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# Kloosterman sums and Fourier coefficients of Eisenstein series

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**Abstract** We derive explicit formulas for some Kloosterman sums on $\Gamma_0(N)$, and for the Fourier coefficients of Eisenstein series attached to arbitrary cusps, around a general Atkin–Lehner cusp.

**Keywords** Kloosterman sums · Atkin–Lehner cusp · Eisenstein series at cusps

**Mathematics Subject Classification** 11L05 · 11F03 · 11M36

## 1 Motivation

Many problems in analytic number theory rely on the spectral theory of automorphic forms for $GL_2$. For instance, it is desirable to obtain cancellation in sums of Kloosterman sums, a goal first achieved by Kuznetsov [12] for $\sum_{c \leq X} \frac{S(m,n;c)}{c}$ with the key tool being his famous spectral decomposition of sums of Kloosterman sums. For arithmetical applications, one typically encounters Kloosterman sums with additional constraints. As an example, for the problem of estimating the fifth moment of modular $L$-functions in the level aspect [11], we encounter

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$$\sum_{\substack{(c,N)=1 \\ c\equiv 0 \ (\mathrm{mod}\ q)}} S(\overline{N}m, n; c) f(c), \tag{1.1}$$

where $f$ is a smooth function on the positive reals with sufficient decay.

Deshouillers and Iwaniec [4] greatly generalized Kuznetsov's arguments and in particular calculated $S_{\mathfrak{a}\mathfrak{b}}(m, n; c)$ for a variety of pairs of cusps $\mathfrak{a}$, $\mathfrak{b}$ for $\Gamma_0(N)$. In order to apply the powerful spectral methods to a sum of Kloosterman sums (such as (1.1)), it is necessary to recognize $S(\overline{N}m, n; c)$ as an $S_{\mathfrak{a}\mathfrak{b}}(m, n; c)$ for some congruence group and choices of cusps. Moreover, the arithmetical conditions on $c$ need to correspond to the lower-left entries occurring in the corresponding double coset decomposition (as in [8, Sect. 2.5]). There is no obvious way to directly solve this recognition problem, and the only known method is to calculate many examples of $S_{\mathfrak{a}\mathfrak{b}}(m, n; c)$. In an unpublished paper, Motohashi [13] showed that (1.1) can be realized as the sum of Kloosterman sums associated to the pair of cusps $\infty$, $1/q$ for the group $\Gamma_0(Nq)$. One goal of this paper is to generalize this calculation, thus providing more flexibility in the sums of Kloosterman sums that one can treat using spectral theory. Our Kloosterman sum formula appears as Theorem 2.7 below.

The Kloosterman sum $S_{\mathfrak{a}\mathfrak{b}}(m, n; c)$ implicitly depends on the scaling matrices one chooses for the cusps $\mathfrak{a}$, $\mathfrak{b}$ (see Sect. 2.1 for definitions). Moreover, on the spectral side of the Bruggeman–Kuznetsov formula, the Fourier coefficients also depend on the choice of scaling matrices. A nice feature of Motohashi's formula for $S_{\infty,1/r}(m, n; c)$ is that he chooses Atkin–Lehner operators as scaling matrices. This is important because it means that for an eigenform of the Atkin–Lehner operators, the Fourier coefficients around the cusp $1/r$ are proportional to the Fourier coefficients around the cusp $\infty$.

For more advanced problems in analytic number theory, it is often important to prove extra cancellation after the usage of the Bruggeman–Kuznetsov formula, aided by additional summation variables involving some combination of $m, n, N, q$. In the context of the fifth moment problem of [11], we encountered sums over $m$ and $n$ weighted by the divisor function. One is therefore led to study the Dirichlet series $\sum_n \tau(n)\rho_{\mathfrak{a}}(n)n^{-s}$, where $\tau(n)$ is the divisor function, and $\rho_{\mathfrak{a}}(n)$ are the Fourier coefficients of a cusp form around the cusp $\mathfrak{a}$. In turn, to relate this Dirichlet series to $L$-functions, one would like $\rho_{\mathfrak{a}}(n)$ to be multiplicative and to be related to $\rho_\infty(n)$. In case $\mathfrak{a}$ is equivalent to $\infty$ under an Atkin–Lehner operator, and provided the cusp form is an eigenform of the Atkin–Lehner operators, then these goals are accomplished. This property was crucially used in [11] when studying (1.1). Motivated by this, we systematically choose Atkin–Lehner operators as scaling matrices.

Another problem in this theme is the explicit evaluation of Fourier coefficients of Eisenstein series attached to general cusps $\mathfrak{a}$ on $\Gamma_0(N)$. These quantities appear on the spectral side of the Bruggeman–Kuznetsov formula, and for some applications (e.g., [11], where the sum (1.1) was spectrally decomposed) one needs rather precise information on these Fourier coefficients. In (3.21) below, we give a fairly compact formula for the Fourier coefficients. However, in analogy to the Dirichlet series mentioned in the previous paragraph, it is preferable to work with multiplicative Fourier coefficients. To this end, we change basis using Dirichlet characters, which leads to Theorem 3.4. Although it is an admittedly large expression, it is nevertheless useful for our intended application in [11]. Much of the work leading up to the derivation of Theorem 3.4, such as Lemma 3.5, should be useful in more general contexts.

This is a companion paper to [11] which relies on the results in this paper. The problem studied in [11] is to estimate $\sum_f L(1/2, f)^5$, where $f$ runs over a Hecke basis of modular cusp forms with fixed small weight and varying prime level. This is an example of a moment that is one larger than the "barrier" moment. In this family, the fourth moment is relatively easy, following from the large sieve inequality for modular forms, and gives back the convexity bound after dropping all but one term. There are very few results in the literature on families of $L$-functions where a sharp bound on a moment is produced that is one larger than the barrier moment. Notable examples include the cubic moments in [3] and [7], and some generalizations.

We expect that variants of the fifth moment problem will lead to more general sums of Kloosterman sums than in (1.1). Such variants could potentially include products of $L$-functions of the form $L(1/2, f \otimes \chi)$ for a fixed Dirichlet character $\chi$, or forms $f$ with nontrivial nebentypus. We therefore hope our results on explicit calculation of Kloosterman sums will be of independent interest.

## 2 Kloosterman sums

### 2.1 Definitions

We mostly follow the notation of [9]. Let $N$ be a positive integer and $\Gamma = \Gamma_0(N)$. An element $\mathfrak{a} \in \mathbb{P}^1(\mathbb{Q})$ is called a cusp. Two cusps $\mathfrak{a}$ and $\mathfrak{a}'$ are equivalent under $\Gamma$ if there is a $\gamma \in \Gamma$ satisfying $\mathfrak{a}' = \gamma\mathfrak{a}$. Let $\mathfrak{a}$ be a cusp and $\Gamma_{\mathfrak{a}} = \{\gamma \in \Gamma : \gamma\mathfrak{a} = \mathfrak{a}\}$ be the stabilizer of the cusp $\mathfrak{a}$ in $\Gamma$. A matrix $\sigma_{\mathfrak{a}} \in \mathrm{SL}_2(\mathbb{R})$, satisfying

$$\sigma_{\mathfrak{a}}\infty = \mathfrak{a}, \quad \text{and} \quad \sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \left\{\pm \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) : n \in \mathbb{Z}\right\} \tag{2.1}$$

is called a scaling matrix for the cusp $\mathfrak{a}$.

*Remarks* For two equivalent cusps $\mathfrak{a}$ and $\mathfrak{a}' = \gamma\mathfrak{a}$, with $\gamma \in \Gamma$, the stabilizers of the cusps are conjugate subgroups in $\Gamma$, namely, $\Gamma_{\mathfrak{a}'} = \gamma\Gamma_{\mathfrak{a}}\gamma^{-1}$. Furthermore, $\sigma_{\mathfrak{a}'} = \gamma\sigma_{\mathfrak{a}}$ is a scaling matrix for the cusp $\mathfrak{a}'$. The matrix $\sigma_{\mathfrak{a}}$ is not uniquely defined by the above properties: for any $\alpha \in \mathbb{R}$,

$$\sigma_{\mathfrak{a}} \left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right)$$

also satisfies (2.1). The choice of the scaling matrix $\sigma_{\mathfrak{a}}$ is important in what follows.

