then Lemma 4.2 implies that u is a critical solution.

Conversely, let $u(\cdot,\xi)$ be a critical solution. Then the corresponding $v(\cdot,\xi)$ satisfies

$$-Av'' + \frac{B^2}{B+1}v = -\frac{1}{B^{-1}+1},\ v'(0) = 0,\ v(\xi_1) = 0, \tag{4.3}$$

on the interval $(0,\xi_1)$. Because $\mu > 0$ in this case, (4.3) admits a unique solution

$$v(x,\xi) = \frac{\cosh \mu x}{B\cosh \mu\xi_1} - \frac{1}{B},\ x \in (0,\xi_1). \tag{4.4}$$

On the interval (ξ_1,ξ_2), v satisfies

$$-Av'' + \frac{B^2}{B+1}v = \frac{1}{B^{-1}+1},\ v(\xi_1) = 0,\ v(\xi_2) = 0, \tag{4.5}$$

which admits a unique solution

$$v(x,\xi) = -\frac{\cosh \mu(x - \frac{\xi_1+\xi_2}{2})}{B\cosh \mu(\frac{\xi_1-\xi_2}{2})} + \frac{1}{B},\ x \in (\xi_1,\xi_2). \tag{4.6}$$

As a solution of (2.11), v is a C^1 function on $(0, 1)$. Then the derivatives of (4.4) and (4.6) must match at ξ_1. Hence

$$\frac{\mu\sinh \mu\xi_1}{B\cosh \mu\xi_1} = -\frac{\mu\sinh \mu(\frac{\xi_1-\xi_2}{2})}{B\cosh \mu(\frac{\xi_1-\xi_2}{2})} \tag{4.7}$$

which implies

$$2\xi_1 = \xi_2 - \xi_1. \tag{4.8}$$

This argument can be continued to yield

$$\xi_2-\xi_1 = \xi_3-\xi_2 = \xi_4-\xi_3 = \cdots = \xi_N-\xi_{N-1} = 2(1-\xi_N), \tag{4.9}$$

namely that ξ is the equidistant partition $\{\frac{2k-1}{2N}\}_{k=1}^N$.

Part 2. Suppose on the contrary that a pre-solution $u(\cdot,\xi)$ is a critical solution. Since $\mu > 0$ in this case, the argument in the proof of part 1 can again be used to show that ξ is an equidistant partition, i.e. $\xi_k = \frac{2k-1}{2N}$. However Theorem 3.2.2 asserts that a pre-solution of an equidistant partition cannot be a solution of (2.4). So $u(\cdot,\xi)$ is not a critical solution.

Part 3. Let $u(\cdot,\xi)$, together with the corresponding $v(\cdot,\xi)$, be the pre-solution of a partition ξ and assume that $u(\cdot,\xi)$ is also a critical solution. Then $v(\xi_j) = 0, j = 1, 2, \ldots, N$ by Lemma 4.2. On $(0,\xi_1)$, v satisfies

$$-Av'' + \frac{B^2}{B+1}v = -\frac{1}{B^{-1}+1},\ v'(0) = 0,\ v(\xi_1) = 0. \tag{4.10}$$

Because μ is imaginary when $B < -1$, (4.10) may not be solvable. If $|\mu|\xi_1 = n\pi + \frac{\pi}{2}$ for some non-negative integer n, then the corresponding homogeneous problem of (4.10) has a nontrivial kernel spanned by $\cos|\mu|x$. However by the self-adjointness of the problem and the fact that

$$\int_0^{\xi_1} \cos|\mu|x\Big(-\frac{1}{B^{-1}+1}\Big)\,dx = \frac{1}{|\mu|}\Big(\sin(n\pi+\frac{\pi}{2}) - \sin 0\Big)\times\Big(-\frac{1}{B^{-1}+1}\Big) = \frac{\pm 1}{|\mu|}\Big(-\frac{1}{B^{-1}+1}\Big) \neq 0,$$

problem (4.10) has no solution. Therefore $|\mu|\xi_1 = n\pi + \frac{\pi}{2}$ cannot occur. Now that $|\mu|\xi_1 \neq n\pi + \frac{\pi}{2}$ for any non-negative integer n, (4.10) admits a unique solution, which is the same as the pre-solution $v(\cdot,\xi)$, and it takes the form

$$v(x) = \frac{\cos|\mu|x}{B\cos|\mu|\xi_1} - \frac{1}{B},\ x \in (0,\xi_1). \tag{4.11}$$

Next consider v on (ξ_1,ξ_2). We have

$$-Av'' + \frac{B^2}{B+1}v = \frac{1}{B^{-1}+1},\ v(\xi_1) = 0,\ v(\xi_2) = 0. \tag{4.12}$$

If $\frac{|\mu|(\xi_2-\xi_1)}{2} = n\pi + \frac{\pi}{2}$ for some non-negative integer n, then the corresponding homogeneous problem of (4.12) has a nontrivial kernel spanned by $\sin(|\mu|(x-\xi_1))$. But

$$\int_{\xi_1}^{\xi_2} \sin(|\mu|(x-\xi_1))\Big(\frac{1}{B^{-1}+1}\Big)\,dx = -\frac{1}{|\mu|}\Big(\cos(2n\pi+\pi) - \cos 0\Big)\Big(\frac{1}{B^{-1}+1}\Big) = \frac{2}{|\mu|}\Big(\frac{1}{B^{-1}+1}\Big) \neq 0,$$

so by the self-adjointness, problem (4.12) has no solution. Hence $\frac{|\mu|(\xi_2-\xi_1)}{2} \neq n\pi + \frac{\pi}{2}$ for any non-negative integer n. Then (4.12) admits a unique solution, same as the pre-solution $v(\cdot,\xi)$, and it takes the form

$$v(x) = -\frac{\cos(|\mu|(x-\frac{\xi_1+\xi_2}{2}))}{B\cos\frac{|\mu|(\xi_1-\xi_2)}{2}} + \frac{1}{B},\ x \in (\xi_1,\xi_2) \tag{4.13}$$

Since v is a C^1 function on $(0, 1)$, the derivatives at ξ_1 computed from (4.11) and (4.13) must match, i.e.

$$-\frac{|\mu|\sin|\mu|\xi_1}{B\cos|\mu|\xi_1} = \frac{|\mu|\sin\frac{|\mu|(\xi_1-\xi_2)}{2}}{B\cos\frac{|\mu|(\xi_1-\xi_2)}{2}} \tag{4.14}$$

This means that

$$\tan|\mu|\xi_1 = \tan\frac{|\mu|(\xi_2-\xi_1)}{2}. \tag{4.15}$$

Similarly, one can show that

$$\tan\frac{|\mu|(\xi_2-\xi_1)}{2} = \tan\frac{|\mu|(\xi_3-\xi_2)}{2} = \cdots = \tan\frac{|\mu|(\xi_N-\xi_{N-1})}{2} = \tan|\mu|(1-\xi_N) \tag{4.16}$$

Then (4.2) follows from (4.15) and (4.16).

Conversely, assume that a partition ξ satisfies the condition (4.2) and the pre-solution of ξ is a jump discontinuous solution of (2.4).

First we claim that none of $|\mu|\xi_1$, $\frac{|\mu|(\xi_2-\xi_1)}{2}$, …, $\frac{|\mu|(\xi_N-\xi_{N-1})}{2}$, $|\mu|(1-\xi_N)$ can be written as $n\pi + \frac{\pi}{2}$ for some non-negative integer

n. Otherwise if one of them takes this form, then all of them will have this form by (4.2), namely

$$|\mu|\xi_1 = n_1\pi + \frac{\pi}{2},\ \frac{|\mu|(\xi_2-\xi_1)}{2} = n_2\pi + \frac{\pi}{2},\ \dots,\\ \frac{|\mu|(\xi_N-\xi_{N-1})}{2} = n_N\pi + \frac{\pi}{2},\ |\mu|(1-\xi_N) = n_{N+1}\pi + \frac{\pi}{2} \quad (4.17)$$

for some non-negative integers $n_1, n_2, \dots, n_{N+1}$. Then

$$\begin{aligned}|\mu| &= |\mu|\xi_1 + |\mu|(\xi_2-\xi_1) + |\mu|(\xi_3-\xi_2)\\ &\quad + \cdots + |\mu|(\xi_N-\xi_{N-1}) + |\mu|(1-\xi_N)\\ &= \Big(n_1\pi + \frac{\pi}{2}\Big) + 2\Big(n_2\pi + \frac{\pi}{2}\Big) + \cdots + 2\Big(n_N\pi + \frac{\pi}{2}\Big)\\ &\quad + \Big(n_{N+1}\pi + \frac{\pi}{2}\Big)\\ &= (n_1 + 2n_2 + \cdots + 2n_N + n_{N+1} + N)\pi\end{aligned}$$

a contradiction to (2.13).