**Definition 2.1** Let $f$ be a Maass form for the group $\Gamma$. The Fourier coefficients of $f$ at a cusp $\mathfrak{a}$, *relative to a choice of* $\sigma_{\mathfrak{a}}$, and denoted $\rho_f(\sigma_{\mathfrak{a}}, n)$, are defined by

$$f(\sigma_{\mathfrak{a}}z) = \sum_{n \neq 0} \rho_f(\sigma_{\mathfrak{a}}, n)e(nx)W_{0,it_j}(4\pi|n|y), \tag{2.2}$$

where

$$W_{0,it_j}(4\pi y) = 2\sqrt{|y|}K_{it_j}(2\pi y).$$

*Remark* If $\sigma_{\mathfrak{a}}$ is replaced with $\sigma_{\mathfrak{a}}\left(\begin{smallmatrix}1 & \alpha\\ 0 & 1\end{smallmatrix}\right)$, then the Fourier coefficients relative to the new scaling matrix are given by

$$\rho_f(\sigma_a\left(\begin{smallmatrix}1 & \alpha\\ 0 & 1\end{smallmatrix}\right), n) = e(n\alpha)\rho_f(\sigma_{\mathfrak{a}}, n).$$

If the Fourier coefficients of $f$ at a cusp $\mathfrak{a}$ are multiplicative with respect to $\sigma_{\mathfrak{a}}$, then for another scaling matrix as above, the Fourier coefficients will typically not be multiplicative. We found the reference [6] useful for its discussions in this context.

Fourier coefficients at equivalent cusps, however, behave more predictably. If $\mathfrak{a}' = \gamma\mathfrak{a}$ and we choose the scaling matrix as $\sigma_{\mathfrak{a}'} = \gamma\sigma_{\mathfrak{a}}$, then due to $\Gamma$-invariance of $f$, we have

$$\rho_f(\sigma_{\mathfrak{a}'}, n) = \rho_f(\sigma_{\mathfrak{a}}, n).$$

When the scaling matrix $\sigma_{\mathfrak{a}}$ is understood, we may write $\rho_f(\sigma_{\mathfrak{a}}, n) = \rho_{\mathfrak{a},f}(n)$.

We now define Kloosterman sums with respect to a pair of cusps, and for general nebentypus. Let $\chi$ be a Dirichlet character modulo $N$. We extend $\chi$ to $\Gamma$ via

$$\chi : \Gamma \longrightarrow S^1$$
$$\gamma = \left(\begin{smallmatrix}a & b\\ Nc & d\end{smallmatrix}\right) \mapsto \chi(d).$$

If $\chi(-1) = (-1)^k$, then $\chi$ can be seen as a multiplier system on $\Gamma$ of weight $k$. Let $\lambda_{\mathfrak{a}}$ be defined by $\sigma_{\mathfrak{a}}^{-1}\lambda_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$. We say that the cusp $\mathfrak{a}$ is *singular* for $\chi$ if $\chi(\lambda_{\mathfrak{a}}) = 1$.

**Definition 2.2** For $\mathfrak{a}$ and $\mathfrak{b}$ singular cusps for $\chi$, the Kloosterman sum associated to $\mathfrak{a}$, $\mathfrak{b}$ and $\chi$ with modulus $c$ is defined as

$$S_{\mathfrak{a}\mathfrak{b}}(m, n; c; \chi) = \sum_{\gamma=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}/\Gamma_\infty} \chi(\operatorname{sgn}(c))\overline{\chi(\sigma_{\mathfrak{a}}\gamma\sigma_{\mathfrak{b}}^{-1})}e\left(\frac{am + dn}{c}\right). \tag{2.3}$$

*Remarks* This definition of the Kloosterman sum slightly differs from that of [9, (2.23)], in that the roles of $m$ and $n$ are reversed, but agrees with [8, (3.13)]. The presence of $-I \in \Gamma_\infty$ means that the lower-left entry, $c$, is only defined up to $\pm$ sign, and the factor $\chi(\operatorname{sgn}(c))$ accounts for this.

The map $\gamma \to \gamma' = -\gamma^{-1}$ gives a bijection between $\Gamma_\infty\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}/\Gamma_\infty$ and $\Gamma_\infty\backslash\sigma_{\mathfrak{b}}^{-1}\Gamma\sigma_{\mathfrak{a}}/\Gamma_\infty$, fixing the lower-left entry, which implies

$$S_{\mathfrak{a}\mathfrak{b}}(m, n; c; \chi) = \chi(-1)\overline{S_{\mathfrak{b}\mathfrak{a}}(n, m; c; \chi)}. \tag{2.4}$$

This corrects a formula in [9, p. 48] which omitted the complex conjugation.

**Definition 2.3** The set of allowed moduli is defined as

$$\mathcal{C}_{\mathfrak{a}\mathfrak{b}} = \left\{\gamma > 0 : \left(\begin{smallmatrix}* & *\\ \gamma & *\end{smallmatrix}\right) \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}\right\}. \tag{2.5}$$

Notice that if $|\gamma| \notin \mathcal{C}_{\mathfrak{a}\mathfrak{b}}$ then the Kloosterman sum of modulus $\gamma$ is an empty sum.

The definition (2.3) is natural, as it occurs in the Fourier expansion around the cusp $\mathfrak{b}$ of the Poincaré series defined by

$$P_n^{\mathfrak{a}}(z, s; \chi, k) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} j(\sigma_{\mathfrak{a}}^{-1}\gamma, z)^{-k} \overline{\chi(\gamma)} \Im(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s e(n\sigma_{\mathfrak{a}}^{-1}\gamma z), \tag{2.6}$$

where $P_n^{\mathfrak{a}}(z, s; \chi, k)$ is $\Gamma$-automorphic of weight $k$ and nebentypus $\chi$ (to be clear, it transforms by $f(\gamma z) = \chi(\gamma) j(\gamma, z)^k f(z)$). See [8, Section 3.2] for more details.

*Remark* The Kloosterman sum associated to the pair of cusps $\mathfrak{a}, \mathfrak{b}$ depends on the choice of pair of scaling matrices $\sigma_{\mathfrak{a}}$ and $\sigma_{\mathfrak{b}}$ (so it might be better to denote it as $S_{\sigma_{\mathfrak{a}}, \sigma_{\mathfrak{b}}}(m, n; c)$). If one changes the choice of the scaling matrix, the Kloosterman sum also changes by

$$S_{\sigma_{\mathfrak{a}}\left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right), \sigma_b\left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right)}(m, n; c; \chi) = e(-\alpha m + \beta n) S_{\sigma_{\mathfrak{a}}, \sigma_{\mathfrak{b}}}(m, n; c; \chi). \tag{2.7}$$

This corrects a formula of [9, p. 48] which has $\alpha$ in place of our $-\alpha$. Changing the cusps $\mathfrak{a}$ and $\mathfrak{b}$ simultaneously to equivalent ones does not alter the Kloosterman sum, provided one also changes the scaling matrices accordingly. More precisely, if $\mathfrak{a}' = \gamma_1 \mathfrak{a}$ and $\mathfrak{b}' = \gamma_2 \mathfrak{b}$ for $\gamma_1, \gamma_2 \in \Gamma$, then

$$S_{\gamma_1 \sigma_{\mathfrak{a}}, \gamma_2 \sigma_{\mathfrak{a}}}(m, n; c; \chi) = \overline{\chi(\gamma_1)} \chi(\gamma_2) S_{\sigma_{\mathfrak{a}}, \sigma_{\mathfrak{b}}}(m, n; c; \chi).$$

If one applies the Bruggeman–Kuznetsov formula to sums of Kloosterman sums associated to the choice of scaling matrices $\sigma_{\mathfrak{a}}, \sigma_{\mathfrak{b}}$, then the Fourier coefficients at cusps $\mathfrak{a}, \mathfrak{b}$ appearing on the spectral side must also be computed using the same scaling matrices.

## 2.2 Atkin–Lehner cusps and scaling matrices

Assume that $N = rs$ with $(r, s) = 1$. We then call a cusp of the form $\mathfrak{a} = 1/r$ an *Atkin–Lehner cusp*. The Atkin–Lehner cusps are precisely those that are equivalent to $\infty$ under an Atkin–Lehner operator, justifying their name. In Proposition 3.3 below, we calculate the stabilizer of a general cusp $\mathfrak{a}$, which when specialized to an Atkin–Lehner cusp shows that $\Gamma_{1/r}$ is generated by $\pm\left(\begin{smallmatrix} 1-N & s \\ -rN & 1+N \end{smallmatrix}\right)$. In particular we see that an Atkin–Lehner cusp is singular with respect to any Dirichlet character $\chi \pmod{N}$.

**Definition 2.4** Let $N = rs$ with $(r, s) = 1$ as above, and let $W = W_s$ be the Atkin–Lehner operator defined by

$$W = W_s = \begin{pmatrix} xs & y \\ zN & ws \end{pmatrix}, \tag{2.8}$$

with $\det(W_s) = s$. On automorphic forms of (even) weight $\kappa$, the Atkin–Lehner operator is defined by $(f|_W)(z) = \det(W)^{\frac{\kappa}{2}} j(W, z)^{-\kappa} f(Wz)$.

Any two choices of Atkin–Lehner operators $W_s$, for the same value of $s$, differ by an element of $\Gamma_0(N)$. Therefore, if the nebentypus $\chi$ is trivial then $f|_{W_s}$ is independent of the choices of $x, y, z, w$. For general nebentypus, one may ensure that $f|_{W_s}$ is independent of the choices under the assumptions $x \equiv 1 \pmod r$ and $z \equiv 1 \pmod s$.

The matrix can be normalized to have determinant 1, without changing the operator, via

$$\frac{1}{\sqrt{s}} W_s = \begin{pmatrix} x\sqrt{s} & y/\sqrt{s} \\ zr\sqrt{s} & w\sqrt{s} \end{pmatrix} = \begin{pmatrix} x & y \\ zr & ws \end{pmatrix} \begin{pmatrix} \sqrt{s} & 0 \\ 0 & 1/\sqrt{s} \end{pmatrix}.$$

Here the determinant condition is $xws - rzy = 1$. We have the freedom to choose $x = z = 1$, and then if $\overline{s}$ is any integer that satisfies $s\overline{s} \equiv 1 \pmod r$, put $w = \overline{s}$ and $y = (\overline{s}s - 1)/r$. Therefore the matrix

$$\sigma_{1/r} = \tau_r \nu_s \quad \text{with} \quad \tau_r = \begin{pmatrix} 1 & (\overline{s}s - 1)/r \\ r & \overline{s}s \end{pmatrix}, \quad \nu_s = \begin{pmatrix} \sqrt{s} & 0 \\ 0 & 1/\sqrt{s} \end{pmatrix}, \tag{2.9}$$

is an acceptable choice for an Atkin–Lehner operator $W_s$. Note $\tau_r \in \Gamma_0(r)$ (in particular, it has integer entries and determinant 1). From the theory of Atkin–Lehner operators, a newform $f$ of weight 0 and trivial nebentypus will satisfy

$$f(\sigma_{1/r} z) = \eta_s(f) f(z), \tag{2.10}$$

where $f|_{W_s} = \eta_s(f) f$ with $\eta_s(f) = \pm 1$.

One may check directly that $\sigma_{1/r}$ also satisfies the conditions in (2.1), i.e., it is a scaling matrix for the cusp $1/r$ and that $\lambda_{1/r} = \left(\begin{smallmatrix} 1-N & s \\ -rN & 1+N \end{smallmatrix}\right)$. If $f$ is a Maass form satisfying (2.10) then its Fourier coefficients at the cusp $\mathfrak{a}$ with respect to the choice (2.9) for its scaling matrix has the Fourier coefficients

$$\rho_f(\sigma_{1/r}, n) = \pm \rho_{\infty, f}(n).$$

For an application in [11], we need a more general version of this, which takes into account the translates $f|_\ell(z) := f(\ell z)$.

**Lemma 2.5** *Suppose $\mathfrak{a}$ is an Atkin–Lehner cusp of $\Gamma_0(N)$, and $f^*$ is a newform of trivial nebentypus and level $M$ with $LM = N$. Then the set of lists of Fourier coefficients $\{(\rho_{\mathfrak{a} f^*|_\ell}(n))_{n \in \mathbb{N}} : \ell | L\}$ is, up to signs, the same as the set of lists of Fourier coefficients $\{(\rho_{\infty f^*|_\ell}(n))_{n \in \mathbb{N}} : \ell | L\}$.*

*Proof* Suppose that $p$ is prime, $p|N$, $p^\alpha || N$, $p^\beta || L$ (so $p^{\alpha-\beta} || M$), and $p^\gamma || \ell$. Let $W_{p^\alpha}$ be the Atkin–Lehner involution for $\Gamma_0(N)$ for the prime $p$. Then the key fact we need is

$$(f^*|_\ell)|_{W_{p^\alpha}} = \eta_{p^{\alpha-\beta}}(f^*)\, f^*|_{\ell'}, \tag{2.11}$$

where $\ell'$ is defined by $\ell' = p^{\beta-\gamma} h$ where $\ell = p^\gamma h$, (so $(h, p) = 1$). Note that the map $\ell \to \ell'$ is an involution, permuting the divisors of $L$. Taking (2.11) for granted for a moment, we may complete the proof of Lemma 2.5, by noting that the Fourier coefficients of $f^*|_\ell$ at an Atkin–Lehner cusp $\mathfrak{a}$ are equal to the Fourier coefficients of

$(f^*|_\ell)|_{W_D}$ for some Atkin–Lehner involution with $D|N$, which is a composition of $W_{p^\alpha}$'s. The lemma follows from repeated usage of (2.11).

Now we prove (2.11). First suppose $\gamma \le \beta/2$, and let $\ell' = p^{\beta-2\gamma}\ell = p^{\beta-\gamma}h$. Then by [1, Lemma 26],

$$(f^*|_\ell)|_{W_{p^\alpha}} = (f^*|_{W'_{p^{\alpha-\beta}}})|_{\ell'} = \eta_{p^{\alpha-\beta}}(f^*)f^*|_{\ell'}, \tag{2.12}$$

where $W'_{p^{\alpha-\beta}}$ is the Atkin–Lehner involution on $\Gamma_0(M)$ (technically, they worked with holomorphic forms but their proof works equally well for Maass forms). This proves the claim under the condition $\gamma \le \beta/2$. If $\gamma > \beta/2$, then one may reverse the roles of $\ell$ and $\ell'$ and apply $W_{p^\alpha}$ to both sides of (2.12) to give the result. □

### 2.3 Kloosterman sums using Atkin–Lehner scaling

**Proposition 2.6** *Let $N = rs$ with $(r, s) = 1$, and choose $\sigma_{1/r}$ as in (2.9). Then the set of allowed moduli for the pair of cusps $\infty$, $\frac{1}{r}$ is*

$$\mathcal{C}_{\infty,1/r} = \left\{\gamma = c\sqrt{s} > 0 : c \equiv 0 \pmod r,\ (c, s) = 1\right\}, \tag{2.13}$$

*and for such $\gamma = c\sqrt{s} \in \mathcal{C}_{\infty,1/r}$, the Kloosterman sum to modulus $\gamma$ is given by*

$$S_{\infty,1/r}(m, n; c\sqrt{s}) = S(\bar{s}m, n; c), \tag{2.14}$$

*where the $S(a, b; c)$ on the right denotes the ordinary Kloosterman sum.*

*Remark* This is the same Kloosterman sum as [13, (14.8)], but differs from the computation in [8, Sect. 4.2] by the presence of an additive character. This difference is due to the differing choices of the scaling matrices. Motohashi's choice of scaling matrix corresponds to the Atkin–Lehner operators as in Sect. 2.2 above. Proposition 2.6 is a special case of the following more general Theorem 2.7, which evaluates the Kloosterman sum that is associated to the pair of Atkin–Lehner cusps $1/r_1$ and $1/r_2$ and with a character $\chi \pmod N$. The evaluations should prove useful for other works.