Next consider the following $N+1$ problems:

$$-Av'' + \frac{B^2}{B+1}v = -\frac{1}{B^{-1}+1},\quad v'(0) = v(\xi_1) = 0 \quad (4.18)$$

$$-Av'' + \frac{B^2}{B+1}v = \frac{1}{B^{-1}+1},\quad v(\xi_1) = v(\xi_2) = 0 \quad (4.19)$$

$$\cdots$$

$$-Av'' + \frac{B^2}{B+1}v = (-1)^N\frac{1}{B^{-1}+1},\quad v(\xi_{N-1}) = v(\xi_N) = 0 \quad (4.20)$$

$$-Av'' + \frac{B^2}{B+1}v = (-1)^{N+1}\frac{1}{B^{-1}+1},\quad v(\xi_N) = v'(1) = 0 \quad (4.21)$$

Each of these problems admits a unique solution because of the claim above, and they take the forms of (4.11) and (4.13), etc. Together they form a continuous function on $(0,1)$, which we denote by v. Moreover (4.2) implies that (4.14) holds, so the derivative of v is continuous on $(0,1)$. Consequently, v solves (2.11), i.e. v is the pre-solution of ξ. The corresponding $u(\cdot,\xi)$ obtained from (2.10), is a jump discontinuous solution of (2.4) by the assumption of part 3. Since $v(\xi_k) = 0$, $k = 1, 2, \dots, N$, $u(\cdot,\xi)$ is also a critical solution. □

5. Minimizing solutions

Lemma 5.1. *Let (1.8), (2.8), and (2.13) hold. Suppose that the pre-solution $u(\cdot,\xi)$ is a jump discontinuous solution of (2.4) at ξ. Then*

$$\frac{\partial^2 I}{\partial\xi_j\partial\xi_k} = \frac{4(-1)^{j+k+1}}{(B^{-1}+1)^2}G(\xi_j,\xi_k), \qquad j \neq k$$

$$\frac{\partial^2 I}{\partial\xi_j\partial\xi_j} = \frac{-4}{(B^{-1}+1)^2}G(\xi_j,\xi_j) + \frac{2(-1)^{j+1}}{B^{-1}+1}v'(\xi_j,\xi).$$

Hence, the Hessian of I is given by

$$H = \frac{4}{(B^{-1}+1)^2}\begin{bmatrix} -G(\xi_1,\xi_1) & G(\xi_1,\xi_2) & -G(\xi_1,\xi_3) & \cdots\\ G(\xi_2,\xi_1) & -G(\xi_2,\xi_2) & G(\xi_2,\xi_3) & \cdots\\ -G(\xi_3,\xi_1) & G(\xi_3,\xi_2) & -G(\xi_3,\xi_3) & \cdots\\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$
$$+\frac{2}{B^{-1}+1}\begin{bmatrix} v'(\xi_1,\xi) & 0 & 0 & \cdots\\ 0 & -v'(\xi_2,\xi) & 0 & \cdots\\ 0 & 0 & v'(\xi_3,\xi) & \cdots\\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Here the prime over v denotes the derivative of $v(x,\xi)$ with respect to x.

Proof. By Lemmas 4.1 and 4.2, for $k \neq j$,

$$\begin{aligned}\frac{\partial^2 I}{\partial\xi_j\xi_k} &= \frac{2(-1)^{j+1}}{B^{-1}+1}\frac{\partial v(\xi_j,\xi)}{\partial\xi_k}\\ &= \frac{2(-1)^{j+1}}{B^{-1}+1}\frac{2(-1)^k}{B^{-1}+1}G(\xi_j,\xi_k)\\ &= \frac{4(-1)^{j+k+1}}{(B^{-1}+1)^2}G(\xi_j,\xi_k).\end{aligned}$$

When $k = j$, by Lemmas 4.1 and 4.2,

$$\begin{aligned}\frac{\partial^2 I}{\partial\xi_j^2} &= \frac{2(-1)^{j+1}}{B^{-1}+1}\frac{\partial v(\xi_j,\xi)}{\partial\xi_j}\\ &= \frac{2(-1)^{j+1}}{B^{-1}+1}\frac{2(-1)^j}{B^{-1}+1}G(\xi_j,\xi_j) + \frac{2(-1)^{j+1}}{B^{-1}+1}v'(\xi_j,\xi)\\ &= \frac{-4}{(B^{-1}+1)^2}G(\xi_j,\xi_j) + \frac{2(-1)^{j+1}}{B^{-1}+1}v'(\xi_j,\xi).\end{aligned}$$

proving the lemma. □

For a partition ξ that satisfies the condition (4.2), define

$$\begin{aligned}\alpha &= |\mu|\cot 2|\mu|\xi_1 = |\mu|\cot|\mu|(\xi_2-\xi_1)\\ &= \cdots = |\mu|\cot|\mu|(\xi_N-\xi_{N-1}) = |\mu|\cot 2|\mu|(1-\xi_N) \quad (5.1)\\ \beta &= |\mu|\csc 2|\mu|\xi_1 = |\mu|\csc|\mu|(\xi_2-\xi_1)\\ &= \cdots = |\mu|\csc|\mu|(\xi_N-\xi_{N-1}) = |\mu|\csc 2|\mu|(1-\xi_N) \quad (5.2)\\ \gamma &= |\mu|\tan|\mu|\xi_1 = |\mu|\tan\frac{|\mu|(\xi_2-\xi_1)}{2}\\ &= \cdots = |\mu|\tan\frac{|\mu|(\xi_N-\xi_{N-1})}{2} = |\mu|\tan|\mu|(1-\xi_N) \quad (5.3)\end{aligned}$$

These three numbers depend on the partition ξ and $|\mu|$. We already know from the proof of Theorem 4.4.3 that none of $2|\mu|\xi_1, |\mu|(\xi_2-\xi_1), \dots, |\mu|(\xi_N-\xi_{N-1})$, and $2|\mu|(1-\xi_N)$ can be of the form $(2n+1)\pi$ for some integer n. Moreover none of them can be written as $2n\pi$ either. Otherwise

$$|\mu| = |\mu|\xi_1 + |\mu|(\xi_2-\xi_1) + |\mu|(\xi_3-\xi_2)\\ + \cdots + |\mu|(\xi_N-\xi_{N-1}) + |\mu|(1-\xi_N)$$

would be an integer multiple of π, a contradiction to (2.13). Hence α, β, and γ are all finite numbers.

Let Q be the $N\times N$ matrix

$$Q = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0\\ 1 & 0 & 1 & 0 & \cdots & 0\\ 0 & 1 & 0 & 1 & \cdots & 0\\ \cdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \quad (5.4)$$

for $N \geq 2$. Denote the eigenvalues of Q by $q_1, q_2, \dots, q_N$. It is known that these q_k's are

$$2,\ \pm 2\cos\frac{\pi(j-1)}{N},\quad j = 2, 3, \dots, \frac{N+1}{2},\quad \text{if } N \text{ is odd};$$
$$2,\ 0,\ \pm 2\cos\frac{\pi(j-1)}{N},\quad j = 2, 3, \dots, \frac{N}{2},\quad \text{if } N \text{ is even}. \quad (5.5)$$

See for instance [12, Appendix B] for a proof. If $N = 1$, our convention is to set Q to be the 1×1 matrix whose entry is 2 and consequently $q_1 = 2$.

Lemma 5.2. *Let (2.13) hold, ξ be a partition that satisfies (4.2), and* **G** *be the $N\times N$ symmetric matrix whose (j,k)-entry is $G(\xi_k,\xi_j)$. Then the eigenvalues of* **G** *are*

$$\frac{1}{A(2\alpha - q_k\beta)},\quad k = 1, 2, \dots, N.$$

Proof. Let Λ be an eigenvalue of $\mathbf{G}$, namely that there exists a nonzero vector $b=(b_1,b_2,\dots,b_N)^T$ such that

$$\sum_{j=1}^{N} G(\xi_k,\xi_j)b_j=\Lambda b_k,\quad k=1,2,\dots,N.$$

First consider the simplest case $N=1$. Then

$$\Lambda=G(\xi_1,\xi_1)=-\frac{\cos(|\mu|(1-\xi_1))\cos(|\mu|\xi_1)}{A|\mu|\sin|\mu|} \tag{5.6}$$

Since our convention is $q_1=2$ when $N=1$,

$$\begin{aligned}\frac{1}{A(2\alpha-q_1\beta)} &= \frac{1}{A(2|\mu|\cot|\mu|2\xi_1-2|\mu|\csc|\mu|2\xi_1)}\\ &=\left(\frac{-1}{2A|\mu|}\right)\cot|\mu|\xi_1\\ &=\left(\frac{-1}{2A|\mu|}\right)\frac{\cos|\mu|\xi_1\cos|\mu|(1-\xi_1)}{\sin|\mu|\xi_1\cos|\mu|(1-\xi_1)}\\ &=\left(\frac{-1}{A|\mu|}\right)\frac{\cos|\mu|\xi_1\cos|\mu|(1-\xi_1)}{\sin|\mu|}\end{aligned} \tag{5.7}$$

Here to reach the last line, we use $\sin|\mu|(1-2\xi_1)=0$, which follows from (4.2) when $N=1$. Since (5.6) and (5.7) agree, the theorem holds when $N=1$.

When $N\ge 2$, let

$$\zeta(x)=\sum_{j=1}^{N} G(x,\xi_j)b_j. \tag{5.8}$$

Hence

$$[-\zeta']_{\xi_k}=\frac{b_k}{A},\quad\text{and}\quad \zeta(\xi_k)=\Lambda b_k.$$

Then for every k,

$$[-\zeta']_{\xi_k}=\frac{1}{A\Lambda}\zeta(\xi_k).$$

Let $\vec{\zeta}=(\zeta(\xi_1),\zeta(\xi_2),\dots,\zeta(\xi_N))^T$. We proceed to find an $N\times N$ matrix T so that

$$[-\zeta']_{\xi_k}=(T\vec{\zeta})_k$$

Then we convert the original eigenvalue problem $\mathbf{G}b=\Lambda b$ to a new eigenvalue problem:

$$T\vec{\zeta}=\left(\frac{1}{A\Lambda}\right)\vec{\zeta}. \tag{5.9}$$

On (ξ_{j-1},ξ_j), $\zeta(x)=c_1\cos|\mu|x+c_2\sin|\mu|x$. From here we write, in the matrix notation,

$$\begin{bmatrix}\zeta(\xi_{j-1})\\ \zeta(\xi_j)\end{bmatrix}=\begin{bmatrix}\cos|\mu|\xi_{j-1} & \sin|\mu|\xi_{j-1}\\ \cos|\mu|\xi_j & \sin|\mu|\xi_j\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}.$$

We denote the 2 by 2 matrix by A_L, for the left of ξ_j. To the right we have similarly, on (ξ_j,ξ_{j+1}), $\zeta(x)=\tilde{c}_1\cos|\mu|x+\tilde{c}_2\sin|\mu|x$ and

$$\begin{bmatrix}\zeta(\xi_j)\\ \zeta(\xi_{j+1})\end{bmatrix}=\begin{bmatrix}\cos|\mu|\xi_j & \sin|\mu|\xi_j\\ \cos|\mu|\xi_{j+1} & \sin|\mu|\xi_{j+1}\end{bmatrix}\begin{bmatrix}\tilde{c}_1\\ \tilde{c}_2\end{bmatrix}$$

with the 2 by 2 matrix denoted by A_R. Hence

$$\begin{bmatrix}c_1\\ c_2\end{bmatrix}=A_L^{-1}\begin{bmatrix}\zeta(\xi_{j-1})\\ \zeta(\xi_j)\end{bmatrix}\ \text{on } (\xi_{j-1},\xi_j),$$
$$\begin{bmatrix}\tilde{c}_1\\ \tilde{c}_2\end{bmatrix}=A_R^{-1}\begin{bmatrix}\zeta(\xi_j)\\ \zeta(\xi_{j+1})\end{bmatrix}\ \text{on } (\xi_j,\xi_{j+1}).$$

Then

$$-\zeta'(\xi_j-)=|\mu|\Big[\sin|\mu|\xi_j,\,-\cos|\mu|\xi_j\Big]A_L^{-1}\begin{bmatrix}\zeta(\xi_{j-1})\\ \zeta(\xi_j)\end{bmatrix}$$
$$-\zeta'(\xi_j+)=|\mu|\Big[\sin|\mu|\xi_j,\,-\cos|\mu|\xi_j\Big]A_R^{-1}\begin{bmatrix}\zeta(\xi_j)\\ \zeta(\xi_{j+1})\end{bmatrix}$$

and

$$\begin{aligned}[-\zeta']_{\xi_j} &= |\mu|\Big[-\sin|\mu|\xi_j,\,\cos|\mu|\xi_j\Big]\\ &\quad\times\left(A_L^{-1}\begin{bmatrix}\zeta(\xi_{j-1})\\ \zeta(\xi_j)\end{bmatrix}-A_R^{-1}\begin{bmatrix}\zeta(\xi_j)\\ \zeta(\xi_{j+1})\end{bmatrix}\right).\end{aligned}$$

We compute

$$A_L^{-1}=\frac{1}{\sin|\mu|(\xi_j-\xi_{j-1})}\begin{bmatrix}\sin|\mu|\xi_j & -\sin|\mu|\xi_{j-1}\\ -\cos|\mu|\xi_j & \cos|\mu|\xi_{j-1}\end{bmatrix}$$

$$A_R^{-1}=\frac{1}{\sin|\mu|(\xi_{j+1}-\xi_j)}\begin{bmatrix}\sin|\mu|\xi_{j+1} & -\sin|\mu|\xi_j\\ -\cos|\mu|\xi_{j+1} & \cos|\mu|\xi_j\end{bmatrix}.$$

So T is a triagonal matrix. For $j=2,3,\dots,N-1$, the three nontrivial entries of the jth row are

$$-|\mu|\csc|\mu|(\xi_j-\xi_{j-1}),$$
$$|\mu|\cot|\mu|(\xi_j-\xi_{j-1})+|\mu|\cot|\mu|(\xi_{j+1}-\xi_j),$$
$$-|\mu|\csc|\mu|(\xi_{j+1}-\xi_j).$$

For the first row of T, since

$$\zeta(x)=\frac{\zeta(\xi_1)}{\cos|\mu|\xi_1}\cos|\mu|x,\ x\in(0,\xi_1),\quad\text{and}$$
$$-\zeta'(\xi_1-)=|\mu|\tan|\mu|\xi_1\zeta(\xi_1),$$

the first two entries are

$$-|\mu|\tan|\mu|\xi_1+|\mu|\cot|\mu|(\xi_2-\xi_1),\quad -|\mu|\csc|\mu|(\xi_2-\xi_1)$$

Similarly the last two entries of the last row of T are

$$-|\mu|\csc|\mu|(\xi_N-\xi_{N-1}),\quad |\mu|\cot|\mu|(\xi_N-\xi_{N-1})-|\mu|\tan|\mu|(1-\xi_N)$$

By (5.1) and (5.2) T is written as

$$T=\begin{bmatrix}2\alpha-\beta & -\beta & 0 & 0 & \cdots & 0\\ -\beta & 2\alpha & -\beta & 0 & \cdots & 0\\ 0 & -\beta & 2\alpha & -\beta & \cdots & 0\\ \cdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 2\alpha-\beta\end{bmatrix} \tag{5.10}$$

We write $T=2\alpha\mathbf{I}-\beta Q$ where $\mathbf{I}$ is the $N\times N$ identity matrix and Q is given in (5.4). Then the eigenvalues of T are $2\alpha-q_k\beta$, $k=1,2,\dots,N$. We claim that $2\alpha-q_k\beta$ is never zero. Note that, with $\eta=|\mu|\xi_1$,

$$2\alpha-q_k\beta=\frac{2|\mu|\cos 2\eta}{\sin 2\eta}-\frac{q_k|\mu|}{\sin 2\eta}=\frac{|\mu|}{\sin 2\eta}(2\cos 2\eta-q_k).$$

Say $2\cos 2\eta-q_k=0$ for $q_k=2\cos\frac{\pi(j-1)}{N}$. Then $2\eta\equiv\frac{\pi(j-1)}{N}$ mod $2\pi\mathbb{Z}$ or $2\eta\equiv-\frac{\pi(j-1)}{N}$ mod $2\pi\mathbb{Z}$. In the first case there exist non-negative integers $n_1,n_2,\dots,n_{N+1}$ such that, by (4.2),

$$2|\mu|\xi_1=\frac{\pi(j-1)}{N}+2n_1\pi,\ |\mu|(\xi_2-\xi_1)=\frac{\pi(j-1)}{N}+2n_2\pi,$$
$$2|\mu|(1-\xi_N)=\frac{\pi(j-1)}{N}+2n_{N+1}$$

Then

$$\begin{aligned}|\mu| &= |\mu|\xi_1+|\mu|(\xi_2-\xi_1)+|\mu|(\xi_3-\xi_2)+\cdots+|\mu|(\xi_N-\xi_{N-1})\\ &\quad+|\mu|(1-\xi_N)\\ &=\left(\frac{\pi(j-1)}{2N}+n_1\pi\right)+\left(\frac{\pi(j-1)}{N}+2n_2\pi\right)\end{aligned}$$

$$+\cdots+\Big(\frac{\pi(j-1)}{N}+2n_N\pi\Big)+\Big(\frac{\pi(j-1)}{2N}+n_{N+1}\pi\Big)$$
$$=(n_1+2n_2+\cdots+2n_N+n_{N+1}+j-1)\pi$$

a contradiction to (2.13). Similar argument shows that for other q_k's $2\alpha-q_k\beta\neq 0$. Then by (5.9) the eigenvalues of $\mathbf{G}$ are

$$\frac{1}{A(2\alpha-q_k\beta)},\ k=1,2,\ldots,N, \tag{5.11}$$

where the q_k's are given in (5.5). □

Theorem 5.3. *Let (1.8) and (2.13) hold, and $H(\xi)$ be the Hessian of I at a critical solution $u(\cdot,\xi)$.*

1. *If $B>0$, then $\xi_k=\frac{2k-1}{2N}$, $k=1,2,\ldots,N$, and the eigenvalues of $H(\xi)$ are*
$$\frac{1}{B+1}\Big(\frac{-4\mu}{\Big(2\coth\frac{\mu}{N}-q_k\operatorname{csch}\frac{\mu}{N}\Big)}+2\mu\tanh(\frac{\mu}{2N})\Big),$$
$$k=1,2,\ldots,N.$$
2. *If $B<-1$, then (4.2) holds and the eigenvalues of $H(\xi)$ are*
$$\frac{1}{B+1}\Big(\frac{4|\mu|^2}{2\alpha-q_k\beta}-2\gamma\Big),\quad k=1,2,\ldots,N.$$
Here α, β, and γ are given in (5.1)–(5.3) and they depend on the partition ξ of the critical solution $u(\cdot,\xi)$.

Proof. In Theorem 4.4 we have already seen that at a critical solution $u(\cdot,\xi)$, ξ is the equidistant partition $\{\frac{2k-1}{2N}\}$ if $B>0$ and ξ satisfies (4.2) if $B<-1$.