**Theorem 2.7** *Let $N = pquv$ with $p, q, u, v$ all pairwise coprime. Put $r_1 = pu$, $s_1 = qv$ and $r_2 = pv$, $s_2 = qu$. The set of allowed moduli for the pair of cusps $1/r_1$, $1/r_2$ is given as*

$$\mathcal{C}_{1/r_1,1/r_2} = \left\{\gamma = c\sqrt{uv} > 0 : c \equiv 0 \pmod{pq},\ (c, uv) = 1\right\}. \tag{2.15}$$

*Let $\chi$ be a Dirichlet character modulo $N$, and factor it as $\chi = \chi_p\chi_q\chi_u\chi_v$, where $\chi_p$ is a character modulo $p$, $\chi_q$ is a character modulo $q$, etc. The Kloosterman sum for this pair of cusps and character $\chi$ with modulus $\gamma = c\sqrt{uv} \in \mathcal{C}_{1/r_1,1/r_2}$ is given as*

$$S_{1/r_1,1/r_2}(m,n;c\sqrt{uv};\chi) = \mathfrak{f}\,\overline{\chi_u}(c)\chi_v(c) \sum_{\substack{a,d \ (\mathrm{mod}\ c)\\ ad\equiv 1 \ (\mathrm{mod}\ c)}} \overline{\chi_p}(d)\overline{\chi_q}(a) e\left(\frac{a\overline{u}\overline{v}m + dn}{c}\right), \tag{2.16}$$

*where*

$$\mathfrak{f} = \mathfrak{f}(p,q,u,v,\chi) = \chi_v(-1)\overline{\chi_p\chi_v}(u)\chi_q\chi_u(v)\chi_u(pq)\overline{\chi_v}(pq). \tag{2.17}$$

*Remarks* One may directly verify that the explicit formula given by (2.16) and (2.17) also satisfies (2.4), which is a nice consistency check. Also, both cusps $1/r_1$, $1/r_2$ are Atkin–Lehner cusps, and hence singular with respect to $\chi$.

*Proof* Our proof closely follows that in [13]. Consider the double coset $\sigma_{1/r_1}^{-1}\Gamma\sigma_{1/r_2}$, and recall the definitions of $\tau_r$ and $\nu_s$ from (2.9). We firstly claim

$$\tau_{r_1}^{-1}\Gamma\tau_{r_2} = \left\{\begin{pmatrix} xv & yq \\ zp & wu \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : x, y, z, w \in \mathbb{Z}\right\}. \tag{2.18}$$

By reducing the entries of the product of matrices implicit in the left-hand side of (2.18) modulo $p$, $q$, $u$, and $v$ respectively, one sees that in each case the lower-left, upper-right, lower-right, and upper-left entries vanish, respectively. Hence the left-hand side of (2.18) is contained in the set on the right-hand side. For the opposite inclusion, we have

$$\tau_{r_1}\begin{pmatrix} xv & yq \\ zp & wu \end{pmatrix}\tau_{r_2}^{-1} = \begin{pmatrix} 1 & \frac{s_1\overline{s_1}-1}{r_1} \\ r_1 & s_1\overline{s_1} \end{pmatrix}\begin{pmatrix} xv & yq \\ zp & wu \end{pmatrix}\begin{pmatrix} s_2\overline{s_2} & \frac{1-s_2\overline{s_2}}{r_2} \\ -r_2 & 1 \end{pmatrix}.$$

Again by reducing modulo $p$, $q$, $u$, and $v$ (the reader may find it easiest to reduce prior to performing matrix multiplication), one obtains an upper triangular matrix in each case, whence the product is an element of $\Gamma_0(N) = \Gamma_0(pquv)$.

Multiplying with the width-normalizing matrices $\nu_s$, we get

$$\sigma_{1/r_1}^{-1}\Gamma\sigma_{1/r_2} = \left\{\begin{pmatrix} x\sqrt{uv} & y/\sqrt{uv} \\ zpq\sqrt{uv} & w\sqrt{uv} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) : x, y, z, w \in \mathbb{Z}\right\}.$$

The determinant condition reads as $xwuv - zpqy = 1$, and for this to be satisfied one needs $(z, uv) = 1$. This shows that (2.15) indeed gives the allowable set of moduli.

Next we wish to decompose this double coset according to the action of $\Gamma_\infty$ on both the left and right, as in [9, Theorem 2.7]. A full set of coset representatives for $\Gamma_\infty\backslash\sigma_{\frac{1}{r_1}}^{-1}\Gamma\sigma_{\frac{1}{r_2}}/\Gamma_\infty$ with a given lower-left entry $zpq\sqrt{uv}$ is given by

$$\begin{aligned} &(\Gamma_\infty\backslash\sigma_{\frac{1}{r_1}}^{-1}\Gamma\sigma_{\frac{1}{r_2}}/\Gamma_\infty) \cap \left\{\left(\begin{smallmatrix} * & * \\ zpq\sqrt{uv} & * \end{smallmatrix}\right)\right\} \\ &\quad = \left\{\begin{pmatrix} x\sqrt{uv} & * \\ zpq\sqrt{uv} & w\sqrt{uv} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) : \begin{matrix} x, w \in (\mathbb{Z}/zpq\mathbb{Z})^* \\ xwuv \equiv 1 \ (\mathrm{mod}\ zpq) \end{matrix}\right\}. \end{aligned} \tag{2.19}$$

Here the condition $xwuv \equiv 1 \pmod{zpq}$ determines $w$ in terms of $x$, and automatically implies $(xw, zpg) = 1$.

Because of the presence of a character, we need to know the lower-right entry of an element of $\Gamma$ in terms of the integers $x, w, z$ from this double coset. Given $\rho = \begin{pmatrix} x\sqrt{uv} & y/\sqrt{uv} \\ zpq\sqrt{uv} & w\sqrt{uv} \end{pmatrix}$, we compute the lower-right entry of $\begin{pmatrix} * & * \\ * & d \end{pmatrix} = \sigma_{\frac{1}{r_1}} \rho \sigma_{\frac{1}{r_2}}^{-1}$ by brute-force calculation as

$$\begin{aligned} d &= \frac{(1 - s_2\overline{s_2})}{r_2}(puxv + s_1\overline{s_1}zp) + puqy + us_1\overline{s_1}w \\ &= (1 - qu\overline{qu})(ux + q\overline{qv}z) + puqy + uqv\overline{qv}w. \end{aligned}$$

Reducing this in each of the moduli $p, q, u, v$, we obtain

$$\begin{aligned} &d \equiv wu \pmod{p}, \quad d \equiv ux \pmod{q}, \\ &d \equiv z\overline{v} \pmod{u}, \quad d \equiv ypuq \equiv -\overline{zpq}puq \equiv -\overline{z}u \pmod{v}. \end{aligned}$$

Alternatively, one may reduce the matrices prior to the matrix multiplication.