Now we prove the more complicated part 2 of the theorem. As we know that when v is a critical solution, since $v(\xi_k)=0$,

$$v(x)=(-1)^{k+1}\frac{\cos(|\mu|(x-\frac{\xi_{k-1}+\xi_k}{2}))}{B\cos\frac{|\mu|(\xi_k-\xi_{k-1})}{2}}+\frac{(-1)^k}{B},\ x\in(\xi_{k-1},\xi_k) \tag{5.12}$$

Then

$$v'(\xi_k)=\frac{(-1)^k|\mu|}{B}\tan\frac{|\mu|(\xi_k-\xi_{k-1})}{2} \tag{5.13}$$

By (5.3) we deduce

$$v'(\xi_k)=\frac{(-1)^k\gamma}{B}. \tag{5.14}$$

By Lemma 5.1 and (5.14), the Hessian of I at a critical solution $u(\cdot,\xi)$ is

$$H(\xi)=\frac{-4}{(B^{-1}+1)^2}\widetilde{\mathbf{G}}-\frac{2\gamma}{B(B^{-1}+1)}\mathbf{I}$$

where $\widetilde{\mathbf{G}}$ is the matrix whose (j,k) entry is $(-1)^{j+k}G(\xi_k,\xi_j)$. The matrix $\widetilde{\mathbf{G}}$ and the matrix $\mathbf{G}$ are similar: $\widetilde{\mathbf{G}}=P\mathbf{G}P^{-1}$ where

$$P=\begin{bmatrix}1&0&0&0&\\0&-1&0&0&\\0&0&1&0&\\0&0&0&-1&\\&&&&\ddots\end{bmatrix} \tag{5.15}$$

and $P^{-1}=P$. Hence $\widetilde{\mathbf{G}}$ has the same eigenvalues as $\mathbf{G}$ does. Then by Lemma 5.2, the eigenvalues of $H(\xi)$ are

$$\frac{-4}{A(B^{-1}+1)^2(2\alpha-q_k\beta)}-\frac{2\gamma}{B(B^{-1}+1)} \tag{5.16}$$

from which part 2 of the theorem follows.

Part 1 of the theorem is easier to prove. Note that when $B>0$ and $u(\cdot,\xi)$ is a critical solution, $\xi_k=\frac{2k-1}{2N}$ by Theorem 4.4.1. Define $\mathbf{G}$ and T in the same way, then Λ is an eigenvalue of $\mathbf{G}$ if and only if $\frac{1}{A\Lambda}$ is an eigenvalue of T. In this case

$$T=\Big(2\mu\coth\frac{\mu}{N}\Big)\mathbf{I}-\Big(\mu\operatorname{csch}\frac{\mu}{N}\Big)Q \tag{5.17}$$

Hence the eigenvalues of $\mathbf{G}$ are

$$\frac{1}{A\Big(2\mu\coth\frac{\mu}{N}-q_k\mu\operatorname{csch}\frac{\mu}{N}\Big)},\ k=1,2,\ldots,N. \tag{5.18}$$

Moreover

$$v'\Big(\frac{2k-1}{2N}\Big)=(-1)^{k+1}\frac{\mu}{B}\tanh(\frac{\mu}{2N})$$

Then

$$H(\xi)=-\frac{4}{(B^{-1}+1)^2}\widetilde{\mathbf{G}}+\frac{2\mu\tanh(\frac{\mu}{2N})}{B(B^{-1}+1)}\mathbf{I}$$

and, since $\widetilde{\mathbf{G}}$ and $\mathbf{G}$ have the same eigenvalues, part 1 follows from (5.18) □

Theorem 5.4. *Let (1.8) and (2.13) hold, and $u(\cdot,\xi)$ be at a critical solution.*

1. *If $B>0$, then $u(\cdot,\xi)$, where ξ is necessarily equidistant, is a maximizing solution.*
2. *If $B<-1$, then the following statements hold.*
   (a) *If $2|\mu|\xi_1\in(\pi-\frac{\pi}{N},\pi)\mod 2\pi\mathbb{Z}$, then $u(\cdot,\xi)$ is a minimizing solution.*
   (b) *If $2|\mu|\xi_1\in(\pi,\pi+\frac{\pi}{N})\mod 2\pi\mathbb{Z}$, then $u(\cdot,\xi)$ is a maximizing solution.*
   (c) *If $2|\mu|\xi_1\in(-\pi+\frac{\pi}{N},\pi-\frac{\pi}{N})\mod 2\pi\mathbb{Z}$, then $u(\cdot,\xi)$ is a saddle solution.*

Proof. Part 1. Because $-2<q_k\leq 2$,

$$0<2\coth\frac{\mu}{N}-2\operatorname{csch}\frac{\mu}{N}\leq 2\coth\frac{\mu}{N}-q_k\operatorname{csch}\frac{\mu}{N}<2\coth\frac{\mu}{N}$$
$$+2\operatorname{csch}\frac{\mu}{N}=2\coth\frac{\mu}{2N} \tag{5.19}$$

By Theorem 5.3.1 and (5.19), all eigenvalues of $H(\xi)$ are negative.

Part 2. Let $\eta=|\mu|\xi_1$. By Theorem 5.3.2 and (5.1)–(5.3), the eigenvalues of $H(\xi)$ are

$$\frac{1}{B+1}\Big(\frac{4|\mu|^2}{2\alpha-q_k\beta}-2\gamma\Big)=\frac{-2|\mu|}{B+1}\Big(\frac{-2}{2\cot 2\eta-q_k\csc 2\eta}+\tan\eta\Big)$$
$$=\Big(\frac{-2|\mu|}{B+1}\Big)\times\frac{(2+q_k)-(2+q_k)\cos 2\eta}{(q_k-2\cos 2\eta)\sin 2\eta} \tag{5.20}$$

Since $-2<q_k\leq 2$, the sign of the eigenvalue in (5.20) is the same as the sign of $(q_k-2\cos 2\eta)\sin 2\eta$.

If $2\eta\in(\pi-\frac{\pi}{N},\pi)\mod 2\pi\mathbb{Z}$, then $\sin 2\eta>0$ and

$$q_k-2\cos 2\eta>q_k-2\cos(\pi-\frac{\pi}{N})\geq 0$$

since the smallest q_k is 2 if $N=1$, and $-2\cos\frac{\pi}{N}$ if $N\geq 2$. Therefore all eigenvalues in (5.20) are positive and $u(\cdot,\xi)$ is a minimizing solution.

If $2\eta\in(\pi,\pi+\frac{\pi}{N})\mod 2\pi\mathbb{Z}$, then $\sin 2\eta<0$ and

$$q_k-2\cos 2\eta>q_k-2\cos(\pi+\frac{\pi}{N})\geq 0$$

as in the last case. Therefore all eigenvalues in (5.20) are negative and $u(\cdot,\xi)$ is a maximizing solution.

If $2\eta \in (-\pi+\frac{\pi}{N}, \pi-\frac{\pi}{N}) \mod 2\pi\mathbb{Z}$, then N is at least 2. Consider the scenario $2\eta \in (0, \pi - \frac{\pi}{N}) \mod 2\pi\mathbb{Z}$. If we take q_k to be the largest, which is $q_N = 2$, then $\sin 2\eta > 0$,

$$q_N - 2\cos 2\eta = 2 - 2\cos 2\eta > 0,$$

and the corresponding eigenvalue in (5.20) is positive; if we take q_k to be the smallest, which is $q_1 = -2\cos\frac{\pi}{N}$, then $\sin 2\eta > 0$,

$$q_1 - 2\cos 2\eta = -2\cos\frac{\pi}{N} - 2\cos 2\eta < 0,$$

and the corresponding eigenvalue in (5.20) is negative. Hence $u(\cdot, \xi)$ is a saddle solution. In the other scenario, $2\eta \in (-\pi + \frac{\pi}{N}, 0) \mod 2\pi\mathbb{Z}$, $q_k - 2\cos 2\eta > 0$ if $q_k = q_N$ and $q_k - 2\cos 2\eta < 0$ if $q_k = q_1$ as before. However $\sin 2\eta < 0$. Again $u(\cdot, \xi)$ is a saddle solution. □

To apply Theorem 5.3.3, one writes (4.2) as

$$\begin{aligned} 2|\mu|\xi_1 &= |\mu|(\xi_2 - \xi_1) + 2\pi d_1, \\ |\mu|(\xi_2 - \xi_1) &= |\mu|(\xi_3 - \xi_2) + 2\pi d_2, \ldots, \\ |\mu|(\xi_N - \xi_{N-1}) &= 2|\mu|(1 - \xi_N) + 2\pi d_N \end{aligned} \tag{5.21}$$

for integers $d_1, d_2, \ldots$,Critical solutions are parametrized by $d_1, d_2, \ldots, d_N$ under the constraint $0 < \xi_1 < \cdots < \xi_N < 1$. It follows that

$$2|\mu|\xi_1 = \frac{|\mu|}{N} + \frac{2\pi}{N}\sum_{k=1}^{N-1}\sum_{j=1}^{k} d_j + \frac{\pi}{N}\sum_{j=1}^{N} d_j. \tag{5.22}$$

Other ξ_k's follow from (5.21) iteratively. Below are some special cases.

Corollary 5.5. *Let (1.8) and (2.13) hold, and $B < -1$.*

1. *When $N = 1$, the partition ξ of a critical solution $u(\cdot, \xi)$ can be written as*
$$\xi_1 = \frac{1}{2} + \frac{d_1}{2}\frac{\pi}{|\mu|}$$
for some $d_1 \in \mathbb{Z}$ as long as $0 < \xi_1 < 1$. This $u(\cdot, \xi)$ is minimizing if $|\mu| + d_1\pi \in (0, \pi) \mod 2\pi\mathbb{Z}$, and $u(\cdot, \xi)$ is maximizing if $|\mu| + d_1\pi \in (\pi, 2\pi) \mod 2\pi\mathbb{Z}$.
2. *When $N = 2$, the partition ξ of a critical solution $u(\cdot, \xi)$ can be written as*
$$\xi_1 = \frac{1}{4} + \Big(\frac{3d_1 + d_2}{4}\Big)\frac{\pi}{|\mu|},\ \xi_2 = \frac{3}{4} + \Big(\frac{d_1 + 3d_2}{4}\Big)\frac{\pi}{|\mu|}$$
for some $d_1, d_2 \in \mathbb{Z}$ as long as $0 < \xi_1 < \xi_2 < 1$. This $u(\cdot, \xi)$ is minimizing if $\frac{|\mu|}{2} + (3d_1 + d_2)\frac{\pi}{2} \in (\frac{\pi}{2}, \pi) \mod 2\pi\mathbb{Z}$, $u(\cdot, \xi)$ is maximizing if $\frac{|\mu|}{2} + (3d_1 + d_2)\frac{\pi}{2} \in (\pi, \frac{3\pi}{2}) \mod 2\pi\mathbb{Z}$, and $u(\cdot, \xi)$ is a saddle point if $\frac{|\mu|}{2} + (3d_1 + d_2)\frac{\pi}{2} \in (-\frac{\pi}{2}, \frac{\pi}{2}) \mod 2\pi\mathbb{Z}$.
3. *If ξ is the equidistant partition, i.e. $\xi_k = \frac{2k-1}{2N}$, then $u(\cdot, \{\frac{2k-1}{2N}\})$ is minimizing if $\frac{|\mu|}{N} \in (\pi - \frac{\pi}{N}, \pi) \mod 2\pi\mathbb{Z}$, $u(\cdot, \{\frac{2k-1}{2N}\})$ is maximizing if $\frac{|\mu|}{N} \in (\pi, \pi + \frac{\pi}{N}) \mod 2\pi\mathbb{Z}$, and $u(\cdot, \{\frac{2k-1}{2N}\})$ is a saddle if $\frac{|\mu|}{N} \in (-\pi + \frac{\pi}{N}, \pi - \frac{\pi}{N}) \mod 2\pi\mathbb{Z}$.*

Proof. When $N = 1$, by (5.22) all the critical ξ's are

$$\xi_1 = \frac{1}{2} + \frac{d_1}{2}\frac{\pi}{|\mu|},\quad d_1 \in \mathbb{Z},\ \text{under the constraint } 0 < \xi_1 < 1. \tag{5.23}$$

Here $2|\mu|\xi_1 = |\mu| + d_1\pi$ and part 1 follows from Theorem 5.4.2.

When $N = 2$, by (5.22) and (5.21) all the critical ξ's are

$$\xi_1 = \frac{1}{4} + \Big(\frac{3d_1 + d_2}{4}\Big)\frac{\pi}{|\mu|},\ \xi_2 = \frac{3}{4} + \Big(\frac{d_1 + 3d_2}{4}\Big)\frac{\pi}{|\mu|},$$
$$d_1, d_2 \in \mathbb{Z},\ \text{under the constraint } 0 < \xi_1 < \xi_2 < 1.$$

Here $2|\mu|\xi_1 = \frac{|\mu|}{2} + (3d_1 + d_2)\frac{\pi}{2}$ and part 2 follows from Theorem 5.4.2.

For the equidistant partition, since $\xi_1 = \frac{1}{2N}$, part 3 again follows from Theorem 5.4.2. □

Note that if N and μ are fixed and μ satisfies (2.13), we can take B to be sufficiently negative, so that by Theorem 3.1.2 all presolutions are jump discontinuous solutions. The critical solutions can be parametrized by integers, like d_1 when $N = 1$ and (d_1, d_2) when $N = 2$. Then the locations of the discontinuities ξ_j are determined by these integer parameters and μ as in (5.22).

Fig. 2 shows the discontinuity location ξ_1 of each critical solution versus $|\mu|$ when $N = 1$. A green curve denotes a minimizing solution and a red curve denotes a maximizing solution according to Corollary 5.5.1.

Fig. 3 shows pairs of (d_1, d_2) when $N = 2$. Some do not yield critical solutions because they violate the constraint $0 < \xi_1 < \xi_2 < 1$; others do give rise to critical solutions if B is chosen sufficiently negative. These critical solutions are classified into minima, maxima and saddle points according to Corollary 5.5.2.

Finally we give the formula of the energy of the equal distance critical solutions in terms of N, the number of discontinuous points. The two cases, $B > 0$ and $B < -1$, again contrast sharply.

Proposition 5.6. *Let (1.8) and (2.13) hold, and $u(\cdot, \xi)$ be a critical solution.*

1. *If $B > 0$, then $\xi_k = \frac{2k-1}{2N}$, $k = 1, 2, \ldots, N$, and*
$$I\Big(u\Big(\cdot, \Big\{\frac{2k-1}{2N}\Big\}\Big)\Big) = \frac{N}{\mu(B+1)}\tanh\frac{\mu}{2N}.$$
This quantity increases as N increases.
2. *If $B < -1$, then*
$$I(u(\cdot, \xi)) = \frac{N}{|\mu|(B+1)}\tan|\mu|\xi_1.$$

Proof. We compute the energy of $u(\cdot, \xi)$ by (4.1). If $B < -1$, using (4.11) and (4.13), we find

$$\begin{aligned} I(u(\cdot, \xi)) &= \frac{B^{-1}}{2(B^{-1}+1)} - \frac{1}{2(B^{-1}+1)}\Big(-\frac{1}{|\mu|B}\tan|\mu|\xi_1 + \frac{\xi_1}{B} \\ &\quad - \frac{2}{|\mu|B}\tan\frac{|\mu|(\xi_2 - \xi_1)}{2} + \frac{\xi_2 - \xi_1}{B} - \cdots\Big) \\ &= \frac{1}{2(B+1)} + \frac{1}{2(B+1)}\Big(\frac{1}{|\mu|}\tan|\mu|\xi_1 \\ &\quad + \sum_{k=2}^{N}\frac{2}{|\mu|}\tan\frac{|\mu|(\xi_k - \xi_{k-1})}{2} \\ &\quad + \frac{1}{|\mu|}\tan|\mu|(1 - \xi_N) - 1\Big) \\ &= \frac{1}{2|\mu|(B+1)}\Big(\tan|\mu|\xi_1 + \sum_{k=2}^{N} 2\tan\frac{|\mu|(\xi_k - \xi_{k-1})}{2} \\ &\quad + \tan|\mu|(1 - \xi_N)\Big) \end{aligned}$$

By (4.2) the above is simplified to

$$I(u(\cdot, \xi)) = \frac{N}{|\mu|(B+1)}\tan|\mu|\xi_1, \tag{5.24}$$

which proves part 2.

For part 1, since $B > 0$, we have $\xi_1 = \frac{1}{2N}$, $\mu > 0$, and

$$I(u(\cdot, \xi)) = \frac{N}{\mu(B+1)}\tanh\frac{\mu}{2N}. \tag{5.25}$$

by a similar computation. This quantity increases as N increases because the function $x \to \frac{\tanh x}{x}$ is decreasing. □

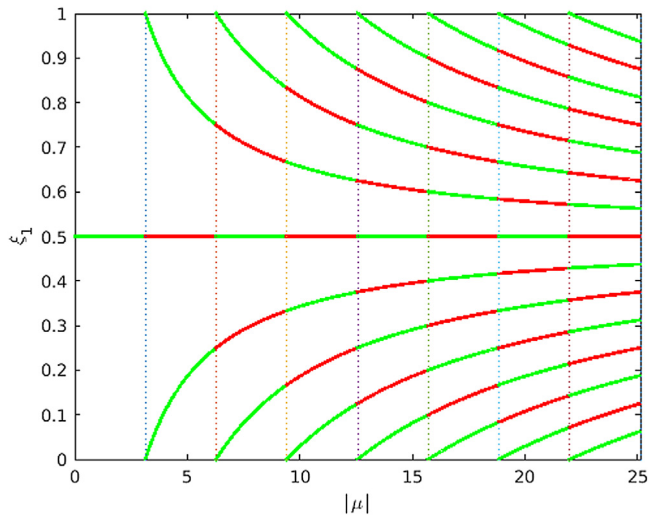

Fig. 2. $B < -1$ and $N = 1$. For each $|\mu|$ satisfying (2.13) if B is sufficiently negative, every pre-solution is a jump discontinuous solution. The discontinuity location ξ_1 is plotted against $|\mu|$ for all critical solutions parametrized by d_1. Minimizing solutions are plotted in green and maximizing ones in red according to Corollary 5.5.1. The dotted vertical lines are at $|\mu| = k\pi$, $k = 0, 1, 2, \ldots$, which are values excluded by (2.13). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

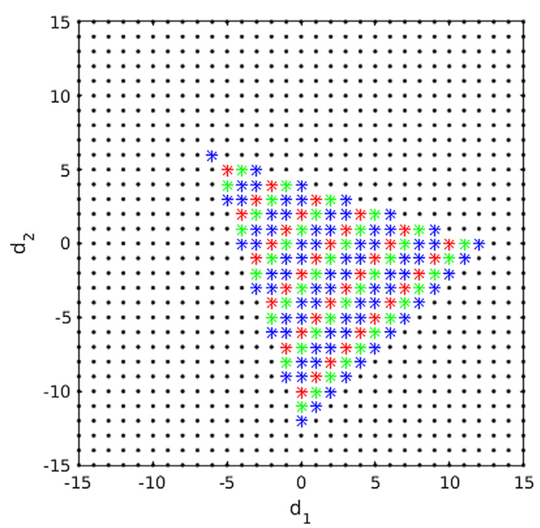

Fig. 3. $B < -1$ and $N = 2$. The lattice of integer pairs (d_1, d_2) with $\mu = 40\,\mathrm{i}$. A dot denotes an integer pair (d_1, d_2) that corresponds to no critical solution; an asterisk represents a pair (d_1, d_2) that yields a critical solution if B is chosen sufficiently negative. A green asterisk represents a minimizing solution, a blue asterisk represents a saddle solution, and a red asterisk represents a maximizing solution . (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

In the case $B < -1$, the energy of the equidistant critical solution is

$$I\Big(u\Big(\cdot, \Big\{\frac{2k-1}{2N}\Big\}\Big)\Big) = \frac{N}{|\mu|(B+1)} \tan \frac{|\mu|}{2N}. \tag{5.26}$$

This quantity is not monotone with respect to N.