The Kloosterman sum is then given by

$$\begin{aligned} &S_{\frac{1}{r_1}, \frac{1}{r_2}}(m, n; zpq\sqrt{uv}; \chi) \\ &= \sum_{\begin{pmatrix} x\sqrt{uv} & * \\ zpq\sqrt{uv} & w\sqrt{uv} \end{pmatrix} \in \Gamma_\infty \backslash \sigma_{\frac{1}{r_1}}^{-1} \Gamma \sigma_{\frac{1}{r_2}} / \Gamma_\infty} \overline{\chi_p}(uw)\overline{\chi_q}(ux)\overline{\chi_u}(z\overline{v})\overline{\chi_v}(-\overline{z}u) e\left(\frac{xm + wn}{zpq}\right). \end{aligned}$$

Using (2.19) and a change of variables $x \mapsto x\overline{uv}$, and with some simplifications, we obtain (2.16). □

**Examples** Specializing Theorem 2.7 to $r_1 = N, r_2 = 1$, we obtain

$$S_{\infty,0}(m, n; c\sqrt{N}; \chi) = \overline{\chi}(c) S(\overline{N}m, n; c), \tag{2.20}$$

with $(c, N) = 1$. More generally, we have

$$S_{\infty,\frac{1}{r}}(m, n; c\sqrt{s}; \chi) = \overline{\chi_r}(s)\chi_s(r)\overline{\chi_s}(c) \sum_{\substack{a,d \pmod{c} \\ ad \equiv 1 \pmod{c}}} e\left(\frac{a\overline{s}m + dn}{c}\right) \overline{\chi_r}(d), \tag{2.21}$$

with $r | c$ and $(c, s) = 1$, and additionally

$$S_{0,\frac{1}{r}}(m, n; c\sqrt{r}; \chi) = \chi_r(-1)\chi_s(r)\overline{\chi_r}(s)\chi_r(c) \sum_{\substack{a,d \pmod{c} \\ ad \equiv 1 \pmod{c}}} e\left(\frac{a\overline{r}m + dn}{c}\right) \overline{\chi_s}(a), \tag{2.22}$$

with $s|c$ and $(c,r)=1$. These formulas should be contrasted with [8, p. 58] or [5, Lemma 4.3], which use a different choice of scaling matrices. In (2.20) the occurrence of the factor $\overline{\chi}(c)$ with the *modulus* of the Kloosterman sum is a nice feature of the pair of cusps $\infty$, 0 as opposed to the case

$$S_{\infty,\infty}(m,n;c;\chi)=\sum_{ad\equiv 1 \pmod c} e\left(\frac{am+dn}{c}\right)\overline{\chi}(d). \tag{2.23}$$

*Remark* A simple application of (2.20) is that for $(a,q)=1$ we have

$$\begin{aligned}\sum_{c\equiv a \pmod q} S(m,n;c)f(c) &= \frac{1}{\phi(q)}\sum_{\chi \pmod q}\chi(a)\sum_{(c,q)=1}\overline{\chi(c)}S(\overline{q}qm,n;c)f(c)\\ &= \frac{1}{\phi(q)}\sum_{\chi \pmod q}\chi(a)\sum_{(c,q)=1}S_{\infty,0}(qm,n;c\sqrt{q};\chi)f(c).\end{aligned} \tag{2.24}$$

The sum over $(c,q)=1$ is the set of all allowed moduli for the group $\Gamma_0(q)$ and the $\infty$, 0 cusp pair, so one may directly apply the Bruggeman–Kuznetsov formula to this type of sum. The analysis of (2.24) was first undertaken by Blomer and Milićević [2]. Their method makes use of $S_{\infty,\infty}(m,n;c;\chi)$, but they require the use of twisted multiplicativity of Kloosterman sums, and the fact that $S_{\infty,\infty}(m,0;c;\chi)$ is a Gauss sum of the character $\chi$ with twist $m$, in order to spectrally decompose (2.24).

## 3 Fourier coefficients of Eisenstein series

### 3.1 Definitions

Let $\mathfrak{c}$ be an arbitrary cusp for $\Gamma_0(N)$. The Eisenstein series attached to $\mathfrak{c}$ (a singular cusp for the nebentypus $\chi$) is defined by

$$E_{\mathfrak{c}}(z,s;\chi)=\sum_{\gamma\in\Gamma_{\mathfrak{c}}\backslash\Gamma}\overline{\chi}(\gamma)\,\mathrm{Im}(\sigma_{\mathfrak{c}}^{-1}\gamma z)^s. \tag{3.1}$$

The Fourier expansion around the cusp $\mathfrak{a}$ takes the form

$$E_{\mathfrak{c}}(\sigma_{\mathfrak{a}}z,u;\chi)=\delta_{\mathfrak{ac}}y^u+\rho_{\mathfrak{ac}}(0,u,\chi)y^{1-u}+\sum_{n\neq 0}\rho_{\mathfrak{ac}}(n,u,\chi)e(nx)W_{0,u-\frac{1}{2}}(4\pi|n|y). \tag{3.2}$$

Consulting [9, Theorem 3.4], we have

$$\rho_{\mathfrak{ac}}(n,u,\chi)=\begin{cases}\phi_{\mathfrak{ac}}(n,u,\chi)\frac{\pi^u}{\Gamma(u)}|n|^{u-1}, & \text{if } n\neq 0\\ \delta_{\mathfrak{ac}}y^u+\phi_{\mathfrak{ac}}(u,\chi)y^{1-u}, & \text{if } n=0,\end{cases} \tag{3.3}$$

where

$$\phi_{\mathfrak{ac}}(n, u, \chi) = \sum_{\substack{(\gamma,\delta) \text{ such that} \\ \rho=\left(\begin{smallmatrix} * & * \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma_\infty \backslash \sigma_{\mathfrak{c}}^{-1}\Gamma\sigma_{\mathfrak{a}}/\Gamma_\infty}} \overline{\chi}(\sigma_{\mathfrak{c}}\rho\sigma_{\mathfrak{a}}^{-1}) \frac{1}{\gamma^{2u}} e\left(\frac{n\delta}{\gamma}\right) = \sum_{\gamma \in \mathcal{C}_{\mathfrak{ca}}} \frac{S_{\mathfrak{ca}}(0, n; \gamma, \chi)}{\gamma^{2u}}, \tag{3.4}$$

and $\phi_{\mathfrak{ac}}(u, \chi) = \phi_{\mathfrak{ac}}(0, u, \chi)$. Note that our ordering of the cusps in the notation $\rho_{\mathfrak{ac}}, \phi_{\mathfrak{ac}}$ is reversed from that of [9], and also that [9, (3.22)] should have $\mathcal{S}_{\mathfrak{ac}}(n, 0; c)$ in place of $\mathcal{S}_{\mathfrak{ac}}(0, n; c)$ to be consistent with [9, (2.23)]. In case $\chi$ is principal, we shall drop it from the notation.

The goal of this section is to evaluate $\phi_{\mathfrak{ac}}(n, u)$ in explicit terms, for which see Theorem 3.4.

### 3.2 Cusps, stabilizers, and scaling matrices

First we write down representatives from the set of $\Gamma$-equivalency classes of cusps. An explicit parametrization may be found in [8, Proposition 2.6]; however, we prefer a different choice as follows.

**Proposition 3.1** *Every cusp of $\Gamma_0(N)$ is equivalent to one of the form $\mathfrak{b} = 1/w$ with $1 \le w \le N$. Two cusps of the form $1/w$ and $1/v$ are equivalent to each other if and only if*

$$(v, N) = (w, N), \quad \textit{and} \quad \frac{v}{(v, N)} \equiv \frac{w}{(w, N)} \left(\operatorname{mod} \left((w, N), \tfrac{N}{(w,N)}\right)\right). \tag{3.5}$$

*A cusp of the form $p/q$ is equivalent to one of the form $1/w$ with $w \equiv p'q \pmod N$ where $p' \equiv p \pmod{(q, N)}$ and $(p', N) = 1$.*

*Proof* Let $\mathfrak{b} = p/q$ be a cusp. We may take $(p, q) = 1$. Using Bezout's lemma choose $a, b \in \mathbb{Z}$ such that $ap + bq = 1$, and $(a, N) = 1$. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. This ensures that $\gamma \cdot \mathfrak{b}$ is a rational number with numerator equal to 1. Replacing $c$ by $c + aNt$ and $d$ with $d + bNt$, we have

$$\begin{pmatrix} a & b \\ c + aNt & d + bNt \end{pmatrix} \cdot \mathfrak{b} = \frac{ap + bq}{(c + aNt)p + (d + bNt)q} = \frac{1}{cp + dq + Nt}.$$

Hence the denominator may be chosen to lie in the interval $[1, N]$. Further note that the denominator is congruent to $dq$ modulo $N$. From

$$d \equiv \overline{a} \pmod N \quad \text{and} \quad a \equiv \overline{p} \pmod q,$$

we deduce that $d \equiv p \pmod{(N, q)}$. Thus we get the last statement in the proposition.

We have established that any cusp is equivalent to one of the form $1/w$ with $1 \leq w \leq N$. Now suppose $1/v$ and $1/w$ are equivalent. Elements of the group $\Gamma$ send relatively prime integer pairs to other such pairs, so if

$$\begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \cdot \frac{1}{w} = \frac{1}{v},$$

then switching the signs on $a, b, c, d$ if necessary, we have

$$a + bw = 1 \quad \text{and} \quad Nc + dw = v. \tag{3.6}$$

The latter equation implies $(v, N) = (dw, N)$. Since $(d, N) = 1$, we get that

$$(v, N) = (w, N). \tag{3.7}$$

The first equation in (3.6) implies $a \equiv 1 \pmod{w}$, which means that $a \equiv 1 \pmod{(w, N)}$. Since the matrix has determinant 1, then $ad \equiv 1 \pmod{N}$, and hence $ad \equiv 1 \pmod{(w, N)}$. Therefore

$$d \equiv 1 \pmod{(w, N)}. \tag{3.8}$$

The second equation in (3.6) gives that $dw \equiv v \pmod{N}$, equivalently

$$d \frac{w}{(w, N)} \equiv \frac{v}{(v, N)} \left( \mathrm{mod}\, \frac{N}{(w,N)} \right).$$

Then (3.8) implies

$$\frac{w}{(w, N)} \equiv \frac{v}{(v, N)} \pmod{\left((w, N), \tfrac{N}{(w,N)}\right)}. \tag{3.9}$$

Thus we have shown if $1/w$ and $1/v$ are equivalent, then (3.5) holds.

Now suppose that $w, v$ satisfy (3.5). Let

$$v' = \frac{v}{(v, N)}, \quad w' = \frac{w}{(w, N)}, \quad N' = \frac{N}{(w, N)}.$$

Then $v' \equiv w' \pmod{(N', (N, w))}$, and $(wv', N') = (w, N') = ((N, w), N')$. Therefore there exist $b, c \in \mathbb{Z}$ so that $v' - w' = bwv' + cN'$. That is, $v - w = bwv + cN$. Define $a = 1 - bw$, $d = 1 + bv$, and let $\gamma = \left(\begin{smallmatrix} a & b \\ cN & d \end{smallmatrix}\right)$, where one may check that $\gamma$ has determinant 1, so $\gamma \in \Gamma_0(N)$. Finally, one may directly verify that $\gamma \cdot \frac{1}{w} = \frac{1}{v}$. □

In the notation of Proposition 3.1, call $(w, N) = f$, and $w = uf$.

**Corollary 3.2** *A complete set of representatives for the set of inequivalent cusps of* $\Gamma_0(N)$ *is given by* $\frac{1}{uf}$ *where* $f$ *runs over divisors of* $N$*, and* $u$ *runs modulo* $(f, N/f)$*, coprime to* $(f, N/f)$*, and where we choose* $u$ *so that* $(u, N) = 1$*, after adding a suitable multiple of* $(f, N/f)$*.*

The cusp 0 is equivalent to $1/1$ and $\infty$ is equivalent to $1/N$, furthermore $1/uf$ is equivalent to the cusp $u/f$ provided $u$ is coprime to $N$.

*Remark* It is not true that cusps of the form $u/f$ and $1/uf$ are always equivalent, even if $(u, f) = 1$. For example, let $N = 72$, and $f = 3$. We have $\frac{2}{3} \nsim_\Gamma \frac{1}{6}$; however, it is true that $\frac{2}{3} \sim_\Gamma \frac{5}{3} \sim_\Gamma \frac{1}{15}$.

We need to compute the stabilizers of various cusps.

**Proposition 3.3** *Let* $\mathfrak{c} = 1/w$ *be a cusp of* $\Gamma = \Gamma_0(N)$*, and set*

$$N = (N, w)N' \quad w = (N, w)w', \quad N' = (N', w)N''. \tag{3.10}$$

*The stabilizer of* $1/w$ *is given as*

$$\Gamma_{1/w} = \left\{ \pm\lambda_{1/w}^t : t \in \mathbb{Z} \right\}, \quad \textit{where} \quad \lambda_{1/w}^t = \begin{pmatrix} 1 - wN''t & N''t \\ -w^2N''t & 1 + wN''t \end{pmatrix}, \tag{3.11}$$

*and one may choose the scaling matrix as*

$$\sigma_{1/w} = \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} \begin{pmatrix} \sqrt{N''} & 0 \\ 0 & 1/\sqrt{N''} \end{pmatrix}. \tag{3.12}$$

*Remark* This is not the choice of scaling matrix we made in Section 2.2 for an Atkin–Lehner cusp. When computing the Fourier coefficients of $E_{\mathfrak{c}}(\sigma_{\mathfrak{a}} z, u)$, with $\mathfrak{a} = 1/r$ Atkin–Lehner and $\mathfrak{c}$ arbitrary, we will choose (3.12) for $\sigma_{\mathfrak{c}}$, and (2.9) for $\sigma_{\mathfrak{a}}$. In addition, one should observe that $N | w^2 N''$ to see that $\lambda_{1/w} \in \Gamma_0(N)$. As an aside, note that $N''$ is the width of the cusp $1/w$.

*Proof* Taking $\gamma = \left(\begin{smallmatrix} a & b \\ cN & d \end{smallmatrix}\right) \in \Gamma_0(N)$ so that $\gamma \cdot \frac{1}{w} = \frac{1}{w}$ means that $a + bw = \pm 1$, and $cN + dw = \pm w$ (with the same choice of $\pm$ sign). Consider the $+$ sign case, the $-$ sign following by symmetry. The former equation determines $a$ in terms of $b$, and the latter is equivalent to $cN' + dw' = w'$. Since $(N', w') = 1$, we have $w' | c$, so define $c = w'r$. Then $d = 1 - rN'$. Finally, the condition that $\gamma$ has determinant 1 means $bw = -rN'$, which means $b = N''t$ (say) and $r = -\frac{w}{(w, N')} t$, giving that $\gamma$ takes the form as stated in (3.11). One may conversely check that any matrix of the form stated in (3.11) stabilizes $1/w$.

To show the final statement, one easily calculates that $\sigma_{1/w}^{-1} \lambda_{1/w} \sigma_{1/w} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, so (2.1) is satisfied. □

### 3.3 Statement of result

We compute the Fourier coefficients of $E_{\mathfrak{c}}(z, u)$ at an Atkin–Lehner cusp $\mathfrak{a} = 1/r$ with $rs = N$, $(r, s) = 1$. Write $\mathfrak{c} = \frac{1}{w} = \frac{1}{vf}$ where $f | N$ and $(v, N) = 1$. Let

$$N' = \frac{N}{f}, \quad N'' = \frac{N'}{(f, N')}, \tag{3.13}$$

and write

$$f_r = (f, r), \quad f_s = (f, s), \quad r = f_r r', \quad s = f_s s'. \tag{3.14}$$

In addition, write

$$f_r = f_r' f_0, \quad \text{where} \quad (f_0, r') = 1, \quad \text{and} \quad f_r' | (r')^\infty,$$

and similarly

$$s' = s'_f s_0, \quad \text{where} \quad (s_0, f_s) = 1, \quad \text{and} \quad s'_f | f_s^\infty.$$