6. $B = 0$

Now we consider kernel J given in (1.12). For this J,

$$j(x) = 0. \tag{6.1}$$

Let $v = J[u]$. Then v satisfies

$$-Av'' = u - \overline{u}, \; v'(0) = v'(1) = 0, \; \overline{v} = 0. \tag{6.2}$$

Here a bar over a function denotes its average: $\overline{u} = \int_0^1 u(x)\,dx$, $\overline{v} = \int_0^1 v(x)\,dx$. If u is a solution of (2.4), we have

$$\begin{cases} -v + u - \mathrm{sgn}(u) = 0 \\ -Av'' = u - \overline{u} \\ v'(0) = v'(1) = 0 \end{cases} \tag{6.3}$$

Given a partition ξ a pre-solution u of the partition ξ is now a solution of

$$-J[u] + u - \sum_{k=1}^{N+1} (-1)^k \chi_{(\xi_{k-1},\xi_k)} = 0 \tag{6.4}$$

With $v = J[u]$, (6.4) implies

$$u = v + \sum_{k=1}^{N+1} (-1)^k \chi_{(\xi_{k-1},\xi_k)} \tag{6.5}$$

and

$$-Av'' - v = \sum_{k=1}^{N+1} (-1)^k \chi_{(\xi_{k-1},\xi_k)} - \sum_{k=1}^{N+1} (-1)^k (\xi_k - \xi_{k-1}),$$
$$v'(0) = v'(1) = 0. \tag{6.6}$$

The solution of (6.6) can be written as

$$v(x,\xi) = \int_0^1 G(x,y)\Big(\sum_{k=1}^{N+1} (-1)^k \chi_{(\xi_{k-1},\xi_k)}(y) - \sum_{k=1}^{N+1} (-1)^k (\xi_k - \xi_{k-1})\Big)\,dy \tag{6.7}$$

where G is the same as (2.16) with

$$\mu = \sqrt{-\frac{1}{A}}, \quad |\mu| = \sqrt{\frac{1}{A}} \tag{6.8}$$

Again we must assume that

$$|\mu| \neq k\pi, \; k = 1, 2, 3, \ldots \tag{6.9}$$

Note that

$$\int_0^1 G(x,y)\,dy = -1. \tag{6.10}$$

We use A_ξ to denote the average of $\sum_{k=1}^{N+1} (-1)^k \chi_{(\xi_{k-1},\xi_k)}$:

$$A_\xi = \int_0^1 \sum_{k=1}^{N+1} (-1)^k \chi_{(\xi_{k-1},\xi_k)}(x)\,dx = \sum_{k=1}^{N+1} (-1)^k (\xi_k - \xi_{k-1}) \tag{6.11}$$

A pre-solution is not necessarily a jump discontinuous solution. For equidistant pre-solutions we have the following.

Proposition 6.1. *Let ξ be the equidistant partition, $\xi_k = \frac{2k-1}{2N}$. Then the pre-solution of $\{\frac{2k-1}{2N}\}_{k=1}^N$ is a jump discontinuous solution of (2.4) if and only if $\frac{|\mu|}{N} \in (0, \pi)$.*

Proof. For the equidistant partition ξ, $A_\xi = 0$. Then the solution v of (6.6) has the symmetry properties as in Corollary 2.2. One finds that

$$v(x) = 1 - \frac{1}{\cos \frac{|\mu|}{2N}} \cos |\mu| x, \quad x \in \Big(0, \frac{1}{2N}\Big). \tag{6.12}$$

By (6.5)

$$u(x) = -\frac{1}{\cos \frac{|\mu|}{2N}} \cos |\mu| x, \quad x \in \Big(0, \frac{1}{2N}\Big). \tag{6.13}$$

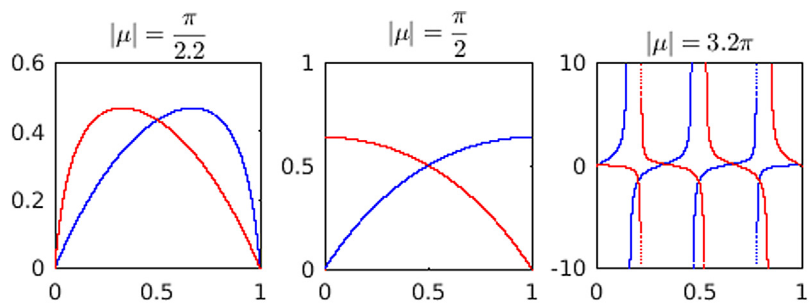

Fig. 4. Curves of $(1-\xi_1)\tan|\mu|\xi_1$ and $\xi_1 \tan|\mu|(1-\xi_1)$. The horizontal coordinate of each intersection point is ξ_1 of a critical solution with $N = 1$. The equidistant one, $\xi_1 = \frac{1}{2}$, is always a critical solution. There are more critical solutions when $|\mu|$ becomes larger.

To be a jump discontinuous solution, u cannot change sign on $(0, \frac{1}{2N})$. Hence $|\mu|x \in (0, \frac{\pi}{2})$ for all $x \in (0, \frac{1}{2N})$. This means $\frac{|\mu|}{N} \in (0, \pi)$. Conversely if $\frac{|\mu|}{N} \in (0, \pi)$, then (6.13) is negative on $[0, \frac{1}{2N}]$ and u is a jump discontinuous solution. □

By (6.7) a pre-solution $v(\cdot, \xi)$ satisfies

$$\frac{\partial v}{\partial \xi_j} = 2(-1)^j\big(G(x, \xi_j) + 1\big) \tag{6.14}$$

Now suppose that the pre-solution $u(\cdot, \xi)$ is also a jump discontinuous solution. Write $I(\xi)$ for $I(u(\cdot, \xi))$. Calculations show

$$\frac{\partial I}{\partial \xi_j} = 2(-1)^{j+1} v(\xi_j, \xi) \tag{6.15}$$

$$\frac{\partial^2 I}{\partial \xi_j \partial \xi_k} = 4(-1)^{j+k+1}\big(G(\xi_j, \xi_k) + 1\big) \tag{6.16}$$

$$\frac{\partial^2 I}{\partial \xi_j^2} = -4\big(G(\xi_j, \xi_j) + 1\big) + 2(-1)^{j+1} v'(\xi_j, \xi) \tag{6.17}$$

Then u is a critical solution if and only if $v(\xi_j, \xi) = 0$ for every j.

Theorem 6.2. *Let (6.9) hold and the pre-solution $u(\cdot, \xi)$ is a jump discontinuous solution. Then $u(\cdot, \xi)$ is a critical solution if and only if none of $|\mu|\xi_1$, $\frac{|\mu|(\xi_2-\xi_1)}{2}$, $\frac{|\mu|(\xi_3-\xi_2)}{2}$, …, $|\mu|(1-\xi_N)$ can be written as $\frac{\pi}{2} + n\pi$ for some non-negative integer n, and*

$$\begin{aligned}(1 + A_\xi)\tan|\mu|\xi_1 &= (1 - A_\xi)\tan\frac{|\mu|(\xi_2 - \xi_1)}{2} \\ &= (1 + A_\xi)\tan\frac{|\mu|(\xi_3 - \xi_2)}{2} = \cdots\end{aligned} \tag{6.18}$$

where $A_\xi = \sum_{k=1}^{N+1}(-1)^k(\xi_k - \xi_{k-1})$.

Proof. Let $u(\cdot, \xi)$ be a critical solution. Then $v(\xi_k, \xi) = 0$, $k = 1, 2, \ldots, N$, by (6.15). On $(0, \xi_1)$ the same argument as in the proof of Theorem 4.4.3 shows that $|\mu|\xi_1$ is not $\frac{\pi}{2} + n\pi$ for some non-negative integer n. Then, since $v'(0) = v(\xi_1) = 0$,

$$v(x) = 1 + A_\xi - \frac{1 + A_\xi}{\cos|\mu|\xi_1}\cos|\mu|x, \quad x \in (0, \xi_1). \tag{6.19}$$

On (ξ_1, ξ_2), $\frac{|\mu|(\xi_2-\xi_1)}{2}$ is not $\frac{\pi}{2} + n\pi$ for some non-negative integer n, and since $v(\xi_1) = v(\xi_2) = 0$,

$$v(x) = -1 + A_\xi - \frac{-1 + A_\xi}{\cos\frac{|\mu|(\xi_2-\xi_1)}{2}}\cos\Big(|\mu|(x - \frac{\xi_1 + \xi_2}{2})\Big), \quad x \in (\xi_1, \xi_2). \tag{6.20}$$

Continue this on each (ξ_{k-1}, ξ_k) and $(\xi_N, 1)$ and use the continuity of v' at ξ_k to deduce (6.18).