**Theorem 3.4** *Let notation be as above. Then $\phi_{\mathfrak{a}\mathfrak{c}}(n, u) = 0$ unless*

$$n = \frac{f_r'}{(f_r', r')} \frac{s'_f}{(s'_f, f_s)} k,$$

*for some integer $k$. In this case, write $k = k_r k_s \ell$, where*

$$k_r = (k, (f_r', r')), \quad k_s = (k, (s'_f, f_s)).$$

*Then*

$$\phi_{\mathfrak{a}\mathfrak{c}}(n, u) = \frac{S(\ell, 0; s_0 f_0)}{(N'' s f_r^2)^u} \frac{f_r'}{(f_r', r')} \frac{s'_f}{(s'_f, f_s)} \sum_{\substack{d|k \\ (d, f_s r')=1}} d^{1-2u} \frac{1}{\varphi(\frac{(f_r', r')}{k_r})} \frac{1}{\varphi(\frac{(s'_f, f_s)}{k_s})}$$
$$\sum_{\chi \ (\mathrm{mod}\ \frac{(f_r', r')}{k_r})} \sum_{\psi \ (\mathrm{mod}\ \frac{(s'_f, f_s)}{k_s})} \times \frac{(\chi\psi)(\ell)\tau(\overline{\chi})\tau(\overline{\psi})}{L(2u, \overline{\chi^2\psi^2}\chi_0)} (\chi\psi)(\overline{s_0 f_0 d^2 v}) \chi(-k_s \overline{(s'_f, f_s)}) \psi(k_r \overline{(f_r', r')}), \tag{3.15}$$

*where $\chi_0$ is the principal character modulo $f_s r'$.*

The proof of Theorem 3.4 takes up the rest of the paper.

### 3.4 Double cosets

To begin, we obtain an explicit formula for the double coset appearing in (3.4).

**Lemma 3.5** *Let $\mathfrak{c} = 1/w$ be any cusp of $\Gamma = \Gamma_0(N)$ and $\mathfrak{a} = 1/r$ an Atkin–Lehner cusp. Let the scaling matrices be as in (2.9) and (3.12). Then*

$$\sigma_{\mathfrak{c}}^{-1} \Gamma \sigma_{\mathfrak{a}} = \left\{ \begin{pmatrix} \frac{A}{N''} \sqrt{N'' s} & \frac{B}{N''} \sqrt{N''/s} \\ C\sqrt{N'' s} & D\sqrt{N''/s} \end{pmatrix} : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL_2(\mathbb{Z}), \begin{matrix} C \equiv -wA \pmod{r} \\ D \equiv -wB \pmod{s} \end{matrix} \right\}. \tag{3.16}$$

*Proof* Let us call $\tau_{\mathfrak{c}} = \left(\begin{smallmatrix} 1 & 0 \\ w & 1 \end{smallmatrix}\right)$, and $\nu_{\mathfrak{c}} = \begin{pmatrix} \sqrt{N''} & 0 \\ 0 & 1/\sqrt{N''} \end{pmatrix}$, so $\sigma_{\mathfrak{c}} = \tau_{\mathfrak{c}}\nu_{\mathfrak{c}}$. Let $\tau_r\nu_s$ denote the decomposition of the scaling matrix in (2.9). Take $\left(\begin{smallmatrix} a & b \\ Nc & d \end{smallmatrix}\right) \in \Gamma$ and compute

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \tau_{\mathfrak{c}}^{-1} \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \tau_r = \begin{pmatrix} 1 & 0 \\ -w & 1 \end{pmatrix} \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \begin{pmatrix} 1 & \frac{s\bar{s}-1}{r} \\ r & s\bar{s} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \tag{3.17}$$

Note that

$$\nu_{\mathfrak{c}}^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \nu_s = \begin{pmatrix} \frac{A}{N''}\sqrt{N''s} & \frac{B}{N''}\sqrt{N''/s} \\ C\sqrt{N''s} & D\sqrt{N''/s} \end{pmatrix}.$$

Considering the product modulo $r$ gives

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ -w & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} a & * \\ -aw & * \end{pmatrix} \pmod{r},$$

and hence $C \equiv -wA \pmod{r}$. Reducing modulo $s$, we obtain

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ -w & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & -\bar{r} \\ r & 0 \end{pmatrix} \equiv \begin{pmatrix} * & -a\bar{r} \\ * & wa\bar{r} \end{pmatrix} \pmod{s},$$

so $D \equiv -wB \pmod{s}$.

Now we check that given $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ satisfying the conditions in (3.16), then it is covered by the products of the form in (3.17). For that purpose we compute

$$\tau_{\mathfrak{c}} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tau_r^{-1} = \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} s\bar{s} & \frac{1-s\bar{s}}{r} \\ -r & 1 \end{pmatrix}.$$

Modulo $r$, the lower-left entry of this product is congruent to $wA + C \equiv 0 \pmod{r}$, and also congruent to $-Brw - Dr = -r(Bw + D) \equiv 0 \pmod{s}$. This implies that the lower-left entry is divisible by $N$. □

The next step towards evaluation of (3.4) is to work out representatives for $\Gamma_\infty\backslash\sigma_{\mathfrak{c}}^{-1}\Gamma\sigma_{\mathfrak{a}}/\Gamma_\infty$. As a consistency check, note that the action of $\Gamma_\infty$ on both the left and right does not affect the congruences linking $A$ to $C$ and $B$ to $D$, which ultimately follows from $w^2N'' \equiv 0 \pmod{N}$. We need to find the set of pairs $C, D$ with $C > 0$, $(C, D) = 1$, $D \pmod{sC}$, for which there exist integers $A, B$ with $AD - BC = 1$ and so that $C \equiv -wA \pmod{r}$ and $D \equiv -wB \pmod{s}$.

Before stating the result, we develop some notation. Suppose that $A, B, C, D$ are as in the right-hand side of (3.16). Recall the notation from (3.13), (3.14), recall $w = vf$, $(v, N) = 1$, and note $f = f_r f_s$. From $(A, C) = 1$ and $C \equiv -wA \pmod{r}$, we derive $(C, r) = (w, r) = f_r$. Similarly, $(D, s) = (w, s) = f_s$. Write

$$C = f_r C', \quad r = f_r r', \quad D = f_s D', \quad s = f_s s', \tag{3.18}$$

where $(C', r') = 1 = (D', s')$. Then we have

$$\begin{array}{ll} C \equiv -wA \pmod r \\ D \equiv -wB \pmod s \end{array} \iff \begin{array}{ll} A \equiv -\overline{(w/f_r)}C' \pmod{r'} \\ B \equiv -\overline{(w/f_s)}D' \pmod{s'}. \end{array} \tag{3.19}$$

The equivalence of congruences in (3.19) only uses the assumptions $(A, C) = (B, D) = 1$.

From $AD - BC = 1$, we have $D \equiv \overline{A} \pmod{|C|}$, which combined with the right-hand side of (3.19) gives

$$D \equiv -\overline{C'}\frac{w}{f_r} \pmod{(f_r, r')},$$

using $(C, r') = (f_r C', r') = (f_r, r')$. Similarly, we have $B \equiv -\overline{C} \pmod{|D|}$, so $D' \equiv \overline{C}\frac{w}{f_s} \pmod{(f_s, s')}$.

**Lemma 3.6** *With notation as in Lemma 3.5 and its following discussion, we have the disjoint union*

$$\Gamma_\infty \backslash \sigma_{\mathfrak{c}}^{-1}\Gamma\sigma_{\mathfrak{a}}/\Gamma_\infty = \delta_{\mathfrak{c}\mathfrak{a}}\Omega_\infty$$
$$\cup \left\{ \begin{pmatrix} * & * \\ C\sqrt{N''s} & \frac{D}{s}\sqrt{N''s} \end{pmatrix} : \begin{array}{l} (C, r) = f_r \\ (D, s) = f_s \end{array}, \quad \begin{array}{l} D \equiv -\overline{C'}\frac{w}{f_r} \pmod{(f_r, r')} \\ D' \equiv \overline{C}\frac{w}{f_s} \pmod{(f_s, s')} \end{array} \right\}, \tag{3.20}$$

*where in addition we have the restrictions* $C > 0$, $(C, D) = 1$, *and* $D$ *runs over representatives* $\pmod{sC}$. *Here* $\delta_{\mathfrak{c}\mathfrak{a}}\Omega_\infty$ *denotes* $\Gamma_\infty$ *if* $\mathfrak{c}$ *is equivalent to* $\mathfrak{a}$, *and denotes the empty set if* $\mathfrak{c}$ *is not equivalent to* $\mathfrak{a}$.