Conversely, if none of $|\mu|\xi_1$, $\frac{|\mu|(\xi_2-\xi_1)}{2}$, $\frac{|\mu|(\xi_3-\xi_2)}{2}$, …, $|\mu|(1-\xi_N)$ can be written as $\frac{\pi}{2} + n\pi$ for some non-negative integer n, and (6.18) holds, then one defines v by (6.19) and (6.20), etc. The resulting function is continuously differentiable on $(0, 1)$ by (6.18), so it is a solution of (6.6) and therefore a pre-solution. By the assumption of the theorem, the corresponding $u(\cdot, \xi)$ from (6.5) is a jump discontinuous solution. Then, since $v(\xi_k) = 0$, u is also a critical solution. □

If ξ is an equidistant partition: $\xi_k = \frac{2k-1}{2N}$, $k = 1, 2, \ldots, N$, then $A_\xi = 0$ and (6.18) holds. For non-equidistant critical solutions, (6.18) implies that

$$\begin{aligned}\xi_1 &\equiv \frac{\xi_3 - \xi_2}{2} \equiv \frac{\xi_5 - \xi_4}{2} \equiv \cdots \mod \Big(\frac{\pi}{|\mu|}\Big)\mathbb{Z}, \\ \frac{\xi_2 - \xi_1}{2} &\equiv \frac{\xi_4 - \xi_3}{2} \equiv \frac{\xi_6 - \xi_5}{2} \equiv \cdots \mod \Big(\frac{\pi}{|\mu|}\Big)\mathbb{Z}.\end{aligned} \tag{6.21}$$

Between adjacent intervals the relationship is more complex. Take $N = 1$ for example. Then (6.18) means $(1 + A_\xi)\tan|\mu|\xi_1 = (1 - A_\xi)\tan|\mu|(1 - \xi_1)$, which is equivalent to

$$(1 - \xi_1)\tan|\mu|\xi_1 = \xi_1 \tan|\mu|(1 - \xi_1). \tag{6.22}$$

One can plot the left and right sides of (6.22) against ξ_1 and a solution of (6.22) corresponds to an intersection point of the two curves; see Fig. 4. As in the $B < -1$ case, the larger $|\mu|$ is the more critical solutions there are.

Now we proceed to find minimizing solutions. We shall only consider equidistant pre-solutions. By Proposition 6.1 we only need to consider $\frac{|\mu|}{N} \in (0, \pi)$ which is the necessary and sufficient condition for a pre-solution to be a jump discontinuous solution.

Proposition 6.3. *Let (6.9) hold, ξ be the equidistant partition $\{\frac{2k-1}{2N}\}_{k=1}^N$, and u be the pre-solution of $\{\frac{2k-1}{2N}\}_{k=1}^N$.*

1. *Let $N = 1$. If $|\mu| \in (0, \pi)$, then u is a minimizing solution.*
2. *Let $N \geq 2$.*
   - *(a) If $|\mu| \in (0, \pi - \frac{\pi}{N})$, then u is a saddle solution.*
   - *(b) If $|\mu| \in (\pi - \frac{\pi}{N}, \pi)$, then u is a minimizing solution*

Proof. Recall the matrix $\mathbf{G}$ and α, β, and γ in (5.1)–(5.3). For equidistant partitions

$$\alpha = |\mu|\cot\frac{|\mu|}{N}, \quad \beta = |\mu|\csc\frac{|\mu|}{N}, \quad \gamma = |\mu|\tan\frac{|\mu|}{2N}. \tag{6.23}$$

The eigenvalues of $\mathbf{G}$ are found in Lemma 5.2. However here we have to consider the matrix $\mathbf{G} + \mathbf{J}$ where $\mathbf{J}$ is the $N \times N$ matrix whose entries are all equal to 1. Let $\vec{p} = (1, 1, \ldots, 1)^T$. Then $\vec{p}$ is an eigenvector of the matrix Q of (5.4) and the associated eigenvalue is 2. Consequently $\vec{p}$ is also an eigenvector of T of (5.10), and an eigenvector of $\mathbf{G}$ with the associated eigenvalue equal to $\frac{1}{A(2\alpha - 2\beta)}$. Moreover $\vec{p}$ is also an eigenvector of $\mathbf{J}$, so it is an eigenvector of $\mathbf{G}+\mathbf{J}$ whose corresponding eigenvalue is

$$\frac{1}{A(2\alpha - 2\beta)} + N. \tag{6.24}$$

Furthermore if $\vec{q} \perp \vec{p}$, then $\mathbf{J}\vec{q} = \vec{0}$. Hence the remaining eigenvectors of $\mathbf{G}$ are also eigenvectors of $\mathbf{G} + \mathbf{J}$ with the same eigenvalues. As in Theorem 5.3.2 we find

$$v'(\xi_j) = (-1)^{j+1}\gamma \tag{6.25}$$

Then the eigenvalues of $H(\xi)$, the Hessian of I at ξ, are

$$-\frac{4}{A(2\alpha - 2\beta)} - 4N + 2\gamma, \quad -\frac{4}{A(2\alpha - q_k\beta)} + 2\gamma, \; k = 1, 2, \ldots, N-1 \tag{6.26}$$

Recall that $q_N = 2$ is the largest among the q_k's.

Let $\eta = \frac{|\mu|}{2N} \in (0, \frac{\pi}{2})$. By (6.23)

$$\begin{aligned} -\frac{4}{A(2\alpha - 2\beta)} - 4N + 2\gamma &= \frac{-2|\mu|}{\cot 2\eta - \csc 2\eta} + 2|\mu|\tan\eta - 4N \\ &= 2|\mu| \frac{2 - 2\cos 2\eta}{(1 - \cos 2\eta)\sin 2\eta} - 4N \\ &= 4|\mu| \left(\frac{1}{\sin 2\eta} - \frac{1}{2\eta} \right) \\ &> 0 \end{aligned}$$

since $\sin 2\eta < 2\eta$ when $2\eta \in (0, \pi)$. So the first eigenvalue in (6.26) is positive. If $N = 1$, this is the only eigenvalue, and u is a minimizing solution.

If $N \geq 2$, consider the other eigenvalues

$$-\frac{4}{A(2\alpha - q_k\beta)} + 2\gamma = 2|\mu| \left(\frac{2 + q_k - (2 + q_k)\cos 2\eta}{(q_k - 2\cos 2\eta)\sin 2\eta} \right), \quad k = 1, 2, \ldots, N-1,$$

as in (5.20). Since $2\eta \in (0, \pi)$ the sign of each eigenvalue here is the same as the sign of $q_k - 2\cos 2\eta$. Take q_k to be $q_1 = -2\cos\frac{\pi}{N}$, the smallest among the q_k's. Then

$$q_1 - 2\cos 2\eta = -2\cos\frac{\pi}{N} - 2\cos 2\eta. \tag{6.27}$$

If $2\eta \in (0, \pi - \frac{\pi}{N})$, (6.27) is negative. Then $H(\xi)$ has one negative eigenvalue in addition to the positive eigenvalue discussed before, so the equidistant critical solution is a saddle solution. If $2\eta \in (\pi - \frac{\pi}{N}, \pi)$, then (6.27) is positive, and all $q_k - 2\cos 2\eta > 0$ since other q_k's are greater than q_1. Hence all eigenvalues of $H(\xi)$ are positive and the equidistant critical solution is a minimizing solution □

7. Discussion

Here we show a connection between the FitzHugh–Nagumo system and (1.1) with a sign changing kernel. We also explain how the fractional Laplacian operator can enter the picture. The kernel (7.14) that arises in this section changes sign, in this respect similar to the kernel (1.9) studied in this paper. On the other hand, (7.14) is much more singular, and hence has a smoothing property. Solutions to its Euler–Lagrange equation cannot be discontinuous.