*Proof* The discussion preceding the statement of the lemma shows that any matrix given on the right-hand side of (3.16) with $C > 0$ gives rise to a double coset of the claimed form. It suffices to show that given integers $C, D$ as in the second line of (3.20), we may find $A, B$ so that $AD - BC = 1$ and satisfying the congruences in (3.19).

From $(C, D) = 1$, we may choose $A_0, B_0$ so that $A_0 D - B_0 C = 1$. From $A_0 \equiv \overline{D} \pmod{C}$, and the congruence on $D$ given in (3.20), we have

$$A_0 \equiv -(\overline{w/f_r})C' \pmod{(f_r, r')}.$$

Hence, there exist integers $x, y$ so that $A_0 + C'\overline{(w/f_r)} = f_r x + r'y$. A similar argument with $B_0$ gives

$$B_0 \equiv -(\overline{w/f_s})D' \pmod{(f_s, s')},$$

and so there exist integers $X, Y$ so that $B_0 + D'\overline{(w/f_s)} = f_s X + s'Y$.

We next want to find $n \in \mathbb{Z}$ so that $A = A_0 + nC = A_0 + nC' f_r$ and $B = B_0 + nD = B_0 + nD' f_s$ satisfies the right-hand side of (3.19). Gathering the above formulas, we have

$$A = A_0 + nC = -\overline{(w/f_r)}C' + f_r(x + nC') + r'y$$
$$B = B_0 + nD = -\overline{(w/f_s)}D' + f_s(X + nD') + s'Y.$$

We may choose $n$ so that $n \equiv -\overline{C'}x \pmod{r'}$ and $n \equiv -\overline{D'}X \pmod{s'}$, since $(C', r') = 1 = (D', s') = (r', s')$, and by the Chinese remainder theorem. With this choice of $n$, then $A$ and $B$ satisfy the congruences on the right-hand side of (3.19). □

Finally we can evaluate $\phi_{\mathfrak{ac}}(n, u)$. First, note that the congruence $D \equiv -\overline{C'}\frac{w}{f_r} \pmod{(f_r, r')}$ is equivalent to $D' \equiv -\overline{C' f_s}\frac{w}{f_r} \pmod{(f_r, r')}$, and that we can write $w = f_r f_s w'$, giving now $D' \equiv -\overline{C'}w' \pmod{(f_r, r')}$. Similarly, the other congruence in (3.20) is equivalent to $D' \equiv \overline{C'}w' \pmod{(f_s, s')}$.

Putting everything together, we have

$$\phi_{\mathfrak{ac}}(n, u) = \frac{1}{(N'' s f_r^2)^u} \sum_{(C', f_s r')=1} \frac{1}{(C')^{2u}} \sum^{*}_{\substack{D' \ (\mathrm{mod}\ s' f_r C') \\ D' \equiv -\overline{C'}w' \ (\mathrm{mod}\ (f_r, r')) \\ D' \equiv \overline{C'}w' \ (\mathrm{mod}\ (f_s, s'))}} e\left(\frac{nD'}{s' f_r C'}\right). \tag{3.21}$$

Now write

$$f_r = f_r' f_0, \quad \text{where} \quad (f_0, r') = 1, \quad \text{and} \quad f_r' | (r')^\infty,$$

and similarly

$$s' = s_f' s_0, \quad \text{where} \quad (s_0, f_s) = 1, \quad \text{and} \quad s_f' | f_s^\infty.$$

Then $(f_r, r') = (f_r', r')$, and $(f_s, s') = (f_s, s_f')$, and so

$$\phi_{\mathfrak{ac}}(n, u) = \frac{1}{(N'' s f_r^2)^u} \sum_{(C', f_s r')=1} \frac{1}{(C')^{2u}} \sum^{*}_{\substack{D' \ (\mathrm{mod}\ s_0 f_0 C' f_r' s_f') \\ D' \equiv -\overline{C'}w' \ (\mathrm{mod}\ (f_r', r')) \\ D' \equiv \overline{C'}w' \ (\mathrm{mod}\ (f_s, s_f'))}} e\left(\frac{nD'}{s_0 f_0 C' f_r' s_f'}\right).$$

By the Chinese remainder theorem, this factors as

$$\phi_{\mathfrak{ac}}(n, u) = \frac{1}{(N'' s f_r^2)^u} \sum_{(C', f_s r')=1} \frac{S(n, 0; C' s_0 f_0)}{(C')^{2u}}$$
$$\times \left( \sum^{*}_{\substack{D' \ (\mathrm{mod}\ f_r') \\ D' \equiv -\overline{C'}w' \ (\mathrm{mod}\ (f_r', r'))}} e\left(\frac{nD'\overline{s_0 f_0 C' s_f'}}{f_r'}\right)\right)$$
$$\times \left( \sum^{*}_{\substack{D' \ (\mathrm{mod}\ s_f') \\ D' \equiv \overline{C'}w' \ (\mathrm{mod}\ (f_s, s_f'))}} e\left(\frac{nD'\overline{s_0 f_0 C' f_r'}}{s_f'}\right)\right).$$

For the sum modulo $f_r'$, note that $(D', f_r') = 1$ if and only if $(D', (f_r', r')) = 1$, since $f_r' | r'^{\infty}$. Therefore, the congruence automatically implies $(D', f_r') = 1$, and so this condition may be dropped. Then after changing variables, it becomes a linear exponential sum which is easy to evaluate. A similar discussion holds for the modulus $s_f'$. In this way, we obtain

$$\phi_{\mathfrak{ac}}(n, u) = \frac{1}{(N'' s f_r^2)^u} \frac{f_r'}{(f_r', r')} \delta\Big(\frac{f_r'}{(f_r', r')} | n\Big) \frac{s_f'}{(s_f', f_s)} \delta\Big(\frac{s_f'}{(s_f', f_s)} | n\Big) \sum_{(C', f_s r')=1} \frac{S(n, 0; C' s_0 f_0)}{(C')^{2u}} e\Big(\frac{-n w' \overline{s_0 f_0 s_f' C'^2}}{f_r'}\Big) e\Big(\frac{n w' \overline{s_0 f_0 f_r' C'^2}}{s_f'}\Big).$$

Now write

$$n = \frac{f_r'}{(f_r', r')} \frac{s_f'}{(s_f', f_s)} k,$$

and note $(\frac{f_r'}{(f_r', r')} \frac{s_f'}{(s_f', f_s)}, C' s_0 f_0) = 1$ giving

$$\phi_{\mathfrak{ac}}(n, u) = \frac{S(k, 0; s_0 f_0)}{(N'' s f_r^2)^u} \frac{f_r'}{(f_r', r')} \frac{s_f'}{(s_f', f_s)} \times \sum_{(C', f_s r')=1} \frac{S(k, 0; C')}{(C')^{2u}} e\Big(\frac{-k w' \overline{s_0 f_0 (s_f', f_s) C'^2}}{(f_r', r')}\Big) e\Big(\frac{k w' \overline{s_0 f_0 (f_r', r') C'^2}}{(s_f', f_s)}\Big).$$

To complete the proof of Theorem 3.4, we perform the following straightforward steps: we evaluate the Ramanujan sum $S(k, 0, C') = \sum_{d|(k,C')} \mu(C'/d) d$ as a divisor sum, and convert from additive to multiplicative characters (see [10, (3.11)]).

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