The FitzHugh–Nagumo system models excitable systems such as neuron fields [13,14]. It is one of the best known examples that include growth and inhibition properties. To see the two properties we need a more sophisticated version of the FitzHugh–Nagumo system where both the membrane voltage variable u and the recovery variable v are allowed to diffuse in space; namely

$$\begin{cases} u_t = d\Delta u - u(u - 1/2)(u - 1) + a - bv \\ \tau v_t = \Delta v - v + u \end{cases} \tag{7.1}$$

where u and v satisfy the zero Neumann boundary condition on the boundary of the domain. Consider the steady states of the system which satisfies

$$\begin{cases} \begin{cases} d\Delta u - u(u - 1/2)(u - 1) + a - bv = 0 \\ \Delta v - v + u = 0 \end{cases} \text{in } D, \\ \begin{cases} \partial_\nu u = 0 \\ \partial_\nu v = 0 \end{cases} \text{on } \partial D \end{cases} \tag{7.2}$$

where D is a domain in $\mathbb{R}^n$ and ∂_ν denotes outward pointing normal derivative. By solving the second equation for v, which yields $v = (-\Delta + 1)^{-1}u$ and inserting it into the first equation one obtains

$$-d\Delta u + u(u - 1/2)(u - 1) - a + b(-\Delta + 1)^{-1}u = 0. \tag{7.3}$$

Eq. (7.3) has a variational structure. Any solution of (7.3) is a critical point of the functional

$$E(u) = \int_D \left(\frac{d}{2}|\nabla u|^2 + \frac{u^2(1-u)^2}{4} - au + \frac{b}{2}u(-\Delta + 1)^{-1}u \right) dx. \tag{7.4}$$

Let G be Green's function of the $-\Delta + 1$ operator with Neumann boundary condition so that

$$\left((-\Delta + 1)^{-1}u \right)(x) = \int_D G(x, y)u(y)\, dy. \tag{7.5}$$

The last term in (7.4) can be rewritten as

$$\int_D u(-\Delta + 1)^{-1}u\, dx = -\frac{1}{2}\int_D \int_D G(x, y)(u(x) - u(y))^2\, dx\, dy + \int_D u^2(x)\, dx, \tag{7.6}$$

and (7.4) becomes

$$E(u) = \int_D \frac{d}{2}|\nabla u|^2\, dx - \int_D \int_D \frac{b}{4}G(x, y)(u(x) - u(y))^2\, dx\, dy + \int_D \left(\frac{u^2(1-u)^2}{4} - au + \frac{b}{2}u^2 \right) dx. \tag{7.7}$$

In applications u is typically a phase field variable and $\mathcal{F}(u)$ is the free energy of the field. The first two terms in (7.7) model two kinds of self-interaction energy of u and the third term is a bulk energy. The two interaction energy terms are effective on different length scales and their preferences are opposite. The first term is local and favors phase fields that are close to constant. The second term is nonlocal and, because of the negative sign in front, benefits from more oscillatory fields, because of the negative sign in front.

Because the two interaction terms are of different mathematical types; the first is a local gradient term and the second is a nonlocal integral term, (7.7) is a hybrid model. In this paper we would like to have a more unified approach where interaction is given by a single term. For this we re-examine the derivation of the gradient term. The gradient term often arises as an approximation of a nonlocal interaction like the second term in (7.7), but with an opposite sign. Let $S(z), z \in \mathbb{R}^n$ be a non-negative and radially symmetric function and consider

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} S(x - y)(u(x) - u(y))^2\, dxdy, \tag{7.8}$$

where we have taken D to be $\mathbb{R}^n$ for simplicity. If we expand $u(y)$ about x and approximate $u(y) \approx u(x) + \nabla u(x) \cdot (y - x)$, then

$$\begin{aligned} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} S(x - y)(u(x) - u(y))^2\, dxdy &\approx \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} S(x - y) \times |\nabla u(x) \cdot (y - x)|^2\, dxdy \\ &= \sigma \int_{\mathbb{R}^n} |\nabla u(x)|^2\, dx \end{aligned}$$

where

$$\sigma = \omega_n \int_0^\infty S(r)r^{n+1}\, dr$$

and ω_n is the volume of the unit ball in $\mathbb{R}^n$. This approximation is more accurate if S has small support.

Thus if we replace the first term in (7.7) by (7.8), then the first two terms can be combined into a single term and we arrive at (1.1). In the case that $D = \mathbb{R}^n$, it is reasonable to assume that J is a function of $|x-y|$, unless the self-interaction of the field u has a preferred direction. If (1.1) is used to replace (7.7), we should let

$$J(z) = \frac{2d}{\sigma} S(z) - b\,\Gamma(z) \tag{7.9}$$

where Γ is the fundamental solution of $-\Delta + 1$ on $\mathbb{R}^n$. It is known that

$$\Gamma(z) = \frac{1}{(2\pi)^{n/2}} |z|^{1-n/2} K_{\frac{n}{2}-1}(|z|) \tag{7.10}$$

where $K_{\frac{n}{2}-1}$ is the $\frac{n}{2}-1$ order modified Bessel's function of the second kind.

We must assume that $S(z)$ is more singular than $\Gamma(z)$ when z is at 0, so $J(z)$ is positive if z is close to 0. When z is away from 0, $J(z)$ may or may not stay positive. This point is illustrated in the following example.

Recent years have seen intense efforts to use the fractional Laplacian operator $(-\Delta)^s$ in place of the usual $-\Delta$ in many elliptic and parabolic problems [15–18]. For $s \in (0,1)$, one defines the singular integral

$$(-\Delta)^s u(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy \tag{7.11}$$

where $c_{n,s}$ is a constant depending on n and s. This expression follows from an interaction energy of the form (7.8) with

$$S(z) = \frac{c_{n,s}}{|z|^{n+2s}}. \tag{7.12}$$

If we are interested in using a fractional Laplacian, the hybrid equation (7.3) should be replaced by

$$d(-\Delta)^s u + b(-\Delta+1)^{-1} u + u(u-1/2)(u-1) - a = 0 \tag{7.13}$$

on $\mathbb{R}^n$. The integral equation (7.13) is precisely the Euler–Lagrange equation of (1.1) with $D = \mathbb{R}^n$,

$$F(u) = \frac{u^2(1-u)^2}{4} - au + \frac{b}{2}u^2,$$

and

$$J(z) = \frac{d c_{n,s}}{|z|^{n+2s}} - \frac{b}{(2\pi)^{n/2}} |z|^{1-n/2} K_{\frac{n}{2}-1}(|z|). \tag{7.14}$$

Note that $K_{\frac{n}{2}-1}(|z|)$ is positive and is of order $|z|^{1-\frac{n}{2}}$ (in the case $n \geq 3$) when $|z|$ is small, so as $|z| \to 0$ the first term in (7.14) tends to ∞ like $|z|^{-(n+2s)}$ and the second term tends to ∞ like $|z|^{-(n-2)}$. The first term dominates and the difference $J(z)$ tends to ∞ like $|z|^{-(n+2s)}$. When $z \to \infty$, $K_{\frac{n}{2}-1}(|z|)$ tends to 0 like $z^{-1/2}e^{-z}$ so $J(z)$ tends to 0 like $z^{-(n+2s)}$ but stays positive when z is sufficiently large. However if b/d is large, $J(z)$ is negative for some z away from 0 and ∞.

References

[1] P.W. Bates, P.C. Fife, X. Ren, X. Wang, Traveling waves in a convolution model for phase transitions, Arch. Ration. Mech. Anal. 138 (2) (1997) 105–136.
[2] G. Alberti, G. Bellettini, A non-local anisotropic model for phase transitions: asymptotic behavior of rescaled energies, Eur. J. Appl. Math. 9 (3) (1998) 261–264.
[3] H. Berestycki, N. Rodríguez, Non-local reaction-diffusion equation with a gap, Discrete Contin. Dyn. Syst. 37 (2) (2017) 685–723.
[4] J. Coville, J. Davila, S. Martínez, Nonlocal anisotropic dispersal with monostable nonlinearity, J. Differential Equations 244 (12) (2006) 3080–3118.
[5] P.C. Fife, X. Wang, A convolution model for interfacial motion: the generation and propagation of internal layers in higher space dimensions, Adv. Differential Equations 3 (1) (1998) 85–110.
[6] X. Ren, M. Winter, Young measures in a nonlocal phase transition problem, Proc. Roy. Soc. Edinburgh Sect. A 127 (3) (1997) 615–637.
[7] S.E. Allen, J.W. Cahn, A microscopic theroy for antiphase boundary motion and its application to antiphase domain coarsening, Acta. Metall. 27 (6) (1979) 1085–1095.
[8] R. Lipton, Cohesive dynamics and brittle fracture, J. Elasticity 124 (2) (2015) 143–191.
[9] S.A. Silling, Reformulation of elasticity theory for discontinuities and long-range forces, J. Mech. Phys. Solids 48 (1) (2000) 175–209.
[10] S.A. Silling, M. Epton, O. Weckner, J. Xu, E. Askari, Peridynamic states and constitutive modeling, J. Elasticity 88 (2) (2007) 151–184.
[11] R.G. Casten, C.J. Holland, Instability results for reaction diffusion equations with Neumann boundary conditions, J. Differential Equations 27 (2) (1978) 266–273.
[12] X. Ren, J. Wei, On the spectra of 3-D lamellar solutions of the diblock copolymer problem, SIAM J. Math. Anal. 35 (1) (2003) 1–32.
[13] R.A. FitzHugh, Impulses and physiological states in theoretical models of nerve membrane, Biophys. J. 1 (1961) 445–466.
[14] J. Nagumo, S. Arimoto, Y. Yoshikawa, An active pulse transmission line simulating nerve axton, Proc. IRE 50 (1962) 2061.
[15] L. Caffarelli, L. Silvestre, Regularity theory for fully nonlinear integro-differential equations, Comm. Pure Appl. Math. 62 (5) (2009) 597–638.
[16] L. Caffarelli, L. Silvestre, The Evans-Krylov theorem for nonlocal fully nonlinear equations, Ann. of Math. 174 (2) (2011) 1163–1187.
[17] L. Caffarelli, C.-H. Chan, A. Vasseur, Regularity theory for parabolic nonlinear integral operators, J. Amer. Math. Soc. 24 (3) (2011) 849–869.
[18] L. Caffarelli, T. Jin, Y. Sire, J. Xiong, Local analysis of solutions of fractional semi-linear elliptic equations with isolated singularities, Arch. Ration. Mech. Anal. 213 (1) (2014) 245–268.